UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

POSGRADO EN CIENCIAS MATEMÁTICAS

FUNCIONES CARDINALES Y ESPACIOS DE FUNCIONES CONTINUAS

TESIS

QUE PARA OPTAR POR EL GRADO DE:

DOCTOR EN CIENCIAS

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MÉXICO, D.F., SEPTIEMBRE 2013

Este trabajo de tesis doctoral recibió el apoyo económico de CONACyT,
beca 213711, y de PAPIIT-DGAPA-UNAM IN 115312.

Introducción

La obtención del primer cúmulo de resultados en C_p-teoría se remonta a los años 60 y 70 del siglo pasado. En su mayoría pertenecen a especialistas en Análisis Funcional y sirven a propósitos específicos en sus áreas, sin embargo, los topólogos pronto se dieron cuenta de que un buen número de teoremas del Análisis Funcional contenían hechos topológicos no triviales muy interesantes. Como consecuencia de ello, un estudio detallado y sistemático de la esencia topológica de los avances mencionados se llevó a cabo por A.V. Arhangel'skii y su escuela. En efecto, algunos avances fundamentales fueron obtenidos por A.V. Arhangel'skii, S.P. Gul'ko, E.G. Pytkeev, O.G. Okunev, D.P. Baturov, E.A. Reznichenko, V.V. Tkachuk y otros autores. Como consecuencia, una buena cantidad de resultados heterogéneos pudo unificarse para formar una teoría coherente y hermosa, que no sólo ofrece un poderoso apoyo para las necesidades topológicas de otras áreas de las matemáticas, como el Análisis Funcional, la Teoría Descriptiva de Conjuntos y el Álgebra Topológica, sino que además tiene un impresionante potencial de desarrollo interno.

La noción de espacio D fue introducida por E.K. van Douwen [**25**]. Estos espacios han atraído gran atención en los últimos años, sin embargo, varios hechos fundamentales acerca de estos espacios permanecen desconocidos. Un hecho muy interesante, que fue observado en el artículo citado, es el siguiente: "Hasta ahora no es conocido ningún ejemplo satisfactorio de un espacio que no sea D, donde por ejemplo satisfactorio nos referimos a un ejemplo con una propiedad tipo cubierta al menos tan fuerte como metacompacidad o submetacompacidad". Probablemente la mejor pregunta abierta conocida acerca de estos espacios es la siguiente: ¿Es cierto que todo espacio Lindelöf regular es un espacio D?

Los espacios D de E.K. van Douwen han sido objeto de estudio intensivo en los últimos años. Ellos han sido estudiados en casi todos los contextos y la C_p-teoría no ha sido una excepción. La noción de monoliticidad monótona introducida por V.V. Tkachuk ha surgido durante el estudio de la propiedad D en los espacios de funciones.

El propósito de este trabajo es describir el estado actual de la investigación en espacios monotónamente monolíticos, en general y en espacios de funciones, y presentar nuestros resultados en este tópico.

En la primera parte de nuestro trabajo se introduce la topología de la convergencia puntual, describimos algunas construcciones básicas y presentamos algunos resultados conocidos que nos serán útiles. Una referencia muy reciente y completa en C_p-teoría, que recomendamos, es el libro de V.V. Tkachuk [**70**]. Después de esto, se describe de forma breve el estado actual de la investigación en espacios D. Sin embargo, recomendamos el artículo de G. Gruenhage [**36**] que proporciona una revisión completa de lo que sabemos y no sabemos sobre espacios D. Posteriormente, se incluye una sección sobre la propiedad D en espacios metalindelöf, donde incluimos un resultado original junto con su correspondiente demostración. En el resto de la primera parte se describe el estado actual de la investigación en espacios monotónamente monolíticos, incluyendo resultados en espacios monotónamente monolíticos, espacios débilmente monótonamente monolíticos, espacios monótonamente κ-monolíticos, espacios fuertemente monótonamente monolíticos y espacios con la propiedad de Collins-Roscoe. Todos estos resultados muestran que la monoliticidad monótona es una propiedad útil e interesante.

En la segunda parte, presentamos nuestros aportes en este tópico, dando pruebas detalladas de todos los resultados. Introducimos los espacios monótonamente estables. Se introduce la estabilidad monótona y se muestra que existe una dualidad entre la estabilidad monótona y la monoliticidad monótona en los espacios de funciones. Presentamos algunos resultados sobre monoliticidad monótona en espacios de funciones sobre productos y Σ-productos. Por otra parte, se da una generalización de un teorema de O.G. Okunev acerca de los espacios de funciones sobre conjuntos conulos. También, probamos algunos resultados sobre espacios monotónamnete κ-monolíticos e introducimos la noción de espacio montónamente $<\kappa$-monolítico. Algunos resultados sobre espacios monótonamente monolíticos se extienden para espacios monotonamente $<\kappa$-monolíticos y se utilizan para encontrar nuevos espacios con la propiedad D. Al final de esta parte, incluimos algunos resultados acerca de la monoliticidad monótona y la propiedad Collins-Roscoe en Σ_s-productos, que generalizan algunos resultados conocidos. Varias preguntas abiertas se responden usando estos resultados.

Por último, se presenta una lista de problemas sobre la propiedad D, la monoliticidad monótona y los espacios de funciones, que permanecen abiertos (para el conocimiento del autor). Se describe cada problema y se proporcionan las referencias correspondientes.

Contents

CHAPTER I

Preliminaries

1. Terminology and Notation

In notation and terminology we follow [**29**]. Every space in this work is a Tychonoff space with more than one point. The letters α, β and γ represent ordinal numbers and the letters λ and κ represent infinite cardinal numbers; ω is the first infinite cardinal, ω_1 is the first non-countable cardinal. The ordinal number α with its order topology will be symbolized by $[0, \alpha)$. We denote the space $[0, \alpha + 1)$ as $[0, \alpha]$. Also, we denote the ordinal number α with its discrete topology simply as α. For a subset A of a topological space X, $\mathrm{cl}_X(A)$ is the closure of A in X, and $\mathrm{int}_X(A)$ is the interior of A in X. If there is no possibility of confusion, we will write simply $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ instead of $\mathrm{cl}_X(A)$ and $\mathrm{int}_X(A)$.

Given a cardinal number, a subset A of a space X is a set of *type* G_κ if it is the intersection of at most κ open sets in X.

By βX we denote the *Stone-Čech compactification* of a space X. Given an infinite cardinal κ, the *Hewitt* κ*-extension* of X is the space $\upsilon_\kappa X = \{x \in \beta X :$ if F is a G_κ-subset of βX and $x \in F$ then $F \cap X \neq \emptyset\}$. Besides, the Hewitt extension $\upsilon_\omega X$ of the space X is canonically homeomorphic to υX.

Let $\mathcal{E}$ and $\mathcal{N}$ be two families of subsets of X. We say that $\mathcal{N}$ is a *network for* X *modulo* $\mathcal{E}$ if for each $E \in \mathcal{E}$ and each open subset U of X with $E \subset U$, there is $N \in \mathcal{N}$ satisfying $E \subset N \subset U$.

A space X is a *Lindelöf* Σ*-space* if X possesses both a cover $\mathcal{K}$ constituted by compact subsets of X and a countable network $\mathcal{N}$ for X modulo $\mathcal{K}$.

A family $\mathcal{N}$ of subsets of X is a *network* in X if it is a network for X modulo $\{\{x\} : x \in X\}$. A space X is cosmic if it has a countable network.

A function $f : X \to Y$ is called *condensation* if it is a continuous bijection; in this case we say that X condenses onto Y. If X condenses onto a subspace of Y we say that X condenses into Y.

A subspace Y of a space X is *C-embedded* in X if each real-valued continuous function can be extended to a real-valued continuous function on all of X.

The *Lindelöf number* $l(X)$ of a space X is the smallest infinite cardinal κ such that any open cover of X contains a subcover of cardinality at most κ.

The *extent* $e(X)$ of X is the smallest infinite cardinal κ such that the cardinality of every closed discrete subspace of X does not exceed κ.

The *density* $d(X)$ of X is the minimal cardinality of a dense set in X.

iw(X) denotes the minimal cardinality of a space Y onto which X can be condensed. The cardinal invariant iw(X) is called the *i-weight* of X.

The *network weight* nw(X) is the minimal cardinality of a network in X.

The *tightness* $t(X)$ of a space X is the smallest infinite cardinal κ such that for any set $A \subset X$ and any point $x \in \mathrm{cl}(A)$, there exists a set $B \subset A$ with $|B| \leq \kappa$ and $x \in \mathrm{cl}(B)$.

For a cardinal invariant ϕ we define the new cardinal invariant ϕ^* as follows: $\phi^*(X) = \sup\{\phi(X^n) : n \in \omega \setminus \{0\}\}$.

Suppose that $\eta = \{X_t : t \in T\}$ is a family of topological spaces, $X = \prod\{X_t : t \in T\}$ is the topological product of the family η, and x^* is a point in X. Then the Σ-*product* of η with basic point x^* is the subspace of X consisting of all points $x \in X$ such that only countably many coordinates $x(\alpha)$ of x are distinct from the corresponding coordinates $x^*(\alpha)$ of x^*. This subspace is denoted by $\Sigma\{X_t : t \in T\}$ or by $\Sigma\eta$. Sometimes $\Sigma\eta$ will be called the Σ-product of η at x^*. Similarly, the σ-*product* of η with basic point x^* is the subspace of X consisting of all points $x \in X$ such that only finitely many coordinates $x(a)$ of x are distinct from the corresponding coordinates $x^*(a)$ of x^*. This subspace is denoted by $\sigma\{X_t : t \in T\}$ or by $\sigma\eta$. Sometimes $\sigma\eta$ will be called the σ-product of η at x^*. Note that the basic point is usually not shown in the notation.

A collection $\mathcal{U}$ of subsets of X is said to be *point-countable* if each point $x \in X$ is an element of at most countably many members of $\mathcal{U}$.

ω_1 is a *caliber* of a space X if every point-countable family of non-empty open subsets of X is countable.

For a space X, $\mathcal{CL}(X)$ denotes the family of all non-empty closed subsets of X, and $\mathcal{K}(X)$ denotes the family of all non-empty compact subsets of X.

For those concepts and notations which appear without definition, consult [**29**] and [**70**].

2. Function spaces

The creation of the C_p-theory or theory of function spaces endowed with the topology of pointwise convergence must be attributed to Alexander Vladimirovich Arhangel'skii. A.V. Arhangel'skii was the first to understand the need to unify and classify a bulk of heterogeneous results from topological algebra, functional analysis and general topology. He was also the first to obtain crucial results that made this unification possible and the first to formulate a critical mass of open problems which showed this theory's huge potential for development. Later, many mathematicians worked hard to contribute to C_p-theory giving it the elegance and beauty that it boasts today.

In this section we introduce the topology of pointwise convergence in spaces of continuous functions. Also, we present some basic constructions and results in function spaces that are well known and will be useful later. All the proofs can be found in [**3**].

For all spaces X and Y we denote by $C(X,Y)$ the set of all continuous functions from X to Y. We set $C(X) = C(X,\mathbb{R})$. By $C_p(X,Y)$ we denote the set $C(X,Y)$ endowed with the topology of pointwise convergence. The standard base of the space $C_p(X,Y)$ consists of all the sets of the form:

$$[x_1,\dots,x_k : U_1,\dots,U_k] = \{f \in C(X,Y) : f(x_i) \in U_i, i = 1,\dots,k\},$$

where $x_1,\dots,x_k$ are elements of X, $U_1,\dots,U_k$ are open sets in Y and $k \in \omega$. Notice that the topology of pointwise convergence in $C(X,Y)$ is the topology induced from the topological product Y^X.

We denote by $C_p(X)$ the space $C_p(X,Y)$. Clearly, the family of all the sets of the form:

$$[f, x_1,\dots,x_k, \epsilon] = \{g \in C(X) : |g(x_i) - f(x_i)| < \epsilon, i = 1,\dots,k\},$$

where $f \in C(X)$, $x_1,\dots,x_k$ are elements of X and ϵ is a positive real number, is a base for the space $C_p(X)$.

Given a space X let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for all $n \in \omega$, i.e., $C_{p,n}(X)$ is the nth iterated function space of X.

Let Y be a subspace of X. By π_Y we denote the function from $C_p(X)$ to $C_p(Y)$ which restricts each element in $C_p(X)$ to Y; that is, $\pi_Y(f) = f \restriction Y$. The function π_Y is called the *restriction function*, and between its properties we have the following:

Proposition 2.1. *For every subspace Y of X the following hold:*

(1) the function π_Y is continuous and $\mathrm{cl}(\pi_Y(C_p(X))) = C_p(Y)$;
(2) if Y is closed in X, then π_Y is an open function from $C_p(X)$ onto the subspace $\pi_Y(C_p(X))$ of $C_p(Y)$;
(3) if Y is compact, then $\pi_Y(C_p(X)) = C_p(Y)$;
(4) if X is normal and Y is closed in X, then $\pi_Y(C_p(X)) = C_p(Y)$;
(5) if Y is everywhere dense in X, then $\pi_Y : C_p(X) \to \pi_Y(C_p(X))$ is a condensation.

Let $f : X \to Y$ be a function between the sets X and Y. The dual function $f^* : \mathbb{R}^Y \to \mathbb{R}^X$ is defined as follows: if $g \in \mathbb{R}^Y$, then $f^*(g) = g \circ f$. The function f^* is called the *dual function* and it has the the following properties:

Proposition 2.2. *Let f be a function from a set X to a set Y, then:*

(1) f^* is a continuous function;
(2) If $f(X) = Y$, then $f^* : \mathbb{R}^Y \to \mathbb{R}^X$ is a homeomorphism from $\mathbb{R}^Y$ onto the closed subspace $f^*(\mathbb{R}^Y)$ of $\mathbb{R}^X$.

Proposition 2.3. *Let $f : X \to Y$ a function from X onto Y, then:*

(1) f is continuous if and only if $f^*(C_p(Y)) \subset C_p(X)$;
(2) if f is a quotient map, then $f^*(C_p(Y))$ is a closed subspace of $C_p(X)$;
(3) f is a condensation if and only if $f^*(C_p(Y))$ is everywhere dense in $C_p(X)$;
(4) f is a homeomorphism if and only if $f^*(C_p(Y)) = C_p(X)$.

Let $f : X \to Y$ be a map from a topological space X onto a set Y. Then the strongest of all completely regular topologies on Y relative to which f is continuous is called the $\mathbb{R}$-*quotient topology* on the set Y. A function from a space X onto a space Y is called an $\mathbb{R}$-*quotient function*, if the topology on Y coincides with the $\mathbb{R}$-quotient topology generated by f.

The statement in 2) Proposition 2.3 has no immediate converse, however, we have the following result.

Proposition 2.4. *A function f from a space X onto a space Y is an $\mathbb{R}$-quotient map if and only if $f^*(C_p(Y))$ is a closed subspace of $C_p(X)$.*

Suppose we are given a set X and a family $\mathcal{F} \subset \mathbb{R}^X$. The function $\Delta\mathcal{F} : X \to \mathbb{R}^{\mathcal{F}}$ given by $[\Delta\mathcal{F}(x)](f) = f(x)$ for each $x \in X$ and $f \in \mathcal{F}$ is called the *diagonal of the family* $\mathcal{F}$. Clearly, if X is a topological space and each $f \in \mathcal{F}$ is a continuous function, then $\Delta\mathcal{F}$ is a continuous function.

Let $\psi = \Delta C_p(X)$. The following results are well known.

Proposition 2.5. *For any space X:*

(1) $\psi(X) \subset C_p(C_p(X))$;

(2) ψ *embeds* X *in* $C_p(C_p(X))$ *as a closed subspace.*

The following results show the C_p-duality between some cardinal functions.

Theorem 2.6. *For any space X:*

(1) $\mathrm{nw}(X) = \mathrm{nw}(C_p(X))$;

(2) $d(X) = \mathrm{iw}(C_p(X))$;

(3) $\mathrm{iw}(X) = d(C_p(X))$.

3. Monolithic and stable spaces in C_p-duality

The concepts of monolithic space and stable space were introduced by A.V. Arhangel'skii in [**4**]. Given an infinite cardinal number κ, a space X is called κ-*monolithic* if $\mathrm{nw}(\mathrm{cl}(A)) \le \kappa$ for every subset A of X of cardinality at most κ. In particular, X is ω-monolithic if the closure of every countable set in X is a space with countable network.

A space X is called *monolithic* if it is κ-monolithic for every cardinal κ, i.e., if for each $A \subset X$ we have $\mathrm{nw}(\mathrm{cl}(A)) \le \max\{|A|, \omega\}$.

A separable space of uncountable network weight (in particular, the Sorgenfrey line) is an example of a space that is not ω-monolithic.

Example 3.1. The following spaces are monolithic:

(1) metrizable spaces;
(2) cosmic spaces;
(3) Σ-products of spaces with a countable base.

The union of two monolithic spaces need not be monolithic. The Nyemitskii plane may serve as an example. It is separable but has no countable network, and hence is not ω-monolithic. At the same time it is the union of two metrizable (hence monolithic) subspaces, one of which is closed and discrete while the other is an open set of type F_σ.

Proposition 3.2. *(1) The union of a locally finite family of closed monolithic subspaces is monolithic;*
(2) the property of (κ-) monolithicity is inherited by arbitrary subspaces;
(3) the product of a countable family of monolithic spaces is monolithic;
(4) every closed and continuous image of a monolithic space is monolithic.

Given an infinite cardinal κ, a space X is called *κ-stable* if for every continuous image Y of X the following conditions are equivalent:

(1) $\mathrm{iw}(Y) \leq \kappa$;
(2) $\mathrm{nw}(Y) \leq \kappa$.

It is well known that $\mathrm{iw}(Y) \leq \mathrm{nw}(Y)$ for every space Y. The converse does not always hold, an example of this is the Sorgenfrey line. A discrete space of cardinality $\mathfrak{c}$ is not ω-stable: it can be condensed onto the space $\mathbb{R}$, which has a countable base.

A space X is called *stable* if it is κ-stable for every infinite cardinal κ. It can be easily seen that X is stable if and only if for every continuous image Y of X we have $\mathrm{iw}(Y) = \mathrm{nw}(Y)$.

Theorem 3.3. *(1) Every compact space is stable;*
(2) every Lindelöf Σ-space is stable;
(3) any product of second countable spaces is stable;
(4) every Σ-product of second countable spaces is stable;
(5) each σ-product of second countable spaces is stable;
(6) any pseudocompact space is ω-stable;
(7) any Lindelöf P-space is ω-stable.

Proposition 3.4. *(1) Every space X is κ-stable for $\kappa \geq \mathrm{nw}(X)$;*
(2) every continuous image of a stable space is stable;
(3) stability is inherited by open-closed subspaces;
(4) the union of a countable set of stable subspaces is stable.

There exists [**32**] a countably compact space X whose square contains an open-closed uncountable discrete subspace Y. Then X is ω-stable, while Y is not. Hence, (see Proposition 3.4, (3)) $X \times X$ is not ω-stable, i.e., ω-stability is not preserved, in general, under the transition of a square of a space.

A discrete space D of cardinality $\mathfrak{c}$ is not ω-stable, but is homeomorphic to a closed subspace of $\mathbb{R}^{\mathfrak{c}}$. Thus, stability is not inherited, in general, by closed subspaces (see Theorem 3.3, (3)).

Theorem 3.5. *If the Hewitt realcompactification υX of a space X is an ω-stable space, then X is ω-stable.*

At first glance, monolithicity and stability appear to be rather unrelated to each other. The following propositions indicate that there is a strong relation between them:

Theorem 3.6. *$C_p(X)$ is κ-monolithic if and only if X is κ-stable.*

Theorem 3.7. *$C_p(X)$ is κ-stable if and only if X is κ-monolithic.*

Corollary 3.8. *For any space X:*

(1) X is monolithic if and only if $C_p(X)$ is stable;
(2) X is stable if and only if $C_p(X)$ is monolithic;
(3) $C_p(C_p(X))$ is monolithic if and only if X is monolithic;
(4) X is stable if and only if $C_p(C_p(X))$ is stable.

4. D-property

A *neighborhood assignment* for a space (X, τ) is a function $\phi : X \to \tau$ with $x \in \phi(x)$ for every $x \in X$. X is said to be a *D-space* if for every neighborhood assignment N, one can find a closed discrete subspace D of X such that $X = \bigcup\{\phi(x) : x \in D\}$.

The notion of a D-space seems to have had its origins in an exchange of letters between E.K. van Douwen and E. Michael in the mid-1970s, but the first paper on D-spaces is a 1979 paper by E.K. van Douwen and W. Pfeffer [**25**]. The D-property is a kind of covering property; it is easily seen that compact spaces and also σ-compact spaces are D-spaces, and that any countably compact D-space is compact. Part of the fascination with D-spaces is that, aside from these easy facts, very little else is known about the relationship between the D-property and many of the standard covering properties. For example, it is not known if a very strong covering property such as hereditarily Lindelöf implies D, and yet for all we know it could be that a very weak covering property such as submetacompact or submetalindelöf implies D. While these questions about covering properties remain unsettled, there nevertheless has been quite a lot of interesting recent work on D-spaces.

The space $[0, \omega_1)$ being countably compact and not compact is not a D-space. It is an open question whether the union of two arbitrary D subspaces is a D-space. Since every discrete space is a D-space, clearly, this property is not preserved by continuous functions. Also, there exist [**23**] a Lindelöf D-space B and a metrizable separable space M (hence a D-space) such that $B \times M$ is not D, so D-property is not preserved, in general, by finite products. Some facts about D-spaces are the following [**12**]:

Theorem 4.1. *(1) Every closed subspace of a D-space is a D-space;*
(2) the closed continuous image of a D-space is a D-space;
(3) the perfect inverse image of a D-space is a D-space.

Recall that the *extent*, $e(X)$ of a space X is the supremum of the cardinalities of its closed discrete subsets, and the *Lindelöf degree*, $l(X)$, is the least cardinal κ such that every open cover of X has a subcover of cardinality κ. Note that $e(X) \leq l(X)$ for any X. It is easy to see that if X is a D-space and $\mathcal{U}$ is an open cover with no subcover of cardinality $< \kappa$, then there must be a closed discrete subset of X of size κ; hence $e(X) = L(X)$. Since closed subspaces of D-spaces are D, we have the following result:

Proposition 4.2. *If X is a D-space and Y is a closed subspace of X, then $e(Y) = l(Y)$.*

Corollary 4.3. *Any countably compact D-space is compact.*

A useful example to know, because it illustrates some of the limits of which properties could imply D, is an old example due to E.K. van Douwen and H. Wicke. There is a space Γ [**24**] which is non-Lindelöf and has countable extent, and is Hausdorff, locally compact, locally countable, separable, first countable, submetrizable, σ-discrete, and realcompact. This space is not D because $e(X) = \omega < l(X)$.

For a time, all known examples of non-D-spaces failed to be D because the conclusion of Proposition 4.2 failed. This was noted by W. Fleissner, essentially bringing up the question: Is X a D-space if and only if $e(Y) = l(Y)$ for every closed subspace Y of X? R. Buzyakova [**16**] asked a related question: Is X hereditarily D if and only if $e(Y) = l(Y)$ for every $Y \subset X$? After some consistent examples P. Nyikos [**47**] found a ZFC example of a non-D-space in which $e(Y) = l(Y)$ for every subspace Y, closed or not.

As we have mentioned, it is clear that compact spaces, in fact σ-compact spaces, are D-spaces. Beyond this it often appears that some base or completeness "structure" is needed in order to prove certain spaces are D-spaces. The following result shows how connections between D-spaces and generalized metric spaces have been extensively studied.

Theorem 4.4. *The following are D-spaces:*

(1) Menger spaces [**8**]*;*
(2) semistratifiable spaces (and hence Moore, semimetric, stratifiable, and σ-spaces) [**12**] *(see also* [**30**]*);*
(3) subspaces of symmetrizable spaces [**15**]*;*
(4) strong Σ-spaces (hence paracompact p-spaces) [**18**]*;*
(5) protometrizable spaces (hence nonarchimedean spaces) [**13**]*;*
(6) spaces having a point-countable base [**6**]*;*
(7) spaces having a point-countable weak base [**15**] *(see also* [**52**]*);*
(8) sequential spaces with a point-countable w-system [**15**] *or spaces with a point-countable k-network* [**51**]*;*
(9) spaces with an ω-uniform base [**8**]*;*
(10) base-base paracompact spaces (hence totally paracompact spaces) [**59**] *(see also* [**57**]*);*
(11) t-metrizable spaces [**39**]*;*

(12) spaces having a σ-cushioned (mod k) pair-network (hence $\Sigma^\sharp$-spaces) [**46**]*;*

(13) spaces satisfying well-ordered (A), linearly semistratifiable spaces, and elastic spaces [**67**].

Lin's result (item (12)) simultaneously generalizes the statements "semistratifiable implies D" and "strong Σ implies D". The result of D. Soukuop and X. Yuming (item (13)) about elastic spaces is an explanation of "protometrizable implies D", and their result about linearly semistratifiable spaces generalizes "semistratifiable implies D". Aurichi's result about Menger (item (1)) spurred much activity, in spite of the fact that it could be considered a corollary of the previously known result that totally paracompact spaces are D (item (10)).

Unfortunately, we still know very little about how the D-property affects a space. Thus, more statements in the form "If X has the D-property then ..." would improve our understanding of this puzzling notion.

Finally, for completeness we present the definitions of the above concepts.

We say that a space X is a *Menger space* if for every sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers there is $\{\mathcal{V}_n : n \in \omega\}$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup\{\bigcup V_n : n \in \omega\}$.

A space X is *semistratifiable* if to each open set U in X we can assign a sequence $\{F(U, n) : n \in \omega\}$ of closed sets in X such that $U = \bigcup\{F(U, n) : n \in \omega\}$ and $F(U, n) \subset F(V, n)$ whenever $U \subset V$.

Suppose X is a topological space and $d : X \times X \to [0, \infty)$ such that, for all $(x, y) \in X \times X$, $d(x, y) = d(y, x)$ and $d(x, y) = 0$ if and only if $x = y$. The function d is said to be a *symmetric* for X provided: for all nonempty $A \subset X$, A is closed in X if and only if $\inf\{d(x, z) : z \in A\} > 0$ for every $x \in X \setminus A$. In this case, one could say X is *symmetrizable* with symmetric d.

A space X is a *strong* Σ space if there exist a σ-locally finite family $\mathcal{N}$ of closed sets in X and a cover $\mathcal{K}$ of X by compact subsets, such that $\mathcal{N}$ is a network for X modulo $\mathcal{K}$.

A base $\mathcal{B}$ for a space X is said to be an *orthobase* if whenever $\mathcal{B}'$ is a subset of $\mathcal{B}$, either $\bigcap \mathcal{B}'$ is open, or $\mathcal{B}'$ is a local base for any point in $\bigcap \mathcal{B}'$. A space X is said to be *proto-metrizable* if it is paracompact and has an orthobase.

A *weak base* for a space X is a collection of subsets $\mathcal{B} = \{\mathcal{B}_x : x \in X\}$ where, for all $x \in X$, $x \in \bigcap \mathcal{B}_x$, $\mathcal{B}_x$ is closed under finite intersections and $\mathcal{B}$ determines the topology on X in the following way: A set $U \subset X$ is open in X if and only if for all $z \in U$, there exists $B \in \mathcal{B}_z$ with $B \subset U$.

A collection $\mathcal{N}$ of subsets of X is a *k-network* if $\mathcal{N}$ is a network for X modulo the family $\mathcal{K}(X)$ of all nonempty compact subsets of X.

Let X be a sequential space. If $x \in W \subset X$ we say W is a *weak neighborhood* of x if whenever $\{x_n : n \in \omega\}$ converges to x then $\{x_n : n \in \omega\}$ is eventually in W, that is, $|\{x_n : n \in \omega\} \setminus W| < \omega$. A collection $\mathcal{W}$ of subsets of X is said to be a *w-system* for the topology if whenever $x \in U \subset X$, with

U open, there exists a subcollection $\mathcal{V} \subset \mathcal{W}$ such that $x \in \bigcap \mathcal{V}$, $\bigcup \mathcal{V}$ is a weak neighborhood of x, and $\bigcup \mathcal{V} \subset U$

We say that a base $\mathcal{B}$ for a topological space X is *ω-uniform* if for every $x \in X$ and every $B \in \mathcal{B}$ such that $x \in B$, the set $\{A \in \mathcal{B} : x \in A \text{ and } A \not\subset B\}$ is countable.

A space X *is base-base paracompact* if X has an open base $\mathcal{B}$ such that every base $\mathcal{B}' \subset \mathcal{B}$ contains a locally finite subcover.

A space (X, τ) is *t-metrizable* if there exists a metrizable topology π on X with $\tau \subset \pi$ and an assignment $H \to J_H$ from $[X]^{<\omega}$ to $[X]^{<\omega}$ such that $\mathrm{cl}_\tau(A) \subset \mathrm{cl}_\pi(\bigcup\{J_H : H \in [A]^{<\omega}\})$.

A collection $\mathcal{P}$ of pairs of subsets of a space X is called a *pair-network* for X if whenever $x \in U$ with U open in X, $x \in P_1 \subset P_2 \subset U$ for some $(P_1, P_2) \in \mathcal{P}$. A collection $\mathcal{P}$ of pairs of subsets of a space X is called *cushioned* if for each $\mathcal{P}' \subset \mathcal{P}$ we have $\mathrm{cl}(\bigcup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}) \subset \mathrm{cl}(\bigcup\{P_2 : (P_1, P_2) \in \mathcal{P}'\})$.

A collection $\mathcal{P}$ of pairs of subsets of a space X is called a *$(modk)$-network for* X if, there is a cover $\mathcal{K}$ of compact subsets of X such that, whenever $K \in \mathcal{K}$ and $K \subset U$ with U open in X, then $K \subset P_1 \subset P_2 \subset U$ for some $(P_1, P_2) \in \mathcal{P}$.

Let X be a space. A collection $\mathcal{P}$ of ordered pairs $P = (P_1, P_2)$ of subsets of X is called a *pair-base* provided that P_1 is open for all $P \in P$ and that for every $x \in X$ and open set U containing x, there is a $P \in \mathcal{P}$ such that $x \in P_1 \subset P_2 \subset U$.

Let X be a space, α an ordinal. We say that X satisfies (αA) if and only if there is $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where $\mathcal{W}(x) = \{W(\beta, x) : \beta < \alpha\}$, such that $x \in W(\beta, x) \subset X$ with the following property. For every open U containing x, there exists an open set $V(x, U)$ containing x and an ordinal $\beta = \varphi(x, U) < \alpha$ such that $x \in W(\beta, y) \subset U$ for all $y \in V(x, U)$. If, in addition, $W(\beta, x) \subset W(\gamma, x)$ whenever $\gamma < \beta < \alpha$, then we say that X satisfies *well-ordered* (αA).

X is said to be *semistratifiable over* α (for some ordinal α) or *linearly semistratifiable* if there exists a mapping $F : \alpha \times \tau(X) \to \mathcal{CL}(X)$ such that:

(1) $U = \bigcup\{F(U, \beta) : \beta < \alpha\}$ for all $U \in \tau(X)$;
(2) if $U \subset W$ then $F(U, \beta) \subset F(W, \beta)$ for all $\beta < \alpha$;
(3) if $\gamma < \beta < \alpha$, then $F(U, \gamma) \subset F(U, \beta)$ for all $U \in \tau(X)$.

A space X is *elastic* if there is a pair-base $\mathcal{P}$ on X and transitive relation $\leq$ on $\mathcal{P}$ such that

(1) if $P, P' \in P$ are such that $P_1 \cap P_1' \neq \emptyset$ then $P \leq P'$ or $P' \leq P$;
(2) if $P \in \mathcal{P}$ and $\mathcal{P}' \subset \{P' \in \mathcal{P} : P' \leq P\}$ then $\mathrm{cl}(\bigcup\{P_1' : P' \in \mathcal{P}'\}) \subset \bigcup\{P_2' : P' \in \mathcal{P}'\}$.

5. D-property and metalindelöf spaces

In their article in Open Problems in Topology II, M. Hrušák and J.T. Moore [**41**] list twenty central problems in set theoretic topology; the question whether or not Lindelöf spaces are D is Problem 15 on this list, and is attributed to E.K. van Douwen. T. Eisworth's survey article [**28**], in the same volume, lists several related questions. In fact, it is not known if any of the following covering properties, even if you add "hereditarily", imply D: Lindelöf, paracompact, ultraparacompact, strongly paracompact, metacompact, metalindelöf, subparacompact, submetacompact, submetalindelöf, paralindelöf, screenable, σ-metacompact (see [**14**] for definitions). Since the non-D-space Γ, mentioned in the previous section, is σ-discrete, it follows that weakly submetacompact does not imply D.

We know that every compact T_1-space is a D-space. Recently, D.T. Soukup and P.J. Szeptycki have shown that under $\diamondsuit$ there is a Hausdorff hereditarily Lindelöf space which is not D [**66**]. However, it is not known whether every regular Lindelöf space is D. In this section, we show that if X is a metalindelöf locally Lindelöf D-space, then X is Lindelöf, this result was proved independently by L.-Xue Peng and Hui Li [**56**].

The following notation will be used throughout: If ϕ is a neighborhood assignment on X, and $D \subset X$, we let $\phi(D) = \bigcup\{\phi(x) : x \in D\}$.

A space X is *metalindelöf* if every open cover of X has point-countable open refinement.

W. Fleissner and A. Stanley [**30**] introduced the notion of ϕ-sticky for a neighborhood assignment ϕ, a tool which simplifies many D-space arguments. G. Gruenhage [**35**] introduced the notions of nearly good relation and ϕ-sticky modulo a relation as tools for proving that spaces have the D-property.

Let X be a space. We say that a relation R on X (respectively, from X to $[X]^{<\omega}$) is *nearly good* if $x \in \mathrm{cl}(A)$ implies $xR\,y$ for some $y \in A$ (respectively, $xR\,\tilde{y}$ for some $\tilde{y} \in [A]^{<\omega}$).

Further, if ϕ is a neighborhood assignment on X, $X' \subset X$, and $D \subset X$, we say *D is ϕ-sticky mod R on X'* if whenever $x \in X'$ and $xR\,y$ for some $y \in D$ (respectively, $xR\,\tilde{y}$ for some $\tilde{y} \in [D]^{<\omega}$), then $x \in \phi(D)$. (In other words, it means that $\phi(D)$ contains all the "relatives" of members (respectively, finite subsets) of D that are in X.) We say more briefly that D is ϕ-sticky mod R if D is N-sticky mod R on X.

The following proposition was essentially proved in [**35**].

Proposition 5.1. *Let ϕ be a neighborhood assignment on X, and let R be a nearly good relation on X. Suppose that given any countable closed discrete D and nonempty closed $F \subset X \setminus \phi(D)$ such that D is ϕ-sticky mod R on F, there is a countable non-empty closed discrete $E \subset F$ such that $D \cup E$ is ϕ-sticky mod R on F. Then there is a closed discrete D^* in X with $\phi(D^*) = X$.*

Lemma 5.2. *If X has a point-countable open cover $\mathcal{U}$ such that $\overline{U}$ is Lindelöf D for each $U \in \mathcal{U}$ then X is a D-space.*

PROOF. Let $\phi : X \to \tau(X)$ be a neighborhood assignment on X. We define a relation R on X as follows:

$$R = \{(x, y) \in X \times X : \{x, y\} \subset U \text{ for some } U \in \mathcal{U}\}.$$

Clearly, R is nearly good. We will show that conditions in Proposition 5.1 are satisfied. Let D be a countable closed discrete subspace of X and let $F \subset X \setminus \phi(D)$ be a nonempty closed set such that D is ϕ-sticky mod R on F. Take a family $\{\Omega_n : n \in \omega\}$ of infinite disjoint subsets of ω such that $\omega = \bigcup\{\Omega_n : n \in \Omega\}$ and $\{0, \ldots, n\} \subset \bigcup\{\Omega_k : k = 1, \ldots, n\}$. We will construct a set E in a recursive process.

Step 0. Let $V^* \in \mathcal{U}$ such that $V^* \cap F \setminus \phi(D) \neq \emptyset$. Let $Y_0 = \mathrm{cl}(V^*) \cap F \setminus \phi(D)$. Since D-property is inherited by closed subspaces, then Y_0 is Lindelöf D. Let E_0 be a countable closed and discrete subspace of Y_0 such that $Y_0 \subset \phi(E_0)$. Since $\mathcal{U}$ is point-countable, then $\mathcal{U}_0 = \{U \in \mathcal{U} : U \cap E_0 \neq \emptyset\}$ is countable. Let $\{U_m : m \in \Omega_0\}$ be a numeration of $\mathcal{U}_0$.

Suppose that for each $k \leq n$ we have constructed countable closed sets $E_k \subset F$ in such form that

1. the families $\mathcal{U}_k = \{U \in \mathcal{G} : U \cap E_k \neq \emptyset\}$ are numbered by $\{U_m : m \in \Omega_k\}$;
2. $\bigcup\{U_i \cap F : i < k\} \subset \phi(\bigcup\{D \cup E_i : i \leq k\})$;
3. $E_k \subset F \setminus \phi(\bigcup\{D \cup E_i : i < k\})$.

Step $n+1$. If for each open set U in the family $\bigcup\{\mathcal{U}_k : k \leq n\}$ we have $U \cap F \subset \phi(\bigcup\{D \cup E_k : k \leq n\})$, take $E = \bigcup\{E_k : k \leq n\}$ and stop the construction. It is not difficult to verify that in this case E satisfies the required conditions. In the other case, let U^* be the first set in $\bigcup\{\mathcal{U}_k : k \leq n\}$ which satisfies $U^* \cap F \setminus \phi(\bigcup\{D \cup E_k : k \leq n\}) \neq \emptyset$. Let $Y_{n+1} = \mathrm{cl}(U^*) \cap F \setminus \phi(\bigcup\{D \cup E_k : k \leq n\})$, then Y_{n+1} is Lindelöf D. Let E_{n+1} be a countable closed and discrete subset of Y_{n+1} such that $Y_{n+1} \subseteq \phi(E_{n+1})$. Since $\mathcal{U}$ is point-countable, then $\mathcal{U}_{n+1} = \{U \in \mathcal{U} : U \cap E_{n+1} \neq \emptyset\}$ is countable. Let $\{U_m : m \in \Omega_{n+1}\}$ be a numeration of $\mathcal{U}_{n+1}$. Notice that by condition (2) in step n, the election of U^* and the fact that $\{0, \ldots, n\} \subset \bigcup\{\Omega_k : k = 1, \ldots, n\}$, we have $\bigcup\{U_i : i < n+1\} \subset \phi(\bigcup\{E_i : i \leq n+1\})$. Therefore (1),(2) and (3) hold in step $n+1$.

In order to finish the construction let $E = \cup\{E_n : n \in \omega\}$. We will show that E satisfies the conditions in Proposition 5.1. Clearly, $E \subset F$. Let $W = \bigcup\{\bigcup\mathcal{U}_n : n \in \omega\} = \bigcup\{U_k : k \in \omega\}$. For each $k \in \omega$, by condition (2) in step $k+1$, $U_k \cap F \subset \phi(E)$. Therefore $W \cap F \subset \phi(E)$. Suppose that $x \in F$ and xRy for some $y \in D \cup E$. If $y \in D$, since D is ϕ-sticky mod R on F, we have $x \in \phi(D) \subset \phi(D \cup E)$. If $y \in E$ then $\{x, y\} \in U \cap F = U_k \cap F$ for some $k \in \omega$. Therefore, $y \in U_k \cap F \subset \phi(E) \subset \phi(D \cup E)$. This shows that $D \cup E$ is ϕ-sticky mod R on F. We shall prove that E is a closed and discrete subset of X.

Let $x \in \mathrm{cl}(E) \subset F$. Since R is nearly good, then xRy for some $y \in E$. Since $D \cup E$ is ϕ-sticky mod R on F, then $x \in \phi(D \cup E) \cap F$. Notice that

$x \notin \phi(D)$. Let n be the first natural number such that $x \in \phi(\bigcup\{D \cup E_k : k \leq n\})$, then by condition (3), $x \in E_n$. Since $E_1 \cup ... \cup E_n$ is closed and discrete we can find an open neighborhood V_x of x with $V_x \cap (E_1 \cup \ldots \cup E_n) = \{x\}$, then by construction, $U_x = V_x \cap \phi(E_1 \cup ... \cup E_n)$ is an open neighborhood of x with $U_x \cap E = \{x\}$. This shows that E is closed and discrete.

Finally, by Proposition 5.1, there exists a closed and discrete subspace D^* of X such that $N(D^*) = X$. Therefore, X is a D-space. □

A space X is called *locally Lindelöf D* if every point x of X has a neighborhood U_x which is a Lindelöf D-space.

Theorem 5.3. *If X is a metalindelöf locally Lindelöf D-space, then X has the D-property.*

Proof. For each $x \in X$ let U_x be an open neighborhood of x such that $\overline{U}_x$ is Lindelöf D. If $\mathcal{U}$ is a point-countable open refinement of $\{U_x : x \in X\}$ then $\mathcal{U}$ satisfies the hypothesis of proposition 5.2. Hence X is D. □

Some spaces which have certain covering properties and scattered property are D-spaces.

Let $\mathbb{K}$ denote a class of spaces which are hereditary with respect to closed subspaces. A space X is called a $\mathbb{K}$-*scattered* space if for any nonempty closed subset F of X, there is some $x \in F$ such that the point x has a neighborhood U_x in the subspace F such that $U_x \in K$.

The class of all D-spaces is denoted by $\mathbb{D}$ and the class of all compact spaces is denoted by $\mathbb{C}$. Clearly, every $\mathbb{D}$-scattered space is $\mathbb{C}$-scattered space and any scattered space is $\mathbb{D}$-scattered. Every submetacompact $\mathbb{D}$-scattered space is a D-space [**54**]. Thus, every submetacompact $\mathbb{C}$-scattered space is D. However, it is not known whether every regular metalindelöf scattered space is a D-space.

In [**53**] L.-Xue Peng, independently, gave a proof of Lemma 5.2 and used this to prove the following:

Theorem 5.4. *Let X be a metalindelöf space.*

(1) If X is $\mathbb{C}$-scattered space with a finite rank, then X is a D-space.
(2) If X is $\mathbb{D}$-scattered space with locally countable extent, then X is a D-space.

It is proved in [**9**] that every T_1 Lindelöf space which is the union of less than $cov(\mathcal{M})$ compact subsets is a D-space, where $\mathcal{M}$ is the ideal of meager subsets of the real line. Since $MA + \neg CH$ implies $\omega_1 < \mathfrak{c} = cov(\mathcal{M})$, it is consistent that every T_1 Lindelöf space of cardinality ω_1 is a D-space. G. Gruenhage [**36**] asked if it is consistent that every paracompact space of cardinality ω_1 is a D-space. Recently Hang Zhang and Wei-Xue Shi have proved that:

Theorem 5.5 ($MA + \neg CH$). [**79**] *Every regular T_1 submetalindelöf space of cardinality ω_1 is D.*

6. D-property in function spaces

Recall that for a compact space X, Baturov's theorem [**10**] states that $l(Y) = e(Y)$ for every subspace Y of $C_p(X)$. M. Matveev asked whether the conclusion in the Baturov theorem for compacta can be strengthened to the D-property. R. Buzyakova answered this question; indeed; she proved the following result [**16**]:

Theorem 6.1. *Let X be compact and $Y \subset C_p(X)$. Then Y is a D-space.*

Later, another proof was obtained by H. Guo and H.J.K. Junnila. They proved that t-metrizable spaces are hereditarily D-spaces (see Theorem 4.4 (11)). Since $C_p(X)$ is t-metrizable for compact K (indeed, the sup-norm topology π and the pointwise topology τ of $C(X)$ witness the t-metrizability of $C_p(X)$, see [**26**]), Buzyakova's result is a consequence of Guo and Junnila's result.

As we have seen, $l(X) = e(X)$ for every D-space X and every countably compact D-space is compact. This simple observation leads us to the following corollaries of Theorem 6.1, which are, in fact, well known classical theorems.

Corollary 6.2 (Baturov's Theorem for compacta). *Let X be compact. Then $l(Y) = e(Y)$ for every subspace Y of $C_p(X)$.*

Corollary 6.3 (Grothendieck's Theorem for Compacta [**34**]). *Let X be compact and Y a countably compact subspace of $C_p(X)$. Then Y is compact.*

Since Baturov's theorem holds not only for compacta but for all Lindelöf Σ-spaces, R. Buzyakova supposed that the following question might have a chance for an affirmative answer (This question was also suggested by M. Matveev, see [**35**]): Let X be a Lindelöf Σ space, it is true that $C_p(X)$ is a hereditary D-space?

In [**35**] G. Gruenhage introduced the concept of nearly good relation and ϕ-sticky modulo a relation (see §5). He proved the next result, using elementary submodels.

Given a neighborhood assignment ϕ on X, let us call a subset Z of X *ϕ-close* if for every $x, y \in Z$ we have $x \in \phi(y)$ (equivalently, $Z \subset \phi(x)$ for every $x \in Z$).

Proposition 6.4. *Let ϕ be a neighborhood assignment on X. Suppose there is a nearly good R on X (respectively, from X to $[X]^{<\omega}$) such that for any $y \in X$ (respectively, $\tilde{y} \in [X]^{<\omega}$), $R^{-1}(y) \setminus \phi(y)$ (respectively, $R^{-1}(\tilde{y}) \setminus \bigcup \phi(\tilde{y})$) is the countable union of ϕ-close sets. Then there is a closed discrete D such that $\phi(D) = X$.*

Remark 6.5. Note that if ϕ and R satisfy the hypotheses of the proposition, then so does their restriction to any subspace. So, if for any ϕ on X we can produce such an R, then X is hereditarily D.

Using Proposition 6.4, G. Gruenhage [**35**] gave a short proof of Theorem 4.4 (6) and Theorem 6.1. Also, using Proposition 6.4, answered the above question posed by M. Matveev by proving that:

Proposition 6.6. *Let X be a Lindelöf Σ space. Then $C_p(X)$ is hereditarily D.*

In another direction, R. Buzyakova proposed the next question [**16**]: Let X be a countably compact space, is it true that every subspace of $C_p(X)$ is a D-space? In [**19**], she described a counterexample. She considered the space $X = \{\alpha \leq \omega_2 : \mathrm{cf}(\alpha) \neq \omega_1\}$ and proved that $l(C_p(X)) = \omega_2$ while $e(C_p(X)) = \omega$. This example also answers Reznichenko's question (whether Baturov's theorem holds for countably compact spaces) in the negative.

R. Buzyakova also proved that:

Theorem 6.7. [**19**] *$C_p(X)$ is Lindelöf for any first countable countably compact subspace of an ordinal.*

Since the question whether or not Lindelöf spaces are D remains open even in the class of C_p-spaces, in search of a counterexample (if there exists one) R. Buzyakova proposed the following question: Is $C_p(X)$ a D-space for any first countable countably compact subspace of an ordinal? L.-Xue Peng answered this question by proving the following result.

Theorem 6.8. [**52**] *Suppose that X is a first countable countably compact subspace of an ordinal. Then $C_p(X)$ is a D-space.*

V.V. Tkachuk also solved the above question with an interesting approach.

Recall that a space X is *Sokolov* if for any sequence $\{F_n : n \in \omega\}$ where F_n is a closed subset of X^n for every $n \in N$, there exists a continuous map $f : X \to X$ such that $\mathrm{nw}(f(X)) = \omega$ and $f^n(F_n) \subset F_n$ for all $n \in \omega$.

V.V. Tkachuk proved the following result and showed that his technique provides a different method to prove Theorem 6.8.

Theorem 6.9. [**74**] *Suppose that X is a countably compact first countable subspace of an ordinal. Then X is a Sokolov space.*

It is known that [**72**]: (1) For any Sokolov first countable space X, the space $C_p(X, E)$ is Lindelöf whenever E is a second countable space. (2) The countable power of a Sokolov space is a Sokolov space. (3) If X is Sokolov, then $C_{p,n}(X)$ is also Sokolov for any $n \in \omega$. (4) For any Sokolov first countable space X, the space $C_{p,n}(X)$ is Lindelöf for any $n \in \omega$. Thus, we obtain the following consequences of Theorem 6.9 which are explanations of Theorem 6.7.

Corollary 6.10. [**74**] *If X is a first countable countably compact subspace of an ordinal, then $(C_p(X))^\omega$ and $C_p(X^\omega)$ are Lindelöf spaces.*

Corollary 6.11. [**74**] *If X is a first countable countably compact subspace of an ordinal, then the iterated function space $C_{p,2n+1}(X)$ is Lindelöf for any $n \in \omega$.*

7. Monotone monolithicity

Monotone (covering) properties have been intensively studied in several contexts. Monotone normality, due to R.W. Heath, D.J. Lutzer, and P. Zenor, is a classical property in the study of generalized metric spaces. Stratifiable spaces, introduced by J. Ceder [**20**], are precisely the class of monotonically perfectly normal spaces. Monotonically compact and monotonically Lindelöf spaces were defined by M. Matveev and were first studied in print by H. Bennett, D. Lutzer and M. Matveev in [**11**]. P.M. Gartside and P.J. Moody [**31**] defined monotonically paracompact spaces and showed that they are exactly the class of protometrizable spaces. S.G. Popvassilev [**58**] introduced the notion of monotonically (countably) metacompact space. Monotonic versions of countable paracompactness and countable metacompactness, that are quite different in spirit from the monotonic properties before mentioned, were introduced independently in [**33**], [**50**], and [**69**].

There is an interesting relation between these properties and D-spaces. As we saw in Theorem 4.4, monotonically perfectly normal (stratifiable) spaces and monotonically paracompact (protometrizable) spaces are D-spaces. Recently, in [**56**], it was proved that every monotonically (countably) metacompact space is hereditarily a D-space. On the other hand, it is an open question if every paracompact monotonically normal space is a D-space [**36**]. This section is devoted to monotone monolithicity, a relatively new monotone property. Of course, we have that any monotonically monolithic space is hereditarily a D-space.

V.V. Tkachuk, introduced the concept of monotonically monolithic space.

Definition 7.1. [**77**] Given a set A in a space X we say that a family $\mathcal{N}$ of subsets of X is an *external network* of A in X if for any $x \in A$ and an open set $U \subset X$ with $x \in U$ there exists $N \in \mathcal{N}$ such that $x \in N \subset U$.

Definition 7.2. [**77**] Say that a space X is *monotonically monolithic* if, for any $A \subset X$ we can assign an external network $\mathcal{O}(A)$ to the set $\mathrm{cl}(A)$ in such a way that the following conditions are satisfied:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if α is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of X such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{O}(\bigcup A_\beta : \beta < \alpha) = \bigcup\{\mathcal{O}(A_\beta) : \beta < \alpha\}$.

The class of monotonically monolithic spaces seems to be interesting in itself. This class has nice categorical properties and contains some important classes of topological spaces.

Theorem 7.3. *Monotone monolithicity is preserved by:*

(1) countable products [**77**]*;*
(2) σ-products [**1**]*;*

(3) arbitrary subspaces [**77**]*;*
(4) closed maps [**77**].

Proposition 7.4. *The following spaces are monotonically monolithic:*
(1) spaces with a unique non-isolated point [**77**]*;*
(2) spaces with a point-countable base [**77**]*;*
(3) t-metrizable spaces [**39**]*;*
(4) stratifiable spaces [**38**].

Among the main results obtained in [**77**] we found the following.

Proposition 7.5. *$C_p(X)$ is monotonically monolithic for X Lindelöf Σ.*

Theorem 7.6. *Any monotonically monolithic space is hereditarily D.*

Notice that Proposition 7.5 is a generalization of Proposition 6.6. From Proposition 7.4 we have, in particular, every metrizable space is monotonically monolithic. Therefore, the class of monotonically monolithic spaces is relatively large. Also, it is easy to see from the definition that cosmic spaces are monotonically monolithic. Notice that every monotonically monolithic space is monolithic. However, not every compact monolithic space is monotonically monolithic; indeed, the space $[0, \omega_1]$ is compact and monolithic but not monotonically monolithic; in fact, its subspace $[0, \omega_1)$ is not D (see Theorem 7.6). Uncountable products of monotonically monolithic spaces not need be monotonically monolithic; $\mathbb{R}^{\omega_1}$ is not monotonically monolithic since $[0, \omega_1)$ embeds as a subspace of it.

We now give a characterization of monotonically monolithic spaces which has served to prove several interesting results.

Theorem 7.7. [**38**] *A space X is monotonically monolithic if and only if one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each $A \subset X$, the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ contains a network for* $\mathrm{cl}(A)$.

It is well known and easy to prove that any compact space of countable tightness is a Fréchet-Urysohn space. The following results show that in the presence of monotonic monolithicity this result can be strengthened.

Theorem 7.8. [**77**] *If a countably compact space X is monotonically monolithic then X is a Fréchet-Urysohn compact space.*

It follows from the previous result and Corollary to 3.25 in [**42**] that any compact monotonically monolithic space is separable, therefore:

Corollary 7.9. [**77**] *If a compact space X is monotonically monolithic and ω_1 is a caliber of X then X is metrizable.*

There exists a Lindelöf monotonically monolithic space of uncountable tightness. Indeed, let L_{ω_1} be the one-point Lindelöfication of a discrete space of cardinality ω_1, then L_{ω_1} is monotonically monolithic by Proposition 7.4 (1) and it is easy to see that it is a Lindelöf space of uncountable tightness.

8. Weakly monotone monolithic spaces

L.-Xue Peng generalized the concept of monotonically monolithic space. He introduced the concept of weakly monotonically monolithic space and show that every such space has the D-property.

Definition 8.1. [**55**] We say that a space X is *weakly monotonically monolithic* if for any $A \subset X$ we can assign an external network $\mathcal{O}(A)$ of A in such a way that the following conditions are satisfied:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if α is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of X such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{O}(\bigcup A_\beta : \beta < \alpha) = \bigcup\{\mathcal{O}(A_\beta) : \beta < \alpha\}$;
(4) If $A \subset X$ is not closed in X then there is some $x \in \mathrm{cl}(A) \setminus A$ such that $\mathcal{O}(A)$ is an external network of $\{x\}$.

Theorem 8.2. [**55**] *If X is a weakly monotonically monolithic space, then X is a D-space.*

G. Gruenhage observed that weakly monotone monolithicity has a characterization similar to Theorem 7.7.

Theorem 8.3. [**38**] *A space X is weakly monotonically monolithic if and only if one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each non-closed set $A \subset X$, there is some $x \in \mathrm{cl}(A) \setminus A$ such that $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is a network for $\{x\}$.*

Let us recall that a family $\mathcal{N}$ of subsets of X is a cs^**-network* of X, if for any sequence $\{x_n : n \in \omega\}$ which converges to a point x and for any open set U which contains x, there is some $N \in \mathcal{N}$ such that $x \in N \subset U$ and $|\{n \in \omega : x_n \in N\}| = \omega$.

Notice that every k-network of X (see §4) is a cs^*-network of X.

Suppose that $\mathcal{N}$ is a point-countable cs^*-network of X. For any $A \subset X$, we let $\mathcal{O}(A) = \{N \in \mathcal{N} : N \cap A \neq \emptyset\}$. If X is sequential, we see that $\mathcal{O}(A)$ satisfies the conditions which appear in Definition 8.1. Thus:

Proposition 8.4. [**55**] *If X is a sequential space with a point-countable cs^*-network, then X is a weakly monotonically monolithic space.*

Corollary 8.5. [**55**] *If X is a sequential space with a point-countable cs^*-network, then X is a D-space.*

Since every k-network of X is a cs^*-network, the second part of Theorem 4.4 (8) is a consequence of Corollary 8.5.

In [**38**] G. Gruenhage used Theorem 7.7, Proposition 6.4 and Remark 6.5 to obtain a quick proof that monotonically monolithic spaces are hereditarily D. We will describe his proof because it is short and elegant: Let X be monotonically monolithic witnessed by operator $\mathcal{N}$ as in Theorem 7.7, and let ϕ be a neighborhood assignment on X. Define a relation R from X to $[X]^{<\omega}$, as

$$x\, R\, F \text{ if and only if } x \in N \subset \phi(x) \text{ for some } N \in \mathcal{N}(F).$$

Since $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ contains a network for $\mathrm{cl}(A)$, it is straightforward to check that this R is nearly good. Now fix $F \in [X]^{<\omega}$. Let $N(x)$ denote any $N \in \mathcal{N}(F)$ such that $x \in N \subset \phi(x)$ (if such N exists). Then $X(N) = \{x \in X : x\, R\, F$ and $N(x) = N\}$ is ϕ-close, and $R^{-1}(F) = \bigcup\{X(N) : N \in \mathcal{N}(F)\}$. So $R^{-1}(F)$ is a countable union of ϕ-close sets, hence, X and R satisfy the hypothesis in Proposition 6.4 and hence X is hereditarily D (see also Remark 6.5).

Let us call a relation R on X (resp., from X to $[X]^{<\omega}$) *nearly OK* if A non-closed implies $x\, R\, y$ for some $x \in \mathrm{cl}(A) \setminus A$ and some $y \in A$ (resp., $x\, R\, \tilde{y}$ for some $x \in \mathrm{cl}(A) \setminus A$ and some $\tilde{y} \in [A]^{<\omega}$).

G. Gruenhage [**38**] introduced the previous concept and described a slight modification of the above argument to give a quick proof of Theorem 8.2.

9. Monotone κ-monolithicity

In [**1**] O. Alas, V.V. Tkachuk and R.G. Wilson continued the study undertaken in [**77**]. They introduced the notion of monotone κ-monolithicity for any infinite cardinal κ and showed that some theorems that were proved for monotonically monolithic spaces also hold for monotonically κ-monolithic spaces.

Definition 9.1. Given an infinite cardinal κ we say that a space X is *monotonically κ-monolithic* if, for any $A \subset X$ with $|A| \leq \kappa$ we can assign an external network $\mathcal{O}(A)$ to the set $\mathrm{cl}(A)$ in such a way that the following conditions are satisfied:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if $\lambda < \kappa$ is a cardinal and we have a family $\{A_\alpha : \alpha < \lambda\} \subset [X]^{\leq\kappa}$ such that $\alpha < \beta < \lambda$ implies $A_\alpha \subset A_\beta$ then $\mathcal{O}(\bigcup A_\alpha : \alpha < \lambda) = \bigcup\{\mathcal{O}(A_\alpha) : \alpha < \lambda\}$.

The following proposition shows some categorical behavior of monotone κ-monolithicity.

Theorem 9.2. [**1**] *Monotone κ-monolithicity is preserved by:*
(1) countable products;
(2) σ-products;
(3) arbitrary subspaces;
(4) closed maps.

In [**1**] an example of a linearly orderable space which is monotonically ω-monolithic but not monotonically ω_1-monolithic was presented (the subspace of $[0, \omega_2)$ of all ordinals of uncountable cofinality). On the other hand, it is easy to see that any monotonically monolithic space is monotonically κ-monolithic for any infinite cardinal κ. We can see that the converse is true.

Proposition 9.3. [**1**] *A space is monotonically monolithic if and only if it is monotonically κ-monolithic for any infinite cardinal κ.*

In particular, all the spaces which appear in Proposition 7.4 are monotone κ-monolithic for any cardinal κ.

As well as weakly monotonic monolithicity, monotonic κ-monolithicity has a characterization similar to Theorem 7.7.

Theorem 9.4. [**56**], [**62**], [**75**] *A space X is monotonically κ-monolithic if and only if one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each $A \subset X$ with $|A| \leq \kappa$, $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ contains a network for* $\mathrm{cl}(A)$.

In some cases κ-monolithicity implies monolithicity. It is a well known fact that an ω-monolithic space of countable tightness is monolithic. In [**56**], [**62**] and [**75**] it was proved independently (using Theorem 9.4) that an analogous result holds for monotonic κ-monolithicity.

Proposition 9.5. *If X is a monotonically κ-monolithic space such that $t(X) \leq \kappa$, then X is monotonically monolithic.*

Let X be a space such that its Hewitt realcompactification υX is ω-stable. By Theorem 3.5 the space X is itself ω-stable and by Theorem 3.6 the space $C_p(X)$ is ω-monolithic. The following result shows that a similar result holds for monotone κ-monolithicity (see Theorem 3.3 (2)).

Theorem 9.6. [**1**] *If X is a space such that its Hewitt κ-extension is Lindelöf Σ, then $C_p(X)$ is monotonically κ-monolithic.*

It is well known [**29**] and easy to proof that if X is a pseudocompact space then the Hewitt realcompactification υX of X and the Stone-Čech compactification βX of X coincide. Thus, $\upsilon X = \beta X$ is Lindelöf Σ for a pseudocompact space X. Also, it is known that υX is Lindelöf Σ whenever $\upsilon(C_p(X))$ is Lindelöf Σ (see Theorem IV 9.5 in [**3**]). Thus, we have:

Corollary 9.7. [**1**] *If X is pseudocompact then $C_p(X)$ is monotonically ω-monolithic.*

Corollary 9.8. [**1**] *If $\upsilon C_p(X)$ is Lindelöf Σ then $C_p(X)$ is monotonically ω-monolithic.*

Since ω-monolithic spaces are more widely used that monolithic ones, it is natural in the study of monotone monolithicity that for the results on ω-monolithicity, there are some strong analogies for its stronger version.

Theorem 9.9. [**1**] *Suppose that a monotonically ω-monolithic space X is the union of at most ω_1-many cosmic spaces. Then X is monotonically monolithic.*

Proposition 9.10. [**75**] *If X is a monotonically ω-monolithic perfectly normal compact space, then X is metrizable.*

Proposition 9.11. [**1**] *If a countably compact space X is monotonically ω-monolithic, then X is compact and has the Fréchet-Urysohn property.*

Notice that Proposition 9.11 is an explanation of Theorem 7.8. In [**1**] an example of a monotonically ω_1-monolithic pseudocompact space which is not compact was presented. Also, an example of a compact scattered monolithic space such that $C_p(X)$ is Lindelöf and X is hereditarily D-space, but such that X is not monotonically ω-monolithic, was presented.

In general, the fact that a space X has a weaker monotonically monolithic topology does not imply that X must be κ-monolithic. To see that, observe that the Sorgenfrey line is not ω-monolithic while it condenses onto a monotonically monolithic space $\mathbb{R}$. Therefore, even the hereditary Lindelöf property does not help to "lift" monolithicity under a condensation. The situation is different if we look at Lindelöf Σ-spaces. Indeed, suppose that $f : X \to Y$ is a condensation and Y is κ-monolithic. If $A \subset X$ and $|A| \leq \kappa$, then $\mathrm{cl}(A)$ is a Lindelöf Σ space which condenses into the space $Z = \mathrm{cl}(f(A))$ with $\mathrm{nw}(Z) \leq \kappa$, so we can apply stability of $\mathrm{cl}(A)$ to conclude that $\mathrm{nw}(\mathrm{cl}(A)) \leq \kappa$, i.e., X is κ-monolithic. Therefore:

Proposition 9.12. [**75**] *If a Lindelöf Σ-space X condenses onto a monolithic space, then X is monolithic.*

The same results hold if we replace monolithicity by monotone monolithicity.

Proposition 9.13. [**75**] *If a Lindelöf Σ-space X condenses onto a monotonically monolithic space, then X is monotonically monolithic.*

10. The Collins-Roscoe property

In [**22**], Collins and Roscoe investigated some conditions for metrizability. In that article they introduced the following condition called *condition (G)* that has been intensively studied in different contexts:

(G) To each $x \in X$, is assigned a countable collection $\mathcal{G}(x)$ of subsets of X such that, whenever $x \in U$, where U is an open subset of X, there is an open V with $x \in V \subset U$ such that, whenever $y \in V$, then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$.

If in addition all elements in $\mathcal{G}(y)$ are open, we say that X satisfies *open* (G). G. Gruenhage observed that condition (G) is equivalent to the following concept that recently has been called the Collins-Roscoe property.

Definition 10.1. A space X has the *Collins-Roscoe property* if for each $x \in X$, one can assign a countable collection $\mathcal{G}(x)$ of subsets of X such that, for any $A \subset X$, $\bigcup\{\mathcal{G}(x) : x \in A\}$ contains an external network for $\mathrm{cl}(A)$.

Since the Collins-Roscoe property was introduced in the study of metrizability, it is interesting to know the relation between this property and covering properties. In [**45**] the following result was proved.

Proposition 10.2. *If the space X has the Collins-Roscoe property, then X is hereditarily metalindelöf.*

It follows immediately from the definition and Theorem 7.7 that every space with the Collins-Roscoe property is monotonically monolithic. G. Gruenhage (see [**35**]) asked if the Collins-Roscoe property is equivalent to monotonic monolithicity, and suggested that $C_p(X)$ for some Lindelöf Σ-space X might be a place to look for an example distinguishing the two concepts (see Proposition 7.5).

V.V. Tkachuk [**78**] has shown that $C_p(\beta D)$ does not have the Collins-Roscoe property whenever D is an uncountable discrete space (but it is monotonically monolithic). Indeed, if D_{ω_1} is a discrete space of cardinality ω_1, then it is easy to see that D_{ω_1} is a continuous image of D and hence $C_p(D_{\omega_1})$ is homeomorphic to a closed subspace of $C_p(D)$ (see Proposition 2.4). A. Dow, H. Junnila, J. Pelant [**27**] proved that $C_p(D_{\omega_1})$ is not metalindelöf; as an immediate consequence, the space $C_p(D)$ is not metalindelöf either. However, the Collins-Roscoe property in a space implies that it is metalindelöf (see Proposition 10.2), so the space $C_p(D)$ does not have this property.

In [**78**] Collins-Roscoe spaces were studied systematically; let us formulate some categorical properties of the class of Collins-Roscoe spaces.

Theorem 10.3. *Collins-Roscoe property is preserved by:*

(1) *arbitrary subspaces* [**22**];
(2) *countable products* [**78**];
(3) *closed maps* [**78**];
(4) *σ-products* [**78**].

Since the Collins-Roscoe property is stronger than monotone monolithicity, it is natural to expect that some questions which are open for monotone monolithicity could be solved positively for the spaces with the Collins-Roscoe property. Also, in the presence of the Collins-Roscoe property several results which are true for monotonically monolithic spaces can be strengthened. Now we present some results of this nature.

V.V. Tkachuk gave a generalization of Theorem 9.9 by showing that:

Theorem 10.4. [**78**] *Suppose that a monotonically ω-monolithic space X is the union of at most ω_1-many cosmic spaces. Then X has the Collins-Roscoe property and, in particular, X is hereditarily metalindelöf.*

Corollary 10.5. [**78**] *Assume that X is a monotonically ω-monolithic space, $d(X) \leq \omega_1$ and $t(X) \leq \omega$. Then X has the Collins-Roscoe property and, in particular, X is hereditarily metalindelöf.*

It is still an open question [**1**] whether any monotonically monolithic space X has to be cosmic when ω_1 is a caliber of X. It turns out that a positive answer can be given for the spaces with the Collins-Roscoe property (see also Corollary 7.9).

Theorem 10.6. [**78**] *If a space X has the Collins-Roscoe property and ω_1 is a caliber of X, then X is cosmic.*

It turns out that, as monotonic monolithicity, the Collins-Roscoe property is also "lifted" by condensations of Lindelöf Σ-spaces (see Proposition 9.13).

Proposition 10.7. [**75**] *Suppose that a Lindelöf Σ-space X condenses onto a Collins-Roscoe space. Then X is a Collins-Roscoe space.*

11. Monotone monolithicity in compact spaces

Recall that a compact space X is said to be *Eberlein compact* if it embeds in $C_p(Y)$ for some compact space Y. A compact space X is *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space. A compact space X is said to be *Corson compact* if X embeds in a Σ-product of real lines, i.e., in $\{x \in \mathbb{R}^\kappa : |\{\alpha \in \kappa : x(\alpha) \neq 0\}| \leq \omega\}$ for some cardinal κ.

It is well known that any Eberlein compact space is Gul'ko compact and that any Gul'ko compact space is Corson compact.

A corollary of Buzyakova's result ($C_p(X)$ is hereditarily a D-space whenever X is compact) is that Eberlein compact spaces are hereditarily D. This prompted the natural question, due to Arhangel'skii, whether Corson compact spaces are hereditarily D. G. Gruenhage showed that the answer is positive.

Theorem 11.1. [**36**] *Every Corson compact space is hereditarily a D-space.*

However, for Gul'ko compact spaces (hence for Eberlein compact spaces) we can explain this result. Indeed, by Proposition 7.5 we have that every Gul'ko compact space is monotonically monolithic. On the other hand, it is well known that Corson compacts are monolithic (see Example 3.1 (3)). The above suggests natural questions about the relationship of monotonically monolithic spaces and Corson compacta.

O. Alas, V.V. Tkachuk and R.G. Wilson proposed the following questions [**1**]:

(1) Is every Corson compact space monotonically ω-monolithic?
(2) Assume that X is a monotonically monolithic compact space, must X be Corson compact?

(3) Must every monotonically ω-monolithic compact space be monolithic?
(4) Is it true that any monotonically monolithic linearly ordered compact space is metrizable?

With respect to the first question, G. Gruenhage [**35**] gave a Corson compact space which is not monotonically ω-monolithic, so the answer to the first question is negative. Gruenhage's construction is very difficult, but if CH is assumed, then it is easy to obtain such an example. Indeed, in [**75**] V.V. Tkachuk gave another method for constructing a Corson compact space which is not monotonically ω-monolithic.

In [**37**], G.Gruenhague considered the following game $G(H, X)$ of length ω played in a space X, where H is a closed subset of X. There are two players, O and P. In the nth round, O chooses an open superset O_n of H, and P chooses a point $p_n \in O_n$. We say O wins the game if $p_n \to H$ in the sense that every open superset of H contains p_n for all but finitely many $n \in \omega$. G. Gruenhage showed that a compact Hausdorff space X is Corson compact if and only if O has a winning strategy in $G(\Delta, X^2)$, where Δ is the diagonal in X^2. Using this game characterization of Corson compact spaces G.Gruenhaghe proved the following result which implies that the answer to the second question above by O. Alas, V.V. Tkachuk and R.G. Wilson is positive.

Proposition 11.2. [**35**] *If X is monotonically ω-monolithic and countably compact, then O has a winning strategy in $G(H, X)$ for any closed subset H of X.*

Corollary 11.3. [**35**] *If X is compact and monotonically ω-monolithic, then X is a Corson compact space.*

In [**39**] it was proved, using thick covers, that any monotonically monolithic compact space is a Corson compact space.

As a consequence of Corollary 11.3, we can obtain an explanation of Proposition 9.11. Let X be a monotonically ω-monolithic countably compact space. By the previous result, X is compact and has the Fréchet-Urysohn property. By Corollary 11.3, X is a Corson compact space. Since every Corson compact has countable tightness, by Proposition 9.5, X is monotonically monolithic. Thus, we have the following result which answers the third question above by O. Alas, V.V. Tkachuk and R.G. Wilson .

Proposition 11.4. [**56**]*,* [**62**]*,* [**75**] *If X is a monotonically ω-monolithic countably compact space, then X is a monotonically monolithic Corson compact space and has the Fréchet-Urysohn property.*

It is well known that any linearly ordered Corson compact space is metrizable, so the previous result also gives a positive answer to the fourth question above by O. Alas, V.V. Tkachuk and R.G. Wilson .

K. Alster [**2**] proved that a scattered compact space is Corson compact if and only if it is Eberlein compact. It was asked in [**77**] whether every scattered monotonically monolithic compact space is Eberlein compact. It follows from Corollary 11.3 and Alster's result that any scattered monotonically monolithic compact space is Eberlein compact.

Now we will present some results for the class of Gul'ko compact spaces and some related conclusions.

Recall that a family $\mathcal{U}$ of subsets of a space X is T_0-separating if for any distinct points $x, y \in X$, there exists $U \in \mathcal{U}$ such that $U \cap \{x, y\}$ is a singleton. A family $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$ of subsets of X is called weakly σ-point-finite if for any point $x \in X$ we have the equality $\mathcal{U} = \bigcup\{\mathcal{U}_n :$ the family $\mathcal{U}_n$ is point-finite at $x\}$.

The following result was proved in [**65**].

Theorem 11.5. *A compact space X is Gul'ko compact if and only if X has a weakly σ-point finite T_0-separating cover $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$ by open F_σ-sets.*

G. Gruenhage used this characterization of Gul'ko compact spaces to prove the following result which improve our understanding of Gul'ko compact spaces.

Theorem 11.6. [**35**] *If X is Gul'ko compact, then X has the Collins-Roscoe property.*

V.V. Tkachuk provides an alternative way to prove Theorem 11.6.

In [**26**], A. Dow, H. Junnila, and J. Pelant considered several properties implied by the existence of a stronger metric topology on function spaces. Among other things, they introduced spaces with point-countably expandable networks. Recall that given a space X and a family $\mathcal{A}$ of subsets of X, we say that a family of open sets $\mathcal{E} = \{O_A : A \in \mathcal{A}\}$ is an *open expansion* of $\mathcal{A}$ if $A \subset O_A$ for any $A \in \mathcal{A}$. A family $\mathcal{A}$ of subsets of a space X is *point-countably expandable* if it has a point-countable open expansion in X.

Suppose that $\mathcal{N}$ is a network in X and $\{O_N : N \in \mathcal{N}\}$ is a point-countable open expansion of $\mathcal{N}$. Given any point $x \in X$, let $\mathcal{G}(x) = \{N \in \mathcal{N} : x \in O_N\}$; it is clear that the collection $G(x)$ is countable. Take any set $A \subset X$ and a point $x \in \mathrm{cl}(A)$; for any open set U which contains x, there exists $N \in \mathcal{N}$ such that $x \in N \subset U$. Pick a point $a \in A \cap O_N$, then $N \in \mathcal{G}(a)$. Thus, the family $\bigcup\{\mathcal{G}(a) : a \in A\}$ contains an external network at all points of A, and therefore:

Proposition 11.7. [**75**] *If a space X has a point-countably expandable network, then X is a Collins-Roscoe space.*

The class of spaces with a point-countably expandable network is important because it contains all Gul'ko compact spaces. Hence, Theorem 11.6 is a corollary of Proposition 11.7.

In view of Theorem 11.6, G. Gruenhage asked about the relationship of monotonically monolithic (Collins-Roscoe) spaces and Gul'ko compacta. Indeed, he proposed the following question: If X is compact, and has the Collins-Roscoe property or is monotonically monolithic, must X be Gul'ko compact? V.V. Tkachuk answered this question. He constructed a Corson compact which fails to be Gul'ko compact and proved that such space has the Collins-Roscoe property [**75**].

Also, V.V Tkachuk obtained another explanation for Theorem 11.6. He proved that compactness in such a result (see Theorem 11.5) can be weakened to the Lindelöf Σ property.

Theorem 11.8. [**78**] *Suppose that X is a Lindelöf Σ-space and there exists a weakly σ-point-finite T_0-separating family of cozero subsets of X. Then the space X has the Collins-Roscoe property and, in particular, it is hereditarily metalindelöf.*

The concept of Σ_s-product was introduced by G. A. Sokolov [**65**]. Given a family of spaces $\{X_t : t \in T\}$, suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T; let $X = \prod\{X_t : t \in T\}$ and fix a point $a \in X$. Given any $x \in X$, let $\operatorname{supp}(x) = \{t \in T : x(t) \neq a(t)\}$ and $\Omega_x = \{n \in \omega : |\operatorname{supp}(x) \cap T_n| < \omega\}$. Then the set $S = \{x \in X : T = \bigcup\{T_n : n \in \Omega_x\}\}$ is called the *Σ_s-product* centered at a with respect to the sequence s.

G.A. Sokolov proved the following result:

Theorem 11.9. [**65**] *A compact space X is Gul'ko compact if and only if X embeds into a Σ_s-product of real lines.*

The following results were proved in [**75**].

Theorem 11.10. *Any Σ_s-product of compact spaces is a Lindelöf Σ-space.*

Proposition 11.11. *Any Σ_s-product S of spaces of countable i-weight has a weakly σ-point-finite family of cozero sets that T_0-separates the points of S.*

If S is a Σ_s-product of compact second countable spaces then by Theorem 11.10 and Proposition 11.11 we have that S is a Lindelöf Σ-space and has a weakly σ-point-finite family of cozero sets that T_0-separates the points of S. It follows from Theorem 11.8 that S has the Collins-Roscoe property. Since any second countable space has a second countable compactification, we have the following consequence (that by Theorem 11.9 is a explanation of Theorem 11.6).

Corollary 11.12. [**75**] *Every Σ_s-product of second countable spaces has the Collins-Roscoe property.*

The Collins-Roscoe property is also "lifted" by condensations of Lindelöf Σ-spaces, this makes it possible to generalize Gruenhage's theorem on Collins-Roscoe property of Gul'ko compact spaces in the context of their mappings into Σ_s-products (see Theorem 11.9). Indeed, the following result is a consequence of Proposition 10.7 and Proposition 11.12, since every Σ_s-product of spaces of countable i-weight can be condensed into a Σ_s-product of spaces of countable weight.

Corollary 11.13. [**75**] *If a Lindelöf Σ-space X condenses into a Σ_s-product of spaces of countable i-weight, then X has the Collins-Roscoe property.*

12. The Collins-Roscoe property in function spaces

G. Gruenhage asked whether the Lindelöf Σ-property of a space X implies that $C_p(X)$ has the Collins-Roscoe property. Although the answer, in general, is negative ($C_p(D_{\omega_1})$ is a counterexample, see §10). In this section we will describe a wide class of spaces for which its function spaces have the Collins-Roscoe property, and in an unexpected form, we also will describe some class of spaces for which the iterated function spaces have the Collins-Roscoe property (and hence are hereditarily metalindelöf, see Proposition 10.2).

A. Dow, H. Junnila, J. Pelant proved [**27**] that if a compact space X has weight not exceeding ω_1 then $C_p(X)$ is hereditarily metalindelöf. Since every space with the Collins-Roscoe property is metalindelöf, the following statement (which follows from Corollary 10.5) is a generalization of their result.

Corollary 12.1. [**78**] *If X is a Lindelöf Σ-space and $nw(X) \leq \omega_1$ then $C_p(X)$ has the Collins-Roscoe property.*

It is well known [**72**] that if X and $C_p(X)$ are Lindelöf Σ-spaces, then $C_{p,n+1}(X)$ and $C_{p,n}(X)$ are Lindelöf Σ for every $n \in \omega$. It was proved in [**73**] that in this case $C_{p,n}(X)$ has a weakly σ-point-finite T_0-separating family of cozero subsets. Then we can apply Theorem 11.8 to see that:

Corollary 12.2. [**78**] *If X and $C_p(X)$ are Lindelöf Σ-spaces then $C_{p,n}(X)$ has the Collins-Roscoe property for all $n \in \omega$.*

If $C_p(X)$ is a Lindelöf Σ-space, since X can be embedded in $C_p(C_p(X))$; it follows from Proposition 7.5 that X is monotonically monolithic. This result can be strengthened to the Collins -Roscoe property. Indeed, if $C_p(X)$ is a Lindelöf Σ-space it was proved in [**49**] that υX is a Lindelöf Σ-space and was proved in [**72**] that $C_p(\upsilon X)$ is Lindelöf Σ, then we can apply Corollary 12.2 to conclude that υX has the Collins-Roscoe property. Finally, since the Collins-Roscoe property is hereditary we obtain:

Corollary 12.3. [**78**] *If $C_p(X)$ is a Lindelöf Σ-space then X has the Collins-Roscoe property.*

E.A. Reznichenko constructed a Corson compact space B [**60**] with a remarkable combination of properties. The space $C_p(B)$ is Lindelöf Σ and

K-analytic. There is a point $b \in B$ such that B is the Stone-Čech compactification of the space $Y = B \setminus \{b\}$. Moreover, the subspace $Y = B \setminus \{b\}$ is pseudocompact and not closed in B. Since $C_p(Y)$ is a continuous image of $C_p(B)$, then $C_p(Y)$ is also a Lindelöf Σ-space. By Corollary 12.3, the space Y has the Collins-Roscoe property. Thus, Y is an example of a pseudocompact space with the Collins-Roscoe property which is not compact (see Proposition 9.11).

It was proved in [**72**] that only the following distributions of the Lindelöf Σ-property in iterated function spaces are possible:

(1) $C_{p,n+1}(X)$ are Lindelöf Σ-spaces for all $n \in \omega$;
(2) no $C_{p,n+1}(X)$ is a Lindelöf Σ space for $n \in \omega$;
(3) only $C_{p,n+1}(X)$ with even $n \in \omega$ are Lindelöf Σ-spaces;
(4) only $C_{p,n+1}(X)$ with odd $n \in \omega$ are Lindelöf Σ-spaces.

In particular, for every Gul'ko compact space X we have that $C_{p,n+1}(X)$ are Lindelöf Σ-spaces for all $n \in \omega$, so Corollary 12.2 and Corollary 12.3 are generalizations of theorem 11.6. Also, from Corollary 12.3 we obtain the following consequences.

Corollary 12.4. [**78**] *If the space $C_p(X)$ is Lindelöf Σ then the space $C_{p,2n}(X)$ has the Collins-Roscoe property for all $n \in \omega$.*

Corollary 12.5. [**78**] *If $C_p(C_p(X))$ is a Lindelöf Σ-space, then the space $C_{p,2n+1}(X)$ has the Collins-Roscoe property for all $n \in \omega$.*

Finally, we can see that under certain conditions the fact that $C_p(X)$ is a Lindelöf Σ-space implies that $C_p(X)$ is a Collins-Roscoe space.

Proposition 12.6. [**78**] *Suppose that X is a space with $nw(X) \leq \omega_1$ and $C_p(X)$ is a Lindelöf Σ-space. Then $C_p(X)$ has the Collins-Roscoe property.*

We know that for any compact space the space $C_p(X)$ is monotonically monolithic. If X is a Gul'ko compact space, we have that $C_p(X)$ has the Collins-Roscoe property. In [**78**], the next question appears: Suppose that X is a Corson compact space, must the space $C_p(X)$ have the Collins-Roscoe property?

In [**26**], A. Dow, H. Junnila, and J. Pelant introduced spaces with point-finitely expandable networks. Recall that given a space X and a family $\mathcal{A}$ of subsets of X, we say that a family of open sets $\mathcal{E} = \{O_A : A \in \mathcal{A}\}$ is an *open expansion* of $\mathcal{A}$ if $A \subset O_A$ for any $A \in \mathcal{A}$. A family $\mathcal{A}$ of subsets of a space X is *point-finitely expandable* if it has a point-finite open expansion in X. A. Dow, H. Junnila, and J. Pelant proved that if X is a Corson compact space, then $C_p(X)$ has a σ-point finitely expandable network. Since any σ-point finitely expandable network is a point-countably expandable network, by Proposition 11.7, we have that:

Proposition 12.7. [**26**] *For any Corson compact space X the space $C_p(X)$ has the Collins-Roscoe property.*

13. Strong monotone monolithicity

Strong monolithic spaces were introduced in [**5**]. A space is *strongly κ-monolithic* if the weight of $\mathrm{cl}(A)$ does not exceed κ whenever $|A| \leq \kappa$. A space is *strongly monolithic* if it is strongly κ-monolithic for any cardinal κ. So, it is natural to introduce a strong version of monotone (κ-)monolithicity.

Definition 13.1. [**77**] Given a set A in a space X we say that a family $\mathcal{B}$ of subsets of X is an *external base* of A in X if for any $x \in A$ and an open set $U \subset X$ with $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Definition 13.2. Say that a space X is *strongly monotonically monolithic* [**77**] if, for any $A \subset X$ we can assign an external base $\mathcal{O}(A)$ to the set $\mathrm{cl}(A)$ in such a way that the following conditions are satisfied:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if α is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of X such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{O}(\bigcup A_\beta : \beta < \alpha) = \bigcup\{\mathcal{O}(A_\beta) : \beta < \alpha\}$.

Further, for an infinite cardinal κ, X is said to be *strongly monotonically κ-monolithic* [**1**], if $\mathcal{O}(A)$ is defined for all sets A with $|A| \leq \kappa$ and satisfies the above conditions.

It is easy to see that any space with a point countable base is strongly monotonically monolithic. Every strong monotonically monolithic space is monotonically monolithic. A cosmic space of uncountable weight is an example of a monotonically monolithic space which is not strongly monotonically monolithic.

The class of strong monotonically monolithic has the following properties.

Proposition 13.3. *Strong monotone monolithicity space is preserved by:*

(1) countable products [**77**]*;*
(2) arbitrary subspaces [**77**]*;*
(3) open and closed functions [**44**]*;*
(4) open functions with separable fibers [**77**]*.*

Closed maps do not preserve strong monotonic monolithicity: The space $\mathbb{R}$ is strongly monotonically monolithic being metrizable; the set $F = \{n : n \in \omega\}$ is closed in $\mathbb{R}$. Let Y be the space obtained by collapsing F to a point. The respective quotient map $\varphi : R \to Y$ is closed and it is standard that Y is not first countable. Therefore, Y is not strongly monotonically monolithic.

An open image of a strongly monotonically monolithic space is not necessarily monolithic: Let Y be the Sorgenfrey line, then Y is not even monolithic. Since $\chi(Y) \leq \omega$, there exists a metrizable space X for which there is an open map $f : X \to Y$. Therefore, X is a strongly monolithic space whose open continuous image fails to be monolithic.

The following characterization is natural (see Theorem 7.7 and Theorem 9.4).

Theorem 13.4. [**75**] *A space X is strongly monotonically monolithic (strongly monotonically κ-monolithic) if and only if, for any finite set $F \subset X$, we can choose a countable family $\mathcal{O}(F)$ of open subsets of X such that, for every $A \subset X$ (with $|A| \le \kappa$), the family $\bigcup\{\mathcal{O}(F) : F \in [A]^{<\omega}\}$ is an external base for A.*

As in the case of monotonically monolithic spaces (see Proposition 9.3), it is easy to prove that a space X is strongly monotonically monolithic if and only if it is strongly monotonically κ-monolithic for every infinite cardinal κ. In fact, the following strong result holds.

Theorem 13.5. [**75**] *A sapce X is strongly monotonically ω-monolithic if and only if it is strongly monotonically monolithic.*

We can also define a strong Collins-Roscoe property, but it is not difficult to see that such a property is precisely open (G) (see §10). For this reason, spaces which satisfied open (G) are also called strong Collins-Roscoe spaces [**75**]. Then, strong monotone monolithicity is a generalization of open (G). It is easily seen that any space with a point-countable base $\mathcal{B}$ satisfies open (G). Indeed, the assignment $\mathcal{G}(x) = \{B \in \mathcal{B} : x \in B\}$ witnesses this fact. Therefore, it is natural to try to extend to strongly monotonically monolithic spaces the results obtained for the spaces with a point-countable base. The following results show that sometimes it is possible.

It is now a classic result of Miscenko's that any compact space with a point-countable base is metrizable. In fact, it is also well known that this result holds for countably compact spaces.

Theorem 13.6. [**77**] *If X is a countably compact strongly monotonically monolithic space then X is metrizable.*

On the other hand, there exists a pseudocompact non-metrizable space X with a point-countable base (see [**63**]). Therefore, X is a non-metrizable pseudocompact strongly monotonically monolithic space.

Any space with a point-countable base and caliber ω_1 is clearly metrizable. For strongly monotonically monolithic spaces we have:

Proposition 13.7. [**1**] *Suppose that X is strongly monotonically monolithic space and ω_1 is a caliber of X. If additionally $d(X) \le \omega_1$, then X is metrizable.*

In [**77**] V.V. Tkachuk asked if a strongly monotonically monolithic space with caliber ω_1 is metrizable. Under the Continuum Hypothesis the answer is positive. Indeed, under CH, the condition of separability in Proposition 13.7 can be omitted. Observe first that $c(X) \le \omega$ and hence $d(X) \le |X| \le 2^{\chi(X)\cdot c(X)} \le \mathfrak{c} = \omega_1$ (see [**40**]). Thus, we have:

Corollary 13.8. [**1**] *Under the Continuum Hypothesis if X is a strongly monotonically monolithic space and ω_1 is a caliber of X, then X is metrizable.*

It follows from a general result of J. Chaber [**21**] that every Lindelöf Σ-space with a point-countable base is second countable. The following theorem shows that the assumption on a point-countable base can be weakened to strong monotonic monolithicity.

Theorem 13.9. [**75**] *If X is a strongly monotonically monolithic Lindelöf Σ-space, then X is metrizable.*

Recall that given a natural number k, a family $\mathcal{A}$ of subsets of a space X is called *k-in-countable* if every set $A \subset X$ with $|A| = k$ is contained in at most countably many elements of $\mathcal{A}$.

Assume that X is a space and $x \in X$. We say that X is *weakly countably tight* at x if there is a countable subset A of $X \setminus \{x\}$ such that x is in the closure of A. A space X is called weakly countably tight if X is *weakly countably tight* at every $x \in X$.

Theorem 13.10. [**80**] *Suppose that X is weakly countably tight (or a k-space) with a k-in-countable base $\mathcal{B}$ for some $k \in N$. Then X is strongly monotonically monolithic.*

As we have seen, any space with a point-countable base is a strong Collins-Roscoe space and any strong Collins-Roscoe space is strongly monotonically monolithic. In [**76**] and [**77**] the following question appears: Is it true that a space X is strongly monotonically monolithic if and only if X has a point-countable base? Assuming Martin's Axiom and $\omega_2 < 2^\omega$, there is a normal Moore space with a 2-in-finite base that has no point-countable base. This space is strongly monotonically monolithic by Theorem 13.10, and hence, the answer to the above question is consistently negative (see [**80**]).

It is one of the more intriguing open problems in the theory of generalized metric spaces whether a strong Collins-Roscoe space has a point countable base, the so called point-countable base problem. Some partial results are known. V.V. Tkachuk proved in [**75**] that if X is a Collins-Roscoe space, then every left-separated subspace $Y \subset X$ has a point-countable open expansion (see §11). Using this fact he proved the following results.

Proposition 13.11. [**75**]

(1) *If X is a Collins-Roscoe space and $\pi_\chi(X) = \omega$, then X has a point-countable π-base.*
(2) *Every strong Collins-Roscoe space has a point-countable π-base.*
(3) *If X is a hereditarily Lindelöf strong Collins-Roscoe space, then X has a point-countable base.*

CHAPTER II

Monotone monolithicity

1. Monotone monolithicity of $C_p(C_p(X))$

It is well known that $C_p(C_p(X))$ is monolithic if and only if X is monolithic (see Theorem I.3.8). Our main goal in this section is to show that a similar result holds for monotone monolithicity and use it to answer some questions proposed in [**77**].

First, we introduce some notations that will be useful.

From now on, we will fix a countable base $\mathcal{B}(\mathbb{R})$ for the usual topology in the set of real numbers $\mathbb{R}$.

Given a space X, for subsets $E_1, \ldots, E_n$ of X and subsets $U_1, \ldots, U_n$ of $\mathbb{R}$, we will use the symbol $[E_1, \ldots, E_n; U_1, \ldots, U_n]$ to denote the set $\{f \in C_p(X) : f(E_i) \subset U_i \text{ for } i = 1, \ldots, n\}$. If $\mathcal{E}$ is a family of subsets of X, then $\mathcal{W}(\mathcal{E})$ will be the family of all the sets of the form $[E_1, \ldots, E_n; B_1, \ldots, B_n]$ where $E_1, \ldots, E_n \in \mathcal{E}$, $B_1, \ldots, Bn \in \mathcal{B}(\mathbb{R})$ and $n \in \omega$.

Remarks 1.1. Let X be space, then:

(1) If $\mathcal{E}$ is a family of subsets of X, then $|\mathcal{W}(\mathcal{E})| \leq \max\{|\mathcal{E}|, \omega\}$;
(2) if $\mathcal{E}$ and $\mathcal{E}'$ are families of subsets of X with $\mathcal{E} \subset \mathcal{E}'$, then $\mathcal{W}(\mathcal{E}) \subset \mathcal{W}(\mathcal{E}')$;
(3) if $\{\mathcal{E}_\beta : \beta < \alpha\}$ is such that each $\mathcal{E}_\beta$ is a family of subsets of X and $\mathcal{E}_\beta \subset \mathcal{E}_{\beta'}$ for $\beta < \beta'$, then $\mathcal{W}\left(\bigcup\{\mathcal{E}_\beta : \beta < \alpha\}\right) = \bigcup\{\mathcal{W}(\mathcal{E}_\beta) : \beta < \alpha\}$.

Definition 1.2. Let $\mathcal{N}$ be a family of subsets of X and let f be a function from X onto Y. We say that $\mathcal{N}$ is a *network for Y modulo f* if for each $x \in X$ and each open subset U of Y with $f(x) \in U$, there is $N \in \mathcal{N}$ such that $x \in N$ and $f(x) \in f(N) \subset U$.

The following two results show that there exists a duality between the notions of network modulo a function and external network (see Definition I.7.1).

Proposition 1.3. *Let $A \subset X$. If $\mathcal{N}$ is an external network of A in X, then the family $\mathcal{W}(\mathcal{N})$ of subsets of $C_p(X)$ is a network for $\pi_A(C_p(X)) \subset C_p(A)$ modulo π_A.*

Proof. Let $f \in C_p(X)$ and suppose that $\pi_A(f) \in U$ for an open subset U in $\pi_A(C_p(X))$. Take $x_1, \ldots, x_n \in A$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ such that $\pi_A(f) \in [x_1, \ldots, x_n; B_1, \ldots, B_n] \cap \pi_A(C_p(X)) \subset U$. Then, there exist $N_1, \ldots, N_n \in \mathcal{N}$

such that $x_i \in N_i \subset f^{-1}(B_i)$ for $i = 1, \dots, n$. It is not difficult to see that if $M = [N_1, \dots, N_n; B_1, \dots, B_n]$, then $M \in \mathcal{W}(\mathcal{N})$, $f \in M$ and $\pi_A(M) \subset U$. $\square$

Proposition 1.4. *Let $f : X \to Y$ be an onto and continuous function and let $\mathcal{N}$ be a family of subsets of X which is a network for Y modulo f. Then $\mathcal{W}(\mathcal{N})$ is an external network of $f^*(C_p(Y))$ in $C_p(X)$.*

Proof. Take a continuous function $g \circ f = f^*(g) \in f^*(C_p(Y))$ and an open subset U of $C_p(X)$ with $g \circ f \in U$. Take $x_1, \dots, x_n \in X$ and $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ such that $g \circ f \in [x_1, \dots, x_n; B_1, \dots, B_n] \subset U$. Thus, $g \in [f(x_1), \dots, f(x_n); B_1, \dots, B_n]$. Because of our hypothesis, we can choose $N_1, \dots, N_n \in \mathcal{N}$ such that $x_i \in N_i$ and $f(N_i) \subset g^{-1}(B_i)$ for $i = 1, \dots, n$. So $g \in [f(N_1), \dots, f(N_n); B_1, \dots, B_n]$. Hence, if $M = [N_1, \dots, N_n; B_1, \dots, B_n]$ then $g \circ f \in M \subset [x_1, \dots, x_n; B_1, \dots, B_n] \subset U$ and $M \in \mathcal{W}(\mathcal{N})$. $\square$

In order to prove our main result of this section we need the following result due to A.V. Arhangel'skii.

Lemma 1.5. [**7**] *Let Y be a dense subspace of the product $X = \prod\{X_t : t \in T\}$, where each space X_t is cosmic. Then, for every continuous real-valued function f on Y, there exist a countable set $S \subset T$ and a continuous real-valued function g on $p_S(Y)$ such that $f = g \circ p_S \upharpoonright Y$.*

Theorem 1.6. *The space $C_p(C_p(X))$ is monotonically monolithic if and only if X is monotonically monolithic.*

Proof. It is a well known fact that X is homeomorphic to a closed subspace of $C_p(C_p(X))$ (see Proposition I.2.5). Since monotone monolithicity is a hereditary property, if $C_p(C_p(X))$ is monotonically monolithic, then X has this property too.

Now, assume that X is a monotonically monolithic space. For each $S \subset X$ take an external network $\mathcal{O}(S)$ of $\mathrm{cl}(S)$ in X in such a way that the assignment $S \to \mathcal{O}(S)$ satisfies (1), (2) and (3) in Definition I.7.2.

If $f \in C_p(C_p(X))$, by Lemma 1.5, we can fix a countable set $S(f) \subset X$ and a continuous function $g(f) : \pi_{S(f)}(C_p(X)) \to \mathbb{R}$ such that $f = g(f) \circ \pi_{S(f)}$. For $A \subset C_p(C_p(X))$, we define $\mathcal{S}(A) = \bigcup\{S(f) : f \in A\}$. The operator $\mathcal{S}$ satisfies the following properties:

(1) if $A \subset C_p(C_p(X))$, then $|\mathcal{S}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset C_p(C_p(X))$, then $\mathcal{S}(A) \subset \mathcal{S}(B)$;
(3) if $\{A_\beta : \beta < \alpha\}$ is a family of subsets of $C_p(C_p(X))$ with $A_\beta \subset A_{\beta'}$ when $\beta < \beta'$, then $\mathcal{S}(\bigcup\{A_\beta : \beta < \alpha\}) = \bigcup\{\mathcal{S}(A_\beta) : \beta < \alpha\}$.

Claim 1. If $A \subset C_p(C_p(X))$ and $\mathcal{S}(A) \subset S$, then $A \subset \pi_S^*(C_p(\pi_S(C_p(X))))$.

We will prove this Claim. If $f \in A$, we can take a set $S(f) \subset X$ and a continuous function $g(f) : \pi_{S(f)}(C_p(X)) \to \mathbb{R}$, as before. Let π_S the restriction function from $C_p(X)$ to $C_p(S)$ and let $\pi^S_{S(f)}$ the restriction function from $C_p(S)$ to $C_p(S(f))$. Then, for $h(f) = g(f) \circ \pi^S_{S(f)} \upharpoonright \pi_S(C_p(X)) : \pi_S(C_p(X)) \to \mathbb{R}$

we have that $f = g(f) \circ \pi_{S(f)} = h(f) \circ \pi_S = \pi_S^*(h(f)) \in \pi_S^*(C_p(\pi_S(C_p(X))))$. This proves the Claim.

Now we are ready to construct a monotonic monolithicity assignment for the space $C_p(C_p(X))$. For $A \subset C_p(C_p(X))$ we take $\mathcal{N}(A) = \mathcal{W}(\mathcal{W}(\mathcal{O}(\mathcal{S}(A))))$. Because of the above properties of the assignment $\mathcal{S}$, the election of $\mathcal{O}$ and Remarks 1.1, it is easy to verify that $\mathcal{N}$ satisfies conditions (1), (2) and (3) in Definition I.7.2 for the space $C_p(C_p(X))$. So, in order to prove that $C_p(C_p(X))$ is monotonically monolithic it is enough to show the following claim.

Claim 2. for any $A \subset X$, $\mathcal{N}(A)$ is an external network of $\mathrm{cl}(A)$ in $C_p(C_p(X))$.

We shall prove this Claim. Let $A \subset X$. Take $S = \mathrm{cl}(\mathcal{S}(A)) \subset X$ and $Y = \pi_S(C_p(X))$. By the election of $\mathcal{O}$, we have that $\mathcal{O}(\mathcal{S}(A))$ is an external network of S in X. By Proposition 1.3, the family $\mathcal{W}(\mathcal{O}(\mathcal{S}(A)))$ of subsets of $C_p(X)$ is a network for Y modulo π_S. Now, by Proposition 1.4, $\mathcal{W}(\mathcal{W}(\mathcal{O}(\mathcal{S}(A)))) = \mathcal{N}(A)$ is an external network of $\pi_S^*(C_p(Y))$ in $C_p(C_p(X))$. Since S is closed, by Proposition I.2.1 (2), the projection π_S is open onto Y and so π_S is a quotient function. By Proposition I.2.3 (2), $\pi_S^*(C_p(Y))$ is closed in $C_p(C_p(X))$. By Claim 1, we know that $A \subset \pi_S^*(C_p(Y))$. Thus, $\mathrm{cl}(A) \subset \pi_S^*(C_p(Y))$. Finally, since $\mathcal{N}(A)$ is an external network of $\pi_S^*(C_p(Y))$ in $C_p(C_p(X))$, $\mathcal{N}(A)$ is also an external network of cl(A) in $C_p(C_p(X))$. □

In the second part of this section we will present some consequences of our main result.

Corollary 1.7. *The following conditions are equivalent:*

(1) X is monotonically monolithic;
(2) $C_{p,2n}(X)$ is monotonically monolithic for some $n \in \omega$;
(3) $C_{p,2n}(X)$ is monotonically monolithic for every $n \in \omega$.

The following result gives a complete description of all possible distributions of the monotone monolithicity property in the spaces $C_{p,n}(X)$. Notice that these distributions are very similar to the distributions of the Lindelöf Σ-property in the spaces $C_{p,n}(X)$ (See §12 in Chapter I).

Proposition 1.8. *One and only one of the following distributions of the monotone monolithicity property in iterated function spaces happen.*

(1) $C_{p,n}(X)$ is monotonically monolithic for every $n \in \omega$;
(2) $C_{p,n}(X)$ is not monotonically monolithic for every $n \in \omega$;
(3) only $C_{p,n}(X)$ with odd $n \in \omega$ are monotonically monolithic spaces;
(4) only $C_{p,n}(X)$ with even $n \in \omega$ are monotonically monolithic spaces;

Proof. All of the four listed cases exclude themselves mutually. Moreover, these are the only possible distributions because of Corollary 1.7. In order to prove that all these possible distributions happen, by Corollary 1.7, it is enough to prove that the following cases must happen: X and $C_p(X)$ are monotonically monolithic, neither X or $C_p(X)$ are monotonically monolithic,

X is not monotonically monolithic but $C_p(X)$ is, and X is monotonically monolithic but $C_p(X)$ is not.

Now we are going to see that, in fact, all these cases must happen. If X has a countable network, then X and $C_p(X)$, being cosmic spaces, are monotonically monolithic; this proves that the first case happens. If S is the Sorgenfrey line, then $d(S) = \omega$ and $d(C_p(S)) = \mathrm{iw}(S) = \omega$. Nevertheless, $\mathrm{nw}(C_p(S)) = \mathrm{nw}(S) > \omega$. Thus, neither S or $C_p(S)$ are monotonically monolithic; this proves that the second case happens. Consider now the space $[0, \omega_1]$. Since its subspace $[0, \omega_1)$ does not have the D-property, $[0, \omega_1]$ is not hereditary D and then $[0, \omega_1]$ is not monotonically monolithic (see Theorem I.7.6). Nevertheless, since $[0, \omega_1]$ is compact, $C_p([0, \omega_1])$ is monotonically monolithic (see Theorem I.6.1), so, the third case happens. Finally, of course, the last case happens when we take $X = C_p([0, \omega_1])$. □

Remark 1.9. Notice that the statements in Theorem 1.6, Corollary 1.7 and Proposition 1.8 remain true if we replace monotone monolithicity by monotone κ-monolithicity.

It is known that $C_p(X)$ is monotonically monolithic whenever X is a Lindelöf Σ-space (see Proposition I.7.5). V.V. Tkachuk asked [**77**] if the converse of this result is true, that is: Suppose that $C_p(X)$ is monotonically monolithic, must X be a Lindelöf Σ-space (Lindelöf space)? He also proposed the following related question: Suppose that the space $C_p(C_p(X))$ is monotonically monolithic. Must $C_p(X)$ be a Lindelöf Σ-space?

Now we will use the results obtained in this section to find a counterexample for the second (hence, for the first) question above.

Example 1.10. Let κ be a cardinal number such that $\kappa > \omega$ and let $\mathcal{D}_\kappa$ the discrete space of cardinality κ. $\mathcal{D}_\kappa$ being metrizable, is monotonically monolithic, so $C_p(C_p(\mathcal{D}_\kappa))$ is monotonically monolithic, but $C_p(\mathcal{D}_\kappa) = \mathbb{R}$ is not even normal.

We can also give an example of a Lindelöf space X such that $C_p(X)$ is not Lindelöf and $C_pC_p(X)$ is monotonically monolithic. Indeed, let L_κ be the one-point Lindelöfication of the discrete space $\mathcal{D}_\kappa$ where $\kappa > \omega$. L_κ is Lindelöf, and its tightness is not countable. By Asanov's Theorem (see Theorem I.4.1 in [**3**]), $C_p(L_\kappa)$ is not Lindelöf. But every space with only one non-isolated point is monotonically monolithic (see Proposition I.7.4). So, L_κ and $C_p(C_p(L_\kappa))$ are monotonically monolithic.

Notice that, as in the last example, in order to construct a non-Lindelöf space $C_p(X)$ with $C_pC_p(X)$ monotonically monolithic, is enough to find a monotonically monolithic space of uncountable tightness.

One of the main characteristics of monotonically monolithic spaces is that they are hereditary D-spaces. Since monotone monolithicity is preserved under arbitrary subspaces and closed continuous functions, we have:

Corollary 1.11. *If X is monotonically monolithic and Y is a closed continuous image of a subspace of $C_{p,2n}(X)$ for some $n \in \omega$, then Y is hereditarily D.*

Finally the next result show that a for a wide class of spaces monotonic monolithicity appear in the iterated function spaces (see Chapter I):

Corollary 1.12. *If $C_{p,2n}(X)$ is monotonically monolithic for any $n \in \omega$ whenever X belong to one of the following classes: Stratifiable spaces, Collins-Roscoe spaces (hence spaces with a pint-countably expandable network, spaces which satisfy open (G), spaces with a point-countable base and metrizable spaces).*

2. When is $C_p(X)$ monotonically monolithic?

In the previous section we saw that $C_p(C_p(X))$ is monotonically monolithic if and only if X is monotonically monolithic. From this result we can deduce that there exists a topological property $\mathcal{P}$ such that $C_p(C_p(X))$ is monotonically monolithic if and only if $C_p(X)$ has $\mathcal{P}$. Indeed, from the proof of Theorem 1.6, we can see that such a property is given by: $C_p(X)$ has $\mathcal{P}$ if for each $A \subset X$ we can assign a countable collection $\mathcal{O}(A)$ of subsets of $C_p(X)$ which is a network for $\pi_{\mathrm{cl}(A)}(C_p(X))$ modulo $\pi_{\mathrm{cl}(A)}$ and the following conditions are satisfied:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset X$ then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if α is an ordinal and we have a family $\{A_\beta : \beta < \alpha\}$ of subsets of X such that $\beta < \beta' < \alpha$ implies $A_\beta \subset A_{\beta'}$ then $\mathcal{O}(\bigcup A_\beta : \beta < \alpha) = \bigcup\{\mathcal{O}(A_\beta) : \beta < \alpha\}$.

Of course, it is interesting to know if we can define such a property for every topological space X, by this reason, in [**62**] the following general problem was formulated: Given a class $\mathcal{C}$ of topological spaces, determine a topological property $\mathcal{P}(\mathcal{C})$ which satisfies: for every space $X \in \mathcal{C}$, $C_p(X)$ is monotonically monolithic if and only if X has the property $\mathcal{P}(\mathcal{C})$. In this section we introduce the notion of monotonically stable space and show that it is the required property. Indeed, our main result in this section states that for every space X, the space $C_p(X)$ is monotonically monolithic if and only if the space X is monotonically stable.

The following facts are easy to verify.

Remark 2.1. Let f be a function from X onto Y.

(1) If a family $\mathcal{N}$ of subsets of X is a network for Y modulo f then $f(\mathcal{N})$ is a network for Y.
(2) If a family $\mathcal{N}$ of subsets of X is a network for Y modulo f and g is a continuous function from Y onto Z then $\mathcal{N}$ is a network for Z modulo $g \circ f$.
(3) If $\mathcal{N}$ is a network for Y, $f^{-1}(\mathcal{N})$ is a network for Y modulo f.

Recall that given a set X and a family $\mathcal{F} \subset \mathbb{R}^X$, $\Delta\mathcal{F} : X \to \mathbb{R}^{\mathcal{F}}$ denotes the the diagonal of the family $\mathcal{F}$. The following result will be useful in this section.

Lemma 2.2. *If f is a function from a set X onto a set Y and $\mathcal{F} \subset \mathbb{R}^Y$, then $\Delta f^*(\mathcal{F})(X)$ is homeomorphic to $\Delta\mathcal{F}(Y)$.*

Proof. Let $A = \mathcal{F}$ and $B = f^*(\mathcal{F})$, both, endowed with the discrete topology. Since $f^* : A \to B$ is a bijection then $b = (f^*)^* : \mathbb{R}^B \to \mathbb{R}^A$ is a homeomorphism (see Propositions 2.2 and 2.3 in Chapter I). Since $\Delta f^*(\mathcal{F})(X) \subset \mathbb{R}^B$ and $\Delta\mathcal{F}(Y) \subset \mathbb{R}^A$, then, in order to finish the proof, it suffices to show that $b(\Delta f^*(\mathcal{F})(X)) = \Delta\mathcal{F}(X)$. But this equality follows from the fact that the function f is surjective and $b(\Delta f^*(\mathcal{F})(x)) = \Delta\mathcal{F}(f(x))$ for every $x \in X$. Indeed, $[b(\Delta f^*(\mathcal{F})(x))](g) = [(f^*)^*(\Delta f^*(\mathcal{F})(x))](g) = [\Delta f^*(\mathcal{F})(x) \circ f^*](g) = \Delta f^*(\mathcal{F})(x)\ (f^*(g)) = \Delta f^*(\mathcal{F})(x)(g \circ f) = (g \circ f)(x) = g(f(x)) = [\Delta\mathcal{F}(f(x))](g)$ for every $g \in \mathcal{F}$. □

Corollary 2.3. *If $f : X \to Y$ is continuous and onto, then the space $\Delta f^*(C_p(Y))(X)$ is homeomorphic to $\Delta C_p(Y)(Y)$.*

Recall that a space X is stable if and only if for every continuous image Y of X we have $\mathrm{iw}(Y) = \mathrm{nw}(Y)$. The following result gives a characterization of stable spaces only in terms of real-valued continuous functions.

Proposition 2.4. *A topological space X is stable if and only if for each subset $A \subset C_p(X)$ we can assign a collection $\mathcal{N}(A)$ of subsets of X such that $|\mathcal{N}(A)| \leq |A|$ and $\mathcal{N}(A)$ is a network for $\Delta\mathrm{cl}(A)(X)$ modulo $\Delta\mathrm{cl}(A)$.*

Proof. First, suppose that X is a stable space and take $A \subset C_p(X)$. Let $\kappa = |A|$, $f = \Delta A$ and $Y = f(X)$. Let $\tilde{Y} = Y$ endowed with the $\mathbb{R}$-quotient topology generated by f (see §2 in Chapter I). Since $\tilde{Y}$ is a continuous image of X and the identity function $i : \tilde{Y} \to Y$ is a condensation, we have that $\mathrm{nw}(\tilde{Y}) \leq \mathrm{iw}(\tilde{Y}) \leq w(Y) \leq \kappa$. Notice that $A \subset f^*(C_p(\tilde{Y}))$; in fact, if $g \in A$ then $g = p_g \circ i \circ f = f^*(p_g \circ i) \in f^*(C_p(\tilde{Y}))$, where $p_g : \mathbb{R}^A \to \mathbb{R}$ is the natural projection. Since $f : X \to \tilde{Y}$ is a $\mathbb{R}$-quotient function then $f^*(C_p(\tilde{Y}))$ is closed in $C_p(X)$ (see Proposition 2.4 in Chapter I) and hence $\mathrm{cl}(A) \subset f^*(C_p(\tilde{Y}))$. By Corollary 2.3, $\Delta f^*(C_p(\tilde{Y}))(X)$ is homeomorphic to $\Delta C_p(\tilde{Y})(\tilde{Y})$ and then $\Delta f^*(C_p(\tilde{Y}))(X)$ is homeomorphic to $\tilde{Y}$. So we have that $\mathrm{nw}(\Delta f^*(C_p(\tilde{Y}))(X)) \leq \kappa$. Since $\Delta\mathrm{cl}(A)(X)$ is a continuous image of $\Delta f^*(C_p(\tilde{Y}))(X)$ under the projection from $\mathbb{R}^{f^*(C_p(Y))}$ to $\mathbb{R}^{\mathrm{cl}(A)}$, then $\mathrm{nw}(\Delta\mathrm{cl}(A)(X)) \leq \kappa$. Let $\mathcal{N}$ be a network for $\Delta\mathrm{cl}(A)(X)$ with $|\mathcal{N}| \leq \kappa$. Now, take $\mathcal{N}(A) = [\Delta\mathrm{cl}(A)(X)]^{-1}(\mathcal{N})$. Clearly, $|\mathcal{N}(A)| \leq |A|$ and by Remark 2.1 (3), we have that $\mathcal{N}(A)$ is a network for $\Delta\mathrm{cl}(A)(X)$ modulo $\Delta\mathrm{cl}(A)$.

Suppose now that X satisfies the second condition in our proposition. Let Y be a continuous image of X and take a continuous function f from X onto Y. Let $\kappa = \mathrm{iw}(Y) = d(C_p(Y))$ (see Theorem 2.6 in Chapter I). Take a dense subset B of $C_p(Y)$ of cardinality κ. Let $A = f^*(B)$. Then $f^*(C_p(Y)) \subset \mathrm{cl}(A)$. By hypothesis, there exists a collection $\mathcal{N}(A)$ of subsets of X such that $|\mathcal{N}(A)| \leq |A| \leq |B| = \kappa$ and $\mathcal{N}(A)$ is a network for $\Delta\mathrm{cl}(A)(X)$ modulo $\Delta\mathrm{cl}(A)$. Since $\Delta f^*(C_p(Y))(X)$ is a continuous image of $\Delta\mathrm{cl}(A)(X)$ under the natural projection $p_{f^*(C_p(Y))}$ from $\mathbb{R}^{\mathrm{cl}(A)}$ onto $\mathbb{R}^{f^*(C_p(Y))}$, by Remark 2.1 (2) we have that $\mathcal{N}(A)$ is a network for $\Delta f^*(C_p(Y))(X)$ modulo $\Delta f^*(C_p(Y)) = p_{f^*(C_p(Y))} \circ \Delta\mathrm{cl}(A)$. By Remark 2.1 (1) we have that $\Delta f^*(C_p(Y))(\mathcal{N})$ is a network for $\Delta f^*(C_p(Y))(X)$ and hence

$$\mathrm{nw}(\Delta f^*(C_p(Y))(X)) \leq |\Delta f^*(C_p(Y))(\mathcal{N}(A))| \leq |\mathcal{N}(A)| \leq \kappa.$$

Finally, since Y is homeomorphic to $\Delta C_p(Y)(Y)$, by Corollary 2.3, Y is homeomorphic to $\Delta f^*(C_p(Y))(X)$. Thus, $\mathrm{nw}(Y) = \mathrm{nw}(\Delta f^*(C_p(Y))(X)) \leq \kappa = \mathrm{iw}(Y)$. This shows that the space X is stable. □

Next, we introduce a property which we will prove is the dual property of being monotonically monolithic between X and $C_p(X)$.

Definition 2.5. We say that a space X is *monotonically stable* if for each $A \subset C_p(X)$, we can assign a family $\mathcal{O}(A)$ of subsets of X which is a network for Δcl(A)(X) modulo Δcl(A) in such a way that the following conditions hold:

(1) $|\mathcal{O}(A)| \leq \max\{|A|, \omega\}$;
(2) if $A \subset B \subset C_p(X)$, then $\mathcal{O}(A) \subset \mathcal{O}(B)$;
(3) if $\{A_\alpha : \alpha < \gamma\}$ is a family of subsets of $C_p(X)$ with $A_\alpha \subset A_\beta$ for $\alpha < \beta$, then $\mathcal{O}(\bigcup\{A_\alpha : \alpha < \gamma\}) = \bigcup\{\mathcal{O}(A_\alpha) : \alpha < \gamma\}$.

Further, for an infinite cardinal κ, X is said to be *monotonically κ-stable* if $\mathcal{O}(A)$ is defined for all sets A with $|A| \leq \kappa$ and satisfies the above conditions.

Proposition 2.6. *A topological space X is monotonically stable if and only if for each finite collection $F \subset C_p(X)$ we can assign a countable collection $\mathcal{N}(F)$ of subsets of X such that for every $A \subset C_p(X)$ the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is a network for $\Delta\mathrm{cl}(A)(X)$ modulo $\Delta\mathrm{cl}(A)$.*

Proof. First, suppose that for each $F \in [X]^{<\omega}$ we assign $\mathcal{N}(F)$ which satisfies the stated conditions. Let $\mathcal{O}(A) = \bigcup\{\mathcal{N}(F) : F \in [X]^{<\omega}\}$. It is easy to check that $\mathcal{O}$ satisfies the conditions of the definition of monotonically stable.

Now we will prove the other implication of our proposition. Suppose X is monotonically stable, witnessed by operator $\mathcal{O}$. Let $\mathcal{N}(F) = \mathcal{O}(F)$ for every finite set $F \subset C_p(X)$. We will show that $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is a network for $\Delta\mathrm{cl}(A)(X)$ modulo $\Delta\mathrm{cl}(A)$ for every $A \subset C_p(X)$. To this end, fix $A \subset C_p(X)$ and let $Y = f(X)$ where $f = \Delta\mathrm{cl}(A)$. Let $x \in X$ and U an open subset of Y with $f(x) \in U$. Then $x \in N$ and $f(N) \subset U$ for some $N \in \mathcal{O}(A)$. Let $F \subset A$ have minimal cardinality such that $N \in \mathcal{O}(F)$. We claim that F is finite. Suppose otherwise, and let $F = \{x_\alpha : \alpha < \kappa\}$ where $\kappa = |F|$. Now let $F_\alpha = \{x_\beta : \beta < \alpha\}$. Since $F = \bigcup\{F_\alpha : \alpha < \kappa\}$ then, by condition 3) in the definition of monotonically stable, we have $\mathcal{O}(F) = \bigcup\{\mathcal{O}(F_\alpha) : \alpha < \kappa\}$; so $N \in \mathcal{O}(F_\alpha)$ for some $\alpha < \kappa$. But $|F_\alpha| < |F|$; this is a contradiction. Thus, $F \in [A]^{<\omega}$ and $N \in \mathcal{O}(F) = \mathcal{N}(F)$. □

Now we will prove one of the main results in this section.

Let X and Y be topological spaces. Suppose that $\mathcal{N}(F)$ is a family of subsets of Y for each $F \in [X]^{<\omega}$. We define the assignment $\mathcal{N}'$ as follows: $\mathcal{N}'(F) = \bigcup\{\mathcal{N}(E) : E \in [F]^{<\omega}\}$ for each $F \in [X]^{<\omega}$. Notice that $\bigcup\{\mathcal{W}(\mathcal{N}'(F)) : F \in [A]^{<\omega}\} = \mathcal{W}(\bigcup\{\mathcal{N}'(F) : F \in [A]^{<\omega}\})$ for every $A \subset X$.

Theorem 2.7. *For every space X, the space $C_p(X)$ is monotonically monolithic if and only if the space X is monotonically stable.*

Proof. First suppose that X is a monotonically stable space and take an assignment $\mathcal{N}$ that witnesses this fact as in Proposition 2.6. For each finite subset F of $C_p(X)$, take $\mathcal{O}(F) = \mathcal{W}(\mathcal{N}'(F))$. Notice that $\mathcal{O}(F)$ is countable. To see that $C_p(X)$ is monotonically monolithic we show that the operator $\mathcal{O}$ satisfies the conditions in Proposition 7.7 in Chapter I. Indeed, take $A \subset C_p(X)$. Let $f = \Delta\mathrm{cl}(A)$ and $Y = f(X)$. By the election of $\mathcal{N}$, the family $\bigcup\{\mathcal{N}'(F) : F \in [A]^{<\omega}\}$ is a network for Y modulo f. By Proposition 1.4, we have that $\bigcup\{\mathcal{W}(\mathcal{N}'(F)) : F \in [A]^{<\omega}\} = \mathcal{W}(\bigcup\{\mathcal{N}'(F) : F \in [A]^{<\omega}\})$ is an external network for $f^*(C_p(Y))$ in $C_p(X)$. Notice that $\mathrm{cl}(A) \subset f^*(C_p(Y))$. In fact, if $g \in \mathrm{cl}(A)$ then $g = p \circ f = f^*(p) \in f^*(C_p(Y))$, where $p = p_g \restriction Y$ is the restriction of the projection $p_g : \mathbb{R}^{\mathrm{cl}(A)} \to \mathbb{R}$. Thus, $\bigcup\{\mathcal{O}(F) : F \in [A]^{<\omega}\} = \bigcup\{\mathcal{W}(\mathcal{N}'(F)) : F \in [A]^{<\omega}\}$ is an external network for $\mathrm{cl}(A)$ in $C_p(X)$.

Suppose now that $C_p(X)$ is a monotonically monolithic space with its respective operator $\mathcal{O}$ as in Proposition 7.7 in Chapter I. Let $\psi = \Delta C_p(X)$ be the canonical embedding of X in $C_p(C_p(X))$ (see §2 in Chapter I). For each finite set $F \subset X$, let $\mathcal{N}(F)$ be equal to $\psi^{-1}(\mathcal{W}(\mathcal{O}'(F)))$. We will show that the operator $\mathcal{N}$ satisfies the conditions in Proposition 2.6 for the space X. Clearly $\mathcal{N}(F)$ is countable. Fix $A \subset C_p(X)$. Since $\bigcup\{\mathcal{O}'(F) : F \in [A]^{<\omega}\}$ is an external network for $\mathrm{cl}(A)$ then, by Proposition 1.3, $\bigcup\{\mathcal{W}(\mathcal{O}'(F)) : F \in [A]^{<\omega}\} = \mathcal{W}(\bigcup\{\mathcal{O}'(F) : F \in [A]^{<\omega}\})$ is a network for $\pi_{\mathrm{cl}(A)}(C_p(C_p(X)))$ modulo $\pi_{\mathrm{cl}(A)}$. Let $f = \Delta\mathrm{cl}(A)$ and $Y = f(X)$. Notice that $f = p_{\mathrm{cl}(A)} \circ \psi$ where $p_{\mathrm{cl}(A)}$ is the projection from $\mathbb{R}^{C_p(X)}$ onto $\mathbb{R}^{\mathrm{cl}(A)}$. Since $\psi(X) \subset C_p(C_p(X))$ then $Y = f(X) = p_{\mathrm{cl}(A)}(\psi(X)) \subset p_{\mathrm{cl}(A)}(C_p(C_p(X))) = \pi_{\mathrm{cl}(A)}(C_p(C_p(X)))$.

We shall verify that $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is a network for Y modulo f. Take $x \in X$ such that $f(x) \in U$ for some open set $U \subset Y$. Let V be an open set in $\pi_{\mathrm{cl}(A)}(C_p(C_p(X)))$ such that $V \cap Y = U$. Since $\pi_{\mathrm{cl}(A)}(\psi(x)) = f(x) \in V$ and $\bigcup\{\mathcal{W}(\mathcal{O}'(F)) : F \in [A]^{<\omega}\}$ is a network for $\pi_{\mathrm{cl}(A)}(C_p(C_p(X)))$ modulo $\pi_{\mathrm{cl}(A)}$, there exist $E \in [A]^{<\omega}$ and $N \in \mathcal{W}(\mathcal{O}'(E))$ such that $\psi(x) \in N$ and $\pi_{\mathrm{cl}(A)}(\psi(x)) \in \pi_{\mathrm{cl}(A)}(N) \subset V$. Thus, for $M = \psi^{-1}(N) \in \psi^{-1}(\mathcal{W}(\mathcal{O}'(E))) = \mathcal{N}(E) \subset \bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$, we have that $x \in M$ and $f(x) \in f(M) = \pi_{\mathrm{cl}(A)}(\psi(M)) \subset \pi_{\mathrm{cl}(A)}(N) \cap Y \subset V \cap Y = U$. □

As we saw in Chapter I, $C_p(X)$ is monolithic if and only if X is stable and X is monolithic if and only if $C_p(X)$ is stable. By Theorems 2.7 and 1.6, we obtain the following two corollaries which show that the duality between monolithic and stable spaces hold for monotonically monolithic and monotonically stable spaces.

Corollary 2.8. *For every space X the space $C_p(X)$ is monotonically stable if and only if the space X is monotonically monolithic.*

Proof. $C_p(X)$ is monotonically stable if and only if $C_p(C_p(X))$ is monotonically monolithic if and only if X is monotonically monolithic. □

Corollary 2.9. *The space $C_p(C_p(X))$ is monotonically stable if and only if X is monotonically stable.*

Remark 2.10. Naturally, statements in Theorem 1.6, Theorem 2.7, Corollary 2.8 and Corollary 2.9 remain true if we replace monotonically monolithic and monotonically stable by monotonically κ-monolithic and monotonically κ-stable, respectively.

Now we preve basic properties of monotonically stable spaces.

Proposition 2.11. *Let $f : X \to Y$ be a continuous function from a space X onto a space Y. If X is monotonically stable, then Y is monotonically stable.*

Proof. By Theorem 2.7, $C_p(X)$ is monotonically monolithic. Since the space $C_p(Y)$ is homeomorphic to $f^*(C_p(Y))$ and monotone monolithicity is hereditary, then $C_p(Y)$ shares this property. Finally, by Theorem 2.7, Y is monotonically stable. □

Proposition 2.12. *A space which is the union of a countable set of monotonically stable subspaces is monotonically stable.*

Proof. Let $Y = \bigcup\{Y_n : n \in \omega\}$ where each Y_n is a monotonically stable space. Then $C_p(Y_n)$ is monotonically stable for every $n \in \omega$. Let us consider the free topological sum $X = \oplus\{Y_n : n \in \omega\}$ of the family $\{Y_n : n \in \omega\}$. By Theorem 7.3 in Chapter I, the space $C_p(X) = \prod\{C_p(Y_n) : n \in \omega\}$ is monotonically monolithic, so X is monotonically stable. Clearly, Y is a continuous image of X, then Y is monotonically stable. □

Proposition 2.13. *Let κ a cardinal. If $C_p(X_\alpha)$ is monotonically stable for any $\alpha < \kappa$, then the space $\prod\{C_p(X_\alpha) : \alpha < \kappa\}$ is monotonically stable.*

Proof. First, notice that X_α is monotonically monolithic for every $\alpha < \kappa$. Let us consider the free topological sum $X = \oplus\{X_\alpha : \alpha < \kappa\}$. It is not difficult to verify that the union of a locally finite family of closed monotonically monolithic subsets of a space is monotonically monolithic. In particular, X is monolithic. Thus $C_p(X) = \prod\{C_p(X_\alpha) : \alpha < \kappa\}$ is monotonically stable. □

Corollary 2.14. *If a space $C_p(X)$ is monotonically stable, then the space $(C_p(X))^\kappa$ is monotonically stable for every cardinal κ.*

It is easy to deduce from Proposition 2.4 that any monotonically stable space is stable. By Proposition 7.5 in Chapter I and Theorem 2.7, Lindelöf Σ-spaces and in particular compact spaces are monotonically stable. As we have seen, $[0,\omega_1]$ is not monotonically monolithic, so $C_p([0,\omega_1])$ is a stable space which is not monotonically stable. Moreover, if X is the subspace of $[0,\omega_2)$ of all ordinals of uncountable cofinality, X is monotonically ω-monolithic but not monotonically ω_1-monolithic [**1**], so $C_p(X)$ is monotonically ω-stable but not monotonically ω_1-stable. A discrete space D of cardinality $\mathfrak{c}$ is not even ω-stable, but is homeomorphic to a closed subspace of $\mathbb{R}^{\mathfrak{c}}$. Thus, monotone stability is not inherited, in general, by closed subspaces and uncountable unions.

3. Monotone monolithicity of $C_p(X)$ when X is a Σ-product

In Example 1.10 we saw that $C_p(C_p(L_\kappa))$ is monotonically monolithic, where L_κ is the one-point Lindelöfication of the discrete space of cardinality κ. It is well known that $C_p(L_\kappa)$ is homeomorphic to the Σ-product in a product of κ copies of the real line. Also, it is known that $C_p(X)$ is monolithic when X is a Σ-product in a product of cosmic spaces and even in a product of Lindelöf Σ-spaces (see [**3**]). So, is natural to ask if these results hold for monotone monolithicity.

In this section we show that, indeed, in the case of products of cosmic spaces the answer is positive by showing a more general result. We use such a result to give another proof of the fact that $C_p(X)$ is monotonically monolithic when X is a compact space (see Theorem 6.1 in Chapter I). On the other hand, we can only prove that $C_p(X)$ is monotonically monolithic when X is the product of a family of Lindelöf Σ-spaces.

The following technical result is well known and easy to prove; we present the proof for completeness.

Lemma 3.1. *Let $X = \prod\{X_\alpha : \alpha < \kappa\}$ be a topological product, F a subset of X, S a subset of κ and $f : F \to \mathbb{R}$ a continuous function. If $p_S \restriction F : F \to p_S(F)$ is a quotient function and $f(x) = f(x')$ whenever $x, x' \in F$ and $p_S(x) = p_S(x')$, then there exists $g : \pi_S(F) \to \mathbb{R}$, continuous, such that $f = g \circ p_S \restriction F$.*

Proof. We define $g : p_S(F) \to \mathbb{R}$ as follows: if $y = p_S(x) \in p_S(F)$ for some $x \in F$, we take $g(y) = f(x)$. Since $f(x) = f(x')$ for each pair $x, x' \in X$ with $p_S(x) = p_S(x')$, g is well defined. In order to prove that g is continuous, observe that for an open subset U of $\mathbb{R}$, the continuity of f implies that the set $f^{-1}(U) = (p_S \restriction F)^{-1}(g^{-1}(U))$ is open in F. Since $p_S \restriction F$ is a quotient function, then $g^{-1}(U)$ is open in $p_S(F)$. Because of the definition of g, we have $f = g \circ p_S \restriction F$. $\square$

Now we are ready to prove one of our main results in this section.

Theorem 3.2. *Let $X = \prod\{X_\alpha : \alpha < \kappa\}$ be a product of cosmic spaces and let Y be a dense subset of X. Suppose that F is a C-embedded subspace of Y such that for each $S \subset \kappa$ the function $q_S = p_S \restriction F : F \to p_S(F)$ is a quotient function. Then $C_p(F)$ is monotonically monolithic.*

Proof. For each $f \in C_p(F)$, since F is a C-embedded subspace of Y, we can fix an extension $\tilde{f} \in C_p(Y)$ of f. By Lemma 1.5, since Y is dense in X and $\tilde{f} : Y \to \mathbb{R}$ is a continuous function, we can fix a countable set $S(f) \subset \kappa$, and a continuous function $g_{S(f)} : \pi_{S(f)}(Y) \to \mathbb{R}$ such that $\tilde{f}(x) = g_{S(f)} \circ p_{S(f)} \restriction Y$. If $A \subset C_p(Y)$, let $\mathcal{S}(A) = \bigcup\{S(f) : f \in A\}$. Observe that operator $\mathcal{S}$ has the following properties:

(1) if $A \subset C_p(Y)$, then $|\mathcal{S}(A)| \le \max\{|A|, \omega\}$;
(2) if $A \subset B \subset C_p(Y)$, then $\mathcal{S}(A) \subset \mathcal{S}(B)$;
(3) if $\{A_\alpha : \alpha < \gamma\}$ is a family of subsets of $C_p(Y)$ with $A_\alpha \subset A_\beta$ when $\alpha < \beta < \gamma$, then $\mathcal{S}(\bigcup\{A_\alpha : \alpha < \gamma\}) = \bigcup\{\mathcal{S}(A_\alpha) : \alpha < \gamma\}$.

For each $\alpha < \kappa$, let $\mathcal{N}(X_\alpha)$ be a countable network of X_α. For each $S \subset \kappa$ we will denote by $\mathcal{N}_F(S)$ the collection of all subsets of X of the form $\prod\{N_\alpha : \alpha < \kappa\} \cap F$ where $N_\alpha \in \mathcal{N}(X_\alpha)$ if $\alpha \in T$ and $N_\alpha = X_\alpha$ for $\alpha \in \kappa \setminus T$, where T is a finite subset of S. Observe that $\mathcal{N}_F(S)$ is a network for $q_S(F)$ modulo q_S for each $S \subset \kappa$, and that the operator $\mathcal{N}_F$ has the following properties:

(1) if $S \subset \kappa$, then $|\mathcal{N}_F(S)| \le \max\{|S|, \omega\}$;
(2) if $S \subset S' \subset \kappa$, then $\mathcal{N}_F(S) \subset \mathcal{N}_F(S')$;
(3) if $\{S_\alpha : \alpha < \gamma\}$ is a family of subsets of κ with $S_\alpha \subset S_\beta$ when $\alpha < \beta$, then $\mathcal{N}_F(\bigcup\{S_\alpha : \alpha < \gamma\}) = \bigcup\{\mathcal{N}_F(S_\alpha) : \alpha < \gamma\}$.

Now we are ready to construct a monotonic monolithicity operator in $C_p(F)$. If $A \subset C_p(F)$, let $\mathcal{O}(A) = \mathcal{W}(\mathcal{N}_F(\mathcal{S}(A)))$. Because of Remark 1.1 and the above properties of the operators $\mathcal{S}$ and $\mathcal{N}_F$, it is easy to verify that the operator $\mathcal{O}$ satisfies conditions (1), (2) and (3) in Definition 7.2. So, in order to finish the proof it is enough to prove:

Claim. $\mathcal{O}(A)$ is an external network of $\mathrm{cl}(A)$ in $C_p(F)$ for every $A \subset C_p(F)$.

We shall prove the Claim. Since $\mathcal{N}_F(\mathcal{S}(A))$ is a network for $q_{\mathcal{S}(A)}(F)$ modulo $q_{\mathcal{S}(A)}$, the family $\mathcal{O}(A) = \mathcal{W}(\mathcal{N}_F(\mathcal{S}(A)))$ is an external network for $q^*_{\mathcal{S}(A)}(C_p(q_{\mathcal{S}(A)}(F)))$ in $C_p(F)$ (see Proposition 1.4). Thus, in order to prove that $\mathcal{O}(A)$ is an external network of $\mathrm{cl}(A)$ in $C_p(F)$, it is enough to show that $\mathrm{cl}(A) \subset q^*_{\mathcal{S}(A)}(C_p(q_{\mathcal{S}(A)}(F)))$.

Let $g \in \mathrm{cl}(A)$. Take $x, x' \in F$ such that $q_{\mathcal{S}(A)}(x) = q_{\mathcal{S}(A)}(x')$. For each $f \in A$, because of the definition of $\mathcal{S}(A)$ we have that $p_{S(f)}(x) = p_{S(f)}(x')$ and so $f(x) = \tilde{f}(x) = g_{S(f)} \circ p_{S(f)}(x) = g_{S(f)} \circ p_{S(f)}(x') = \tilde{f}(x') = f(x')$. If $g(x) \neq g(x')$, we can choose disjoint open subsets $B, B' \in \mathcal{B}(\mathbb{R})$ with $g(x) \in B$ and $g(x') \in B'$. Then, the open set $U = \{f \in C_p(F) : f(x) \in$

B and $f(x') \in B'\}$ contains g and has an empty intersection with A; which is not possible since $g \in \mathrm{cl}(A)$. Thus $g(x) = g(x')$. Since the projection $q_{\mathcal{S}(A)} : F \to q_{\mathcal{S}(A)}(F)$ is a quotient map, by Lemma 3.1, there is a continuous function $g_{\mathcal{S}(A)} : q_{\mathcal{S}(A)}(F) \to \mathbb{R}$ such that $g = g_{\mathcal{S}(A)} \circ q_{\mathcal{S}(A)} = q^*_{\mathcal{S}(A)}(g_{\mathcal{S}(A)}) \in q^*_{\mathcal{S}(A)}(C_p(q_{\mathcal{S}(A)}(F)))$. Therefore, $\mathrm{cl}(A) \subset q^*_{\mathcal{S}(A)}(C_p(q_{\mathcal{S}(A)}(F)))$. □

Remark 3.3. Let us observe that if in the above result we only have that for each $S \subset \kappa$ with cardinality at most λ, the function $q_S = p_S \restriction F : F \to p_S(F)$ is a quotient function, then we can conclude that $C_p(F)$ is monotonically λ-monolithic.

Suppose that $\eta = \{X_t : t \in T\}$ is a family of topological spaces, $X = \prod\{X_t : t \in T\}$ is the topological product of the family η, x^* is a point in X and κ is a cardinal number. Then the Σ_κ*-product* of η with basic point x^* is the subspace of X consisting of all points $x \in X$ such that only less than κ coordinates $x(t)$ of x are distinct from the corresponding coordinates $x^*(t)$ of x^*. This subspace is denoted by $\Sigma_\kappa\{X_t : t \in T\}$ or by $\Sigma_\kappa\eta$. Sometimes $\Sigma_\kappa\eta$ will be called the Σ_κ-product of η at x^*.

Let us observe that given a family η of topological spaces, then $\Sigma\eta = \Sigma_{\omega_1}\eta$ and $\sigma\eta = \Sigma_\omega\eta$.

We know that if $X = \prod\{X_t : t \in T\}$ is the topological product of the family $\prod\{X_t : t \in T\}$, x^* is a point in X, and κ is a cardinal number; then for $Y = \Sigma_\kappa\{X_t : t \in T\}$ and every $S \subset T$ the function $p_S \restriction Y : Y \to p_S(Y)$ is open and hence is a quotient function. So we can apply Theorem 3.2 to obtain:

Corollary 3.4. *Let κ be an infinite cardinal number and Y a Σ_κ-product of a family of cosmic spaces, then $C_p(Y)$ is monotonically monolithic.*

Corollary 3.5. *Let κ be an infinite cardinal number and Y a Σ_κ-product of a family of cosmic spaces, then Y is monotonically stable.*

Corollary 3.6. *If X is a σ-product of a family of spaces, each of them with countable network, then $C_{p,n}(X)$ is monotonically monolithic for every $n \in \omega$.*

Proof. Since every space with countable network is monotonically monolithic, by Theorem I.7.3 (2), we have that X is monotonically monolithic. Because of Corollary 3.4, the space $C_p(X)$ is monotonically monolithic. By Corollary 1.8, it happens that $C_{p,n}(X)$ is monotonically monolithic for any $n \in \omega$. □

Now we give another proof of Theorem 6.1 for compact spaces.

Corollary 3.7. *$C_p(X)$ is monotonically monolithic for any compact space X.*

Proof. Let X be a compact space. Take an embedding $f : X \to \mathbb{R}^\kappa$ for some cardinal κ. Take $F = f(X)$ and $Y = \mathbb{R}^\kappa$. Clearly, $p_S \restriction F : F \to p_S(F)$

being closed is a quotient function, for any $S \subset \kappa$. By Theorem 3.2, $C_p(F)$ is monotonically monolithic. Thus $C_p(X)$ is monotonically monolithic. $\square$

Lindelöf Σ-spaces are a natural generalization of cosmic spaces. Indeed, cosmic spaces are exactly the Lindelöf σ-spaces, so it is interesting to know when the results obtained until now hold for Lindelöf Σ-spaces. We will work in this direction.

We will need the following results.

Lemma 3.8. [**3**] *Let $X = \prod\{X_\alpha : \alpha < \kappa\}$ be a topological product, $T = \sigma \prod\{X_\alpha : \alpha < \kappa\}$, Y a subspace of X which contains T and λ an infinite cardinal. Assume that for each finite set $K \subset \kappa$, $l(\prod\{X_\alpha : \alpha \in K\}) \leq \lambda$. If $f : Y \to Z$ is a continuous and* $\mathrm{iw}(Z) \leq \lambda$, *then there is a set $S \subset \kappa$ with $|S| \leq \lambda$ such that if $x, x' \in Y$ and $p_S(x) = p_S(x')$ then $f(x) = f(x')$.*

Lemma 3.9. [**29**] *Let $X = \prod\{X_\alpha : \alpha < \kappa\}$ be a product of topological spaces. Let K_α be a compact subset of X_α for each $\alpha < \kappa$. If U is an open subset of X such that $K = \prod\{K_\alpha : \alpha < \kappa\} \subset U$, then there exist a finite set $S \subset \kappa$ and a family $\{U_\alpha : \alpha \in S\}$ with $K_\alpha \subset U_\alpha \subset X_\alpha$ for each $\alpha \in S$ such that $K \subset \bigcap\{p_\alpha^{-1}(U_\alpha) : \alpha \in S\} \subset U$.*

The following result gives a generalization of Proposition 7.5 in Chapter I.

Theorem 3.10. *Let $X = \prod\{X_\alpha : \alpha < \kappa\}$ be a product of Lindelöf Σ-spaces. Then, $C_p(X)$ is monotonically monolithic.*

Proof. For each $\alpha < \kappa$, let $\mathcal{K}_\alpha$ be a compact cover of X_α and let $\mathcal{N}_\alpha$ be a countable network for X_α modulo $\mathcal{K}_\alpha$. For $S \subset \kappa$, we denote by $\mathcal{N}_S$ the family of all subsets of X of the form $\bigcap\{p_\alpha^{-1}(N_\alpha) : \alpha \in S_0\}$ where $N_\alpha \in \mathcal{N}_\alpha$ for each $\alpha \in S_0$ and S_0 is a finite subset of S. Note that $|\mathcal{N}_S| \leq \max\{|S|, \omega\}$; in particular, if S is countable then $\mathcal{N}_S$ is countable. For each family $\mathcal{E}$ of subsets of X, let $\mathcal{F}(\mathcal{E})$ be the collection $\{\bigcup \mathcal{A} : \mathcal{A} \subset \mathcal{E}$ and $|\mathcal{A}| < \omega\}$ and $\mathcal{C}_S(\mathcal{E}) = \{N \setminus E : N \in \mathcal{N}_S, E \in \mathcal{E}\}$.

Let $f : X \to \mathbb{R}$ be a continuous function. The product of a finite collection of Lindelöf Σ-spaces is a Lindelöf Σ-space. Moreover, $\mathrm{iw}(\mathbb{R}) = \omega$. So, by Lemma 3.8, we can fix a countable set $S(f) \subset \kappa$ such that if $x, x' \in X$ and $p_S(x) = p_S(x')$ then $f(x) = f(x')$. Since the projection $p_S : X \to X_S = \prod\{X_\alpha : \alpha \in S\}$ is open for each $S \subset \kappa$, there exists a continuous function $g(f) : X_{S(f)} \to \mathbb{R}$ such that $f = g(f) \circ p_{S(f)}$ (see Proposition 4.2). For a finite set $F \subset C_p(X)$, let $S(F) = \bigcup\{S(g) : g \in F\}$, $\mathcal{E}_F = \{g^{-1}(B) : g \in F, B \in \mathcal{B}(\mathbb{R})\}$ and $\mathcal{N}(F) = \mathcal{W}(\mathcal{C}_{S(F)}(\mathcal{F}(\mathcal{E}_F)))$. Observe that the family $\mathcal{N}(F)$ is countable.

In order to prove that $C_p(X)$ is monotonically monolithic, it is enough to prove, because of Proposition I.7.7, that if $A \subset C_p(X)$ then $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network of $\mathrm{cl}(A)$ in $C_p(X)$. Take $A \subset C_p(X)$ and $f \in \mathrm{cl}(A)$.

Claim. If $x \in X$, $B \in \mathcal{B}(\mathbb{R})$ and $f(x) \in B$, then there are $F \subset A$, finite, and $P \in \mathcal{C}_{S(F)}(\mathcal{F}(\mathcal{E}_F))$ such that $x \in P$ and $f \in [P\,; B]$.

We are going to prove the Claim. For each $\alpha < \kappa$, let $K_\alpha \in \mathcal{K}_\alpha$ with $x(\alpha) \in K_\alpha$. Thus, $x \in K = \prod\{K_\alpha : \alpha < \kappa\}$. For every $y \in K \setminus f^{-1}(B)$, take a $B_y \in \mathcal{B}(\mathbb{R})$ such that $f(y) \in B_y$ and $f(x) \notin \mathrm{cl}(B_y)$, and take a function $g_y \in A$ such that $g_y(x) \in B \setminus \mathrm{cl}(B_y)$ and $g_y(y) \in B_y$. The family $\{g_y^{-1}(B_y) : y \in K \setminus f^{-1}(B)\} \cup \{f^{-1}(B)\}$ covers K. Since K is compact, there is a finite subset T_0 of $K \setminus f^{-1}(B)$ such that $\{g_y^{-1}(B_y) : y \in T_0\} \cup \{f^{-1}(B)\}$ covers K. If $U = \bigcup\{g_y^{-1}(B_y) : y \in T_0\} \cup f^{-1}(B)$, then $K \subset U$. By Lemma 3.9, there are a finite subset $S_0 \subset \kappa$ and a family $\{U_\alpha : \alpha \in S_0\}$ of open subsets with $K_\alpha \subset U_\alpha \subset X_\alpha$ for each $\alpha \in S_0$, such that $K \subset \bigcap\{p_\alpha^{-1}(U_\alpha) : \alpha \in S_0\} \subset U$. Let $S(A) = \bigcup\{S(g) : g \in A\}$ and $S = S_0 \cap S(A)$. Hence, $K \subset \bigcap\{p_\alpha^{-1}(U_\alpha) : \alpha \in S\} \subset U$. In fact, this follows from the fact that for each $g \in A$, $g \restriction E$ is constant for every set E of the form $p_{S(A)}^{-1}(p_{S(A)}(x))$. Thus, each $h \in \mathrm{cl}(A)$ satisfies this property too. For each $\alpha \in S$, take $g_\alpha \in A$ with $a \in S(g_\alpha)$. Since for each $\alpha \in S$ we have that $K_\alpha \in \mathcal{K}_\alpha$, $K_\alpha \subset U_\alpha$ and $\mathcal{N}_\alpha$ is a network for X_α modulo $\mathcal{K}_\alpha$, there exists $N_\alpha \in \mathcal{N}_\alpha$ with $K_\alpha \subset N_\alpha \subset U_\alpha$. Let $N = \bigcap\{p_\alpha^{-1}(N_\alpha) : \alpha \in S\}$. Then $K \subset N \subset U$. Finally, let $P = N \setminus \bigcup\{g_y^{-1}(B_y) : y \in T_0\}$, then $x \in P$, $P \in \mathcal{C}_{S(F)}(\mathcal{F}(\mathcal{E}_F))$ for $F = \{g_y : y \in T_0\} \cup \{g_\alpha : \alpha \in S\} \subset A$ and $f \in [P\,; B]$. This ends the proof of the Claim.

Now suppose that $f \in U$ for an open subset U of $C_p(X)$. Take $x_1, \dots, x_n \in X$ and $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ such that $f \in [x_1, \dots, x_n; B_1, \dots, B_n] \subset U$. Because of the Claim, for each $i = 1, \dots, n$ there are $F_i \subset A$, finite, and $P_i \in \mathcal{C}_{S(F_i)}(\mathcal{F}(\mathcal{E}_{F_i}))$ such that $x_i \in P_i$ and $f \in [P_i\,; B_i]$. Take $F = \bigcup\{F_i : i = 1, \dots, n\} \subset A$; then $P_i \in \mathcal{C}_{S(F)}(\mathcal{F}(\mathcal{E}_F))$ for each $i = 1, \dots, n$. Finally, if $M = [P_1, \dots, P_n; B_1, \dots, B_n]$, then $f \in M \subset [x_1, \dots, x_n; B_1, \dots, B_n] \subset U$, where $M \in \mathcal{W}(\mathcal{C}_{S(F)}(\mathcal{F}(\mathcal{E}_F))) = \mathcal{N}(F)$. Therefore, $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network of $\mathrm{cl}(A)$ in $C_p(X)$. □

Corollary 3.11. *Any product of Lindelöf Σ-spaces is monotonically stable.*

4. Monotone monolithicity of $C_p(X)$ and cozero sets

In [**17**] R. Buzyakova investigated how the Lindelöf property of the function space $C_p(X, Y)$ is influenced by slight changes in X or Y. In particular, it is interesting to know what happens to $C_p(X)$ if we remove one point from X. It is known that removing a point of countable tightness from a zero-dimensional compact space may destroy the Lindelöf property of a function space. For example, let $X = \mathcal{D} \cup \{\infty\}$ be the one-point compactification of an uncountable discrete space $\mathcal{D}$. It is known that X is an Eberlein compact space (see [**3**]). Therefore, $C_p(X)$ is Lindelöf. The space $C_p(\mathcal{D})$ is not Lindelöf because it is homeomorphic to $\mathbb{R}^{\mathcal{D}}$. However, R. Buzyakova proved that removing a point of countable character from a zero-dimensional compact space does not affect the Lindelöf property of the function space. She asked if the same holds for every compact space, or for any space X.

O.G. Okunev [**48**] proved that if Y is a cozero set in X, then $C_p(Y)$ is a continuous image of a closed subset of $C_p(X)^\omega$. Okunev's result generalizes the result of R. Buzyakova. Indeed, he also proved that if $C_p(X)$ is Lindelöf, then $C_p(X)^\omega$ shares this property whenever X is a σ-compact zero-dimensional space. Thus, removing even a zero set from a zero-dimensional σ-compact space does not affect the Lindelöf property of the function space.

In this section we will improve the first of the above of Okunev's results by showing that: if Y is a cozero set in X, then $C_p(Y)$ is homeomorphic to a closed subset of $C_p(X)^\omega$. As a consequence we have that if $C_p(X)$ is monotonically monolithic and Y is a cozero subset of X, then $C_p(Y)$ is monotonically monolithic.

Theorem 4.1. *Let X be a space and let Y be a cozero subset of X. Then, $C_p(Y)$ is homeomorphic to a closed subspace of $C_p(X)^\omega$.*

Proof. We began the proof of our theorem following Theorem 2.1 in [**48**]. Let $h : X \to [0,1]$ be a continuous function such that $Y = h^{-1}((0,1])$. For each $n \in \omega$, define $F_n = h^{-1}([1/(n+1),1])$ and $F = X \setminus Y$. Of course, F and F_n, $n \in \omega$, are zero sets. Note that $F_n \subset \operatorname{int}(F_{n+1})$ and $Y = \bigcup\{F_n : n \in \omega\}$. Consider the closed subset $Z = \{f \in C_p(X) : f(F) \subset \{0\}\}$ of $C_p(X)$. Consider the set

$$P = \{G \in Z^\omega : G(n) \restriction F_n = G(m) \restriction F_n \text{ for each } n, m \in \omega, m \geq n\}.$$

Then

$$P = \bigcap_{n\in\omega} \bigcap_{m\geq n} \bigcap_{x\in F_n} \{G \in Z^\omega : G(n)(x) = G(m)(x)\}.$$

So, P is closed in Z^ω and hence in $C_p(X)^\omega$. Now, we are going to prove that $C_p(Y)$ is a continuous image of P. Take the function $T : P \to \mathbb{R}^Y$ defined as

$$T(G)(x) = G(n)(x) \text{ if } x \in F_n.$$

It is clear that T is well defined. Let $G \in P$ and $x \in Y$. Thus, $x \in F_n \subset \operatorname{int}(F_{n+1})$ for some $n \in \omega$. Since $T(G) \restriction F_{n+1} = G(n+1) \restriction F_{n+1}$, $T(G)$ coincides with the continuous function $G(n+1)$ in F_{n+1} which is a neighborhood of x. Therefore, $T(G)$ is continuous at x. So, we obtain that $T(G)$ is continuous on Y, and so $T(P) \in C_p(Y)$.

Now, we are going to show that $C_p(Y) \subset T(P)$. Fix a continuous function $\theta : [0,1] \to [0,1]$ such that $\theta(1) = 1$ and $\theta([0,1/2]) = \{0\}$. Since F and F_n are zero sets for every $n \in \omega$, we can also fix, for each $n \in \omega$, a continuous function $h_n : X \to [0,1]$ such that $h_n(F) \subset \{0\}$ and $h_n(F_n) \subset \{1\}$. Take $s_n = \theta \circ h_n$. Then $s_n : X \to [0,1]$ is continuous, $s_n(F_n) = \{1\}$ and s_n is equal to 0 in some neighborhood of F. If $f \in C_p(Y)$, then the function

$g(f,n) : X \to \mathbb{R}$ defined by the rule

$$g(f,n)(x) = \begin{cases} f(x)s_n(x) & \text{if } x \in Y; \\ 0 & \text{if } x \in F, \end{cases}$$

is continuous on X and coincides with f on F_n. Note that if $G_f \in C_p(X)^\omega$ is defined as $G_f(n) = g(f,n)$ for each $n \in \omega$, then $G_f \in P$ and $T(G_f) = f$. This concludes the proof that $T(P) = C_p(Y)$. Now, we are going to prove that T is a continuous function. Take $[x; B]$ in $C_p(Y)$ where $x \in Y$ and $B \in \mathcal{B}(\mathbb{R})$. Let $m \in \omega$ such that $x \in F_m$. Hence $x \in F_n$ for each $n \geq m$; so $G(n)(x) = G(m)(x)$ for each $G \in P$ and $n \geq m$. Therefore, the set

$$T^{-1}([x, B]) = \{G \in P : G(m)(x) \in B\} = P \cap \{H \in C_p(X)^\omega : H(m)(x) \in B\}$$

is an open subset of P. Thus $C_p(Y)$ is a continuous image of the subspace P of $C_p(X)^\omega$.

Until here, we follow Okunev's proof. Now, we are going to go a little further, we are going to prove that, in fact, $C_p(Y)$ is homeomorphic to a closed subset of $C_p(X)^\omega$.

For each $n \in \omega$, consider the function $S_n : C_p(Y) \to C_p(X)$ given by $S_n(f) = g(f,n)$ for each $f \in C_p(Y)$. We now prove that S_n is a continuous function. Let $A \subset C_p(Y)$ and $f \in \mathrm{cl}_{C_p(Y)}(A)$. Let $W = [S_n(f); K; \epsilon] = \{t \in C_p(X) : |S_n(f)(x) - t(x)| < \epsilon,$ for each $x \in K\}$ be a canonical neighborhood of $S_n(f)$, where $K \subset X$ is finite and $\epsilon > 0$. Take $K_0 = K \cap Y$ and $r \in [f; K_0; \epsilon] \cap A$. If $x \in K \cap F$, then $|S_n(r)(x) - S_n(f)(x)| = 0 < \epsilon$. If $x \in K \cap Y = K_0$, then

$$|S_n(r)(x) - S_n(f)(x)| = |r(x)s_n(x) - f(x)s_n(x)| \leq |r(x) - f(x)| < \epsilon.$$

This proves that $S_n(r) \in W \cap S_n(A)$. Thus, $S_n(f) \in \mathrm{cl}_{C_p(X)}(S_n(A))$. So, we have proved that each S_n is continuous. Take the diagonal of the family of functions $\{S_n : n \in \omega\}$: $S = \Delta\{S_n : n \in \omega\} : C_p(Y) \to C_p(X)^\omega$. Then, S is a continuous function. Moreover, $S(f) = G_f$; so, $S(C_p(Y)) \subset P$ and $T(S(f)) = f$.

Since $T(S(f)) = f$ for each $f \in C_p(Y)$, S is injective. Hence, from the continuity of S and T we can deduce that S is a homeomorphism from $C_p(Y)$ to the subspace $S(C_p(Y))$ of $C_p(X)^\omega$.

Finally we are going to prove that $S(C_p(Y))$ is a closed subset of $C_p(X)^\omega$. Since P is closed in $C_p(X)^\omega$, it is enough to prove that $S(C_p(Y))$ is closed in P. We will first prove that if $G \in P$ and, if for each $n, m \in \omega$ with $n < m$, and $x \in F_m$ we have $G(n)(x) = G(m)(x)s_n(x)$, then $G \in S(C_p(Y))$. It is sufficient to prove that $G = S(T(G))$. Let $n \in \omega$ and $x \in X$. If $x \in F$ then $G(n)(x) = 0 = S(T(G))(n)(x)$. If $x \in Y$, we can choose $m \in \omega$ such that $n < m$ and $x \in F_m$. Hence

$$G(n)(x) = G(m)(x)s_n(x) = T(G)(x)s_n(x) = S(T(G))(n)(x).$$

This concludes the proof of the equality $G = S(T(G))$. On the other hand, if $G = G_f \in S(C_p(Y))$ for some $f \in C_p(Y)$, then for each $n, m \in \omega$ with $n < m$ and $x \in F_m$ we have $G(n)(x) = g(f, n)(x) = f(x)s_n(x) = G(m)(x)s_n(x)$. Therefore, $G \in S(C_p(Y))$ if and only if $G \in P$ and $G(n)(x) = G(m)(x)s_n(x)$ for each $n, m \in \omega$ with $n < m$ and $x \in F_m$; that is,

$$S(C_p(Y)) = P \cap \bigcap_{m\in\omega} \bigcap_{n<m} \bigcap_{x\in F_m} \{G \in C_p(X)^\omega : G(n)(x) = G(m)(x)s_n(x)\}.$$

So, $S(C_p(Y))$ is closed in P. □

Corollary 4.2. *Let $\mathcal{P}$ be a class of spaces which is invariant with respect to countable powers and closed subsets. If $C_p(X) \in \mathcal{P}$ and Y is a cozero subset of X, then $C_p(Y) \in \mathcal{P}$.*

Since every open F_σ subset of a normal space is a cozero set, we obtain:

Corollary 4.3. *Let $\mathcal{P}$ be a class of spaces invariant with respect to countable powers and closed subsets. If X is normal, $C_p(X) \in \mathcal{P}$ and Y is an F_σ open subset of X, then $C_p(Y) \in \mathcal{P}$.*

As we saw in Chapter I, the class of monotonically monolithic spaces and the class of Collins-Roscoe spaces are invariant with respect to countable powers and arbitrary subsets. Also, X is monotonically stable if and only if $C_p(X)$ is monotonically monolithic (see Theorem 2.7). So:

Corollary 4.4. *If $C_p(X)$ is a monotonically monolithic space (resp., has the Collins-Roscoe property) and Y is a cozero subset of X, then $C_p(Y)$ is monotonically monolithic (resp., has the Collins-Roscoe property).*

Corollary 4.5. *If X is a monotonically stable space and Y is a cozero subset of X, then Y is monotonically stable.*

5. Monotone κ-monolithicity and function spaces

In this section we use the ideas in [**38**] to give a characterization of monotonically κ-monolithic spaces and use it to prove that any monotonically κ-monolithic space of tightness κ is monotonically monolithic. As a consequence, we explain one result in [**1**] by proving that: If X is a monotonically ω-monolithic countably compact space, then X is a monotonically monolithic Corson compact space and has the Fréchet-Urysohn property. Finally, we find some spaces X for which $C_p(X)$ is monotonically κ-monolithic. These spaces will be used in the next section to obtain spaces $C_p(X)$ with the hereditary D-property.

Proposition 5.1. *A space X is monotonically κ-monolithic if and only if one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that, for every set $A \subset X$ with $|A| \leq \kappa$, $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ contains an external network for* $\mathrm{cl}(A)$.

Proof. If one can assign to each finite subset F of X a countable collection $\mathcal{N}(F)$ of subsets of X such that $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ contains a network for $\mathrm{cl}(A)$ for every subset A of X with $|A| \leq \kappa$, then we denote $\mathcal{O}(A) = \bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ for every subset A of X with $|A| \leq \kappa$. It is easy to see that operator $\mathcal{O}$ is a monotonically κ-monolithic operator for X, as in Definition 9.1 in Chapter I.

Let X be a monotonically κ-monolithic space and let $\mathcal{O}$ be a monotonically κ-monolithic operator for X as in Definition 9.1 in Chapter I. If $F \subset X$ and $|F| < \omega$, then $\mathcal{N}(F) = \mathcal{O}(F)$ is a countable family of subsets of X. We show that $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ contains a network for $\mathrm{cl}(A)$ for every subset A of X with $|A| \leq \kappa$. Let $A \subset X$ with $|A| \leq \kappa$. If $x \in \mathrm{cl}(A)$ and U is an open neighborhood of x, then $x \in N \subset U$ for some $N \in \mathcal{O}(A)$. Let $F \subset A$ have minimal cardinality such that $N \in \mathcal{O}(F)$. We claim that F is finite. Suppose otherwise, and let $F = \{x_\alpha : \alpha < \lambda\}$ where $\lambda = |F|$. Now let $F_\alpha = \{x_\beta : \beta < \alpha\}$. Since $F = \bigcup\{F_\alpha : \alpha < \lambda\}$ then, by condition 3) in the Definition 9.1 of Chapter I, we have $\mathcal{O}(F) = \bigcup\{\mathcal{O}(F_\alpha) : \alpha < \lambda\}$; so $N \in \mathcal{O}(F_\alpha)$ for some $\alpha < \lambda$. But $|F_\alpha| < |F|$; this is a contradiction. Thus, $F \in [A]^{<\omega}$ and $N \in \mathcal{O}(F) = \mathcal{N}(F)$. □

Proposition 5.2. *If X is monotonically κ-monolithic and $t(X) \leq \kappa$, then X is monotonically monolithic.*

Proof. By Proposition 5.1, we can assign to each finite subset $F \subset X$, a countable collection $\mathcal{N}(F)$ such that, for each subset A of X with $|A| \leq \kappa$, the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network for $\mathrm{cl}(A)$. By Theorem 7.7 in Chapter I, in order to get a proof of our proposition, it is enough to prove that for each $A \subset X$ the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network of $\mathrm{cl}(A)$. So, take $A \subset X$ and $x \in \mathrm{cl}(A)$. Let U be an open subset of X with $x \in U$. Since $t(X) \leq \kappa$, there is $B \subset A$ with $|B| \leq \kappa$ such that $x \in \mathrm{cl}(B)$. Because $\bigcup\{\mathcal{N}(F) : F \in [B]^{<\omega}\}$ is an external network of $\mathrm{cl}(B)$, we can take a finite set $F \subset B \subset A$, and $N \in \mathcal{N}(F)$ such that $x \in N \subset U$. This proves that $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network for $\mathrm{cl}(A)$. □

O. Alas, V.V. Tkachuk and R.G. Wilson asked in [**1**] if X is monolithic whenever X is compact and monotonically ω-monolithic. The following result shows that the answer is positive.

Corollary 5.3. *If X is a Corson compact monotonically ω-monolithic space, then X is monotonically monolithic.*

Proof. Let X be a Corson compact space. Then X is contained in a Σ-product of real lines, that is, X is contained in $C_p(L_\kappa)$, where L_κ is the one-point Lindelöfication of a discrete space of cardinality κ. It is easy to se that L_κ^n is Lindelöf for each $n \in \omega$. By the Theorem of Arhangel'skii-Pykteev we have that $t(C_p(L_\kappa)) \leq \omega$ and so $t(X) \leq \omega$. Since X is monotonically ω-monolithic, Proposition 5.1 guarantees that X is monotonically monolithic. □

Since every compact monotonically ω-monolithic space is a Corson compact space (see Corollary 11.3 in Chapter I), we obtain:

Corollary 5.4. *If X is a compact monotonically ω-monolithic space, then X is a monotonically monolithic Corson compact space.*

Theorem 5.5. *If X is a monotonically ω-monolithic countably compact space, then X is a monotonically monolithic Corson compact space and has the Fréchet-Urysohn property.*

PROOF. Let X be a monotonically ω-monolithic countably compact space. By Proposition 9.11 in Chapter I, X is compact and has the Fréchet-Urysohn property. By Corollary 5.4, X is a monotonically monolithic Corson compact space. $\square$

In the second part of this section we find some spaces for which $C_p(X)$ is monotonically κ-monolithic, but before we need to prove some results.

Proposition 5.6. *Given an infinite cardinal κ, suppose that X is a monotonically κ-monolithic space and f is a continuous function from X onto Y such that $f(\mathrm{cl}(A)) = \mathrm{cl}(f(A))$ for any set $A \subset X$ with $|A| \leq \kappa$. Then Y is monotonically κ-monolithic.*

PROOF. Fix a point $x_y \in f^{-1}(y)$ for each $y \in Y$ and let $\mathcal{O}$ be an operator which witnesses monotone κ-monolithicity of X, as in Definition 9.1 in Chapter I. Given a set $B \subset Y$ with $|B| \leq \kappa$ let $B_X = \{x_y : y \in B\}$ and $\mathcal{N}(B) = \{f(N) : N \in \mathcal{O}(B_X)\}$. It is routine to see that $\mathcal{N}$ satisfies the conditions (1)$-$(3) of the definition of monotone κ-monolithicity for the space Y.

To see that $\mathcal{N}(B)$ is an external network of cl(B) in Y; take any $y \in \mathrm{cl}(B)$ and U an open set in Y with $y \in U$. The set B_X being of cardinality at most κ we have $f(\mathrm{cl}(B_X)) = \mathrm{cl}(B)$, so there is a point $x \in \mathrm{cl}(B_X)$ with $f(x) = y$. If $N \in \mathcal{O}(B_X)$ is such that $x \in N \subset f^{-1}(U)$, then $f(N) \in \mathcal{N}(B)$ and $y \in f(N) \subset U$. Thus, Y is monotonically κ-monolithic. $\square$

Proposition 5.7. *Let X be a space and let Y be a G_κ-dense subset of X which is C-embedded in X. Then for every $A \subset C_p(X)$ with $|A| \leq \kappa$ the function $\pi_Y \restriction A : A \to \pi_Y(A)$ is a homeomorphism.*

PROOF. The function $\pi = \pi_Y \restriction A : A \to \pi_Y(A)$ is bijective and continuous. We shall prove that π is open. Fix an open subset B of A. Let $f \in B$. Take $W = [x_1, \ldots, x_n; U_1 \ldots, U_n]$ a canonical open set in $C_p(X)$ with $f \in (W \cap A) \subset B$. Notice that $G^i_f = f^{-1}(f(x_i))$ and $G^i_g = g^{-1}(g(x_i))$ for each $g \in A \backslash B$, are sets of type G_δ in X. Thus, the sets $G^i = \bigcap\{G^i_f \cap G^i_g : g \in A\backslash B\}$ are nonempty sets of type G_κ in X. Since Y is a G_κ-dense subset of X, we can choose $y_i \in Y \cap G^i$ for $i = 1, \ldots, n$. Finally, $V = [y_1, \ldots, y_n; U_1 \ldots, U_n]$ is canonical open set in $C_p(Y)$ with $\pi(f) \in (V \cap \pi(A)) \subset \pi(B)$. $\square$

Proposition 5.8. *If $C_p(X)$ is a monotonically κ-monolithic space and Y is a G_κ-dense subset of X which is C-embedded in X, then $C_p(Y)$ is monotonically κ-monolithic.*

Proof. By Proposition 5.7, the function $\pi_Y \mid A : A \to \pi_Y(A)$ is a homeomorphism for every $A \subset C_p(X)$ with $|A| \leq \kappa$. It is easy to see in this case that $\pi_Y(\mathrm{cl}(A)) = \mathrm{cl}(\pi_Y(A))$ for any set $A \subset C_p(X)$ with $|A| \leq \kappa$. It follows from Proposition 5.6, that $C_p(Y)$ is monotonically κ-monolithic. □

Given a space X and an infinite cardinal κ, recall that the Hewitt-κ-extension of X that we denote by $\upsilon_\kappa X$ is the subspace of βX consisting of all points $x \in \beta X$ for which every set of type G_κ in βX containing x intersects X. It is well known and easy to prove that any continuous function $f : X \to \mathbb{R}$ can be extended continuously on $\upsilon_\kappa X$ (see [**29**]).

Theorem 5.9. *If X is a space such that its Hewitt κ-extension is monotonically κ-stable, then X is monotonically κ-stable, that is, $C_p(X)$ is monotonically κ-monolithic.*

Proof. Since $\upsilon_\kappa X$ is monotonically κ-stable, it follows from Remark 2.10 for Theorem 2.7 that $C_p(\upsilon_\kappa X)$ is monotonically κ-monolithic. Since X is G_κ-dense and is C-embedded in $\upsilon_\kappa X$, we can apply Proposition 5.8 to see that $C_p(X)$ is monotonically κ-monolithic. It follows from Remark 2.10 for Theorem 2.7 that X is monotonically κ-stable. □

Lemma 5.10. *Let λ be an infinite cardinal number and $\kappa = \lambda^+$. If X is a product of Lindelöf Σ-spaces and Y is a Σ_κ-product in X then $\upsilon_\lambda Y = X$.*

Proof. Suppose $X = \prod\{X_t : t \in T\}$ where each X_α is a Lindelöf Σ-space and $Y = \Sigma_\kappa \prod\{X_t : t \in T\}$. Clearly Y contains $\sigma \prod\{X_t : t \in T\}$. We shall prove that Y is C-embedded in X. Indeed, if $f : Y \to \mathbb{R}$ is a continuous function. Since $l(\prod\{X_\alpha : \alpha \in K\}) \leq \omega$ for each finite set $K \subset T$ and $\mathrm{iw}(\mathbb{R}) \leq \omega$, by Lemma 3.8 there exists a countable set $S \subset \kappa$ such that if $x, x' \in Y$ and $p_S(x) = p_S(x')$ then $f(x) = f(x')$. Since $p_S \restriction Y : Y \to p_S(Y)$ is a quotient function, by Lemma 3.1 there exists $g : \pi_S(Y) \to \mathbb{R}$, continuous, such that $f = g \circ p_S \restriction Y$. Thus $\tilde{f} = g \circ p_S : X \to \mathbb{R}$ is continuous and $f = \tilde{f} \restriction Y$.

Since Y is C-embedded in X, $\beta Y = \beta X$. It is well known that any Lindelöf space is realcompact and that an arbitrary product of realcompact spaces is realcompact (see [**29**]). So X is realcompact, that is, $X = \upsilon_\omega X$. Therefore, for any $x \in \beta X \setminus X$ there exists an open set $U \subset \beta X$ such that $U \cap Y \subset U \cap X = \emptyset$. This shows that $\upsilon_\lambda Y \subset X$. Now we shall prove the other contention. Take $x \in X$ and let U be a set of type G_λ in βX such that $x \in U$. Since $U \cap X$ is a non-empty set of type G_λ in X and Y is G_λ-dense in X, $(U \cap Y) = (U \cap X) \cap Y \neq \emptyset$. Thus $x \in \upsilon_\lambda Y$. Therefore $X \subset \upsilon_\lambda Y$. □

It follows from Proposition 9.3 in Chapter I and Remark 2.10 for Theorem 2.7 that a space X is monotonically stable if and only if it is λ-stable for any cardinal λ. The following result follows directly from Corollary 3.11, Theorem 5.9 and Lemma 5.10.

Corollary 5.11. *Let λ be an infinite cardinal number and $\kappa = \lambda^+$. If X is a product of Lindelöf Σ spaces and Y is a Σ_κ-product in X, then $C_p(Y)$ is monotonically λ-monolithic.*

Given a cardinal κ, recall that a topological space X is *κ-pseudocompact* if it is G_κ-dense in βX. It is well known that a space X is κ-pseudocompact if and only if for each continuous function $f : X \to \mathbb{R}^\kappa$ the space $f(X)$ is compact.

Since any κ-pseudocompact space X is pseudocompact, then it is C-embedded in βX. The following result is a consequence of Proposicion 5.8 or Theorem 5.9.

Corollary 5.12. *For every κ-pseudocompact space X, the space $C_p(X)$ is monotonically κ-stable.*

Corollary 5.13. *Let λ be an infinite cardinal number and $\kappa = \lambda^+$, then $C_p([0,\kappa))$ is monotonically λ-monolithic.*

In [**77**], the following questions were proposed: (1) Suppose that $C_p(X)$ has the Collins-Roscoe property, must X be Lindelöf? (2) Suppose that $C_p(X)$ has the Collins-Roscoe property, must X be Lindelöf Σ? Now we are going to use the results obtained in this section to find some counterexamples for the first (hence for the second) above question.

Example 5.14. $C_p([0,\omega_1))$ is a Collins-Roscoe space (see Definition 10.1 in Chapter I). Indeed, for each $\alpha < \omega_1$, let $C_p^\alpha([0,\omega_1))$ be the set of all the real-valued continuous functions defined on $[0,\omega_1)$ which are constant on (α,ω_1). It is a well known fact that $C_p([0,\omega_1)) = \bigcup\{C_p^\alpha([0,\omega_1)) : \alpha < \omega_1\}$. Moreover, for each $\alpha < \omega_1$, $C_p^\alpha([0,\omega_1))$ is homeomorphic to $C_p([0,\alpha]) \times \mathbb{R}$. Thus, $\mathrm{nw}(C_p^\alpha([0,\omega_1))) = \mathrm{nw}(C_p([0,\alpha])) = \mathrm{nw}([0,\alpha]) = \omega$. Therefore, $C_p([0,\omega_1))$ is the union of ω_1 subspaces with countable network. Since ω_1 is pseudocompact, by Proposition 5.12, the space $C_p([0,\omega_1))$ is monotonically ω-monolithic. So, by Theorem 10.4 in Chapter I, $C_p([0,\omega_1))$ has the Collins-Roscoe property.

Proposition 5.15. *If X is a monotonically ω-monolithic space of cardinality ω_1, then $C_p(C_p(X))$ is a Collins-Roscoe space.*

Proof. Let $X = \{x_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$ let $X_\alpha = \{x_\beta : \beta \le \alpha\}$. For each $\alpha < \omega_1$, let $Z_\alpha = \pi^*_{X_\alpha}(C_p(X_\alpha)) \subset C_p(C_p(X))$ the set of all continuous functions in $C_p(C_p(X))$ which have a factorization trough of $C_p(X_\alpha)$. Let $f \in C_p(C_p(X))$, by Lemma 1.5, there exist a countable set $S \subset X$ and a continuous real-valued function g on $\pi_S(C_p(X))$ such that $f = g \circ \pi_S$. Since S is countable, there exists $\alpha < \omega_1$ such that $S \subset X_\alpha$. Let $\pi_S^{X_\alpha} : C_p(X_\alpha) \to C_p(S)$ the restriction function. Then we have $f = h \circ \pi_{X_\alpha} = \pi^*_{X_\alpha}(h)$ where $h =$

$g \circ \pi_S^{X_\alpha} \in C_p(X_\alpha)$. Hence $f \in Z_\alpha$. Therefore $C_p(C_p(X)) \subset \bigcup\{Z_\alpha : \alpha < \omega_1\}$. Moreover, for each $\alpha < \omega_1$, Z_α is homeomorphic to $C_p(X_\alpha)$ (see Proposition I.2.2). Since X_α is countable, $\mathrm{nw}(Z_\alpha) = \mathrm{nw}(C_p(X_\alpha)) = \mathrm{nw}(X_\alpha) = \omega$. Therefore, $C_p(C_p(X))$ is the union of ω_1 subspaces with countable network. Moreover, by Remark 1.9 for Theorem 1.6, the space $C_p(C_p(X))$ is monotonically ω-monolithic. So, by Theorem 10.4 in Chapter I, the space $C_p(C_p(X))$ has the Collins-Roscoe property. □

Example 5.16. Let L_{ω_1} be the one-point Lindelöfication of a discrete space of cardinality ω_1. As we have seen in Example 1.10, L_{ω_1} is monotonically monolithic and $C_p(L_{\omega_1})$ is not Lindelöf. On the other hand, since the cardinality of L_{ω_1} is ω_1, by Proposition 5.15, the space $C_p(C_p(L_{\omega_1}))$ has the Collins-Roscoe property.

6. Monotone $<\kappa$-monolithicity

In this section, we are going to introduce the concept of monotonically $<\kappa$-monolithic space when $\kappa \geq \omega_1$. This concept is weaker than that of monotonically κ-monolithic space. In the previous section we have proved that a monotonically κ-monolithic space of tightness κ is monotonically monolithic. Thus, we can ask if there is a relation between monotone κ-monolithicity (or monotone $<\kappa$-monolithicity) and other cardinal functions. We know that any monotonically monolithic space is hereditarily D. Our main result in this section states that if X is a monotonically $<\kappa$-monolithic space and $\mathrm{nw}(X) \leq \kappa$, then X is hereditarily D. Actually, this has been the purpose of introducing the notion of monotonically $<\kappa$-monolithic space. We will use our main result to obtain spaces $C_p(X)$ with the hereditary D-property. In particular, we are going to show that $C_p(X)$ is hereditarily D whenever X is a pseudocompact space with network weight at most ω_1.

Definition 6.1. For a cardinal number $\kappa > \omega_1$, we say that a space X is *monotonically $<\kappa$-monolithic* if for each finite subset F of X we can assign a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each subset $A \subset X$ with $|A| < \kappa$, the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network for $\mathrm{cl}(A)$.

Let X be a monotonically $<\kappa$-monolithic space with operator $\mathcal{N}$, and let ϕ be a neighborhood assignment on X. We define the relation $R = R(\mathcal{N}, \phi)$ from X to $[X]^{<\omega}$ as follows: $(x, F) \in R$ if and only if there is $N \in \mathcal{N}(F)$ with $x \in N \subset \phi(x)$. For any $A \subset X$ we will use the following notation $\mathcal{R}(A) = \bigcup\{R^{-1}(F) : F \in [A]^{<\omega}\} = R^{-1}([A]^{<\omega})$. Clearly, $\mathcal{R}(A) \subset \mathcal{R}(B)$ whenever $A \subset B \subset X$.

Given a neighborhood assignment ϕ on X, recall that for $A \subset X$ we denote by $\phi(A)$ the set $\bigcup\{\phi(x) : x \in A\}$, moreover, a subset Z of X is *ϕ-close* if $Z \subset \phi(x)$ for each $x \in Z$.

Remarks 6.2. Let X be a monotonically $<\kappa$-monolithic space with operator N and let ϕ be a neighborhood assignment on X, then:

(1) If A is a subset of X with $|A| < \kappa$ and $x \in \mathrm{cl}(A)$, then there is $F \in [A]^{<\omega}$ such that $(x, F) \in R$; that is, $\mathrm{cl}(A) \subset \mathcal{R}(A)$.
(2) If $F \in [X]^{<\omega}$, then we have that $R^{-1}(F) = \bigcup\{Z_N : N \in \mathcal{N}(F)\}$ where $Z_N = \{x \in X : x \in N \subset \phi(x)\}$. Observe that Z_N is ϕ-close for each $N \in \mathcal{N}(F)$. Therefore, $R^{-1}(F)$ is the union of a countable collection of ϕ-close subsets.
(3) If $\{E_\beta : \beta < \alpha\}$ is a collection of subsets of X, then $\mathcal{R}(\bigcup\{E_\beta : \beta < \alpha\}) = \bigcup\{\mathcal{R}(\bigcup\{E_\gamma : \gamma \leq \beta\}) : \beta < \alpha\}$.
(4) Let $\{E_\beta : \beta < \alpha\}$ be a collection of subsets of X, let $\{U_\beta : \beta < \alpha\}$ be a family of open subsets of X. Assume that for each $\beta < \alpha$ we have that $\mathcal{R}(\bigcup\{E_\gamma : \gamma \leq \beta\}) \subset \bigcup\{U_\gamma : \gamma \leq \beta\}$, then for each $\beta \leq \alpha$ we have $\mathcal{R}\left(\bigcup\{E_\gamma : \gamma < \beta\}\right) \subset \bigcup\{U_\gamma : \gamma < \beta\}$. Indeed, by the previous remark, if $\beta \leq \alpha$ we can see that: $\mathcal{R}\left(\bigcup\{E_\gamma : \gamma < \beta\}\right) = \bigcup\{\mathcal{R}(\bigcup\{E_\delta : \delta \leq \gamma\}) : \gamma < \beta\} \subset \bigcup\{\bigcup\{U_\delta : \delta \leq \gamma\} : \gamma < \beta\} = \bigcup\{U_\gamma : \gamma < \beta\}$. Moreover, by the first remark, if $\beta \leq \alpha$ and $|\bigcup\{E_\gamma : \gamma < \beta\}| < \kappa$, then $\mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\}) \subset \bigcup\{U\gamma : \gamma < \beta\}$.

In order to prove our main result we need the following lemmas.

Lemma 6.3. *Let X be a topological space. If α is an ordinal number, $\{E_\beta : \beta < \alpha\}$ is a sequence of closed discrete subsets of X and $\{U_\beta : \beta < \alpha\}$ is a sequence of open sets in X such that $E_\beta \subset U_\beta \setminus \bigcup\{U_\gamma : \gamma < \beta\}$ for each $\beta < \alpha$, and $\mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\}) \subset \bigcup\{U_\gamma : \gamma < \beta\}$ for $\beta \leq \alpha$, then $\bigcup\{E_\beta : \beta < \alpha\}$ is closed and discrete in X.*

Proof. Let x in $E = \mathrm{cl}(\bigcup\{E_\beta : \beta < \alpha\}) \subset \bigcup\{U_\beta : \beta < \alpha\}$. Let β the first ordinal number such that $x \in U_\beta$. Then $U = U_\beta \setminus \mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\})$ is an open neighborhood of x such that $U \cap E \subset E_\beta$. Since E_β is closed and discrete, it follows that $x \in E_\beta$. Then we can select an open neighborhood V of x such that $V \cap E_\beta = \{x\}$. Thus $W = U \cap V$ is an open neighborhood of x such that $W \cap E = \{x\}$. This shows that $\bigcup\{E_\beta : \beta < \alpha\}$ is closed and discrete in X. □

Lemma 6.4. *Let X be a monotonically $<\kappa$-monolithic space with operator $\mathcal{N}$, and let ϕ be a neighborhood assignment on X. If $D \subset X$ is countable, closed and discrete with $D \subset V$ for an open subset $V \subset X$ and if $x^* \in X \setminus V$, then there exists a countable closed and discrete subset $E \subset X \setminus V$ of X with $x^* \in E$ and such that $\mathcal{R}(D \cup E) \subset V \cup \phi(E)$.*

Proof. Take a family $\{\Omega_n : n \in \omega\}$ of infinite disjoint subsets of ω such that $\omega = \bigcup\{\Omega_n : n \in \omega\}$ and $\{0, \dots, n\} \subset \Omega_1 \cup \dots \cup \Omega_n$. We construct the set E in a recursive process.

Step 0. Let $E_0 = \{e_0\}$ where $e_0 = x^*$ and let $\{Z_i : i \in \Omega_0\}$ be a countable collection of ϕ-close subsets of X such that $\mathcal{R}(D \cup E_0) \subset \bigcup\{Z_i : i \in \Omega_0\}$ (see Remark 6.2 (2)).

Suppose that for every $k \le n$ we have chosen countable closed and discrete sets E_k and constructed families of ϕ-close subsets $\{Z_i : i \in \Omega_k\}$ such that for $U_k = V \cup \phi(E_k)$ we have:

$H(k) : E_k \subset U_k \setminus \bigcup\{U_i : i < k\}$;

$I(k) : \mathcal{R}(D \cup \bigcup\{E_i : i \le k\}) \subset \bigcup\{Z_i : i \in \Omega_k\}$;

$J(k) : \bigcup\{Z_i : i < k\} \subset \bigcup\{U_i : i \le k\}$.

Step $n+1$. If $Z_i \subset \bigcup\{U_k : k \le n\}$ for each $i \in \bigcup\{\Omega_k : k \le n\}$, then $E = \bigcup\{E_k : k \le n\}$ satisfies the requested conditions and so we would finish the proof. Otherwise, let m be the first natural number in $\bigcup\{\Omega_k : k \le n\}$ such that $Z_m \setminus \bigcup\{U_k : k \le n\} \neq \emptyset$. Notice that by $J(n)$, $m \ge n$. Fix $e_{n+1} \in Z_m \setminus \bigcup\{U_k : k \le n\}$. Take $E_{n+1} = e_{n+1}$. Because of Remark 6.2 (2), we can take a collection $\{Z_i : i \in \Omega_{n+1}\}$ of ϕ-close subsets of X such that $\mathcal{R}(D \cup \bigcup\{E_i : i \le n+1\}) \subset \bigcup\{Z_i : i \in \Omega_{n+1}\}$. Observe that conditions $H(n+1)$ and $I(n+1)$ hold. We prove that $J(n+1)$ holds. Since $J(n)$ holds, it is enough to prove that $Z_n \subset \bigcup\{U_k : k \le n+1\}$. If $n < m$, since $n \in \bigcup\{\Omega_k : k \le n\}$ by the choice of m, we have $Z_n \subset \bigcup\{U_k : k \le n\}$. If $n = m$, since Z_n is ϕ-close and by the choice of E_{n+1}, we have $Z_n \subset \phi(E_{n+1}) \subset V \cup \phi(E_{n+1})$. Therefore $J(n+1)$ holds.

Assume that we cannot finish the recursive process in a finite step, then we have, for each $n \in \omega$, a non-empty set E_n and a family $\{Z_i : i \in \Omega_n\}$ of ϕ-close sets such that $H(n)$, $I(n)$ and $J(n)$ hold. Take $E = \bigcup\{E_n : n \in \omega\}$. Notice that $x^* = e_0 \in E$. Since $H(n)$ holds for every $n \in \omega$, we have $E \subset X \setminus V$. Because of $I(n)$, $J(n)$ and Remark 6.2 (3) we can see that $\mathcal{R}(D \cup E) = \bigcup\{\mathcal{R}(D \cup \bigcup\{E_k : k \le n\}) : n \in \omega\} \subset \bigcup\{Z_n : n \in \omega\} \subset \bigcup\{U_n : n \in \omega\} = V \cup \phi(E)$. We shall prove that E is closed and discrete. Because of Remark 6.2 (1), we have $\text{cl}(E) \subset \mathcal{R}(E) \subset \mathcal{R}(D \cup E) \subset \bigcup\{U_n : n \in \omega\}$. Using this fact and $H(n)$ for $n \in \omega$, we conclude that $\text{cl}(\bigcup\{E_n : n < \beta\}) \subset \bigcup\{U_n : n < \beta\}$ for $\beta \le \omega$. Therefore, by Lemma 6.3 we conclude that the set $E = \bigcup\{E_n : n < \omega\}$ is closed and discrete. □

Lemma 6.5. *Let X be a monotonically $<\kappa$-monolithic space with operator $\mathcal{N}$ and let ϕ be a neighborhood assignment on X. If $D \subset X$ is closed and discrete, $|D| < \kappa$, $D \subset V$ for an open set $V \subset X$ and $x^* \in X \setminus V$, then there exists $E \subset X \setminus V$, closed and discrete in X with $|E| \le |D|$, $x^* \in E$ and such that $\mathcal{R}(D \cup E) \subset V \cup \phi(E)$.*

Proof. We proceed by transfinite induction on the cardinality of D. If $|D| = \omega$, the result follows from Lemma 6.4. Suppose that there is a cardinal number $\lambda < \kappa$ such that the result is true for every subset D with cardinality less than λ. We are going to prove that the Lemma holds when we consider subsets D of cardinality λ. So, take a closed and discrete subset $D = \{x_\alpha : \alpha < \lambda\}$ of X such that $D \subset V$ for some open subset $V \subset X$ and $x^* \in X \setminus V$. We construct the set E by transfinite recursion.

Step 0. Let $D_0 = \{x_0\}$, and take $x_0^* = x^*$. By Lemma 6.4, there is a countable closed and discrete subset E_0 of X with $E_0 \subset X \setminus V$, $x_0^* \in E_0$ and such that $\mathcal{R}(\{x_0\} \cup E_0) \subset V \cup \phi(E_0)$.

Assume that $\alpha < \lambda$ and for each $\beta < \alpha$, we have constructed closed and discrete subsets E_β such that for $U_\beta = V \cup \phi(E_\beta)$, the following conditions hold:

$H(\beta) : |E_\beta| \leq \max\{\omega, |\beta|\}$;

$I(\beta) : E_\beta \subset U_\beta \setminus \bigcup\{U_\gamma : \gamma < \beta\}$;

$J(\beta) : \mathcal{R}(\{x_\gamma : \gamma \leq \beta\} \cup \bigcup\{E_\gamma : \gamma \leq \beta\}) \subset \bigcup\{U_\gamma : \gamma \leq \beta\}$.

Step α. Since $H(\beta)$ and $J(\beta)$ hold for each $\beta < \alpha$, by Remark 6.2 (4) we have $\mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\}) \subset \bigcup\{U_\gamma : \gamma < \beta\}$ for each $\beta \leq \alpha$. Using this fact and $I(\beta)$ for each $\beta < \alpha$, by Lemma 6.3, we conclude that $\bigcup\{E_\beta : \beta < \alpha\}$ is closed and discrete in X. Then $D_\alpha = \{x_\beta : \beta \leq \alpha\} \cup \bigcup\{E_\beta : \beta < \alpha\}$ is closed and discrete in X. Besides, $|D_\alpha| \leq \max\{\omega, |\alpha|\}$. If $V \cup \phi(\bigcup\{E_\beta : \beta < \alpha\}) = X$, then the set $E = \bigcup\{E_\beta : \beta < \alpha\}$ satisfies the required conditions and the proof would be finished. In the other possible case, fix $x_\alpha^* \in X \setminus \bigcup\{V \cup \phi(E_\beta) : \beta < \alpha\}$. By hypothesis, there exists $E_\alpha \subset X \setminus \bigcup\{V \cup \phi(E_\beta) : \beta < \alpha\}$, closed and discrete in X with $|E_\alpha| \leq |D_\alpha| \leq \max\{\omega, |\alpha|\}$, $x_\alpha^* \in E_\alpha$ and such that $\mathcal{R}(D_\alpha \cup E_\alpha) \subset \bigcup\{V \cup \phi(E_\beta) : \beta \leq \alpha\}$. Note that, for $U_\alpha = V \cup \phi(E_\alpha)$, conditions $H(\alpha)$, $I(\alpha)$ and $J(\alpha)$ hold.

Finally, if we do not finish the process in a step $\alpha < \lambda$, we take $E = \bigcup\{E_\alpha : \alpha < \lambda\}$. Clearly, $E \subset X \setminus V$, $|E| \leq \lambda = |D|$ and $x^* \in E$. Moreover, using $J(\beta)$ for $\beta < \lambda$ and the first part of Remark 6.2 (3) we have $\mathcal{R}(D \cup E) = \bigcup\{\mathcal{R}(\{x_\gamma : \gamma \leq \beta\} \cup \bigcup\{E_\gamma : \gamma \leq \beta\}) : \beta < \lambda\} \subset \bigcup\{U_\beta : \beta < \lambda\} = V \cup \phi(E)$. So, we only need to prove that E is closed and discrete. Since $H(\beta)$ and $J(\beta)$ hold for each $\beta < \lambda$, by Remark 6.2 (4) we have $\mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\}) \subset \bigcup\{U_\gamma : \gamma < \beta\}$ for each $\beta \leq \lambda$. Using this fact and $I(\beta)$ for each $\beta < \lambda$, by Lemma 6.3, we conclude that $E = \bigcup\{E_\beta : \beta < \lambda\}$ is closed and discrete in X. □

Every monotonically monolithic space is monotonically κ-monolithic for any infinite cardinal κ (see Proposition 9.3 in Chapter I). It is clear from Definition 6.1 and Proposition 5.1 in Chapter I that every monotonically κ-monolithic space is monotonically $<\kappa$-monolithic for all $\kappa \geq \omega_1$. So, the following result is a generalization of Theorem 7.6 in Chapter I.

Theorem 6.6. *Let X be a monotonically $<\kappa$-monolithic space where $\kappa \geq \mathrm{nw}(X)$. Then, X is hereditarily D.*

Proof. It is sufficient to prove that X has property D. Let ϕ be a neighborhood assignment in X, and let $\mathcal{N} = \{N_\alpha : \alpha < \kappa\}$ be a network for X. We will construct by transfinite recursion a closed and discrete subset E of X such that $\phi(E) = X$ as follows:

Step 0. Let $D_0 = V = \emptyset$ and ξ_0 be the first ordinal $\xi < \kappa$ such that there exists $x \in X$ with $x \in N_{\xi_0} \subset \phi(x)$. Fix $x_0^* \in X$ such that $x_0^* \in N_{\xi_0} \subset \phi(x_0^*)$. Because of Lemma 6.4, there is a countable closed and discrete set $E_0 \subset X$ with $x_0^* \in E_0$ and such that $\mathcal{R}(E_0) \subset \phi(E_0)$.

Let $\alpha < \kappa$ and assume that for each $\beta < \alpha$ we have chosen a point $x^*_\beta \in X$, an ordinal $\xi_\beta < \kappa$ and a closed discrete subset $E_\beta \subset X$; in such form that for $U_\beta = \phi(E_\beta)$ the following conditions are satisfied:

$H(\beta)$: $\xi_\gamma < \xi_\beta$ whenever $\gamma < \beta < \alpha$;

$I(\beta)$: $|E_\beta| \leq \max\{\omega, |\beta|\}$;

$J(\beta)$: $E_\beta \subset U_\beta \setminus \bigcup\{U_\gamma : \gamma < \beta\}$;

$K(\beta)$: $\mathcal{R}(\bigcup\{E_\gamma : \gamma \leq \beta\}) \subset \bigcup\{U_\gamma : \gamma \leq \beta\}$;

$L(\beta)$: ξ_β is the first element of κ such that there is $x \in X \setminus \bigcup\{U_\gamma : \gamma < \beta\}$ with $x \in N_{\xi_\beta} \subset \phi(x)$. Also, x^*_β is a fixed point in $X \setminus \bigcup\{U_\gamma : \gamma < \beta\}$ with $x^*_\beta \in E_\beta$ and $x^*_\beta \in N_{\xi_\beta} \subset \phi(x^*_\beta)$.

Step α. Since $I(\beta)$ and $K(\beta)$ hold for each $\beta < \alpha$, by Remark 6.2 (4) we have $\mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\}) \subset \bigcup\{U_\gamma : \gamma < \beta\}$ for each $\beta \leq \alpha$. Using this fact and $J(\beta)$ for $\beta < \alpha$, by Lemma 6.3, the set $D_\alpha = \bigcup\{E_\beta : \beta < \alpha\}$ is closed and discrete in X. If $\phi(D_\alpha) = X$, then the set $E = D_\alpha$ satisfies the required conditions and we would finish the recursive process. Otherwise, observe that $|D_\alpha| \leq \max\{\omega, |\alpha|\}$. Let ξ_α be the first ordinal ξ such that there is $x \in X \setminus \phi(D_\alpha)$ with $x \in N_\xi \subset \phi(x)$. Choose $x^*_\alpha \in X \setminus \phi(D_\alpha)$ such that $x^*_\alpha \in N_{\xi_\alpha} \subset \phi(x^*_\alpha)$. By Lemma 6.5, there exists a closed and discrete set $E_\alpha \subset X \setminus \phi(D_\alpha)$ which satisfies $|E_\alpha| \leq |D_\alpha| \leq \max\{\omega, |\alpha|\}$, $x^*_\alpha \in E_\alpha$ and such that $\mathcal{R}(D_\alpha \cup E_\alpha) \subset \phi(D_\alpha \cup E_\alpha)$. Note that conditions $H(\alpha)$, $I(\alpha)$, $J(\alpha)$, $K(\alpha)$ and $L(\alpha)$ hold.

Finally, if we do not finish the process in a step $\alpha < \kappa$, take $E = \bigcup\{E_\alpha : \alpha < \kappa\}$. If $x \in X \setminus \phi(E) = X \setminus \{U_\gamma : \gamma < \kappa\}$, then there exists $\xi < \kappa$ with $x \in N_\xi \subset \phi(x)$ and there is $\beta < \kappa$ with $\xi_\beta > \xi$; this contradicts the choice of ξ_β. Therefore, we must have that $\phi(E) = X$. So $\mathrm{cl}(\bigcup\{E_\gamma : \gamma < \beta\}) \subset \bigcup\{U_\gamma : \gamma < \beta\}$ for each $\beta \leq \kappa$. Using this fact and $J(\beta)$ for $\beta < \kappa$, by Lemma 6.3, the set E is closed and discrete in X. □

We can also to define monotone $<\kappa$-stability as follows:

Definition 6.7. For a cardinal number $\kappa > \omega_1$, we say that a topological space X is *monotonically* $<\kappa$*-stable* if and only if for each finite collection $F \subset C_p(X)$ we can assign a countable collection $\mathcal{N}(F)$ of subsets of X such that for every $A \subset C_p(X)$ with $|A| < \kappa$ the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is a network for $\Delta\mathrm{cl}(A)(X)$ modulo $\Delta\mathrm{cl}(A)$.

Remark 6.8. As in Remark 1.9 we can see that statements in Theorem 1.6, Corollary 1.7 and Proposition 1.8 remain true if we replace monotone monolithicity by monotone $<\kappa$-monolithicity. Also, as in Remark 2.10, statements in Theorem 1.6, Theorem 2.7, Corollary 2.8 and Corollary 2.9 remain true if we replace monotonically monolithic and monotonically stable by monotonically $<\kappa$-monolithic and monotonically $<\kappa$-stable, respectively.

As a consequence of Theorem 6.6 and the previous remark we have:

Corollary 6.9. *Let X be a monotonically $<\kappa$-stable space where $\kappa \geq$* $\mathrm{nw}(X)$. *Then, $C_p(X)$ is hereditarily D.*

Corollary 6.10. *Let X be a monotonically $<\kappa$-monolithic space where $\kappa \geq \mathrm{nw}(X)$. Then, $C_{p,2n+1}(X)$ is hereditarily D for any $n \in \omega$.*

Corollary 6.11. *Let X be a monotonically $<\kappa$-monolithic space where $\kappa \geq \mathrm{nw}(X)$ and $n \in \omega$. Then, every closed continuous image Y of a subspace of $C_{p,2n}(X)$ satisfies $l(Y) = e(Y)$.*

Corollary 6.12. *Let X be a monotonically $<\kappa$-monolithic space where $\kappa \geq \mathrm{nw}(X)$, $n \in \omega$ and let Y be a closed continuous image of a subspace of $C_{p,2n}(X)$. Then Y is is compact whenever it is countably compact.*

Given an infinite cardinal λ, recall that a space X is *initially λ-compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \lambda$ has a finite subcover. Given a cardinal κ, a topological space X is called *initially $<\kappa$-compact* if X is initially λ-compact for every $\lambda < \kappa$.

Theorem 6.13. *If X has a cover $\mathcal{K}$ and a countable network $\mathcal{N}$ modulo $\mathcal{K}$ such that each $K \in \mathcal{K}$ is initially $<\kappa$-compact, then $C_p(X)$ is monotonically $<\kappa$-monolithic.*

Proof. For every family $\mathcal{E}$ of subsets of X, we take $\mathcal{C}(\mathcal{E}) = \{N \setminus E : E \in \mathcal{E}, N \in \mathcal{N}\}$ and $\mathcal{F}(\mathcal{E}) = \{\bigcup \mathcal{E}_0 : \mathcal{E}_0 \subset \mathcal{E}, |\mathcal{E}_0| < \omega\}$. For each finite set $F \subset C_p(X)$, take $\mathcal{E}_F = \{g^{-1}(B) : g \in F, B \in \mathcal{B}(\mathbb{R})\}$ and $\mathcal{N}(F) = \mathcal{W}(\mathcal{C}(\mathcal{F}(\mathcal{E}_F)))$ (see Remark 1.1). Observe that, since F is finite, $\mathcal{N}(F)$ is countable. We are going to prove that for each $A \subset C_p(X)$ satisfying $|A| < \kappa$, the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network of $\mathrm{cl}(A)$ in $C_p(X)$. Take $A \subset C_p(X)$ with $|A| < \kappa$ and $f \in \mathrm{cl}(A)$.

Claim. If $x \in X$, $B \in \mathcal{B}(\mathbb{R})$ and $f(x) \in B$, then there are a finite set $F \subset A$ and $P \in \mathcal{C}(\mathcal{F}(\mathcal{E}_F))$ such that $x \in P$ and $f \in [P; B]$.

We will prove the Claim. Let $K \in \mathcal{K}$ which contains x. For each $y \in K \setminus f^{-1}(B)$, we take a set $B_y \in \mathcal{B}(\mathbb{R})$ such that $f(y) \in B_y$ and $f(x) \notin \mathrm{cl}(B_y)$, and we take a function $g_y \in A$ such that $g_y(x) \in B \setminus \mathrm{cl}(B_y)$ and $g_y(y) \in B_y$. The family $\{g_y^{-1}(B_y) : y \in K \setminus f^{-1}(B)\} \cup \{f^{-1}(B)\}$ covers K and has cardinality less than κ. By hypothesis, there is a finite set $K_0 \subset K \setminus f^{-1}(B)$ such that $\{g_y^{-1}(B_y) : y \in K_0\} \cup \{f^{-1}(B)\}$ covers K. Take $N \in \mathcal{N}$ with $K \subset N \subset (\bigcup\{g_y^{-1}(B_y) : y \in K_0\}) \cup f^{-1}(B)$. By construction, if $P = N \setminus \bigcup\{g_y^{-1}(B_y) : y \in K_0\}$, then $x \in P$, $P \in \mathcal{C}(\mathcal{F}(\mathcal{E}_F))$ where $F = \{g_y : y \in K_0\} \subset A$ and $f \in [P; B]$. This proves the Claim.

Now, assume that $f \in U$ for an open subset U of $C_p(X)$. Take $x_1, \dots, x_n \in X$ and $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ such that $f \in [x_1, \dots, x_n; B_1, \dots, B_n] \subset U$. Because of the Claim, for each $i = 1, \dots, n$, there exist $F_i \subset A$, finite, and $P_i \in \mathcal{C}(\mathcal{F}(\mathcal{E}_{F_i}))$ such that $x_i \in P_i$ and $f \in [P_i; B_i]$. Take $F = \bigcup\{F_i : i = 1, \dots, n\} \subset A$. So, $P_i \in \mathcal{C}(\mathcal{F}(\mathcal{E}_F))$ for each $i = 1, \dots, n$. Finally, if $M = [P_1, \dots, P_n; B_1, \dots, B_n]$, then $f \in M \subset [x_1, \dots, x_n; B_1, \dots, B_n] \subset U$, where $M \in \mathcal{W}(\mathcal{C}(\mathcal{F}(\mathcal{E}_F))) = \mathcal{N}(F)$. □

Corollary 6.14. *Let X be a countably compact and initially $<\mathrm{nw}(X)$-compact space. Then $C_p(X)$ is hereditarily a D-space.*

Given a cardinal λ, recall that a topological space X is *λ-bounded* if each $A \subset X$ which satisfies $|A| \leq \lambda$ is contained in a compact subset of X. Every λ-bounded space is initially λ-compact [**68**].

Example 6.15. Let κ be a regular cardinal. Clearly, $\mathrm{nw}([0,\kappa)) = \kappa$. Let $A \subset \kappa$ with $|A| < \kappa$. Then, A is contained in the compact subset $[0, \sup(A)]$ of $[0,\kappa)$. This shows that $[0,\kappa)$ is λ-bounded and hence initially λ-compact, for each cardinal $\lambda < \kappa$. So, $[0,\kappa)$ is initially $<\kappa$-compact. Therefore, $C_p([0,\kappa))$ is monotonically $<\kappa$-monolithic (see Theorem 6.13). Since $\mathrm{nw}(C_p([0,\kappa))) = \mathrm{nw}([0,\kappa)) = \kappa$, by Theorem 6.6, $C_p([0,\kappa))$ is hereditarily a D-space.

It follows from Definition 6.1 and Proposition I.5.1 that every monotonically κ-monolithic space is monotonically $<\kappa^+$-monolithic. Then:

Corollary 6.16. *If X is monotonically κ-monolithic and* $\mathrm{nw}(X) \leq \kappa^+$*, then X is hereditarily D.*

In [**16**], R. Buzyakova asked if for every countably compact space X, $C_p(X)$ is hereditarily D. Afterward, she answered this question in the negative [**19**] by showing that there is a countably compact space X of cardinality ω_2 for which $l(C_p(X)) > e(C_p(X))$, and then $C_p(X)$ is not a D-space. Nevertheless, we have:

Proposition 6.17. *If X is a pseudocompact space with network weight at most ω_1, then $C_p(X)$ is hereditarily D.*

Proof. Let X be a pseudocompact space with $\mathrm{nw}(X) \leq \omega_1$. By Corollary 5.12, we have that $C_p(X)$ is monotonically ω-monolithic. Because of Corollary 6.16 and since $\mathrm{nw}(C_p(X)) = \mathrm{nw}(X) \leq \omega_1$, we conclude that $C_p(X)$ is hereditarily a D-space. □

Recall that a collection $\mathcal{A}$ of subsets of the natural numbers ω is an *almost disjoint family* if each A in $\mathcal{A}$ is infinite, and for two different elements $A, B \in \mathcal{A}$, $|A \cap B| < \omega$. A *maximal almost disjoint family* (mad family) is a maximal element in the family of all the almost disjoint families with the containment order.

A topological space X is a *Mrówka space* if it has the form $\omega \cup \mathcal{A}$, where $\mathcal{A}$ is an almost disjoint family, and its topology is generated by the following base: each $\{n\}$ is open for every $n \in \omega$, and an open canonical neighborhood of $A \in \mathcal{A}$ is of the form $\{A\} \cup B$ where $B \subset \omega$ and $A \setminus B$ is finite. In this case, we denote X by $\Psi(\mathcal{A})$. For every almost disjoint family $\mathcal{A}$, $\Psi(\mathcal{A})$ is a 0-dimensional locally compact first countable space, A is a closed discrete subspace of $\Psi(\mathcal{A})$ and A is dense. Moreover, $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is maximal (see [**32**]). So, $\Psi(\mathcal{A})$ is not normal if $\mathcal{A}$ is an infinite mad family.

Example 6.18. For a maximal almost disjoint family $\mathcal{A}$, the Mrówka space $\Psi(\mathcal{A})$ is pseudocompact and not countably compact. Under CH,

$\mathrm{nw}(C_p(\Psi(\mathcal{A}))) = \mathrm{nw}(\Psi(\mathcal{A})) \leq \omega_1$. Hence, by Corollary 6.17, $C_p(\Psi(\mathcal{A}))$ is hereditarily D.

7. Monotone monolithicity and Σ_s-products

As we saw in Chapter I: Every Collins-Roscoe space is monotonically monolithic. It is clear that if X is a Σ-product of an uncountable family of monotonically monolithic spaces with more than one point, then X is not a monotonically monolithic space, since it contains a closed subspace homeomorphic to the Σ-product of the product of ω_1 copies of the discrete space of cardinality two. On the other hand, every σ-product of a family of monotonically monolithic spaces is monotonically monolithic, and every σ-product of a family of Collins-Roscoe spaces has the Collins-Roscoe property. The concept of Σ_s-product was introduced by G.A. Sokolov who proved that a compact space X is a Gul'ko compact space if and only if X embeds into a Σ_s-product of real lines. It is easy to see that every Σ_s-product of a family $\{X_t : t \in T\}$ of spaces based on a point $a \in \prod\{X_t : t \in T\}$ contains the respective σ-product based on a, and is contained in the respective Σ-product based on a. G. Gruenhage proved that every Gul'ko compact space has the Collins-Roscoe property and V.V. Tkachuk gave a generalization of this result showing that every Σ_s-product of a family of second countable spaces has the Collins-Roscoe property.

Taking into account the above results, in [**75**] the following questions were posed: Is it true that every Σ_s-product of monotonically monolithic spaces is monotonically monolithic? Is it true that every Σ_s-product of Collins-Roscoe spaces has the Collins-Roscoe property? In this section, we answer these two questions in the affirmative.

Let $\{X_t : t \in T\}$ be a family of spaces and suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T. Let $X = \prod\{X_t : t \in T\}$ and fix a point $a \in X$. Given any $x \in X$, $A \subset X$ and $E \subset T$, let $\mathrm{supp}(x) = \{t \in T : x(t) \neq a(t)\}$, $\mathrm{supp}(x, E) = \mathrm{supp}(x) \cap E$ and $\mathrm{supp}(A, E) = \bigcup\{\mathrm{supp}(x, E) : x \in A\}$. For each $x \in X$, let Ω_x be the set $\{n \in \omega : |\mathrm{supp}(x, T_n)| < \omega\}$. Then the subspace $Z = \{x \in X : T = \bigcup\{T_n : n \in \Omega_x\}\}$ of X is called the Σ_s-product centered in a with respect to the sequence s.

Remark 7.1. Let $X = \prod\{X_t : t \in T\}$ be a topological product, let a be one fixed point in X and $s = \{T_n : n \in \omega\}$ a sequence of subsets of T.

(1) If x is an element of the Σ_s-product centered in a with respect to the sequence s, then $|\mathrm{supp}(x)| \leq \omega$; in fact, $\mathrm{supp}(x) = \bigcup\{\mathrm{supp}(x, T_n) : n \in \Omega_x\}$.
(2) If s^* is a sequence of subsets of T with $s \subset s^*$, then the Σ_s-product centered in a with respect to the sequence s is contained in the Σ_{s^*}-product centered in a with respect to the sequence s^*.

If s is a sequence of subsets of a set T, we define a relation R on T as follows: we say that $t_1 R\, t_2$ if for every $E \in s$ we have that $t_1 \in E$ if and only if $t_2 \in E$.

Lemma 7.2. *Let s be a sequence of subsets of T closed under complements and finite intersections. If $H_1, \ldots, H_n \in [T]^{<\omega}$ are nonempty sets such that for $t_i \in H_i$ and $t_j \in H_j$ we have $t_i R\, t_j$ if and only if $i = j$, then we can find a disjoint family $\{E_1, \ldots, E_n\} \subset s$ such that $H_i \subset E_i$ for $i = 1, \ldots, n$.*

Proof. If $n = 2$, for $t_1 \in H_1$ and $t_2 \in H_2$ we can find $E \in s$ such that $t_1 \in E$ and $t_2 \in T \setminus E$. Let $E_1 = E$ and $E_2 = T \setminus E$. Then, $\{E_1, E_2\}$ satisfies the required conditions. For $n > 2$ take $H_1, \ldots, H_n$ as in the Lemma. For every $i, j \leq n$ with $i \neq j$, take a disjoint family $\{E_{ij}, E'_{ij}\} \subset s$ such that $H_i \subset E_{ij}$ and $H_j \subset E'_{ij}$. Now take $E_i = \bigcap\{E_{ij} \cap E'_{ji} : j \leq n \text{ and } j \neq i\}$ for $i = 1, \ldots, n$. Then the elements in the family $\{E_1, \ldots, E_n\} \subset s$ are pairwise disjoint and $H_i \subset E_i$ for $i = 1, \ldots, n$. □

Theorem 7.3. *Every Σ_s-product of monotonically monolithic spaces is monotonically monolithic.*

Proof. Suppose that X_t is monotonically monolithic and fix the respective operator $\mathcal{N}_t$ for every $t \in T$. Suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T. Let $X = \prod\{X_t : t \in T\}$ and fix a point $a \in X$. We must prove that the Σ_s-product Z centered in a with respect to the sequence s is monotonically monolithic. Notice that since monotone monolithicity is a hereditary property, by the Remark 7.1 (2), we can suppose that the family s is closed under complements and finite intersections. Let $\mathcal{E}(s) = \{\{E_1, \ldots, E_n\} \in [s]^{<\omega} : E_i \cap E_j = \emptyset \text{ for } i \neq j\}$ be the family of all finite subfamilies of s such that its elements are pairwise disjoint.

Now we are ready to construct a monotonic monolithicity operator in Z. Fix a set $A \subset Z$. Take $E \subset T$, then by the Remark 7.1 (1) the set $\operatorname{supp}(A, E)$ has cardinality at most λ, where $\lambda = \max\{|A|, \omega\}$. Let $\mathcal{N}_E(A)$ be the family of all sets of the form $\prod\{N_t : t \in E\}$, where $N_t \in \mathcal{N}_t(p_t(A))$ for $t \in F$, $N_t = \{a(t)\}$ for $t \in E \setminus F$ and F is a finite subset of $\operatorname{supp}(A, E)$. Notice that the cardinality of $\mathcal{N}_E(A)$ does not exceed λ. Finally, let

$$\mathcal{N}(A) = \left\{ Z \cap \bigcap_{E \in \mathcal{F}} p_E^{-1}(N_E) : N_E \in \mathcal{N}_E(A) \text{ for every } E \in \mathcal{F} \text{ and } \mathcal{F} \in \mathcal{E}(s) \right\}.$$

Since $\mathcal{E}(s)$ is countable, the cardinality of $\mathcal{N}(A)$ is also at most λ. We will show that the operator $\mathcal{N}$ has the required properties.

Claim 1. If $A \subset B \subset Z$ then $\mathcal{N}(A) \subset \mathcal{N}(B)$.

Let $E \subset T$. Clearly $\operatorname{supp}(A, E) \subset \operatorname{supp}(B, E)$. By the election of $\mathcal{N}_t$ and since $p_t(A) \subset p_t(B)$ we have $\mathcal{N}_t(p_t(A)) \subset \mathcal{N}_t(p_t(B))$ for every $t \in T$. Therefore, $\mathcal{N}_E(A) \subset \mathcal{N}_E(B)$. Now take $N = Z \cap \bigcap\{p_E^{-1}(N_E) : E \in \mathcal{F}\} \in \mathcal{N}(A)$ where $N_E \in \mathcal{N}_E(A)$ for every $E \in \mathcal{F}$ and $\mathcal{F} \in \mathcal{E}(s)$. Since $N_E \in$

$\mathcal{N}_E(A) \subset \mathcal{N}_E(B)$ for every $E \in \mathcal{F}$, we have that $N \in \mathcal{N}(B)$. Hence, $\mathcal{N}(A) \subset \mathcal{N}(B)$.

Claim 2. If $A = \bigcup\{A_\alpha : \alpha < \gamma\} \subset Z$ where $A_\beta \subset A_\gamma$ for every $\alpha < \beta < \gamma$, then $\mathcal{N}(A) = \bigcup\{\mathcal{N}(A_\alpha) : \alpha < \gamma\}$.

By Claim 1, it suffices to show that $\mathcal{N}(A) \subset \bigcup\{\mathcal{N}(A_\alpha) : \alpha < \gamma\}$. First we will show that if $E \subset T$ and $N_E \in \mathcal{N}_E(A)$ then $N_E \in \mathcal{N}_E(A_{\alpha_E})$ for some $\alpha_E < \gamma$. In fact, let $E \subset T$ and take $N_E \in \mathcal{N}_E(A)$. Then $N_E = \prod\{N_t : t \in E\}$ where $N_t \in \mathcal{N}_t(p_t(A))$ for $t \in F$, $N_t = \{a(t)\}$ for $t \in E \setminus F$ and F is a finite subset of $\operatorname{supp}(A, E)$. Since $\operatorname{supp}(A, E) = \bigcup\{\operatorname{supp}(A_\alpha, E) : \alpha < \gamma\}$ and $\mathcal{N}_t(p_t(A)) = \mathcal{N}_t(\bigcup\{p_t(A_\alpha) : \alpha < \gamma\}) = \bigcup\{\mathcal{N}_t(p_t(A_\alpha)) : \alpha < \gamma\}$ for every $t \in F$, then we can choose $\alpha_E < \gamma$ such that $N_t \in \mathcal{N}_t(p_t(A_{\alpha_E}))$ for every $t \in F$ and F is a finite subset of $\operatorname{supp}(A_{\alpha_E}, E)$. Thus, $N_E \in \mathcal{N}_E(A_{\alpha_E})$. Now take $N = Z \cap \bigcap\{p_E^{-1}(N_E) : E \in \mathcal{F}\} \in \mathcal{N}(A)$ where $N_E \in \mathcal{N}_E(A)$ for every $E \in \mathcal{F}$ and $\mathcal{F} \in \mathcal{E}(s)$. For every $E \in \mathcal{F}$ we can take $\alpha_E < \gamma$ with $N_E \in \mathcal{N}_E(A_{\alpha_E})$. Let $\alpha = \max\{\alpha_E : E \in \mathcal{F}\}$. Then $N_E \in \mathcal{N}_E(A_\alpha)$ for each $E \in \mathcal{F}$ and therefore $N \in \mathcal{N}(A_\alpha)$. So, $\mathcal{N}(A) \subset \bigcup\{\mathcal{N}(A_\alpha) : \alpha < \gamma\}$.

Claim 3. For every $A \subset Z$ the family $\mathcal{N}(A)$ is an external network for $\operatorname{cl}_Z(A)$.

Fix $A \subset Z$, $x \in \operatorname{cl}_Z(A)$ and U an open set in Z with $x \in U$. Let $K \subset T$ a finite set and let $\{W_t : t \in K\}$ be a family such that: W_t is open in X_t for every $t \in K$, $a(t) \notin W_t$ if $x(t) \neq a(t)$ and $x \in W \subset U$ where $W = \{y \in Z : y(t) \in W_t \text{ for each } t \in K\}$. Let $\{H_1, \ldots, H_n\}$ be a partition of K such that for $t_i \in H_i$ and $t_j \in H_j$, we have that $t_i R\, t_j$ if and only if $i = j$. By Lemma 7.2 we can obtain a disjoint family $\{E'_1, \ldots, E'_n\} \in \mathcal{E}(s)$ such that $H_i \subset E'_i$ for $i = 1, \ldots, n$.

Take $i \in \{1, \ldots, n\}$. Let $t_i \in H_i$. Since $x \in Z$, then $T = \bigcup\{T_m : m \in \Omega_x\}$ and so $|\operatorname{supp}(x, T_{m_i})| < \omega$ for some $T_{m_i} \in s$ with $t_i \in T_{m_i}$. For $E_i = E'_i \cap T_{m_i}$, we have that $F_i = \operatorname{supp}(x, E_i)$ is a finite set and $H_i \subset E_i$. For every $t \in F_i \setminus H_i$ let $W_t = X_t \setminus \{a(t)\}$. Notice that $x(t) \in W_t \setminus \{a(t)\}$ for every $t \in F_i$. For an arbitrary $t \in F_i$, since $x(t) \in \operatorname{cl}_{X_t}(p_t(A))$ and $\mathcal{N}_t(p_t(A))$ is a network for $\operatorname{cl}_{X_t}(p_t(A))$, we can choose $N_t \in \mathcal{N}_t(p_t(A))$ such that $x(t) \in N_t \subset W_t$. If $t \in E_i \setminus F_i$ let $N_t = \{a(t)\}$. Thus, since F_i is a finite subset of $\operatorname{supp}(A, E_i)$, we have $N_{E_i} = \prod\{N_t : t \in E_i\} \in \mathcal{N}_{E_i}(A)$.

Finally, $\mathcal{F} = \{E_1, \ldots, E_n\} \in \mathcal{E}(s)$. Let $N = Z \cap \bigcap\{p_E^{-1}(N_E) : E \in \mathcal{F}\}$. Since $N_E \in \mathcal{N}_E(A)$ for every $E \in \mathcal{F}$, then $N \in \mathcal{N}(A)$. For $G = \bigcup\{F_i \cup H_i : i = 1, \ldots, n\}$ and $V = \{y \in Z : y(t) \in W_t \text{ for each } t \in G\}$ we have $x \in N \subset V \subset W \subset U$. □

Corollary 7.4. [**1**] *Any σ-product of monotonically monolithic spaces is monotonically monolithic.*

Corollary 7.5. [**77**] *Any countable product of monotonically monolithic spaces is monotonically monolithic.*

Corollary 7.6. *If X is a Σ_s-product of cosmic spaces, then $C_{p,n}(X)$ is monotonically monolithic for any $n \in \omega$.*

PROOF. Since every space with countable network is monotonically monolithic, we have that X is monotonically monolithic. Because of Corollary 3.4, the space $C_p(X)$ is monotonically monolithic. By Corollary 1.8, it happens that $C_{p,n}(X)$ is monotonically monolithic for any $n \in \omega$. $\square$

We will finish this section by proving that every Σ_s-product of a family of Collins-Roscoe spaces shares this property.

Recall that a space X has the Collins-Roscoe property if and only if for each $x \in X$, a countable collection $\mathcal{G}(x)$ of subsets of X is assigned such that, whenever $x \in U$ and U open, there is an open set V with $x \in V \subset U$ such that, whenever $y \in V$, then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$.

Theorem 7.7. *Every Σ_s-product of Collins-Roscoe spaces has the Collins-Roscoe property.*

PROOF. Suppose that, for each $t \in T$, X_t is Collins-Roscoe and fix the respective operator $\mathcal{G}_t$ for every $t \in T$. Suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T. Let $X = \prod\{X_t : t \in T\}$ and fix a point $a \in X$. We must prove that the Σ_s-product Z centered in a with respect to the sequence s has the Collins-Roscoe property. Since the Collins-Roscoe property is a hereditary property then, by Remark 7.1 (2), we can suppose that the family $\{T_n : n \in \omega\}$ is closed under complements and finite intersections. As before, we associate sequence s with the family $\mathcal{E}(s) = \{\{E_1, \ldots, E_n\} \in [s]^{<\omega} : E_i \cap E_j = \emptyset \text{ for } i \neq j\}$.

Now we are ready to construct an operator that witnesses the Collins-Roscoe property in Z. Fix a point $x \in Z$. Let $E \subset T$. By the Remark 7.1 (1), the set $\operatorname{supp}(x, E)$ is countable. Let $\mathcal{G}_E(x)$ be the family of all sets of the form $\prod\{N_t : t \in E\}$ where $N_t \in \mathcal{G}_t(x(t))$ for $t \in F$, $N_t = \{a(t)\}$ for $t \in E \setminus F$ and F is a finite subset of $\operatorname{supp}(x, E)$. Since each $\mathcal{G}_t(x(t))$ is countable, the family $\mathcal{G}_E(x)$ is also countable. Finally, let

$$\mathcal{G}(x) = \left\{ Z \cap \bigcap_{E \in \mathcal{F}} p_E^{-1}(N_E) : N_E \in \mathcal{G}_E(x) \text{ for every } E \in \mathcal{F} \text{ and } \mathcal{F} \in \mathcal{E}(s) \right\}.$$

Since $\mathcal{E}(s)$ is countable, the family $\mathcal{G}(x)$ is countable. We will show that the operator $\mathcal{G}$ has the required properties.

Fix $x \in Z$ and an open set U in Z with $x \in U$. We shall prove that there is an open set V with $x \in V \subset U$ such that, whenever $y \in V$, then $x \in N \subset U$ for some $N \in \mathcal{G}(y)$. Let $K \subset T$ be a finite set and let $\{W_t : t \in K\}$ be a family such that: W_t is an open set in X_t for each $t \in K$, $a(t) \notin W_t$ if $x(t) \neq a(t)$, and $x \in W \subset U$, where $W = \{y \in Z : y(t) \in W_t \text{ for each } t \in K\}$. Let $\{H_1, \ldots, H_n\}$ be a partition of K such that for $t_i \in H_i$ and $t_j \in H_j$ we have that $t_i R\, t_j$ if and only if $i = j$. By Lemma 7.2 we can obtain a disjoint family $\{E'_1, \ldots, E'_n\} \in \mathcal{E}(s)$ such that $H_i \subset E'_i$ for $i = 1, \ldots, n$. Take $i = \{1, \ldots, n\}$. Let $t_i \in H_i$. Since $x \in Z$, then $T = \bigcup\{T_n : n \in \Omega_x\}$ and so $|\operatorname{supp}(x, T_{m_i})| < \omega$ for some $T_{m_i} \in s$ with $t_i \in T_{m_i}$. Then for $E_i = E'_i \cap T_{m_i}$ we have that $H_i \subset E_i$

and $G_i = H_i \cup \operatorname{supp}(x, E_i) \subset E_i$ is finite. For each $t \in \operatorname{supp}(x, E_i) \setminus H_i$ let $W_t = X_t \setminus \{a(t)\}$. Clearly $x(t) \in W_t \setminus \{a(t)\}$ for every $t \in \operatorname{supp}(x, E_i)$.

Observe that $\mathcal{F} = \{E_1, \ldots, E_n\} \in \mathcal{E}(s)$. Let $G = \bigcup\{G_i : i = 1, \ldots, n\}$. Let t be an arbitrary element of G. By the election of $\mathcal{G}_t$, and because $x(t)$ belongs to W_t, there is an open set V_t in X_t with $x(t) \in V_t \subset W_t$ such that whenever $y_t \in V_t$, then $x(t) \in N_t \subset W_t$ for some $N_t \in \mathcal{G}_t(y_t)$. Let $V = \{y \in Z : y(t) \in V_t$ for each $t \in G\}$. Clearly $x \in V \subset W$. Fix $y \in V$. Take $i = \{1, \ldots, n\}$. Let $F_i = G_i \cap \operatorname{supp}(y, E_i) \subset E_i$. Notice that $\operatorname{supp}(x, E_i) \subset F_i$. For an arbitrary $t \in F_i$, since $y(t) \in V_t$, we can choose $N_t \in \mathcal{G}_t(y(t))$ such that $x(t) \in N_t \subset W_t$. For $t \in E_i \setminus F_i$, let $N_t = \{a(t)\}$. Thus, since F_i is a finite subset of $\operatorname{supp}(y, E_i)$ we have $N_{E_i} = \prod\{N_t : t \in E_i\} \in \mathcal{G}_{E_i}(x)$.

Finally notice that if $N = Z \cap \bigcap\{p_E^{-1}(N_E) : E \in \mathcal{F}\}$, then $N \in \mathcal{G}(y)$. Furthermore, for $W' = \{y \in Z : y(t) \in W_t$ for each $t \in G\}$, we have that $x \in N \subset W' \subset W \subset U$. □

Corollary 7.8. [**78**] *Every σ-product of Collins-Roscoe spaces has the Collins-Roscoe property.*

Corollary 7.9. [**78**] *Every countable product of Collins-Roscoe spaces has the Collins-Roscoe property.*

Since a compact space X is a Gul'ko compact space if and only if X embeds into a Σ_s-product of real lines, we have:

Corollary 7.10. [**38**] *Any Gul'ko compact space is a Collins-Roscoe space.*

Any space X which has a weakly-point-finite T_0-separating family of cozero subsets can be condensed into some Σ_s-product of real lines. Thus, by Proposition I.10.7 we have:

Corollary 7.11. [**78**] *Suppose that X is a Lindelöf Σ-space and there exists a weakly σ-point-finite T_0-separating family of cozero subsets of X. Then the space X has the Collins-Roscoe property and, in particular, it is hereditarily metalindelöf.*

Open problems and questions

Now we have a considerable amount of information about monotonically monolithic spaces and Collins-Roscoe spaces. However, the unsolved problems are numerous. In this section we give the list of some of the more intriguing open questions in this topic.

One of the central questions on D-spaces posed by E.K. van Douwen and W. Pfeffer is whether every Lindelöf space is a D-space. As we have seen, the concept of a D-space was studied a great deal ever since in almost every context and C_p-theory was not an exception. The following question by A.V. Arhangel'skii shows that the C_p-version of the question by E.K. van Douwen and W. Pfeffer is also open.

Question 1.1. [**28**] *Given a space X, suppose that $C_p(X)$ is Lindelöf. Must $C_p(X)$ be a D-space?*

For an arbitrary compact space we have that X is D and $C_{p,2n+1}(X)$ is hereditarily a D-space for any $n \in \omega$. However, we do not know:

Question 1.2. [**74**],[**76**] *Is it true that $C_p(C_p(X))$ is a D-space for any compact space X?*

If in addition X is a Corson compact space, we have that X is hereditarily a D space and $C_{p,2n+1}(X)$ is hereditarily a D-space for any $n \in \omega$. Furthermore, Gul'ko proved that, for any Corson compact X, the odd iterated function spaces are Lindelöf and the even ones are normal. This result was strengthened by Sokolov [**64**] who established that all iterated function spaces of a Corson compact space are Lindelöf. V.V. Tkachuk extracted from Sokolov's method the notion of a Sokolov space. It is known that for any Sokolov space X with $l^*(X) \cdot t^*(X) \leq \omega$ the space $C_{p,n}(X)$ is Sokolov and $l^*(C_{p,n}(X)) \cdot t^*(C_{p,n}(X)) \leq \omega$ for any $n \in \omega$ [**71**]. Of course, any Corson compact space X is Sokolov and satisfies $l^*(X) \cdot t^*(X) \leq \omega$ [**71**]. So, the following question might have a chance for an affirmative answer.

Question 1.3. [**74**] *Let X be a Corson compact space. Given any $n \in \omega$, the iterated function space $C_{p,n}(X)$ is Sokolov and Lindelöf, but must $C_{p,2n}(X)$ be a D-space for any $n \geq 1$?*

A compact Sokolov space X need not be a Corson compact space, but satisfies $l^*(X) \cdot t^*(X) \leq \omega$ [**71**]. So we can ask in a more general form.

Question 1.4. [**74**] *Let X be a Sokolov compact space. Given any $n \in \omega$, the iterated function space $C_{p,n}(X)$ is Sokolov and Lindelöf, but must $C_{p,2n}$ be a D-space for any $n > 1$?*

As we have seen, Sokolov compact spaces are a natural generalization of Corson compact spaces. On the one hand it is natural to extend to the class of Sokolov compact spaces the results obtained for Corson compact spaces. On the other hand, Sokolov property has a strong relationship with the class of D-spaces. In particular, all Sokolov spaces have countable extent so they are Lindelöf whenever they have the D-property. It is not clear whether the converse is true.

Question 1.5. [**74**] *Let X be a Sokolov compact space. Must X be a hereditarily D-space?*

Question 1.6. [**74**] *Suppose that X is a Sokolov Lindelöf space. Must X be a D-space?*

It is easy to see that the ordinal space ω_1 embeds in a Σ-product of real lines as a closed subspace, so Gul'ko's results are applicable to prove that $C_p(\omega_1)$ is Lindelöf. R. Buzyakova generalized this result establishing that, for any first countable countably compact subspace X of an ordinal, the space $C_p(X)$ is Lindelöf. It was proved in [**16**] that $C_p(\omega_1)$ is a D-space. In [**52**] and [**74**] this result was generalized by proving that: if X is a first countable countably compact subspace of an ordinal, then $C_p(X)$ is a D-space. In addition, in [**74**], was established that if X is a first countable countably compact subspace of an ordinal then X is a Sokolov space. As a consequence $C_p(X)^\omega$, $C_p(X^\omega)$ and $C_{p,2n+1}$ for $n \in \omega$ are Lindelöf whenever X is a first countable countably compact subspace of an ordinal.(see §6, Chapter I). The questions below are inspired by these results and by the relationship between Sokolov spaces and D-spaces.

Question 1.7. [**74**] *Suppose that X is a first countable subspace of an ordinal and $ext(X) = \omega$. Must the space X be Sokolov?*

Question 1.8. [**74**] *Suppose that X is a first countable subspace of an ordinal and $ext(X) = \omega$. Must $C_p(X)$ be a D-space?*

Question 1.9. [**74**] *Suppose that X is a Sokolov first countable space. Then $C_p(X)$ is Lindelöf and Sokolov, but must it be a D-space?*

Any first countable countably compact subspace of an ordinal is a Sokolov space. The space ω_1 is Sokolov, first countable and countably compact; besides, it embeds in a Σ-product of real lines. Therefore, it is a natural hypothesis whether every first countable Sokolov space can be condensed into a Σ-product of real lines. However, in [**74**] an example of a Sokolov first countable countably compact space of cardinality $\mathfrak{c}^+$ which cannot be condensed into a Σ-product of real lines was constructed. So, the hypothesis which follows are a natural change:

Question 1.10. [**74**] *Let X be the set of all countably cofinal ordinals in ω_2. Is it true in ZFC that X cannot be condensed (or, equivalently, embedded) into a Σ-product of real lines?*

A famous theorem of Gul'ko says that if X is compact and $C_p(X)$ is Lindelöf Σ then X is Corson compact, i.e., there exists an embedding of X into a Σ-product of real lines. In [**73**] a general fact about Lindelöf Σ-spaces was established which implies that if both spaces X and $C_p(X)$ are Lindelöf Σ then there is a rich family of retractions on the space X. As a consequence, for any Tychonoff space X, if $C_p(X)$ is Lindelöf Σ then X can be condensed into a Σ-product of real lines. This gave an essential strengthening of Gul'ko's theorem.

We know that if $C_p(X)$ is Lindelöf Σ then X has the Collins-Roscoe property. Also, a compact space with the Collins-Roscoe property is a Corson compact and hence can be embedded into a Σ-product of real lines. The next question is open.

Question 1.11. [**78**] *Let X be a Lindelöf Σ-space with the Collins-Roscoe property. Is it true that X can be condensed into a Σ-product of real lines?*

G. Gruenhage [**35**] asked if the Collins-Roscoe property is equivalent to monotonic monolithicity, and suggested that $C_p(X)$ for some Lindelöf Σ-space X might be a place to look for an example distinguishing the two concepts. V.V. Tkachuk [**78**] has shown that $C_p(\beta D)$ does not have the Collins-Roscoe property whenever D is an uncountable discrete space, but it is monotonically monolithic. However, this question for compact spaces and Lindelöf Σ-spaces remains open:

Question 1.12. [**75**] *Suppose that X is a monotonically monolithic compact space. Must X have the Collins-Roscoe property?*

Question 1.13. [**75**] *Suppose that X is a monotonically monolithic Lindelöf Σ-space. Must X have the Collins-Roscoe property?*

It is known that if X is a compact monotonically ω-monolithic space then X is monotonically monolithic. Since Lindelöf Σ-spaces are a natural generalization of compact spaces, it is also natural to ask if this result can be extended to the class of Lindelöf Σ-spaces.

Question 1.14. [**75**] *Suppose that X is a monotonically ω-monolithic Lindelöf Σ-space. Must X be monotonically monolithic?*

In [**1**] the problem of classifying monotonically monolithic generalized ordered spaces was proposed. It seems to be a difficult problem. The main conjecture is that any monotonically monolithic compact linearly ordered topological space is metrizable. Any space $S \subset \omega_1$ has countable tightness, so it is monotonically ω-monolithic if and only if it is monotonically monolithic. It was proved in [**1**] that a subspace $S \subset \omega_1$ is monotonically ω-monolithic if and only if it is not stationary, if and only if it is metrizable. The following problem is open.

Problem 1.15. [**1**] *Given an ordinal α, give a description of all monotonically (ω-)monolithic subspaces of α.*

It is well known that the Lindelöf Σ-property of $C_p(X)$ usually has very strong implications even if nothing is assumed about X. In this case, the Lindelöf Σ-property of $C_p(X)$ implies that $C_{p,2n}(X)$ has the Collins-Roscoe property for any $n \in \omega$. But we do not know if in this case $C_p(X)$ has itself the Collins-Roscoe property. In the case when the network weight of X is almost ω_1 we know that the answer is positive, but in the general case we do not know if $C_p(X)$ is at least hereditarily metalindelöf or has the hereditary D-property.

Question 1.16. [**78**] *Suppose that $C_p(X)$ is a Lindelöf Σ-space. Is it true that $C_p(X)$ has the Collins-Roscoe property?*

Question 1.17. [**78**] *Suppose that $C_p(X)$ is a Lindelöf Σ-space. Is it true that every subspace $Y \subset C_p(X)$ is metalindelöf?*

Question 1.18. [**76**], [**78**] *Suppose that $C_p(X)$ is a Lindelöf Σ-space. Must $C_p(X)$ itself be a hereditary D-space?*

Recall that ω_1 is a caliber of a space X if every point-countable family of non-empty open subsets of X is countable. Clearly, if ω_1 is a caliber of X then any disjoint family of open sets in X is countable; that is, $c(X) \leq \omega_1$. It can be seen that sometimes being ω_1 a caliber of X implies very strong restrictions on the topological structure of X or $C_p(X)$. It is known that if a space X has the Collins-Roscoe property and ω_1 is a caliber of X then X is cosmic [**78**]. However, it is not known if this result holds when we only assume monotone monolithicity of X, even when X is Lindelöf or has countable tightness.

Question 1.19. [**1**] *Assume that X is a monotonically monolithic space and ω_1 is a caliber of X. Must X have a countable network?*

Question 1.20. [**1**] *Assume that X is a monotonically monolithic Lindelöf space and ω_1 is a caliber of X. Must X have a countable network?*

Question 1.21. [**1**] *Assume that X is a monotonically monolithic space, $t(X) \leq \omega$, and ω_1 is a caliber of X. Must X have a countable network?*

In the presence of compactness the above result can be strengthened. Indeed: If a compact space X is monotonically monolithic and ω_1 is a caliber of X, then X is metrizable [**77**]. It would be interesting to improve this result as follows (recall that we do not know if monotone monolithicity and the Collins-Roscoe property coincide in the class of compact spaces):

Question 1.22. [**75**] *Suppose that X is a monotonically monolithic compact space with $c(X) \leq \omega$. Must X be metrizable?*

Question 1.23. [**78**] *Suppose that a compact space X has the Collins-Roscoe property and $c(X) = \omega$. Must X be metrizable?*

Furthermore, in the presence of strong monotone monolithicity we can also obtain strong restrictions over X. Indeed: If X is a strongly monotonically monolithic space, ω_1 is a caliber of X, and $d(X) \leq \omega_1$, then X is metrizable. Furthermore, under CH, if X is a strongly monotonically monolithic space and ω_1 is a caliber of X, then X is metrizable [**1**]. Of course, it would be interesting improve these results by answering the following questions.

Question 1.24. [**1**] *It is true in ZFC that any strongly monotonically monolithic space with caliber ω_1 is second countable?*

Question 1.25. [**1**] *Assume that X is a strongly monotonically monolithic Lindelöf space and ω_1 is a caliber of X. Is it true in ZFC that X must be second countable?*

It is still an open question whether every strong Collins-Roscoe space has a point-countable base. The Corollary to Theorem 5 of [**45**] establishes that every strong Collins-Roscoe space X has a dense subspace Y with a point-countable base. As is noted in [**75**] it is possible to extract from the proof of Theorem 5 of [**45**] that Y actually has a point-countable external base $\mathcal{B}$. In [**75**] it was proved that if X is a Collins-Roscoe space, then every left-separated subspace $Y \subset X$ has a point-countable open expansion. It is not known if these results can be strengthened as follows, when compactness of the space X is assumed.

Question 1.26. [**75**] *Suppose that X is a Collins-Roscoe compact space. Must X have a dense metrizable subspace?*

Question 1.27. [**75**] *Suppose that X is a monotonically monolithic compact space. Must X have a dense metrizable subspace?*

For a maximal almost disjoint family $\mathcal{A}$ of subsets of ω, under CH the space $C_p(\Psi(\mathcal{A}))$ is a hereditarily D-space. Recall that a maximal almost disjoint family $\mathcal{A}$ is a *Mrówka mad family* if the one-point compactification of $\Psi(\mathcal{A})$ coincides with its Stone-Čech compactification. If we do not assume CH and $\mathcal{A}$ is a Mrówka mad family, it is not difficult to see, using a slight modification of Proposition II.5.8, that $C_p(\Psi(\mathcal{A}))$ is monotonically $<\kappa$-monolithic when $\mathrm{nw}(C_p(\Psi(\mathcal{A}))) = \mathrm{nw}(\Psi(\mathcal{A})) = |\mathcal{A}|$. Hence, in this case $C_p(\Psi(\mathcal{A}))$ is a hereditarily D-space. Furthermore, it was proved in [**43**] that if $\mathcal{A}$ is an infinite maximal almost disjoint family on ω, then the space $C_p(\Psi(\mathcal{A}))$ is not normal, and its extent and Lindelöf number are all equal to $|\mathcal{A}|$.

We think that the following question might have a chance for an affirmative answer.

Question 1.28. [**62**] *Is it true in ZFC that for every almost disjoint family $\mathcal{A}$, that space $C_p(\Psi(\mathcal{A}))$ is a hereditarily D-space?*

If κ is an infinite cardinal number of countable cofinality, then κ is a σ-compact space, so $C_p([0,\kappa))$ is monotonically monolithic and hence a hereditarily D-space. If κ is regular we know that $C_p([0,\kappa))$ is monotonically $<\kappa$-monolithic where $\mathrm{nw}(C_p([0,\kappa))) = \kappa$. Hence, in this case $C_p([0,\kappa))$ is also a hereditarily D-space. However, we do not know if this result holds for singular cardinals.

Question 1.29. [**62**] *Let λ be a singular cardinal of uncountable cofinality. Is it true that $C_p([0,\lambda))$ is a D-space (resp., a hereditarily D-space)?*

It is a classical result that any product (σ-product or Σ-product) of Lindelöf Σ-spaces is stable. In [**61**] the concept of monotonically stable space was introduced and it was proved that any product (σ-product or Σ-product) of cosmic spaces is monotonically stable. However, for Lindelöf Σ-spaces it was only proved that any product of such spaces is monotonically stable. Furthermore, for an infinite cardinal λ, if $\kappa = \lambda^+$ and Y is a Σ_κ-product of a family Lindelöf Σ-spaces, then Y is monotonically λ-stable. Therefore, the next question arises in a natural form.

Question 1.30. [**61**] *Is it true that every σ-product or Σ-product of an arbitrary family of Lindelöf Σ-spaces is monotonically stable?*

Any σ-product of compact spaces is σ-compact and hence, monotonically stable. In a more general form, any Σ_s-product of compact spaces is Lindelöf Σ and thus, monotonically stable. But we do not know:

Question 1.31. [**61**] *Is it true that every Σ-product of an arbitrary family of compact spaces is monotonically stable?*

Finally, the definition of monotonically stable is in terms of real-valued continuous functions and the pointwise convergence topology, so it would be interesting to obtain some intrinsec characterization of such concept.

Problem 1.32. [**61**] *Find an internal characterization of monotone stability.*

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