Universidad Nacional Autónoma de México
Programa de Posgrado en Astrofísica
Instituto de Astronomía

Gravedad extendida: aplicaciones y consecuencias astrofísicas

Tesis
que para optar por el grado de
Doctora en Ciencias (Astrofísica)
presenta:

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México, D.F. Junio, 2013

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Extending gravity:

applications and astrophysical consecuences

Thesis
presented to obtain the degree of
Doctor of Sciences (Astrophysics)
by:

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Mexico City, June, 2013

Resumen

Con el fin de evitar la necesidad de los componentes oscuros en sistemas astrofísicos y cosmológicos, a lo largo de esta tesis se muestra que es posible construir una teoría de gravedad modificada consistente con diferentes observaciones astrofísicas a partir de considerar la escala de aceleración de MOND a_0 como una constante fundamental en la gravitación. Esto, sin introducir materia oscura desconocida para ajustar las observaciones.

Se construyó primero la teoría de gravedad extendida no–relativista modificando la fuerza gravitacional Newtoniana a partir de una función únicamente de la variable x, que depende de la masa y el radio del sistema y a_0. De este modo se ajustaron diversos fenómenos astrofísicos dentro de los errores observacionales y se probó la consistencia con el límite máximo a las desviaciones a la gravedad Newtoniana dentro del Sistema Solar.

Luego se muestra de dos maneras que el régimen de aceleración MONDiano es el límite de campo débil de una teoría métrica extendida $f(\chi) = \chi^{3/2}$: primero, a partir de un análisis a orden de magnitud de las ecuaciones de campo resultantes para una métrica estática esféricamente simétrica, perturbando la componente temporal de la métrica hasta segundo orden en potencias de v/c y a orden cero las componentes radial y angular; después, con el análisis completo de perturbaciones hasta segundo orden en las componentes temporal y radial de dicha métrica y las componentes angulares a orden cero.

A partir de los componentes métricos obtenidos del análisis perturbativo completo, se muestra que la teoría ajusta la fenomenología de las curvas de rotación planas y la relación Tully–Fisher asociada, así como los lentes gravitacionales observados en galaxias y grupos de galaxias.

Finalmente, también exploré en mi trabajo de tesis la posibilidad de que la constante de aceleración de MOND sea una cantidad fundamental relacionada con algunas constantes universales y no únicamente una constante fenomenológica. Obtuvimos que es posible recuperar relaciones que inicialmente parecían “coincidencias” numéricas, pero que pudieran ser manifestaciones de relaciones fundamentales en la física.

Abstract

In the aim to avoid the necessity of dark components in astrophysical and cosmological systems, throughout this thesis it is shown that it is possible to construct a modified theory of gravity consistent with many astrophysical observations considering that MOND's acceleration scale a_0 is a fundamental constant that enters into the gravitation. This, without introducing unknown dark matter to fit the observations.

First, we constructed the extended non–relativistic theory of gravity modifying the Newtonian force through a function only of the variable x, which depends on the mass and the radius of the system. In this way, we adjusted diverse astrophysical phenomena within the observational errors and checked the consistency with the upper limits of the deviations to Newtonian gravity in the Solar System.

Then, it is shown in two ways that the weak field limit of the theory $f(\chi) = \chi^{3/2}$ gives us the MONDian acceleration regime: first, from an order of magnitude approach of the resulting field equations, for a static spherically symmetric metric, perturbing the time metric component up to the second order in powers of v/c and up to zeroth order in the radial and angular components; then, performing the complete perturbation analysis up to the second order in the time and radial components of the metric and up to zeroth order in the angular components.

From the metric components obtained from the complete perturbation analysis, it is shown that our theory adjusts the phenomenology of the flat rotation curves for disc galaxies and the associated Tully–Fisher relation, also the gravitational lenses observed in galaxies and groups of galaxies.

Finally in this thesis, I also explored the possibility of MOND's acceleration constant to be a fundamental quantity related to some universal constants and not only a phenomenological constant. I obtained that it is possible to recover some relations initially thought as numerical "coincidences", but that may be manifestations of some fundamental relations in physics.

Publicaciones producto del trabajo de tesis

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  aceptado para su publicación en MNRAS (doi:10.1093)
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Agradecimientos

Agradezco profundamente el apoyo, tanto académico como personal, de mis tutores Sergio Mendoza y Juan Carlos Hidalgo. Gracias a ustedes he aprendido a adentrarme en el mundo científico con un enfoque crítico y creativo. Además por su invaluable amistad y seis años de vivencias, de nuevo, gracias!

Agradezco a los miembros del jurado, Salvatore Capozziello, Francisco Guzmán, Roberto Sussman y Tonatiuh Matos, por sus valiosos comentarios y aportes a este trabajo.

Agradezco también el apoyo siempre a mi formación de Xavier Hernández, de quien tengo muchas enseñanzas. Por el aprendizaje y hospitalidad en Nápoles de Salvatore Capozziello, Cosimo Stornaiolo y Gerardo Cristofano, gracias!

Muy especialmente a Bertha Vázquez, siempre dedicada y preocupada por todos los estudiantes. Por todas las veces que me recordaste lo que olvidaba entregar, por toda tu ayuda y amistad.

A mis amigos y compañeros del instituto, que aquí siguen o se encuentran lejos terminando su doctorado, con quienes compartí clases, presiones, desvelos y fiestas. Muy especialmente a Rodrigo, porque sin tí el instituto se siente solitario.

A mis amigos de la facultad, que han tomado diferentes caminos que convergen gracias a nuestra imperecedera amistad: Edith, Pamela, Tonantzin, Arturo, Cristopher, Ismael, Iván y Paco.

Sobre todo agradezco a mi familia, a mis papás Tula y Raúl, a mis hermanos Tania y Raúl, a Noemí y Ángel, que siempre han estado junto a mi y comparten mis sueños y alegrías.

A mi nueva familia, Esteban y Bruno. A mi Esteban que comparte cada aventura, triunfo y fracaso con su característico sentido del humor y gruñonez.

Finalmente, agradezco al Instituto de Astronomía y todo el personal que me ha ayudado en muchos momentos. Agradezco el apoyo económico de DGAPA-UNAM (PAPIIT IN116210-3, IN111513-3 y IN103011-3) y CONACyT (207529 y SNI 729-9806) para la realización de esta tesis.

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I gratefully acknowledge to Instituto de Astronomía and the support of a DGAPA-UNAM grant (PAPIIT IN116210-3, IN111513-3 and IN103011-3) and CONACyT (207529 and SNI 729-9806).

A mis papás
y con especial cariño a mi abuelita Ana.

Índice general

Índice de figuras

Índice de tablas

Capítulo I

Introducción

Las observaciones de las supernovas tipo Ia, las anisotropías observadas en la radiación cósmica de fondo en microondas, las oscilaciones acústicas de la materia bariónica, el espectro de potencias de materia, los lentes gravitacionales, entre otras, representan fuertes evidencias para el modelo cosmológico estándar, el llamado *modelo de concordancia* ΛCDM. De las más recientes observaciones de la sonda espacial europea *Planck*, se supone una componente bariónica que contribuye sólo un $4\,\%$ al contenido actual de la densidad de materia–energía en el universo, junto con un $1\,\%$ de radiación, mientras que las componentes "oscuras" son aproximadamente el $95\,\%$, alrededor del $27\,\%$ materia oscura fría (Cold Dark Matter, CDM) y un $68\,\%$ energía oscura o contribución de la constante cosmológica positiva Λ (Planck Collaboration et al., 2013).

La materia oscura fue postulada para explicar las curvas de rotación observadas en galaxias espirales, así como los cocientes masa–luminosidad en galaxias gigantes y cúmulos de galaxias, los lentes gravitacionales y la formación de estructura en el universo temprano, entre otros fenómenos astrofísicos y cosmológicos (ver la sección §1). Por su parte, la energía oscura o la constante cosmológica fue postulada para explicar la expansión acelerada del universo (Riess et al., 1998; Perlmutter et al., 1999).

Hasta ahora el modelo ΛCDM es el que mejor ajusta a las observaciones existentes. Sin embargo, la búsqueda directa o indirecta de partículas candidato a materia oscura ha arrojado resultados nulos. Aunado a esto, la falta de un fundamento teórico sólido para la energía oscura, abren la posibilidad de que no existan entes oscuros en el universo sino que, en alguna parte, la teoría asociada a estos fenómenos astrofísicos y cosmológicos tenga que ser modificada.

Los modelos de materia y energía oscuras se basan en que la teoría de gravitación de Newton y la relatividad general de Einstein son las teorías de gravedad válidas a todas las escalas. Sin embargo, su validez sólo ha sido probada con excelente precisión en sistemas cuyas escalas no son mayores a las del Sistema Solar. En este sentido es concebible que ambas, tanto expansión acelerada del universo como mayor fuerza gravitacional requerida en diferentes sistemas, representen un cambio en nuestro entendimiento de las interacciones gravitacionales.

Desde un punto de vista geométrico, las teorías de gravedad modificada son una alternativa viable a los problemas astrofísicos y cosmológicos que la materia y energía oscuras tratan de resolver. De hecho, la idea de modificar las leyes de gravedad vigentes es casi tan antigua como la gravitación de Newton misma. Casi 200 años después de que Newton publicara los *Philosophiæ naturalis principia mathematica* (Newton, 1687), en su ensayo "On the Hypotheses which lie at the Bases of Geometry" (1856) (Sobre las hipótesis que sirven de fundamento a la geometría), Riemann afirmaba que las relaciones métricas del espacio no pueden ser deducidas más que por la experiencia, que hay que explorar la probabilidad de las hipótesis, "dentro de los límites de observación, y juzgar por esto el grado de seguridad de la extensión de estos hechos fuera de estos límites, tanto en el sentido de los inconmensurablemente grandes, como en el de los inconmensurablemente pequeños" (Riemann, 1873a,b).

Para finales del siglo XIX, se creía que era posible resolver todos los problemas de la física que se presentaran por medio de la mecánica Newtoniana y el electromagnetismo clásico de Maxwell. Fue entonces cuando surgieron experimentos y observaciones que desafiaron la teoría establecida a dos escalas diversas a las que se estaba acostumbrado a tratar: a velocidades comparables a la de la luz y a escalas de longitud comparables con su longitud de onda de de Broglie.

Los problemas a los que se enfrentaba la física entonces representaron un cambio en el entendimiento de la naturaleza como se conocía. El tener la posibilidad de explorar velocidades y energías grandes llevó a Einstein a formular su teoría de la relatividad, la especial (Einstein, 2005) y la general (Einstein, 1916). El explorar las escalas más pequeñas de la materia derivó en la construcción de la mecánica cuántica. Para 1887, se llevó a cabo uno de los experimentos más bellos e importantes en la historia de la física: el experimento de Michelson & Morley (1887). Éste descartó la existencia del éter, que suena ahora como un "ente oscuro", y sentó las bases para la relatividad especial de Einstein.

Antes, Bouvard (1821) había observado un movimiento anómalo de Urano en su órbita y pensaba que podría deberse a que la gravedad del Sol a esa distancia pudiera ser diferente de la Newtoniana o debido a un octavo planeta no descubierto aún. En 1845, Le Verrier (1846) y Adams (ver Waff & Kollerstrom, 2001) predijeron matemáticamente dónde debería observarse Neptuno. Exactamente en esa posición fue observado por Galle en 1846 (ver Waff & Kollerstrom, 2001). Posteriormente, las observaciones de la precesión anómala de Mercurio en su órbita llevaron a Le Verrier (ver des sciences, 1859) a postular la existencia de un planeta desconocido entre el Sol y Mercurio. Él llamó a este planeta faltante Vulcano. Le Verrier murió en 1877 creyendo en la existencia de este planeta, luego de que un astrónomo aficionado reportara su tránsito sobre el disco solar (ver Baum & Sheehan, 1997).

Como sabemos este planeta nunca fue observado, y el problema se resolvió únicamente con una extensión a la teoría de gravitación de Newton: la relatividad general de Einstein. Finalmente, otro experimento fue definitivo para probar la validez de la relatividad general: la deflexión de la luz de las estrellas lejanas por el disco solar. La relatividad general predice un ángulo de deflexión dos veces mayor al sugerido por la mecánica de Newton, y esto fue lo que se Dyson y Eddington observaron (Dyson et al., 1920).

Todos estos experimentos y observaciones cambiaron la manera en que entendemos el universo. La lección más importante que hay que aprender de la historia es que han sido las observaciones y experimentos los que han llevado a trascender los paradigmas y a cambiar o extender las leyes de la física. El camino que tomó la física a partir de las observaciones que desafiaron directamente a la mecánica clásica y al electromagnetismo podría indicarnos el que podría tomar la teoría actualmente aceptada. Debido a que la materia y energía oscuras no han sido detectadas directamente, debemos buscar modificaciones a la teoría gravitacional que ha sido usada para postular su existencia.

La sonda espacial europea *Planck* ha obtenido el más preciso y detallado mapa de la luz más antigua del universo. Así mismo, otras sondas y exploraciones que ya están en operación, como DES (Dark Energy Survey), *Gaia*, SDSS (Sloan Digital Sky Survey) IV, han comenzado a arrojar las primeras luces al modelo cosmológico aceptado y en un futuro continuarán haciéndolo otras como *Euclid* y SKA (Square Kilometre Array)[†].

§1. Gravedad modificada no–relativista

La materia oscura se postuló a principios de los 1930s cuando comenzaron a surgir discrepancias en las curvas de rotación observadas en galaxias de disco y en las dispersiones de velocidades observadas en cúmulos de galaxias. En 1932, Jan Oort descubrió que las estrellas en la vecindad solar se movían más rápido de lo esperado (resultado que después se confirmó era incorrecto; ver Kuijken & Gilmore, 1989a). Por su parte, Zwicky (1933) notó el mismo comportamiento con la velocidad de las galaxias en cúmulos. A la materia que podría explicar las observaciones le llamaron "materia faltante" o "materia oscura". Más tarde, Babcock (1939) sugirió que el cociente masa–luminosidad se incrementaba con el radio para observaciones de la curva de rotación de la galaxia de Andrómeda; sin embargo él no lo atribuyó a ninguna forma de materia faltante, sino a absorción de la luz dentro de la galaxia o a "dinámica modificada" en las regiones externas. Finalmente, Rubin et al. (1980) presentaron evidencia contundente que confirmaba que un gran número de galaxias están dominadas por materia oscura, contenida en un "halo oscuro" no visible.

En un principio se pensaba que la materia faltante eran objetos que no podían verse a través de un telescopio, como enanas cafés, agujeros negros, polvo, planetas, etc., pero las abundancias observadas de estos objetos no alcanzaban a cuantificar el total de masa esperado por la dinámica galáctica. De este modo se postuló la materia oscura "no bariónica": partículas elementales que únicamente interactúan gravitacionalmente con la materia conocida. Desde entonces se han propuesto decenas de partículas nuevas que han sido descartadas (ver, por ejemplo, Bertone et al., 2005). Actualmente el principal candidato es el neutralino, una partícula supersimétrica resultado de una extensión del modelo estándar

[†]En los próximos años, la gran cantidad de datos de telescopios y detectores a bordo de sondas espaciales ayudarán a ajustar mejor los modelos de materia/energía oscuras y las teorías de gravedad modificada. Con esto, será posible hacer una elección más realista de una sobre la otra. Es importante notar que estas misiones detectan radiación electromagnética, no prueban directamente la existencia de materia oscura o si la teoría gravitacional debe ser modificada. La detección directa de una partícula de materia oscura con la densidad astrofísica requerida (90 % de toda la materia), sin duda significará el final de esta discusión. Sin embargo en el futuro cercano, sólo se contará con la detección indirecta de las partículas de materia oscura, así que la coherencia de las teorías de gravedad modificada con los datos servirá como fuerte evidencia de la no existencia de la materia oscura, pues no es posible probar con ningún experimento la no existencia de cualquier entidad.

de partículas elementales.

Como solución alternativa a los problemas de materia faltante y expansión acelerada del universo, se han formulado diversas propuestas que modifican las teorías de gravitación aceptadas a distintas escalas, tanto en el límite no–relativista como en el relativista. La idea principal detrás de algunas de estas teorías es el planteamiento de una modificación a las teorías de gravedad Newtoniana y a la relatividad general de Einstein.

Los requisitos que una teoría de gravedad debe satisfacer en el régimen no–relativista son: (1) no sobrepasar los límites máximos observacionales de la desviación a la ley de gravitación de Newton dentro del Sistema Solar, (2) ajustar las curvas de rotación planas en galaxias de disco y cumplir con la relación Tully–Fisher para galaxias soportadas por momento angular, así como con la relación Faber–Jackson para galaxias soportadas por presión y, recientemente, ajustar las regiones externas de cúmulos globulares y de sistemas binarios extendidos y (3) reproducir las relaciones de escala observadas en diferentes sistemas, desde las galaxias esferoidales enanas del Grupo Local hasta galaxias elípticas gigantes y cúmulos de galaxias, entre muchas otras pruebas donde tradicionalmente se requiere de materia oscura.

La primera modificación exitosa en este régimen es la Dinámica Newtoniana Modificada (Modified Newtonian Dynamics, MOND), descrita brevemente en la siguiente sección (ver Sanders & McGaugh, 2002, para un review sobre MOND). El estudio de esta propuesta es importante ya que, debido a su naturaleza fenomenológica y a su éxito en el límite no-relativista, se entiende que cualquier teoría fundamental de gravedad modificada debería ajustarse a ella en escalas galácticas en el régimen de bajas aceleraciones. Como se discutirá en el Capítulo II, la teoría de gravedad extendida que proponemos es equivalente a la descripción MONDiana en algunos sistemas, por ejemplo en los que son esféricamente simétricos, pero con ventajas notables.

MOND

En el límite no–relativista, Milgrom (1983b,a) introdujo la Dinámica Newtoniana Modificada para explicar las curvas de rotación planas observadas en galaxias espirales. Utilizando una constante fenomenológica de aceleración a_0, Milgrom formuló un criterio tal que para aceleraciones superficiales mucho mayores a a_0, la dinámica es Newtoniana y para aceleraciones mucho menores que a_0, la dinámica se modifica para producir órbitas circulares con velocidades independientes del radio, a escalas suficientemente grandes.

Tal modificación a la dinámica en simetría esférica o cilíndrica está dada por (Milgrom, 1983b)

$$a\,\mu(a/a_0) = |\nabla\phi_N| = \frac{GM(r)}{r^2}\,, \tag{1.1}$$

donde ϕ_N es el potencial gravitacional Newtoniano, G es la constante de gravitación de Newton, $M(r)$ es la masa encerrada al radio r y la *función de interpolación* $\mu(a/a_0)$ tiene los siguientes límites:

$$\mu(a/a_0) = \begin{cases} 1, & \text{para } a \gg a_0\,, \\ a/a_0, & \text{para } a \ll a_0\,. \end{cases} \tag{1.2}$$

En estos límites, el régimen de aceleraciones muy grandes respecto a a_0 corresponde a gravitación Newtoniana y el de aceleraciones muy pequeñas respecto a a_0 al llamado *régimen profundo* de MOND ("deep MOND") o MONDiano. De las ecuaciones anteriores se deduce que en este régimen la aceleración está dada por

$$a = -\frac{(GMa_0)^{1/2}}{r}. \tag{1.3}$$

Formalmente en un sistema con simetría cilíndrica, como es el caso de una galaxia de disco, la suposición de simetría esférica necesariamente introduce errores en los cálculos. Sin embargo, en analogía con el caso Newtoniano, donde la curva de rotación de un disco infinitamente delgado que decrece exponencialmente se desvía tan sólo un 20 % del caso esférico a radios grandes (ver, por ejemplo, Binney & Tremaine, 2008), el error producido se encuentra dentro del rango de error observacional. Entonces, dado que las observaciones muestran que las regiones externas de galaxias de disco caen dentro del régimen profundo de MOND y que además a estos radios la geometría es casi esférica, la velocidad circular tiende a una constante a radios grandes:

$$v_{\mathrm{flat}} = (GM_b a_0)^{1/4} \approx \text{const.}, \tag{1.4}$$

ya que la aceleración centrífuga es $a_c = v^2/r$ y se asume que la contribución principal a la masa bariónica M_b se da a radios menores. De esta manera se recuperan las curvas de rotación planas observadas en galaxias de disco. Más aún, esta última prescripción es consistente con la relación Tully–Fisher bariónica, entre la velocidad en la parte plana de la curva de rotación y la masa bariónica para un cociente masa–luminosidad constante, como sigue (McGaugh, 2005):

$$v_{\text{flat}} \propto M_b^{1/4}. \tag{1.5}$$

Existen en la literatura diferentes funciones de interpolación que se han propuesto para caracterizar la aceleración en rangos intermedios. La función clásica fue propuesta por Milgrom (1983b):

$$\mu(a/a_0) = \frac{a/a_0}{\sqrt{1+(a/a_0)^2}}. \tag{1.6}$$

Otras funciones ajustan con mayor precisión las curvas de rotación, tanto de nuestra Galaxia como la de otras galaxias de distintos tamaños y masas (ver, por ejemplo, la Figura II.2). Con las funciones de interpolación estándar es posible ajustar con gran precisión las curvas de rotación de las galaxias de disco a todos los radios. Del mejor ajuste a curvas de rotación de galaxias espirales en la vecindad Galáctica se tiene que (Begeman et al., 1991)

$$a_0 \approx 1.2 \times 10^{-10} \mathrm{m/s^2}. \tag{1.7}$$

Hasta aquí MOND es sólamente una prescripción fenomenológica, no una teoría completa, ya que viola el principio de conservación de momento. Estas leyes de conservación se pueden satisfacer automáticamente para una teoría física derivada de principios variacionales. En este sentido, Bekenstein & Milgrom (1984) construyeron una primera generalización no–relativista de MOND a partir del siguiente lagrangiano acuadrático, llamada AQUAL (AQUAdratic Lagrangian):

$$\mathcal{L} = -\frac{a_0^2}{8\pi G} g\left(\frac{|\nabla\phi_N|^2}{a_0^2}\right) - \rho\phi_N, \tag{1.8}$$

donde ρ es la densidad de masa y $g(y)$ es una función adimensional, tal que se recupera la aceleración Newtoniana (o la ecuación de Poisson) para $g(y) = y$. De este lagrangiano se obtiene la ecuación de Poisson modificada:

$$\nabla \cdot \left[\mu\left(\frac{|\nabla\phi_N|}{a_0}\right)\nabla\phi_N\right] = 4\pi G\rho, \tag{1.9}$$

donde $\mu(\sqrt{y}) = dg(y)/dy$. En los límites Newtoniano y profundo de MOND se asume que:

$$g(y) = \begin{cases} y, & y \gg 1\,, \\ \frac{2}{3}y^{3/2}, & y \ll 1\,. \end{cases} \tag{1.10}$$

Para sistemas con geometría esférica, cilíndrica o plana, la ecuación (1.9) se integra directamente. Con la siguiente prescripción para la aceleración:

$$a = -\nabla\phi_N, \tag{1.11}$$

la solución corresponde a la fórmula de MOND (1.1). Para sistemas que no tienen esta simetría, la integración numérica muestra que se puede utilizar la misma fórmula con buena precisión.

MOND ha sido una modificación a la gravedad muy exitosa al ajustar las curvas de rotación planas y la relación Tully–Fisher para galaxias de disco. En este sentido, cualquier teoría de gravedad modificada que pretenda reemplazar a la materia oscura debe converger a la prescripción MONDiana en el límite $a \ll a_0$. Sin embargo, del estudio de grupos y cúmulos de galaxias se ha mostrado que, aún en el régimen profundo de MOND, se requiere una componente dominante de materia oscura en estos sistemas (del 60 al 80 % de la masa dinámica o virial). Angus et al. (2008) mostraron que la región central de cúmulos podría explicarse para un halo de neutrinos con masa de 2 eV (que es aproximadamente el límite superior experimental). Sin embargo en la escala de grupos de galaxias, esta contribución central no puede explicarse por una contribución con un halo de neutrinos de esa masa. Además, MOND/AQUAL no es capaz de reproducir los lentes gravitacionales observados en diferentes sistemas (Takahashi & Chiba, 2007; Natarajan & Zhao, 2008); esto se debe principalmente a ser una descripción no–relativista, por tanto no puede explicar fenómenos de lentes gravitacionales y cosmológicos donde se requiere de una teoría relativista. Discutiré estos problemas en la siguiente sección.

§2. Gravedad modificada relativista

En el régimen relativista, el modelo ΛCDM supone la validez de la relatividad general desde escalas galácticas hasta cosmológicas e introduce materia y energía oscuras para ajustar las observaciones. Sin embargo, la relatividad general no ha sido probada independientemente en estas escalas, así que una alternativa natural es pensar que la teoría de Einstein más el modelo estándar de partículas fallan a estas escalas. En el régimen relativista, las teorías alternativas de gravedad modificada deben ajustar los lentes gravitacionales observados, el espectro de potencias de materia, las anisotropías observadas en la radiación cósmica de fondo en microondas, la formación de estructura en el universo temprano, entre otros, todo esto sin asumir la existencia de materia oscura, además de la expansión acelerada del Universo sin recurrir a la energía oscura (ver, por ejemplo, Jain & Zhang, 2008).

A continuación se resume la teoría de gravitación relativista de Einstein junto con algunas propuestas relevantes de gravedad modificada en esta dirección.

Ecuaciones de campo en relatividad general

Consideremos la acción total S de un sistema gravitacional dada por

$$S = S_m + S_g \, , \tag{2.1}$$

donde S_m es la acción de la materia y S_g es la acción del campo gravitacional. En relatividad general (ver, por ejemplo, Landau & Lifshitz, 1975):

$$S_m = -\frac{1}{2c} \int \mathcal{L}_m \sqrt{-g} \, d^4x \, , \tag{2.2}$$

para c la velocidad de la luz en el vacío, $\mathcal{L}_m$ la densidad Lagrangiana del sistema, g el determinante del tensor métrico $g_{\mu\nu}$[†] y d^4x el elemento de volumen en el espacio–tiempo. El sector de gravedad está descrito por la acción de Hilbert S_H:

$$S_H = -\frac{c^3}{16\pi G} \int R \sqrt{-g} \, d^4x \, , \tag{2.3}$$

donde R es el escalar de Ricci. Las ecuaciones de campo de Einstein, que se obtienen de la variación nula de la acción completa, es decir $\delta \left(S_g + S_m \right) = 0$, con respecto a la métrica,

[†]De aquí en adelante utilizamos la signatura $(+,-,-,-)$ para la métrica y la convención de Einstein para suma sobre índices repetidos.

están dadas por

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2.4}$$

para $R_{\mu\nu}$ el tensor de Ricci y el tensor de energía–momento $T_{\mu\nu}$ definido a través de la relación estándar

$$\delta S_m = -\frac{1}{2c} T_{\mu\nu}\, \delta g^{\mu\nu}. \tag{2.5}$$

Estas ecuaciones son diferenciales no–lineales hasta de segundo orden en la métrica, con soluciones exactas derivadas principalmente con restricciones de simetría. De hecho, Einstein creía que no había soluciones exactas, además de la trivial para un espacio–tiempo plano, hasta que Schwarzchild mostró el contraejemplo, al resolver las ecuaciones de campo para un espacio–tiempo estático esféricamente simétrico en vacío (ver, por ejemplo, Misner et al., 1973).

Al incluir la constante cosmológica Λ en el lagrangiano del campo gravitacional, la acción (2.3) queda

$$S_g = -\frac{c^3}{16\pi G} \int (R - 2\Lambda) \sqrt{-g}\, d^4x\,, \tag{2.6}$$

y las ecuaciones de campo de Einstein resultantes con constante cosmológica son

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{2.7}$$

Estas son las ecuaciones del modelo ΛCDM. Históricamente, la constante cosmológica fue introducida por Einstein para tener un universo estacionario; a este modelo se le conoce como el *modelo estático de Einstein*. El introducir esta constante en las ecuaciones de Einstein ha sido algo que ha cambiado según la época. Las observaciones actuales de supernovas Ia indican que $\Lambda \neq 0$. Es muy importante notar que la adición de Λ fue la primera modificación a las ecuaciones de campo de Einstein. Su aparición dentro del lagrangiano que describe al espacio–tiempo significa que éste adquiere una curvatura que no puede ser eliminada de ninguna manera (ver Landau & Lifshitz, 1975). Esta curvatura se asocia generalmente al espacio vacío y las fluctuaciones cuánticas que ocurren debido al principio de incertidumbre de Heisenberg. Este resultado fue demostrado por Zeldovich, quien demostró que el vacío genera un tipo de fuerza antigravitacional o *fuerza respulsiva del vacío* (Landau & Lifshitz, 1975).

Alternativas a la relatividad general

Como una extensión natural a la relatividad general, tenemos un tipo de teorías que consideran un lagrangiano que depende, no sólo del escalar de Ricci R, sino de una función general de él. A estas teorías se les denomina *teorías de gravedad* $f(R)$. En este caso la acción gravitacional generalizada es (Capozziello & Faraoni, 2011)

$$S_g = -\frac{c^3}{16\pi G}\int f(R)\sqrt{-g}\,d^4x, \tag{2.8}$$

y la acción de la materia (2.2) permanece intacta. El caso $f(R) = R$ es el más simple y corresponde a la relatividad general.

En la presente tesis nos enfocamos en un conjunto de teorías $f(R)$ construidas como una generalización de MOND. Esta clase de teorías será descrita con detalle en la siguiente sección.

La acción (2.8) se puede generalizar más incluyendo invariantes del tensor de Ricci $R_{\mu\nu}$ y del tensor de Weyl $C_{\mu\nu\alpha\beta}$[†]. Esta acción del campo gravitacional puede escribirse como

$$S_g = -\kappa\int f\left(R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right)\sqrt{-g}\,d^4x\,, \tag{2.9}$$

donde κ es una constante de acoplamiento que mantiene las unidades correctas en la acción. Inicialmente, estas teorías se propusieron esperando obtener una teoría de gravedad renormalizable agregando a la acción términos de grado mayor en la curvatura.

Las teorías de gravedad $f(R)$ son las más simples de este tipo de teorías (éstas se discuten en la siguiente sección). Las que contienen al tensor de Weyl en el lagrangiano gravitacional se denominan teorías de *gravedad conforme*. Otras, llamadas teorías de *gravedad de Gauss–Bonnet* (Lovelock, 1971) contienen al término del mismo nombre en el lagrangiano ($R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$). Otro caso especial está dado por la *gravedad de Lovelock* (Lovelock, 1971), que incluye polinomios del tensor de Riemann en el lagrangiano. Estas teorías conducen a ecuaciones conservadas de 2do. orden para la métrica, en cualquier dimensión, y se consideran la generalización de la relatividad general de Einstein para más de cuatro dimensiones.

[†] El tensor de Weyl es la componente libre de traza de tensor de Riemann. Es invariante bajo transformaciones conformes de la métrica.

Otras teorías de gravedad modificada a nivel relativista son las teorías $f(R,T)$ (Harko et al., 2011), para R el escalar de Ricci y T la traza del tensor de energía–momento, es decir, $T := T^{\alpha}_{\alpha}$.

Otro tipo de teorías de campo clásicas, además de la relatividad general de Einstein, son las teorías *escalares–tensoriales (Brans & Dicke, 1961). En esta aproximación la interacción gravitacional está mediada por un campo escalar así como por el tensor métrico (ambas teorías se denominan métricas).* El tensor de energía–momento dado por la ecuación (2.5) es la fuente del campo gravitacional, al igual que en relatividad general, y ambas satisfacen el principio de equivalencia de Einstein. Sin embargo, lo que cambia es la forma en que se da la interacción por el campo escalar al hacer que la constante gravitacional G pueda variar como función del espacio–tiempo. Las ecuaciones de campo de la teoría contienen un parámetro libre, ω (constante de acoplamiento de Brans–Dicke), que puede ser ajustado de las observaciones. Datos recientes del experimento Cassini–Huygens, muestran que $\omega > 40,000$ (Bertotti et al., 2003). La teoría además cumple con el principio de Mach (ver, por ejemplo, Misner et al., 1973) y además predice la deflexión de la luz y la precesión de la órbita de Mercurio. Como veremos en la sección §2 hay una equivalencia entre las teorías métricas $f(R)$ y las teorías de Brans–Dicke para un cierto valor de la constante de acoplamiento ω.

Además de estas propuestas hay toda una gama de teorías de gravedad modificada relativista. Por ejemplo, RAQUAL (Relativistic AQUAL; Bekenstein & Milgrom, 1984), la primera generalización relativista de AQUAL. El problema con esta modificación es que permite propagación superlumínica de las ondas gravitacionales, además de dejar invariante la deflexión de la luz predicha por relatividad general (Bekenstein, 2004).

Otra propuesta de Bekenstein (2004) es TeVeS (Tensor–Vector–Scalar), que describe la gravedad mediante una combinación del tensor métrico más un campo vectorial y dos campos escalares. Ésta, al ser derivada de una acción, cumple con conservación de momento y energía, además de converger a MOND en su límite de campo débil para una métrica estática esféricamente simétrica. Sin embargo, es una teoría muy complicada (con varios grados de libertad indeterminados) y tiene problemas al ajustar los lentes gravitacionales en diferentes sistemas astrofísicos (Zhao et al., 2006; Mavromatos et al., 2009).

Hay otras propuestas más recientes, como la *Gravedad Escalar–Tensorial–Vectorial* (Scalar–Tensor–Vector Gravity, STVG) o también llamada *Gravedad Modificada* (Modified Gravity, MOG) (Moffat, 2006). STVG/MOG está basada en un principio variacional y postula la existencia de un campo vectorial y tres campos escalares que son constantes de la teoría. En la aproximacion de campo débil, la teoría produce una modificación tipo–Yukawa en la fuerza gravitacional, lo que resulta en mayor gravedad que la Newtoniana lejos de la fuente. Se mantiene además dentro de las desviaciones a la aceleración Newtoniana en el Sistema Solar. Esta modificación ha sido exitosa en reproducir las curvas de rotacion y la relacion Tully–Fisher, los perfiles de masa de cumulos de galaxias, los lentes gravitacionales del Bullet Cluster y observaciones cosmológicas sin la necesidad de materia oscura.

Hay otra gran variedad de propuestas de gravedad modificada. Sin embargo, por la simplicidad de las teorías $f(R)$ y su reducción directa a la relatividad general, nos enfocamos en éstas para tratar de explicar las observaciones sin componentes oscuras. A continuación se discuten dichas teorías.

Teorías de gravedad $f(R)$

Las teorías de gravedad $f(R)$ fueron propuestas primero por Buchdahl (1970) y luego retomadas por Starobinsky (1980). En estas la acción gravitacional está dada por la ecuación (2.8).

Hay dos formas de variar la acción (2.3) en relatividad general, que conducen a las mismas ecuaciones de campo (2.4). En el llamado *formalismo métrico,* la acción es una función únicamente de la métrica y derivadas de ella, esto es $S = S[g]$, y la conexión Γ es la de Levi–Civita (dependiente de la métrica). La otra forma es el denominado *formalismo de Palatini*, propuesto por Einstein en 1925 (Ferraris et al., 1982), en el que la conexión y la métrica se tratan de manera independiente. En este caso la acción depende de ambos, es decir, $S = S[g, \Gamma]$. De las dos formas la acción de la materia depende solamente de la métrica[†].

En relatividad general no hay distinción entre ambos formalismos en cuanto a que las ecuaciones de campo son equivalentes, pero en teorías de gravedad $f(R)$ conducen a dinámicas diferentes. Las teorías métricas $f(R)$ son hasta de cuarto orden en la métrica (ver,

[†]En una tercera opción se considera, además de la variación independiente de la acción con respecto a la métrica y a la conexión, que el lagrangiano de materia $\mathcal{L}_m$ depende también de la conexión. A estas teorías se les conoce como teorías $f(R)$ en el *formalismo métrico–afín*.

por ejemplo, Capozziello, 2002; Nojiri & Odintsov, 2003; Carroll et al., 2004; Capozziello & Faraoni, 2011), mientras que la ventaja de variar a la conexión independientemente de la métrica en el formalismo de Palatini es que las ecuaciones resultantes son hasta de segundo orden en la métrica, al igual que las ecuaciones de campo de Einstein (ver, por ejemplo, Olmo, 2005). Debido a esto son más sencillas de tratar que las teorías métricas $f(R)$. Los primeros modelos, sin embargo, presentaban conflictos a escalas del Sistema Solar y con la física a escalas pequeñas (ver Olmo, 2005, y referencias dentro). Recientemente hay toda una serie de trabajos que buscan probar la viabilidad de estas teorías.

Se ha mostrado que las teorías $f(R)$ en la aproximación de Palatini son equivalentes a las de Brans–Dicke con el parámetro $\omega = -3/2$ y en la aproximación métrica son equivalentes con $\omega = 0$ (ver, por ejemplo, de Felice & Tsujikawa, 2010; Olmo, 2011).

Más generalmente, esta equivalencia existe entre las teorías $f(R)$ y las teorías escalares–tensoriales[†]. Usualmente, las teorías $f(R)$ se trabajan en el llamado *marco de Jordan* ("Jordan frame")[‡]. La transformación entre un marco y otro se logra definiendo una transformación conforme entre la métrica $g_{\mu\nu}$ y un tensor métrico $\tilde{g}_{\mu\nu}$ definido en el *marco de Einstein* ("Einstein frame")[§] de la forma:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu}, \\ \Omega^2 &= f'(R), \end{aligned} \tag{2.10}$$

donde la prima denota derivada respecto al argumento y el factor conforme Ω puede depender del escalar de curvatura y de los campos de materia. Eligiendo adecuadamente ese factor, es posible mapear una teoría de gravedad no–estándar, formulada en el marco de Jordan, a otra que es estándar en el marco de Einstein, donde la gravedad toma la forma

[†]Dos teorías son consideradas "dinámicamente equivalentes" si, bajo una redefinición adecuada de los campos gravitacional y de materia, sus ecuaciones de campo y/o sus acciones coinciden.

[‡]El lagrangiano en las teorías escalares–tensoriales se expresa en el marco de Jordan, en el cual el tensor de energía–momento se conserva covariantemente y las partículas de prueba se mueven en geodésicas de la métrica del espacio–tiempo, o en el marco de Einstein, en el cual el tensor de energía–momento de los campos de materia no siempre se conserva covariantemente y las partículas de prueba no necesariamente siguen geodésicas.

[§]Hay un debate grande acerca de cuál marco tiene significado físico y cómo comparar cada formulación con las observaciones y experimentos. Algunos autores argumentan que la equivalencia conforme no necesariamente indica equivalencia física (ver, por ejemplo, Faraoni et al., 1999; Capozziello et al., 2010c), mientras que otros aseguran que si los relojes y reglas estándares se ajustan apropiadamente, ambas teorías son matemática y físicamente equivalentes (Flanagan, 2004).

usual de Einstein.

Se define un campo escalar como $\Phi := \sqrt{3/2}\kappa^{-1} \ln f'(R)$, donde κ es una constante. La acción gravitacional generalizada en el marco de Einstein resulta

$$S_E = -\kappa \int \left[\frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} (\tilde{\nabla}\Phi)^2 - V(\Phi) \right] \sqrt{-\tilde{g}}\, d^4x \,, \tag{2.11}$$

donde la tilde denota cantidades en el marco de Einstein y el potencial $V(\Phi)$ está representado por:

$$V(\Phi) = \frac{f'(R)\tilde{R} - f(R)}{2\kappa^2 (f'(R))^2}. \tag{2.12}$$

Esta acción corresponde a la de relatividad general mínimamente acoplada a un campo escalar real. En esta descripción las ecuaciones de campo son de segundo orden en la métrica, lo que hace más sencillo su manejo. Hay algunas pruebas cosmológicas de teorías de gravedad $f(R)$ en el marco de Jordan y su correspondiente teoría escalar–tensorial en el marco de Einstein (ver, por ejemplo, Bamba et al., 2012, y referencias dentro). Sin embargo, hay algunas indicaciones de que la transformación entre marcos de referencia no es biunívoca, es decir, no hay certeza de que regresar los resultados del marco de Einstein al de Jordan sea correcto (Faraoni et al., 1999; Capozziello et al., 2010c).

Teorías métricas $f(R)$

Las teorías métricas $f(R)$ satisfacen diferentes pruebas astrofísicas y cosmológicas y presentan una alternativa sólida al sector oscuro (ver, por ejemplo, Sotiriou & Faraoni, 2010; Capozziello & Faraoni, 2011). La dificultad que presentan es que presentan genéricamente ecuaciones diferenciales no–lineales de cuarto orden en la métrica.

En el formalismo métrico, la variación de la acción (2.3) para la acción gravitacional (2.8), da las siguientes ecuaciones de campo:

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = T_{\mu\nu} + f'(R)_{;\mu\nu} - \Delta f'(R) g_{\mu\nu} \,, \tag{2.13}$$

donde la prima denota derivada con respecto al argumento y Δ al D'Alambertiano.

Se ha mostrado que estas teorías juegan un papel importante a escalas galácticas, de cúmulos de galaxias y escalas cosmológicas. Por ejemplo, se ha mostrado que una teoría

$f(R)$ ajusta el diagrama de Hubble con las observaciones más recientes de supernovas Ia (Capozziello et al., 2003; Torres, 2009). Además es posible tener más lentes gravitacionales que los esperados en relatividad general (Mendoza & Rosas-Guevara, 2007; Mendoza et al., 2012). A escalas galácticas es posible ajustar curvas de rotación de galaxias de disco (Capozziello et al., 2007a; Sobouti, 2007; Mendoza et al., 2012) y obtener la relación Tully–Fisher (Sobouti, 2007; Mendoza et al., 2012). También se han ajustado satisfactoriamente los perfiles de masa de cúmulos de galaxias (Capozziello et al., 2009a), entre otras pruebas.

§3. Esquema de la tesis

A lo largo del trabajo de tesis hemos construido una teoría de gravedad modificada que extiende las teorías de gravitación de Newton y Einstein a escalas donde las observaciones requieren clásicamente de componentes oscuras.

En el Capítulo II se muestra la construcción de una teoría de gravedad extendida no–relativista compatible con las desviaciones a la gravitación de Newton en el Sistema Solar y que es capaz de reproducir las observaciones desde escalas galácticas hasta extragalácticas sin materia oscura. Esta teoría asume que la constante de aceleración de MOND, a_0, es fundamental en la gravitación.

En el Capítulo III se encuentra una generalización relativista de MOND a partir de un tipo especial de teorías métricas $f(R)$, construida asumiendo que a_0 es fundamental en la gravitación. Es por los resultados discutidos en la sección §2 que preferimos trabajar con las teorías $f(R)$ en el marco de Jordan y no con las escalares–tensoriales en el de Einstein, a pesar de que las ecuaciones son más sencillas. Y en particular en la aproximación métrica, ya que como se dijo antes, las teorías $f(R)$ en la aproximación de Palatini han presentado problemas con observaciones a nivel del Sistema Solar y con la física del modelo estándar de partículas elementales. Un número considerable de estudios en la literatura se ha enfocado a derivar la función $f(R)$ del ajuste con los datos sin utilizar materia y/o energía oscuras exóticas. Vale la pena notar que experimentos en el Sistema Solar muestran la validez de la relatividad general a estas escalas, así que la teoría métrica $f(R)$ debe converger a ésta en este límite (Will, 1993).

En el Capítulo IV se muestra que a partir del análisis de perturbaciones en una métrica estática, esféricamente simétrica hasta segundo orden en potencias de v/c de la teoría descrita en el Capítulo III es posible reproducir las observaciones a nivel galáctico de curvas de rotación planas y la relación Tully–Fisher y de lentes gravitacionales en grupos y cúmulos de galaxias.

En el Capítulo V se explora la posibilidad de que la constante de aceleración a_0 sea una cantidad fundamental relacionada con algunas constantes universales y no únicamente una constante fenomenológica.

Finalmente se encuentran las conclusiones del trabajo de tesis.

* * *

Capítulo I

Introduction

The observation of type Ia supernovae, the anisotropies observed in the microwave background, the acoustic oscillations in the baryonic matter, the power–law spectrum of galaxies and gravitational lenses among others, represent strong evidences for the standard cosmological model, the so–called ΛCDM *concordance model*. From recent observations of the European space mission *Planck*, the contribution of the baryonic matter to the present content of the matter–energy density of the universe was inferred to be only $4\,\%$, while the dark sector constitutes $\sim 95\,\%$, of which $27\,\%$ is Cold Dark Matter (CDM) and $68\,\%$ is dark energy or a positive cosmological constant Λ (Planck Collaboration et al., 2013).

The dark matter component was postulated in order to explain the observed rotation curves of spiral galaxies, as well as the mass to light ratios in giant galaxies and galaxy clusters, the observed gravitational lenses and the structure formation in the early universe, among other astrophysical and cosmological phenomena (see section §1). On the other hand, the dark energy or a cosmological constant has been postulated to explain the accelerated expansion of the universe (Riess et al., 1998; Perlmutter et al., 1999).

Until now, the ΛCDM model adjusts best most of the observations. However, the direct or indirect search of dark matter candidates has yielded null results. In addition, the lack of any further evidence for dark energy, opens up the possibility that there are no dark entities in the universe but instead, the theory associated to these astrophysical and cosmological phenomena needs to be modified.

Current models of dark matter and dark energy are based on the assumption that Newtonian gravity and Einstein's general relativity are valid at all scales. However, their validity has only been demonstrated with high precision for systems which scales are no larger than the Solar System scale. In that sense is conceivable that both, the accelerated expansion of the universe and the stronger gravitational force required in different systems, represent a change in our understanding of gravitational interactions.

From the geometrical point of view, modified theories of gravity are viable alternatives to solve the astrophysical and cosmological problems that dark matter and dark energy are trying to solve. In fact, the idea of modified gravity laws is almost as old as the Newtonian gravity itself. Almost 200 years after the publication of Newton's *Philosophiæ naturalis principia mathematica* (Newton, 1687), in his essay "On the Hypotheses which lie at the Bases of Geometry" (1856), Riemann asserted that the metric–relations of space are only to be deduced from experience, that we may therefore investigate the probability of the hypotheses "within the limits of observation...and enquire about the justice of their extension beyond the limits of observation, on the side both of the infinitely great and of the infinitely small" (Riemann, 1873a,b).

At the end of the XIX century it was believed that all physics could be explained with the Newtonian mechanics and the classical electromagnetic theory of Maxwell. It was then when experiments and observations challenged the established theory in two diverse scales: for velocities comparable to the speed of light and for scales close to its de Broglie wavelength.

The problems that physics faced by then triggered a change in understanding nature as it was known before. Considering the possibility of exploring large velocities and huge energies, yielded Einstein to develop his theory of relativity, special (Einstein, 2005) and general (Einstein, 1916). Exploring the smallest scales of matter results in the construction of quantum mechanics. In 1887, one of the most beautiful and important experiments in physics was carried by Michelson & Morley (1887). This experiment discarded the existence of the ether, and set the basis of Einstein's special theory of relativity.

Formerly, Bouvard (1821) observed an anomalous motion of Uranus in its orbit and thought it could be due that the Sun's gravity at that distance is different to the Newtonian one or due to an eight planet not discovered by then. In 1845, Le Verrier (1846) & Adams (see Waff & Kollerstrom, 2001) mathematically predicted where Neptune should be observed. Exactly at that predicted position it was observed by Galle in 1846 (see Waff & Kollerstrom, 2001). Later, observations of the anomalous precession of Mercury's orbit led Le Verrier (see des sciences, 1859) to postulate the existence of an unknown planet between Mercury and the Sun. He called that missing planet Vulcan. Le Verrier died in 1877 believing in the existence of that planet, after an amateur astronomer reported its transit over the solar disc (see Baum & Sheehan, 1997).

Vulcan has in fact never been observed, and the problem was solved only with an extension of Newton's theory of gravity: Einstein's general relativity. There is another experiment that conclusively demonstrated the validity of general relativity: the deflection of light of distant stars by the solar limb. General relativity predicts a deflection angle twice the value expected by Newtonian mechanics, and it was exactly what Dyson and Eddington observed (Dyson et al., 1920).

All these experiments and observations changed the way we understand the universe. The most important lesson to learn from history is that observations and experiments transcend the paradigms and lead to changes or extensions in the laws of physics. The road that physics took from the observations that directly challenged classical mechanics and electromagnetism, could tell us the path that the nowadays accepted theory must take. Since dark matter and dark energy have not been directly detected, we should look for modifications of the gravitational theory that has been used to postulate their existence.

The European space mission *Planck* has obtained the most precise and detailed map of the oldest light of the universe. Similarly, other missions and surveys already in operation, like DES (Dark Energy Survey), *Gaia*, SDSS (Sloan Digital Sky Survey) IV, have started to highlight some aspects of the accepted cosmological model and other will continue doing it for the next years, like *Euclid* and SKA (Square Kilometre Array)[†].

[†]In the upcoming years, large amount of data from telescopes and detectors on board of spatial missions will help to better shape both dark matter/energy models and modified theories of gravity. With this, it may be possible to make a realistic selection of one above the other. It is important to note that these missions detect electromagnetic radiation, they do not directly prove the existence of dark matter or if the gravitational theory should be modified. Direct detection of a dark matter particle with the required astrophysical density (90 % of all matter) will certainly be the end of this longstanding discussion. It is non–direct detection in the not so distant future, and further data coherence of modified theories of gravity will serve as strong evidence of the non–existence of dark matter particles, since it is no possible by any experiment to prove the non–existence of a certain entity.

§1. Non–relativistic modified gravity

Dark matter was postulated in the early 1930's when discrepancies began to emerge in the observed rotation curves of disc galaxies and the observed velocity dispersions in clusters of galaxies. In 1932, Jan Oort found that stars in the solar neighbourhood were moving faster than expected (later, this result was confirmed wrong; see Kuijken & Gilmore, 1989a). Meanwhile, Zwicky (1933) noted the same behaviour in the speed of galaxies in clusters. They called "missing matter" or "dark matter" to the mass that could explain the observations. Later, Babcock (1939) suggested that the mass-to-light ratio increased with the radius for observations of the rotation curve of the Andromeda galaxy; he attributed it to absorption of light within the galaxy or "modified dynamics" in the outer regions, not to any form of missing matter. Finally, Rubin et al. (1980) presented strong evidence confirming that a large number of galaxies are dominated by dark matter, contained in a invisible "dark halo".

Initially, it was thought that the missing matter were objects that could not be seen through a telescope, as brown dwarfs, black holes, dust, planets, etc., but the expected amount of these objects failed to quantify the total expected mass to correctly explain the galactic dynamics. Thereby the "non baryonic" dark matter was postulated: elementary particles interacting only gravitationally with known matter. Since then, dozens of new proposed particles have been discarded (see,e.g., Bertone et al., 2005). At the present time preferred candidate is the neutralino, a supersymmetric particle result of an extension of the standard model of particle physics.

As an alternative to the problem of missing matter and accelerated expansion of the Universe, various proposals that modify the accepted theories of gravity at different scales have been formulated, in both the non–relativistic and the relativistic limits. The main idea behind some of these theories is the proposal of a modification to Newton's gravity and Einstein's general relativity.

The requirements that a theory of gravity must satisfy in the non–relativistic regime are: (1) do not exceed the maximum observational deviations to Newton's gravitation law within the Solar System, (2) to fit the flat rotation curves of disc galaxies and satisfy the Tully–Fisher relation for angular momentum–supported galaxies, as well as the Faber–Jackson relation for pressure–supported galaxies and, recently, to fit the outer regions of globular clusters and wide binary systems and (3) to reproduce the observed scaling relations for different systems, from dwarf spheroidal galaxies in the Local Group to giant elliptical galaxies and clusters of galaxies, among many other tests that traditionally require dark matter.

The first successful modification in this regime was the Modified Newtonian Dynamics (MOND), described briefly in the next section (see Sanders & McGaugh, 2002, for a review). It is important to study this proposal because, due to its phenomenological nature and its success in the non-relativistic limit, it is understood that any fundamental theory of modified gravity should adapt to it on galactic scales in the low accelerations regime. As discussed in Chapter II, the extended gravity theory we propose is equivalent to the MONDian description in some systems, for example in spherical symmetric ones, but with remarkable advantages.

MOND

In the non–relativistic limit, Milgrom (1983b,a) introduced the Modified Newtonian Dynamics to explain the observed flat rotation curves of spiral galaxies. Using a phenomenological acceleration constant a_0, Milgrom formulated a criterion such that for accelerations much greater than a_0, the dynamics is Newtonian and for accelerations much smaller than a_0, the dynamics is modified to produce circular orbits with velocities independent of the radial coordinate, at sufficiently large scales.

Such modification to the dynamics in spherical or cylindrical symmetry is given by (Milgrom, 1983b)

$$a\,\mu(a/a_0) = |\nabla\phi_N| = \frac{GM(r)}{r^2}\,, \tag{1.1}$$

where ϕ_N is the Newtonian gravitational potential, G is Newton's gravitational constant, $M(r)$ is the enclosed mass at the radius r and the *interpolation function* $\mu(a/a_0)$ has the limits

$$\mu(a/a_0) = \begin{cases} 1, & \text{for } a \gg a_0\,, \\ a/a_0, & \text{for } a \ll a_0\,. \end{cases} \tag{1.2}$$

In these limits, the regime of accelerations much greater than a_0 corresponds to Newtonian gravity and the limit of accelerations much smaller than a_0 corresponds to the so–called "*deep MOND*" or MONDian regime. From the last equations it follows that in this regime the acceleration is given by

$$a = -\frac{(GMa_0)^{1/2}}{r}. \tag{1.3}$$

Formally in a system with cylindrical symmetry, like a disc galaxy, the spherical symmetry assumption necessarily introduces errors in the calculations. However, analogue to the Newtonian case, where the rotation curve of an infinitely thin disc decreasing exponentially deviates only 20 % from the spherical case at large radii (see, e.g., Binney & Tremaine, 2008), the resulting error is within the range of observational error. Then, given that the observations show that the external regions of disc galaxies are in the deep MOND regime and at these radii we have almost spherical symmetry, the circular velocity tends to a constant to large radii:

$$v_{\rm flat} = (GM_b a_0)^{1/4} \approx \text{const.}, \tag{1.4}$$

since the centrifugal acceleration is $a_c = v^2/r$ and it is assumed that the main contribution to the baryonic mass M_b is given at smaller radii. In this way the flat rotation curves observed in disc galaxies are recovered. Moreover, this last prescription is consistent with the baryonic Tully–Fisher relation, between the velocity in the flat part of the rotation curve and the baryonic mass for a constant mass to light ratio, as (McGaugh, 2005)

$$v_{\rm flat} \propto M_b^{1/4}. \tag{1.5}$$

We can find in the literature different interpolation functions which have been proposed to characterise the acceleration at intermediate ranges. The classical function was proposed by Milgrom (1983b):

$$\mu(a/a_0) = \frac{a/a_0}{\sqrt{1+(a/a_0)^2}}. \tag{1.6}$$

Others functions fit with better precision the rotation curves of our Galaxy and others, of different sizes and masses (cf. Figure II.2). With the standard interpolation functions it is possible to fit with high precision the rotation curves of disc galaxies at all radii. From the best fit to rotation curves of spiral galaxies in the Galactic neighbourhood we have (Begeman et al., 1991)

$$a_0 \approx 1.2 \times 10^{-10} \text{m/s}^2. \tag{1.7}$$

So far, MOND is only a phenomenological prescription, not a complete theory, because it violates momentum conservation. These conservation laws can be automatically satisfied by a physical theory derived from variational principles. In this sense, Bekenstein & Milgrom (1984) constructed a first non–relativistic generalisation of MOND from the following 'AQUAdratic Lagrangian' (AQUAL):

$$\mathcal{L} = -\frac{a_0^2}{8\pi G} g\left(\frac{|\nabla\phi_N|^2}{a_0^2}\right) - \rho\phi_N, \tag{1.8}$$

where ρ is the mass density and $g(y)$ is an dimensionless function, such that the Newtonian acceleration is recovered (or the Poisson's equation) for $g(y) = y$. From this Lagragian, the following modified Poisson's equation is obtained:

$$\nabla \cdot \left[\mu\left(\frac{|\nabla\phi_N|}{a_0}\right)\nabla\phi_N\right] = 4\pi G\rho, \tag{1.9}$$

where $\mu(\sqrt{y}) = dg(y)/dy$. In the Newtonian and deep MOND limits it is assumed that:

$$g(y) = \begin{cases} y, & y \gg 1\,, \\ \frac{2}{3}y^{3/2}, & y \ll 1\,. \end{cases} \tag{1.10}$$

For systems with spherical, cylindrical or flat geometry, equation (1.9) integrates directly. With the following prescription for the acceleration:

$$a = -\nabla\phi_N, \tag{1.11}$$

the solutions corresponds to MOND's formula (1.1). For systems that have not this symmetry, the numerical integration reveals that the same formula can be used with good precision.

MOND is a very successful modification of gravity in that it can fit the flat rotation curves and the Tully–Fisher relation for disc galaxies. In this sense, any modified theory of gravity, pretending to replace the dark matter, must converge to the MONDian prescription in the limit $a \ll a_0$. However, from the study of groups and clusters of galaxies it has been shown that, even in the deep MOND regime, a dominant dark matter component is required in these systems (60 to 80 % of the dynamical or virial mass). Angus et al. (2008) showed that the central region of galaxy clusters could be explained with a halo of neutrinos with mass of 2 eV (which is about the value of the experimental upper limit). But on the scale of groups of galaxies, the central contribution can not be explained by a contribution of neutrinos with that mass. Moreover, MOND/AQUAL is not able to reproduce the observed gravitational lensing for different systems (see, e.g., Takahashi & Chiba, 2007; Natarajan & Zhao, 2008), mainly because it is a non–relativistic description, and as such it can not explain gravitational lensing and cosmological phenomena, which require a relativistic theory of gravity. I will discuss these problems in the next section.

§2. Relativistic modified gravity

In the relativistic regime, the ΛCDM model assumes the validity of general relativity from galactic to cosmological scales and introduces dark matter and dark energy in order to fit the observations. However, general relativity has not been tested independently on those scales, therefore a natural alternative is the possibility that Einstein's general relativity and the standard model of particles fail at these scales. In the relativistic regime, the alternative theories of modified gravity must fit the observed gravitational lenses, the power spectrum of matter, the observed anisotropies in the cosmic microwave background, the structure formation in the early universe, among others, without assuming the existence of dark matter, and at cosmological scales without the need of dark energy (see, e.g., Jain & Zhang, 2008).

The next section is a summary of the Einstein's relativistic gravitational theory and of relevant theories proposed to modified gravity in this direction.

Field equations in general relativity

Consider the total action S of a gravitational system given by

$$S = S_m + S_g\,, \tag{2.1}$$

where S_m is the action of matter and S_g is the action of the gravitational field. In general relativity (see, e.g., Landau & Lifshitz, 1975):

$$S_m = -\frac{1}{2c}\int \mathcal{L}_m \sqrt{-g}\, d^4x\,, \tag{2.2}$$

with c the speed of light in vacuum, $\mathcal{L}_m$ the Lagrangian density of the system, g the determinant of the metric tensor $g_{\mu\nu}$[†] and d^4x the volume element in space–time. The gravity sector is described by the Hilbert's action S_H:

$$S_H = -\frac{c^3}{16\pi G}\int R\sqrt{-g}\, d^4x\,, \tag{2.3}$$

where R is the curvature or Ricci's scalar. The Einstein's field equations, obtained from the null variation of the complete action, i.e., $\delta\left(S_g + S_m\right) = 0$, with respect to the metric, are given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.4}$$

for $R_{\mu\nu}$ the Ricci tensor and the energy–momentum tensor $T_{\mu\nu}$ defined through the standard relation

$$\delta S_m = -\frac{1}{2c}T_{\mu\nu}\,\delta g^{\mu\nu}. \tag{2.5}$$

These are second order non–linear differential equations in the metric, with exact solutions mostly derived with symmetry restrictions. In fact, Einstein believed that there were no exact solutions, besides the trivial one for a flat space–time, until Schwarzchild showed the counterexample, solving the field equations for a static spherically symmetric space–time in vacuum (see, e.g., Misner et al., 1973).

By including the cosmological constant Λ in the Lagrangian of the gravitational field, the action (2.3) gives

$$S_g = -\frac{c^3}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\, d^4x\,, \tag{2.6}$$

and the resulting Einstein's field equations with cosmological constant are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{2.7}$$

[†] From now on we use the signature $(+,-,-,-)$ for the metric and the Einstein's index convention for the summation over repeated indices.

These are the equations of the ΛCDM model. Historically, the cosmological constant was introduced by Einstein to have a stationary universe; this model is known as *Einstein's static model*. Introducing this constant into the Einstein field equations has changed depending on the epoch. From the actual observations of supernovae Ia, we know that $\Lambda \neq 0$. It is very important to note that the introduction of Λ was the first modification to the Einstein's field equations. Its introduction in the Lagrangian that describes the spacetime, implies that the spacetime acquires an intrinsic curvature (see Landau & Lifshitz, 1975). Such curvature is associated to the vacuum of the universe and the quantum fluctuations due to the Heinsenberg uncertainty principle. This last result was demonstrated by Zeldovich, who demonstrated that vacuum generates an antigravitational force or *repulsive force of the vacuum* (Landau & Lifshitz, 1975).

Alternatives to general relativity

As a natural extension to general relativity, we have a kind of the theories that consider a Lagrangian depending, not only on the Ricci scalar R, but on a general function of it. These theories are called $f(R)$ *theories of gravity*. In this case the generalised gravitational action is (Capozziello & Faraoni, 2011)

$$S_g = -\frac{c^3}{16\pi G} \int f(R)\sqrt{-g}\, d^4x, \tag{2.8}$$

and the action of matter (2.2) remains intact. The case $f(R) = R$ is the simpler one and corresponds to general relativity.

In this thesis we focus our attention in a class of $f(R)$ theories proposed as a relativistic generalisation of MOND. These theories will be described in detail in the next section.

The action (2.8) can be generalised further including invariants of the Ricci tensor $R_{\mu\nu}$ and of the Weyl tensor $C_{\mu\nu\alpha\beta}$[†]. This action of the gravitational field can be written as

$$S_g = -\kappa \int f\left(R, R^{\mu\nu}R_{\mu\nu}, R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right) \sqrt{-g}\, d^4x\,, \tag{2.9}$$

where κ is a coupling constant that keeps the correct units in the action. Initially, these

[†]The Weyl tensor is the trace-free component of the Riemann tensor. It is invariant under conformal transformations of the metric.

theories were proposed in the pursuit of a re-normalisable theory of gravity adding to the action higher order terms in the curvature.

The $f(R)$ theories of gravity are the simpler among this kind of theories (I will discuss them in the next section). Theories containing the Weyl tensor in the gravitational Lagrangian are called *conformal gravity* theories. Others, called *Gauss–Bonnet gravity* theories (Lovelock, 1971) contain the Gauss–Bonnet term $(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})$ in the Lagrangian. Another special case is the *Lovelock gravity* (Lovelock, 1971), which includes in the Lagrangian polynomials of the Riemann tensor. These theories lead to conserved second order equations for the metric, in any dimension, and they are considered the generalisation of the Einstein's general relativity in more than four dimensions.

Other relativistic theories of modified gravity are the $f(R,T)$ theories (Harko et al., 2011), for the Ricci scalar R and T the trace of the energy–momentum tensor, i.e., $T := T^{\alpha}_{\alpha}$.

Other kind of classical field theories, besides Einstein's general relativity, are the *scalar–tensor* theories of gravity (Brans & Dicke, 1961). In this approach, the gravitational interaction is mediated by a scalar field and the metric tensor (both are called *metric theories)*. The energy–momentum tensor given by equation (2.5) is the source of the gravitational field, as in general relativity, and both satisfy the Einstein's equivalence principle. However, the interaction changes with the scalar field allowing the gravitational constant G to vary as a function of the space–time. The field equations of the theory contain a free parameter, ω (Brans–Dicke coupling constant), that can be adjusted from the observations. Recent data from the Cassini–Huygens experiment, show that $\omega > 40,000$ (Bertotti et al., 2003). This theory also holds the Mach's principle (see, e.g., Misner et al., 1973) and also predicts the deflection of light and the precession of Mercury's orbit. As we will see in section §2 there is an equivalence between the $f(R)$ metric theories and the Brans–Dicke theories for a specific value of the coupling constant ω.

Additionally, there is a huge variety of modified theories of gravity in the relativistic regime. For example, RAQUAL (Relativistic AQUAL; Bekenstein & Milgrom, 1984), the first relativistic generalisation of AQUAL. The problem with this modification is that it allows for superluminal propagation of gravitational waves and maintains invariant the deflection of light predicted by general relativity (Bekenstein, 2004).

Another proposal of Bekenstein (2004) is TeVeS (Tensor–Vector–Scalar), which describes the gravity through a combination of the metric tensor plus a vector field and two scalar fields. This theory meets the momentum and energy conservation principles, because it is derived from a variational principle, and converges to MOND in its weak field limit for a static spherically symmetric metric. However, is a very intrincated theory (with several, undetermined degrees of freedom) and presents some problems when fitting the gravitational lenses in astrophysical systems (Zhao et al., 2006; Mavromatos et al., 2009).

There are other more recent proposals, like the *Scalar–Tensor–Vector Gravity* (STVG) or the *Modified Gravity* (Moffat, 2006, MOG;). STVG/MOG is based on a variational principle and postulates the existence of a vector field and three scalar fields, which are constants of the theory. In the weak field approximation, this theory produces a Yukawa–like modification to the gravitational force, or equivalently, a greater one than the Newtonian gravitational attraction far from the source. It also lies within the deviations to the Newton's acceleration in the Solar System. This modification has succeeded in reproducing rotation curves and the Tully–Fisher relation, mass profiles in clusters of galaxies, gravitational lensing in the Bullet Cluster and the cosmological observations, all without requiring dark matter.

There are many other proposals of modified gravity. However, we have focused our attention on the $f(R)$ theories to explain the observations without dark components, because of its simplicity and its direct reduction to general relativity. I will discuss such theories in the next section.

$f(R)$ theories of gravity

The class of $f(R)$ theories of gravity were initially proposed by Buchdahl (1970) and Starobinsky (1980). For these theories, the gravitational action is given by equation (2.8).

In general relativity, there are two ways of varying the action (2.3) that lead to the same field equations (2.4). In the so–called *metric formalism*, the action is a function only of the metric tensor and its derivatives, this is $S = S[g]$, and the connection is the Levi–Civita one (dependent on the metric). Another so–called *Palatini formalism*, proposed by Einstein in 1925 (Ferraris et al., 1982), in which the connection Γ and the metric are independent between them. In this case, the action depends on both of them, i.e., $S = S[g, \Gamma]$. In both ways, the action of matter depends only on the metric tensor [†].

[†]A third option considers, besides the independent variation of the action with respect to the metric

In general relativity there is no distinction between both formalisms because the field equations are equivalent, but in $f(R)$ theories they yield different results. The $f(R)$ metric theories yield equations which contain terms up to the fourth order in the metric (see, e.g., Capozziello, 2002; Nojiri & Odintsov, 2003; Carroll et al., 2004; Capozziello & Faraoni, 2011), meanwhile the advantage of varying independently the connection and the metric in the Palatini formalism is that the resulting equations contain only second order metric terms, just like the Einstein's field equations (see, e.g., Olmo, 2005). These equations are simpler than those derived from $f(R)$ metric theories. The first models, however, presented problems at the Solar System and small scales (see Olmo, 2005, and references therein). Recently, a whole variety of works are trying to prove the viability of these theories.

It has been shown that $f(R)$ theories in the Palatini formalism are equivalent to Brans–Dicke theories with the parameter $\omega = -3/2$ and in the metric approach are equivalent with $\omega = 0$ (see, e.g., de Felice & Tsujikawa, 2010; Olmo, 2011).

This equivalence establishes a correspondence between $f(R)$ theories and scalar–tensor theories[†]. Usually, $f(R)$ theories are formulated in the so–called *Jordan frame*[‡]. To transform between frames, a conformal transformation between the metric $g_{\mu\nu}$ and the metric tensor $\tilde{g}_{\mu\nu}$, defined in the *Einstein frame*[§], is performed:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= \Omega^2 g_{\mu\nu}, \\ \Omega^2 &= f'(R), \end{aligned} \tag{2.10}$$

where the prime denotes derivative with respect to the argument and the conformal factor Ω can depend on the scalar curvature and on the matter fields. By carefully choosing such

and the connection, that the Lagrangian of matter $\mathcal{L}_m$ depends also on the connection. These theories are called $f(R)$ theories in the *metric–affine formalism*.

[†]Two theories are considered "dinamically equivalent" if, under a suitable redefinition of the gravitational and matter fields, their field equations and/or their actions coincide.

[‡]The Lagrangian in scalar–tensor theories is expressed in the Jordan frame, in which the energy–momentum tensor is covariantly conserved and in which test particles follow geodesics of the space–time metric, or in the Einstein frame, in which the energy–momentum tensor of the matter fields is not always covariantly conserved and test particles do not necessarily follow geodesics.

[§]There is a big discussion about the physical meaning of each frame and how each formulation is compared with the observations and experiments. Some authors argue that conformal equivalence does not necessarily indicate physical equivalence (see, e.g., Faraoni et al., 1999; Capozziello et al., 2010c), while others argue that if standard clocks and rulers are adjusted appropiately, both theories are mathematically and physically equivalent (Flanagan, 2004).

factor, it is possible to map a non–standard theory of gravity, formulated in the Jordan frame, to another that is standard in the Einstein frame, where gravity takes the usual Einstein form.

A scalar field is defined as $\Phi := \sqrt{3/2}\kappa^{-1}\ln f'(R)$. The generalised gravitational action in the Einstein frame is

$$S_{\mathbb{E}} = -\kappa \int \left[\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}(\tilde{\nabla}\Phi)^2 - V(\Phi) \right] \sqrt{-\tilde{g}}\, d^4x\,, \tag{2.11}$$

where the tilde denotes quantities in the Einstein frame and the potential $V(\Phi)$ is represented by

$$V(\Phi) = \frac{f'(R)\tilde{R} - f(R)}{2\kappa^2 (f'(R))^2}. \tag{2.12}$$

This action corresponds to general relativity minimally coupled to a real scalar field. In this description the field equations are of the second order in the metric and therefore, they are simpler. There are some cosmological tests of $f(R)$ gravity in the Jordan frame and its corresponding scalar–tensor theory in the Einstein frame (see, e.g., Bamba et al., 2012, and references therein). However, there are some indications that the transformation between frames is not biunivocal, i.e., there is no certainty that going back with the results from the Einstein frame to the Jordan frame is correct (Faraoni et al., 1999; Capozziello et al., 2010c).

$f(R)$ metric theories

$f(R)$ metric theories satisfy different astrophysical and cosmological tests and present a strong alternative to the dark sector (see, e.g., Sotiriou & Faraoni, 2010; Capozziello & Faraoni, 2011). The difficulty with these theories is that they generically present non–linear differential equations up to the fourth order in the metric.

In the metric formalism, the null variation of the action (2.3) for the gravitational action (2.8), leads the following field equations:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = T_{\mu\nu} + f'(R)_{;\mu\nu} - \Delta f'(R) g_{\mu\nu}\,, \tag{2.13}$$

where the prime denotes derivative with respect to the argument and Δ is the D'Alambertian.

It has been shown that these theories play an important role in galactic scales, in clusters of galaxies and in cosmological scales. For example, it has been shown that an $f(R)$ theory adjust the Hubble diagram with the recent observations of type Ia supernovae (Capozziello et al., 2003; Torres, 2009). Moreover, it is possible to obtain more gravitational lensing that the expected in general relativity (Mendoza & Rosas-Guevara, 2007; Mendoza et al., 2012). In the galactic scales it is possible to fit the rotation curves of disc galaxies (Capozziello et al., 2007a; Sobouti, 2007; Mendoza et al., 2012) and the Tully–Fisher relation (Sobouti, 2007; Mendoza et al., 2012). Also, the mass profiles of clusters of galaxies have been adjusted successfully (Capozziello et al., 2009a), among other tests.

§3. Thesis outline

Throughout this thesis we have constructed a modified theory of gravity which extends Newton and Einstein's theories at scales where observations classically require dark components.

Chapter II shows the construction of a non–relativistic extended theory of gravity compatible with the deviations to Newton's gravitational acceleration in the Solar System and capable of reproduce the observations, from galactic to extragalactic scales, without dark matter. Such theory assumes that MOND's acceleration constant a_0, is fundamental in the gravitation.

Chapter III shows a relativistic generalisation of MOND from a special kind of $f(R)$ metric theories of gravity, assuming that a_0 is fundamental into the gravitation. Due to the works discussed in section §2, we prefer to work with $f(R)$ theories in the Jordan frame instead of the scalar–tensor ones in the Einstein frame. And in particular, in the metric approximation because, as was discussed in section §2, $f(R)$ theories in the Palatini approach present problems with the observations at the Solar System level and with the standard model of elementary particles. A considerable number of works in the literature has focus its attention to derive the $f(R)$ function from the adjust of the data without using dark matter and/or dark energy.

It is worth noting that experiments at Solar System scales show the validity of general relativity; in this sense the $f(R)$ metric theory must converge to Einstein's theory in this limit (Will, 1993).

Chapter IV shows that the metric coefficients obtained for the theory described in Chapter III, from perturbation analysis developed up to the second order in powers of v/c for a static spherical symmetric space–time, reproduces exactly the observations of flat rotation curves and the Tully–Fisher relation for disc galaxies and the gravitational lenses observed in groups and clusters of galaxies.

Chapter V shows that MOND's acceleration constant a_0 is a fundamental quantity related with some universal constants, and not only a phenomenological quantity.

Finally, I show the conclusions of this thesis.

Capítulo II

Gravedad Newtoniana extendida

En la búsqueda de una formulación general para una teoría de gravedad modificada en el régimen no–relativista y como alternativa a la hipótesis de materia oscura, construimos un modelo válido para una amplia variedad de escalas astrofísicas. En este Capítulo se muestra cómo a través de la inclusión de la constante de aceleración de Milgrom en la gravitación, se pueden construir fórmulas muy generales para la aceleración sentida por una partícula. El análisis dimensional muestra que esta inclusión conduce naturalmente a la aparición de una escala de masa–longitud en la gravedad, rompiendo su invariancia de escala.

En esta parte del trabajo de tesis, construimos una forma particular de fuerza gravitacional modificada y probamos su consistencia con las observaciones en un amplio rango de entornos astrofísicos, desde el Sistema Solar hasta escalas extragalácticas. Mostramos que sobre cualquier rango limitado de parámetros físicos, que definen una clase específica de objetos astrofísicos, la dispersión de velocidades de un sistema debe estar dada por una ley de potencias de su masa y tamaño. Estas potencias aparecen ligadas entre sí a través de una relación natural de constricción de la teoría. De aquí obtenemos una relación de equilibrio gravitacional generalizada, válida para todos los sistemas astrofísicos. Se presenta un esquema general para tratar distribuciones de densidad esféricamente simétricas, que en particular muestra que el plano fundamental de las galaxias elípticas, el equilibrio virial Newtoniano, las relaciones Tully–Fisher y Faber–Jackson, así como las relaciones de escalamiento observadas en galaxias locales esferoidales enanas, no son mas que casos particulares de esa relación cuando se aplica a las escalas de masa–longitud apropiadas. Además, discutimos las implicaciones de esta aproximación como teoría de gravedad modificada y destacamos las ventajas de trabajar con la fuerza, en la formulación de una teoría gravitacional, en lugar de alterar la segunda ley del movimiento de Newton.

* * *

Extended Newtonian gravity

In the pursuit of a general formulation for a modified gravitational theory at the non-relativistic level and as an alternative to the dark matter hypothesis, we construct a model valid over a wide variety of astrophysical scales. Through the inclusion of Milgrom's acceleration constant into a gravitational theory, we show that very general formulae can be constructed for the acceleration felt by a particle. Dimensional analysis shows that this inclusion naturally leads to the appearance of a mass-length scale in gravity, breaking its scale invariance. A particular form of the modified gravitational force is constructed and tested for consistency with observations over a wide range of astrophysical environments, from solar system to extragalactic scales. We show that over any limited range of physical parameters, which define a specific class of astrophysical objects, the velocity dispersion of a system must be a power law of its mass and size. These powers appear linked together through a natural constraint relation of the theory. This yields a generalised gravitational equilibrium relation valid for all astrophysical systems. A general scheme for treating spherical symmetric density distributions is presented, which in particular shows that the fundamental plane of elliptical galaxies, the Newtonian virial equilibrium, the Tully-Fisher and the Faber-Jackson relations, as well as the scalings observed in local dwarf spheroidal galaxies, are nothing but particular cases of that relation when applied to the appropriate mass-length scales. We discuss the implications of this approach for a modified theory of gravity and emphasise the advantages of working with the force, instead of altering Newton's second law of motion, in the formulation of a gravitational theory.

§4. Introduction

The dynamical mass to light ratios derived for spiral galaxies are usually much greater than expected for their stellar components. This is often interpreted as indicating the gravitational dominance of hypothetical dark matter. Alternatively, one could argue that the discrepancy between dynamical mass and baryonic mass is telling us that the Newtonian law of gravity is not the one governing the dynamics. In particular, the Modified Newtonian Dynamics (MOND) proposed by Milgrom (1983b) has been proved to be successful in explaining how galaxies rotate, without any dark matter (see e.g. Sanders & McGaugh, 2002, for a review).

Recently, the range of astrophysical problems treated under the MOND hypothesis has increased significantly. Abundant recent publications on velocity dispersion measurements for stars in the local dwarf spheroidal galaxies, the extended and flat rotation curves of spiral galaxies, the large dispersion velocities of galaxies in clusters, the gravitational lensing due to massive clusters of galaxies, and even the cosmologically inferred matter content for the universe, have been successfully modelled under MOND. These, not as indirect evidence for the existence of a dominant dark matter component, but as direct evidence for the failure of the current Newtonian and general relativistic theories of gravity, in the large scale or low acceleration regimes relevant for the above (see e.g. Sanders & Noordermeer, 2007; Nipoti et al., 2007; Famaey et al., 2007; Gentile et al., 2007; Tiret et al., 2007; Sánchez-Salcedo et al., 2008; Bekenstein, 2004; Capozziello et al., 2007a; Sobouti, 2007; Mendoza & Rosas-Guevara, 2007, for recent examples). MOND has proved successful on many astrophysical situations, though difficult on others (see e.g. Milgrom, 2008, 2010a; Bekenstein, 2006; Zhao, 2005, for a good review on these points).

The key feature of Milgrom's Modified Newtonian Dynamics is the introduction of a fundamental acceleration scale $a_0 = 1.2 \times 10^{-10} \mathrm{m\,s^{-2}}$ (see e.g. Milgrom, 2008) into gravitation. The introduction of a_0 can alternatively be regarded as grounded upon direct empirical evidence, as the observed dynamics of large spiral discs attest. Additionally, non-relativistic gravity due to a point mass M, results in a force on any test particle such that it is pulled towards M with an acceleration a. Fundamentally, Newton's constant of gravity G completes the description of the problem and so, by means of Buckingham's theorem of dimensional analysis (cf. Sedov, 1959), the absolute value of the attractive acceleration felt by the test particle located at a distance r from the point mass M is given by

$$a = a_0\, f(x), \tag{4.1}$$

where

$$x := l_M / r, \tag{4.2}$$

with,

$$l_M := \left(\frac{GM}{a_0}\right)^{1/2}. \tag{4.3}$$

The acceleration expressed in equation (4.1) converges to Newton's gravitational acceleration when the function $f(x) = x^2$ and to MOND's acceleration when $f(x) = x$. These two examples of functions f represent the gravitational approach to an extended non-relativistic theory of gravity. The main problem is how to find a function $f(x)$ which, for the appropriate limits, converges to the Newtonian and MONDian regimes.

Hernandez et al. (2010) showed that the Bekenstein (2004) function $f(x) = x + x^2$ serves quite well when applied to dwarf spheroidal (dSph) galaxies and to the rotation curves at large radii of spiral galaxies, but is inconsistent with measured limits on departures from Newtonian gravity at solar system scales. Also, Famaey & Binney (2005) showed that this particular form of the function $f(x)$ does not work well when applied to our Galaxy. Despite the fact that this prescription has the corresponding limits as expected ($f(x) \to x$ when $x \ll 1$ and $f(x) \to x^2$ when $x \gg 1$), a more general function must be constructed.

Note that pure dimensional analysis, with the introduction of an acceleration scale a_0, determines exactly the dimensional form that the acceleration must have. In very general terms, it also shows that the introduction of a_0 means that gravity has a characteristic mass-length scale l_M which makes possible the construction of equation (4.1). With all these, the acceleration turns out to be a function of the variable x only, which as we will show later, gives a robust way of working with an extended theory of gravity at the non-relativistic level.

It is important to note that Milgrom has already introduced the length l_M (see e.g. Milgrom, 1983b,a, 1986; Milgrom & Sanders, 2008; Milgrom, 2008, where it appears as a transition radius). In these studies it is shown that this mass-length scale serves as a transition point where the MONDian regime passes to the Newtonian one. Milgrom & Sanders (2008) stressed the points that a mass distribution whose length is much greater than its associated mass-length l_M is in the MONDian regime (since $x \ll 1$) and a mass distribution whose length is much smaller than its mass-length scale is in the Newtonian regime (since $x \gg 1$). The case $x = 1$ can roughly be thought of as the point where the transition from the Newtonian to the MONDian regimes occurs.

We now show that there is a connection between this approach and the one commonly used in the implementation of the MOND theory. The physical form of MOND is given by (see e.g. Bekenstein & Milgrom, 1984; Milgrom, 2001) an Aquadratic Lagrangian (AQUAL) and so, its variation reproduces the equation of motion. For relevant symmetries in the problem, this approach gives the important result that the absolute value of the acceleration felt by a test particle in the presence of a point mass is given by

$$a \; \mu\left(a/a_0\right) = \left|\nabla \phi_N\right| = \frac{GM}{r^2}. \tag{4.4}$$

In this equation, the Newtonian scalar potential is represented by ϕ_N and the interpolation function $\mu(a/a_0)$ is such that $\mu(a/a_0) = 1$ in the Newtonian limit, which corresponds to $a \gg a_0$ and $\mu(a/a_0) = a/a_0$ in the MONDian regime, with $a \ll a_0$. Equating the acceleration in relation (4.4) with that of equation (4.1), it follows that

$$\mu(a/a_0) = \frac{x^2}{f(x)}, \tag{4.5}$$

which implicitly shows that the MOND formalism can be equivalently expressed through the modification of the gravitational force (4.1). We note here that the MOND formulation (4.4) refers to a modification of the dynamical sector of the theory, whereas equation (4.1) is completely based on the modification of the gravitational force. Both are operationally equivalent formulations. The MOND formulation has always been tackled through dynamical modifications. However, we show in this work that there are many advantages when choosing the modification in the gravitational sector. Therefore we only use the constant a_0 for consistency with the dynamical modifications. It is important to emphasise that in the gravitational modifications it is more natural to frame the problem in terms of the mass-length scale l_M defined in equation (4.3). Furthermore, it is the use of dimensional analysis which tells us the very important fact that the dimensionless force $f(x)$ in equation (4.1) only depends on the ratio l_M/r.

The Chapter is organised as follows. Section §5 introduces a particular form of the function $f(x)$. This is used in the subsequent sections for applications in different astrophysical environments, from solar system to galaxy cluster scales. Finally, in Section §7 we discuss the advantages of such a general function $f(x)$.

§5. The force model

The dimensionless gravitational force $f(x)$ in equation (4.1) felt by a given test particle must be analytic, and as such it can be written as

$$f(x) = \sum_{n=-\infty}^{n=\infty} c_n x^n. \tag{5.1}$$

We now show how to obtain a reasonable $f(x)$ by simple exploration of the Newtonian and MONDian regimes. First of all, notice that in the Newtonian and deep MOND limits, the function $f(x)$ is such that $f(x) = c_N x^2$ and $f(x) = c_M x$ respectively, with $c_N = c_M = 1$. We now focus on the Newtonian $x \gg 1$ regime and explore an expansion about that limit of the form

$$\begin{aligned} \left(\frac{a}{a_0}\right) &= \left(\frac{l_M}{r}\right)^2 + \left(\frac{l_M}{r}\right) + \left(\frac{l_M}{r}\right)^0 + \left(\frac{l_M}{r}\right)^{-1} + \ldots, \\ &= \left(\frac{l_M}{r}\right)^2 \left\{1 + \left(\frac{l_M}{r}\right)^{-1} + \left(\frac{l_M}{r}\right)^{-2} + \left(\frac{l_M}{r}\right)^{-3} + \ldots\right\}, \\ &= x^2 \left(1 + x^{-1} + x^{-2} + x^{-3} + \ldots\right). \end{aligned} \tag{5.2}$$

The limit $x \to \infty$ gives the Newtonian acceleration, so taking into account all the terms of the geometric series for $x > 1$ then,

$$\left(\frac{a}{a_0}\right) = x^2 \left(1 + \frac{1}{x-1}\right) = \frac{x^3}{x-1}. \tag{5.3}$$

We now put special emphasis on the MONDian $x \ll 1$ regime and explore the corresponding expansion, given by

$$\begin{aligned} \left(\frac{a}{a_0}\right) &= \left(\frac{l_M}{r}\right) + \left(\frac{l_M}{r}\right)^2 + \left(\frac{l_M}{r}\right)^3 + \left(\frac{l_M}{r}\right)^4 + \ldots, \\ &= \left(\frac{l_M}{r}\right) \left\{1 + \left(\frac{l_M}{r}\right) + \left(\frac{l_M}{r}\right)^2 + \left(\frac{l_M}{r}\right)^3 + \ldots\right\}, \\ &= x \left(1 + x + x^2 + x^3 + \ldots\right). \end{aligned} \tag{5.4}$$

Note that the deep MOND regime is obtained in the limit $x \to 0$, and so the geometric series of equation (5.4) for $x < 1$ gives

$$\left(\frac{a}{a_0}\right) = \frac{x}{1-x}. \tag{5.5}$$

Equation (5.2) can be thought of as the series for the negative powers of relation (5.1) and equation (5.4) as the one for the positive powers of the same relation. The interesting thing to note is that both of them can be analytically continued for all values of x. For the limit cases, the minus sign on the denominator of both equations (5.3) and (5.5) can be changed for a positive sign.

Since we are interested in the complete analytic series let us propose a general acceleration formula given by the addition or substraction of equations (5.3) and (5.5) as follows:

$$\left(\frac{a}{a_0}\right)_{\pm} = \frac{x \pm x^3}{1 \pm x}. \tag{5.6}$$

Note that this last equation tends to the Newtonian acceleration regime when $x \to \infty$ and to the MONDian acceleration limit when $x \to 0$. In fact, due to the symmetry of the numerator and denominator of equation (5.6), a more general relation can be postulated:

$$\left(\frac{a}{a_0}\right)_{\pm} = x\,\frac{1 \pm x^{n+1}}{1 \pm x^n}. \tag{5.7}$$

This satisfies the Newtonian and MONDian acceleration limits for $x \to \infty,\ 0$ respectively. Note also that the case $n = 1$ with a minus sign is the same as two times the case $n = 0$ with a plus sign, and both correspond to the *Bekenstein ground state* acceleration formula (Bekenstein, 2004). This has proved to be useful for the dynamical modelling of dSph galaxies (Hernandez et al., 2010), but not for our own Galaxy (Famaey & Binney, 2005).

The acceleration function (5.7) has no singularities, since according to l'Hôpital's rule, $a/a_0 \to (n+1)/n$ as $x \to 1$. In fact, to see this directly, notice that for the minus sign it follows from equation (5.7) that

$$\begin{aligned} \left(\frac{a}{a_0}\right)_{-} &= x\,\frac{(1-x)(1+x+x^2+x^3+\ldots+x^n)}{(1-x)(1+x+x^2+\ldots+x^{n-1})}, \\ &= x\,\frac{(1+x+x^2+x^3+\ldots+x^n)}{(1+x+x^2+\ldots+x^{n-1})}. \end{aligned} \tag{5.8}$$

For further applications we note that the right hand side of equation (5.7) with a minus sign can be Taylor expanded as follows:

$$\left(\frac{a}{a_0}\right)_- = x + x^{n+1} - x^{n+2} + x^{2n+1} - x^{2n+2} + \dots, \tag{5.9}$$

for $x < 1$, and

$$\left(\frac{a}{a_0}\right)_- = x^2 - x^{1-n} + x^{2-n} - x^{1-2n} + x^{2-2n} + \dots, \tag{5.10}$$

for $x > 1$. Choosing the positive sign we obtain:

$$\left(\frac{a}{a_0}\right)_+ = x - x^{n+1} + x^{n+2} + x^{2n+1} - x^{2n+2} + \dots, \tag{5.11}$$

for $x < 1$, and

$$\left(\frac{a}{a_0}\right)_+ = x^2 + x^{1-n} - x^{2-n} - x^{1-2n} + x^{2-2n} + \dots, \tag{5.12}$$

for $x > 1$. This shows that the Newtonian and MONDian regimes are reached in the correct limit regardless of the value of n.

The extreme limiting case of $n \rightarrow \infty$ corresponds to the function

$$\left(\frac{a}{a_0}\right)_e = \begin{cases} x, & \text{for } 0 \leq x \leq 1 \quad \text{(MONDian regime).} \\ x^2, & \text{for } x \geq 1 \quad \text{(Newtonian regime).} \end{cases} \tag{5.13}$$

This acceleration formula is of no use due to the discontinuity on the first derivative at $x = 1$, but serves as a reference to understand that the real acceleration must smoothly pass from the Newtonian to the MONDian regime. Also, as noted by Milgrom & Sanders (2008), the point where $x = 1$ represents approximately the transition from the Newtonian to the MONDian regimes. In the proposed model, this is strictly valid in the extreme case, with $n \rightarrow \infty$. This point is relevant, as any function moving away from the value $a = a_0$ at $x = 1$ seems to have a better chance at modelling a more real astrophysical situation, since it will smoothly transit from the MONDian regime to the Newtonian one.

Figure II.1 shows a plot of a/a_0 as a function of x for various values of n and choice of

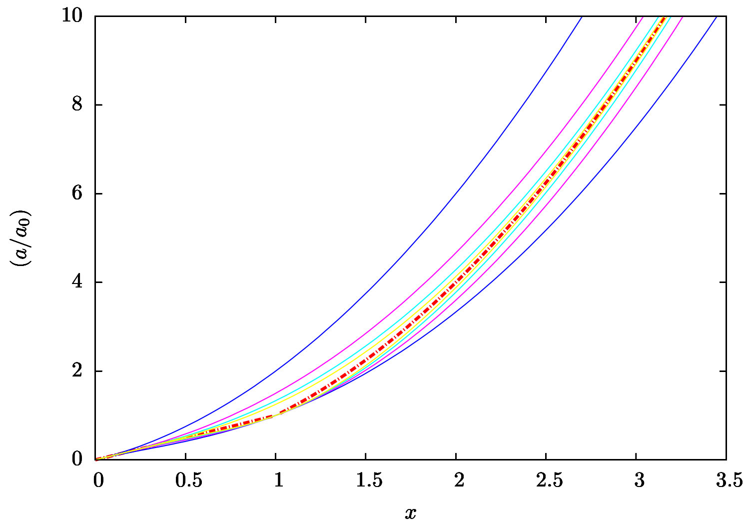

Figura II.1: The figure shows the acceleration function a in units of Milgrom's constant a_0 as a function of the parameter x. The thick dash-dot curve corresponds to the extreme limiting value $n \to \infty$, i.e. $a/a_0 = x$ for $x \leq 1$ and $a/a_0 = x^2$ for $x \geq 1$. The curves above and below this extreme acceleration line represent values of $n = 4,\ 3,\ 2,\ 1$, for the minus and plus signs of equation (5.7) respectively. The extreme limiting curve has a kink at $x = 1$ and is of no physical interest due to the undefined derivative at that point.

signs in equation (5.7). From this figure it is seen that the curves with $n \sim 3 - 4$ are very close to the extreme limiting case, but preserve a good soft transition region between the MONDian and Newtonian regimes. However, it is through fits with observations that an optimal real number n is to be calibrated. In fact, the fit with observations must also give us a way to decide between the plus or minus sign in equation (5.7).

In order to fix the parameter n and the choice of sign in equation (5.7), we construct different corresponding MOND interpolation functions $\mu(a/a_0)$ for this acceleration formula and compare them with the best model for our Galaxy presented by Famaey & Binney (2005). To do that, we must substitute equations (5.7) and (4.1) into (4.5), yielding

$$\mu(a/a_0) = x\,\frac{1 \pm x^n}{1 \pm x^{n+1}}. \tag{5.14}$$

The function $x(a/a_0)$ that appears in equation (5.14) is obtained by solving numerically equation (5.7) for a fixed value of n. Figure II.2 shows that the best fit to the optimal

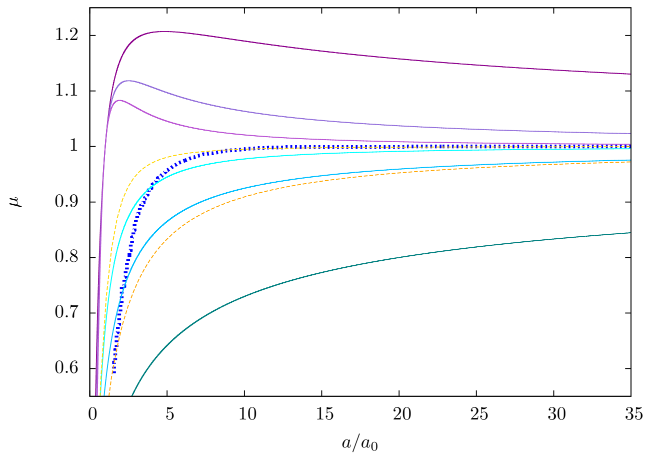

Figura II.2: The figure shows MOND's interpolation function $\mu(a/a_0)$ for our Galaxy (dotted curve) as inferred by Famaey & Binney (2005). From bottom to top (not counting the dotted curve) the plots represent the following models: (1) Bekenstein ground state acceleration model (Bekenstein, 2004), which corresponds to our equation (5.7) with $n = 1$ and a minus sign. (2) The interpolation formula $\mu(\chi) = \chi/(1+\chi)$ (Famaey & Binney, 2005, dashed). (3) $n = 2$ for a minus sign. (4) $n = 3$ for a minus sign. (5) The interpolation formula $\mu(\chi) = \chi/\left(1+\chi^2\right)^{1/2}$ (Milgrom, 1983b, dashed). (6) $n = 3$, plus sign. (7) $n = 2$, plus sign. (8) $n = 1$, plus sign.

$\mu(a/a_0)$ obtained by Famaey & Binney (2005) for our Galaxy, is reproduced with the minus sign and with $n = 3.13 \approx 3$. The effective gravitational acceleration formula is hence chosen as

$$\left(\frac{a}{a_0}\right) = f(x) = x\,\frac{1-x^4}{1-x^3} = x\,\frac{1+x+x^2+x^3}{1+x+x^2}. \qquad (5.15)$$

Given observational errors, and uncertainties in the mass to light ratios and their radial variability in our galaxy, the constrains of Figure II.2 are subject to considerable uncertainties. As such, we intend to show merely that functions of the proposed family are clearly consistent with available estimates of $\mu(a/a_0)$, in this rather uncertain parameter range. In the following section we show this particular $f(x)$ as consistent with the much more stringent constraints available and the quasi–Newtonian scale of the solar system. Also, simple explanations for the observed structural relations of elliptical and dwarf spheroidal galaxies will be shown to appear naturally in Section §6. Note that for this particular case, the function $\mu(a/a_0)$ has an analytic solution, since the function $x(a/a_0)$ from equation

(5.7) with $n = 3$ is the root of a fourth order polynomial in x. However, due to its complicated form, we omit it here. It is interesting that for this value of n, the expansion about $x \gg 1$ begins with the Newtonian term, and then skips the following two terms according to equation (5.10). This guarantees that dynamics will remain extremely close to Newtonian for a large range of values of $x > 1$ and sheds light on the extended Newtonian character of the force of gravity. Similarly, for $x \ll 1$, the leading term of equation (5.9) gives the deep MOND regime x with the following one being x^4. The absence of the x^2 and x^3 terms implies that physics close to the deep MOND regime will not present any strong variations for a considerable range of values in $x < 1$.

We also note that the acceleration function (4.1) is such that Newton's theorems, i.e. the acceleration field at distance r from the centre of a spherical system depends only on the total mass $M(r)$ interior to r, while external shells result in no force, are valid for any analytic function (5.1) which depends on the parameter x only. In order to see this, suppose that $a \propto x^p$ with p an integer number. Assume that the test particle is placed at position r inside a spherically symmetric shell. If we trace a cone with solid angle $\delta\Omega$ and vertex at the test particle, the shell is intersected at two opposite points $\mathbf{r}_1$ and $\mathbf{r}_2$. The masses δM_1 and δM_2 contained within the solid angle $\delta\Omega$ at these points keep the proportion $\delta M_2/\delta M_1 = (r_2/r_1)^2$. This relation means that $(\delta M_1/r_1^2)^{p/2} = (\delta M_2/r_2^2)^{p/2}$, and so $\delta x^p(r_1) = \delta x^p(r_2)$. In other words, the acceleration exerted by the outer shell at position r cancels out. Since we can do this for any integer p, it follows that any analytical function of x (cf. equation (5.1)) guarantees Newton's theorems. This is an encouraging property of the acceleration (4.1).

We now extend equation (4.1) to a vector form. As noted above, the AQUAL formulation is satisfied by our model and as such its field equation (Poisson's generalisation formula) is given by (Bekenstein & Milgrom, 1984)

$$\nabla \cdot [\mu(a/a_0)\,\nabla\phi] = 4\pi G\rho = \nabla^2\phi_N, \tag{5.16}$$

where the scalar potential ϕ satisfies the condition $\mathbf{a} = -\nabla\phi$. For systems with a high degree of symmetry, Bekenstein & Milgrom (1984) showed that

$$\left(\frac{\mathbf{a}}{a_0}\right)\mu\,(a/a_0) = -\frac{1}{a_0}\nabla\phi_N = -\frac{G}{a_0}\int\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\rho(\mathbf{r}')\,dV', \tag{5.17}$$

where we can now express the interpolation function μ through expression (4.5), and $\rho(\mathbf{r})$ represents the matter density at the coordinate point $\mathbf{r}$. On the other hand, the vector form of equation (4.1) must necessarily be of the form

$$\boldsymbol{a} = a_0 f(x) \boldsymbol{e}_a, \tag{5.18}$$

where the unit vector $\boldsymbol{e}_a$ points in the direction of the acceleration $\boldsymbol{a}$. Substitution of equation (5.18) into (5.17) with help of relation (4.5) yields

$$x^2 \boldsymbol{e}_a = -\frac{G}{a_0} \int \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') dV'. \tag{5.19}$$

This description generalises the previous relations in the sense that a point mass M can be directly substituted for $M(r)$ in all relevant equations.

The magnitude of the vectorial equation (5.19) is the formula to calculate the variable x for a given mass distribution density $\rho(\mathbf{r})$. As an example, for a point mass M the density is given by $\rho(\mathbf{r}) = M\delta(\mathbf{r})$, where δ represents Dirac's delta function. Consequently, the variable x is given by equation (4.2) as expected. For the case of a spherically symmetric distribution of matter, the density depends only on the radial coordinate r. This means that (Binney & Tremaine, 2008) $x^2 \boldsymbol{e}_a = -GM(r)\boldsymbol{e}_r / r^2$, where $M(r)$ is the mass contained within the radius r and $\boldsymbol{e}_r$ is a unit vector in the direction of the radial coordinate. With this, it follows that the variable x depends only on the mass contained within the radius r. Since the general acceleration function $f(x)$ is analytic (cf. equation (5.1)), then it is also clear from this point of view that Newton's theorems are valid for the class of models presented in this Chapter.

Recently Milgrom (2010b) has developed a quasi-linear formulation of MOND, which in particular can by applied to our spherical symmetric case (see also Zhao & Famaey, 2010). We now show that this is equivalent to our formulation. In this theory, a general gravitational potential $\Phi := \phi_N + \varphi$ is proposed, where φ satisfies the equation

$$\nabla^2 \varphi = \nabla \cdot \left[\nu \left(|\nabla \phi_N| / a_0 \right) \nabla \phi_N \right], \tag{5.20}$$

where $\nu(y)$ represents a new interpolation function which tends to $y^{-1/2}$ far away from the strongest gravity regime. Our approach is equivalent to their results, since the connection between $f(x)$ and ν is given by

$$f(x) = x^2 \left[\nu(x^2) + 1\right]. \tag{5.21}$$

Comparing this last result with equation (4.5) it follows that $\mu = (\nu + 1)^{-1}$ and so, for $\nu \gg 1$, then $\mu = \nu^{-1}$. For the specific value $\nu(y) = y^{-1/2}$ then

$$f(x) = x + x^2. \tag{5.22}$$

The above two equations show that the function f is specifically a function of x^2 only, i.e. it depends exclusively on the Newtonian force x^2.

In the following sections, we discuss several astrophysical systems where the proposed gravity law (5.15) can be tested over a wide range of values of x, and explore the various predictions which emerge. We perform comparisons mostly through expected scalings related to velocity dispersions, masses and equilibrium radii, derived from very general dimensional arguments. We consider mostly spherical mass distributions $M(r)$, making use of the validity of Newton's theorems for spherical symmetry already discussed.

Solar system consistency and an illustrative rotation curve of our Galaxy

As a first test of the proposed force law we compare the variations with respect to Newtonian acceleration which would be introduced at solar system scales, to the exquisitely measured upper limits on this quantity.

For solar system scales with distances of between 0.1 and ~ 50 AU and taking $M = 1\,M_{\odot}$, we get $10^2 \lesssim x \lesssim 10^4$ (cf. Figure II.7). Due to these large values, we note immediately that the deviations from Newtonian dynamics are negligible. Figure II.3 shows that our model lies well within the observational upper limits for the departures from Newtonian radial acceleration for the planets in our solar system, as reported by Sereno & Jetzer (2006). In consequence, under the proposed force law, MONDian type dynamics have no relevance at solar system scales. In light of our results, we argue in favour of studies by Anderson et al. (2002); Turyshev & Toth (2009); Toth & Turyshev (2009), that regard the anomaly of the recession velocity of the Pioneer probes as an effect of the thermal radiation of the spacecrafts (see however, Bekenstein, 2006; Milgrom, 2009b, for interpretations of

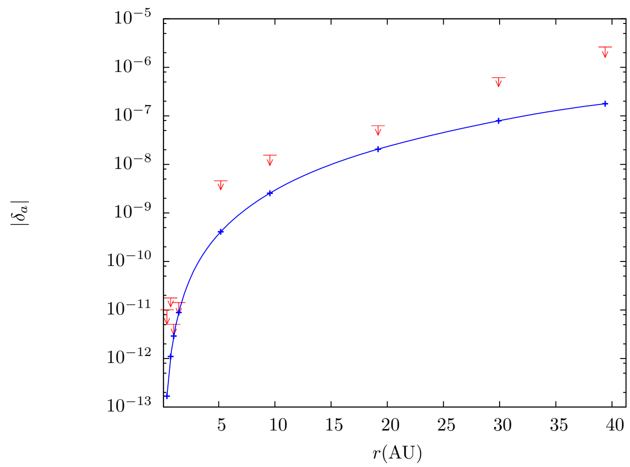

Figura II.3: The curve shows the fractional modifications $\delta a := (a - a_{Nt})/a_{Nt}$ our model introduces when compared to purely Newtonian acceleration a_{Nt}. This is compared with the upper limits on the deviations from radial Newtonian acceleration as reported by Sereno & Jetzer (2006). The diagram shows that the modifications introduced by our model are always within the observed upper limits. From left to right the crosses represent the calculated values δa at the distance of Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune and Pluto respectively.

the anomaly as a signature of MOND).

As a particular example, we now turn to the rotation velocity curve of our Galaxy. The circular rotation velocity $V(r)$ associated to equation (5.15) is given by

$$V(r) = \left(a_0 l_M \frac{1 - x^4}{1 - x^3} \right)^{1/2}. \tag{5.23}$$

For the case of our Galaxy, the mass of the bulge and the mass of the disc as a function of the radial distance are respectively given by (see e.g. Allen & Santillan, 1991)

$$m(r)\Big|_{\text{bulge}} = m\, r^3 \left(a^2 + r^2\right)^{-3/2}, \tag{5.24}$$

$$m(r)\Big|_{\text{disc}} = 2\pi\Sigma_0 r_\star^2 \left[1 - \left(\frac{r}{r_\star} + 1\right) e^{-r/r_\star}\right], \tag{5.25}$$

where $m = 1.4 \times 10^{10}\, M_\odot$, $a = 0.387\,\text{kpc}$, $\Sigma_0 = 7 \times 10^8\, M_\odot\, \text{kpc}^{-2}$ and $r_\star = 3.5\,\text{kpc}$. The effective mass at a distance r from the centre of our Galaxy is then given by

$$M(r) = m(r)\Big|_{\text{bulge}} + m(r)\Big|_{\text{disc}}, \tag{5.26}$$

and so, by substitution of equation (5.26) into (5.23) the theoretical rotation curve of our Galaxy, shown in Figure II.4, is obtained. Given the validity of Newton's theorems for spherical mass distributions in our case, the inner rotation curve of Figure II.4 is fully self consistent. This occurs since in that region, the dynamics are completely dominated by the galactic bulge, which has the same spherical symmetry as is assumed in the calculation. The same holds to a large degree for the calculated rotation curve beyond a few disc scale radii, where the mass distribution has essentially converged. In the intermediate regime, the assumption of spherical symmetry in the calculation necessarily introduces some error. However, by analogy with the Newtonian case, where the rotation curve of an exponentially decreasing infinitely thin disc deviates only about $20\,\%$ from that of the corresponding spherical mass distribution (see e.g Binney & Tremaine, 2008), this error is probably small. The validation of this analogy through a full 3D implementation of our model, lies beyond the scope of this first presentation of our model. In the light of this, Figure II.4 simply provides a graphical representation of the fact that our $f(x)$ was calibrated from our Galaxy's best fit $\mu(a/a_0)$ as given by Famaey & Binney (2005).

This rotation curve represents well the observed features of our Galaxy's rotation curve (see e.g Kuijken & Gilmore, 1989b). This was actually built into our model, as the value of n in the proposed force law was calibrated from the numerical MOND interpolation function estimated by Famaey & Binney (2005), which was obtained precisely from calibrations against the rotation curve of our Galaxy.

§6. Equilibrium radii of bound configurations

If we think of an arbitrary astrophysical equilibrium structure with total mass M, with characteristic equilibrium radius r_e and internal velocity dispersion σ, we can estimate the relation between the above three quantities by equating the kinematic pressure to the mean gravitational force. Note that in this section we use M to denote the total mass of an extended astrophysical system. Writing the kinematic pressure per unit mass as $C\rho\sigma^2/M$,

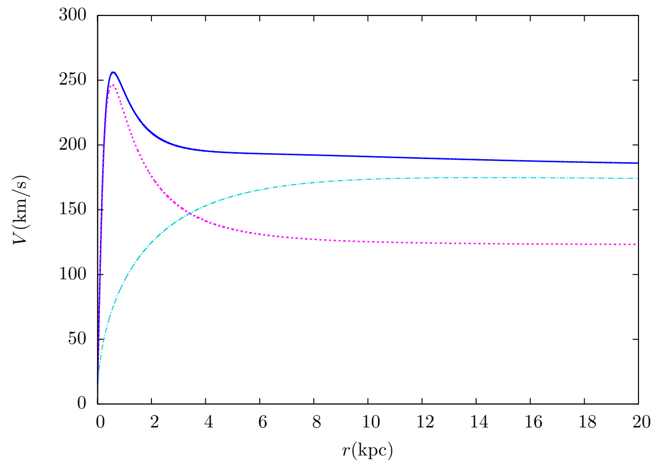

Figura II.4: The figure shows the rotation curve (continuous line) of our Galaxy using the theory of gravity proposed in this Chapter (cf. equation 5.15). We have assumed that only disc (dashed-dotted line) and bulge (dotted line) contributions to the rotation curve are present.

replacing ρ for $M r_e^{-3}$ and equating this to the proposed force law of equation (5.15) we obtain

$$C\tilde{\sigma}^2 = \frac{R_e^4 - 1}{R_e^4 - R_e}, \tag{6.1}$$

where we have introduced a dimensionless velocity dispersion $\tilde{\sigma} := \sigma/(G a_0 M)^{1/4}$ and an equilibrium radius $R_e := r_e/l_M$ for the problem. The constant C is expected to be of order 1.

We notice that for small values of R_e, corresponding to large values of the variable x, i.e. the Newtonian regime, the right hand side of equation (6.1) reduces to $1/R_e$. In this case equation (6.1) leads directly to the virial equilibrium relation of Newtonian gravity $r_e = C^{1/2}\sigma/(G\rho)^{1/2}$. At the opposite MONDian limit, with small values of x and so, large values of R_e, the right hand side of equation (6.1) tends to 1. In this limit, equation (6.1) yields the Faber-Jackson relation $C^{1/2}\sigma = (G a_0 M)^{1/4}$. Allowing for a proportionality between isotropic velocity dispersions in pressure supported systems and rotation velocities in angular momentum supported ones, this last relation could also be seen as the baryonic Tully-Fisher relation which also has an index close to 4 (see e.g. Puech et al., 2010). It is interesting that in terms of the equilibrium gravitational radii of the proposed force law, the galactic Faber-Jackson relation appears as the "Jeans mass" solution for large values of R_e. Intermediate cases should be well described by the appropriate points along the transition between the two limits of equation (6.1).

If we think of a gravitationally bound system with an associated velocity dispersion σ, the above results can also be easily understood from the point of view of dimensional analysis. Indeed, employing the same procedure we used in constructing equation (4.1), the problem is now characterised by the dispersion velocity σ of the system, the radius r, the total mass M, Newton's constant of gravity G and Milgrom's constant a_0. With these, Buckingham's theorem of dimensional analysis demands that the velocity dispersion must have the following form

$$\tilde{\sigma} = g(x), \tag{6.2}$$

with $g(x)$ an arbitrary function to be determined. A natural approximation often used, is to model observations over a limited parameter range through power law representations. As such, we can explore the consequences of imposing a power law for the function $g(x)$, i.e.,

$$\tilde{\sigma} \propto x^{\alpha} = l_M^{\alpha} r^{-\alpha}. \tag{6.3}$$

By comparing equations (6.1) and (6.3), we see that equation (6.3) is merely a power law approximation to the full relation supplied by the proposed force law. One is lead to expect that over any limited range of values of x, equation (6.1) should be accurately approximated by equation (6.3), with a suitable choice of the parameter α, which will more generally be a function of x.

The strong prediction of equation (5.15), as seen in equation (6.3), is that over any constrained range of values of x, e.g. corresponding to any well defined class of astrophysical objects, a relation of the type

$$\sigma = C r^{\beta} M^{\gamma}, \tag{6.4}$$

will always appear. Comparison with equation (6.3) yields directly $\beta = -\alpha$ and shows that the two power law indices of equation (6.4) are not independent, but will obey the necessary constraint

$$\gamma = \frac{1}{4} - \frac{\beta}{2}. \tag{6.5}$$

Again, we see that the two limits explored above correspond to $\alpha = 1/2$ and $\alpha = 0$, for the Newtonian and deep MOND regimes respectively. It is clear that the constraint of equation (6.5), obtained through dimensional analysis, is satisfied in these two cases. Note that this constraint can also be derived through a power law approximation to the full force model of equation (6.1). In what follows it will become evident that this power law approximation and the corresponding predicted constraint (6.5) accurately reproduce the empirical scalings observed in elliptical galaxies and the well studied dwarf spheroidals of the local group. We shall also present an exact expression for $\alpha(x)$ at all scales as a strong testable prediction.

The fundamental plane of elliptical galaxies

The clearest correlations between the structural parameters of elliptical galaxies are expressed by the fundamental plane relations, first described by Djorgovski & Davis (1987); Dressler et al. (1987), as scaling relations between the observed effective radius r of the Galaxy, the central line of sight velocity dispersion σ, and the mean surface brightness I within the effective radius (see e.g. Bernardi et al., 2003; Desroches et al., 2007, for recent observations). Assuming a constant mass to light ratio and using the fact that the mean surface brightness is given by the luminosity L within the effective radius, such that $I = L/2r^2$, measurements over large samples, now in the thousands of galaxies, give a relation identical to equation (6.4).

In what follows we present a treatment for elliptical galaxies in isolation. Many such systems reside in galaxy clusters where external field effects might have some relevance, e.g. in accounting for the escape of high velocity stars (Wu et al., 2008). However, the well established constancy of the basic scaling relations for ellipticals across different density environments, implies that these external field effects are by far not the physical causes driving the structure of the fundamental plane. As such, we shall not consider such environmental effects further at this point.

A broad agreement in the literature yields values close to the ones reported by Bernardi et al. (2003) of $\beta = -0.3356 \pm 0.023$ and $\gamma = 0.503 \pm 0.025$. The first interesting point to note is that the above measured values deviate for β, slightly but noticeably, from the Newtonian expectations of $\beta = -1/2, \gamma = 1/2$. This feature is termed the tilt in the fundamental plane, and has traditionally been explained in terms of mass to light ratios which vary as a function of mass for elliptical galaxies. Although such explanations are not unreasonable (see e.g. Desroches et al., 2007), we shall see that this tilt in the fundamental plane can be accounted for, at constant mass to light ratios, by the slight deviations from the Newtonian limit expected from equation (6.1) at the values of R_e which correspond to elliptical galaxies.

As already seen from equation (6.1) under the power law approximation of equation (6.4), we can compare the observed values for the power law scalings of the fundamental plane of equation (6.4) against the theoretical expectations of our model through equation (6.5). For the observed value of $\beta = -0.3356 \pm 0.023$, equation (6.5) implies $\gamma = 0.418 \pm 0.012$. This is marginally consistent with recent measurements of $\gamma = 0.503 \pm 0.025$ at a two sigma level, under the hypothesis of a constant average mass to light ratio for elliptical galaxies. We therefore see that a power law approximation through a dimensional analysis approach to the problem furnishes the relation between the indices β and γ in equation (6.4), given by relation (6.5). The actual value of both indices will be obtained theoretically through the use of the full force model in what follows. It is noteworthy that practically all of the tilt in the fundamental plane can be accounted for directly by the expectations of equation (6.5). The fullest explanation might well include a certain degree of mass dependence for the average mass to light ratios, as pointed out by various studies (see e.g Desroches et al., 2007).

In order to evaluate the parameter α we proceed by fitting a power law function of the form (6.3) to equation (6.1). This can be done locally to the smooth function (6.1) by calculating the tangent line to the function $\log \tilde{\sigma}(\log x)$ at a given point $P_0(x_0, \tilde{\sigma_0})$, which satisfies the relation

$$\log \tilde{\sigma} = \alpha(x_0) \log x + \log \tilde{\sigma_0} - \alpha(x_0) \log x_0. \tag{6.6}$$

The slope of the tangent line is given by

$$\alpha(x) = \frac{\partial \log \tilde{\sigma}}{\partial \log x} = \frac{x^3}{2} \frac{3 + 2x + x^2}{(1 + x + x^2)(1 + x + x^2 + x^3)}, \tag{6.7}$$

which simply means that the constant α depends on the scale x_0 of the system as shown by Figure II.5. Notice that $\alpha \to 1/2$ for the Newtonian $x \to \infty$ regime and $\alpha \to 0$ for the MONDian $x \to 0$ limit.

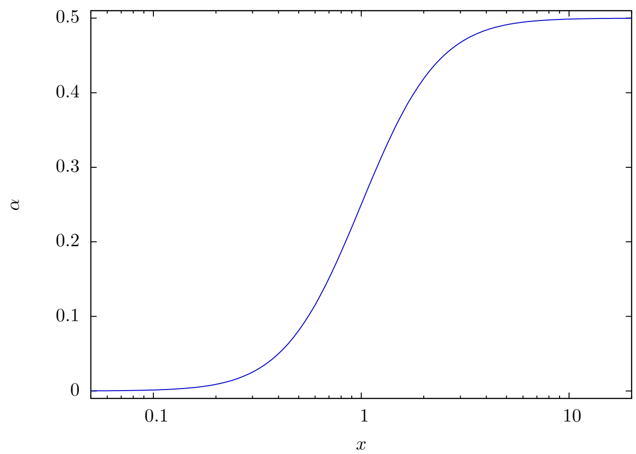

Figura II.5: The plot shows the variation of the power index α with respect to x, in the general gravitational equilibrium relation (see text) valid for all systems and at all scales. For large values of the parameter x, the Newtonian limit $\alpha = 1/2$ is reached, and for small ones convergence to the MONDian one $\alpha = 0$ is obtained. As shown in Figure II.7 many astrophysical systems of interest fall in the intermediate regions about $x = 1$.

For any given class of astrophysical objects, the parameter α can now be obtained by evaluating x and then solving for α from equation (6.7).

Note that typical values for elliptical galaxies with radii of order a few kpc, and masses one order of magnitude around $10^{10}M_{\odot}$ (see Figure II.7), we get $x \sim 1$, which from equation (6.7) yields $\alpha \sim 0.3$. The tilt in the fundamental plane is hence naturally explained by the force law presented, requiring only minimum departures from constant average values for the mass to light ratios of elliptical galaxies.

Recent studies spanning larger ranges of velocity dispersion and mass values for elliptical galaxies (e.g. Gargiulo et al., 2009; Desroches et al., 2007), have begun to identify slight trends in the power indexes of the fundamental plane. This is exactly what one would expect from the developments presented here, as when one broadens the range of physical parameters, the power law approximation to equation (6.1) ceases to be valid, and a warp will necessarily develop. Being elliptical galaxies very close to the Newtonian limit of $\alpha = 1/2$, we can estimate the first order trend of this warp against mass by considering that typical densities for elliptical galaxies tend to drop as one goes to more massive systems, which would move the fit further away from the Newtonian regime, leading to a gradual decrease in α with increasing total mass. By comparing with Figure 8 of Gargiulo et al. (2009), we see precisely this very trend, with the measured value for their index a in the $\log r_e = a \log \sigma_0$ fit (notice that $a \propto \alpha^{-1}$) increasing as the limit mass of the galaxies included in the sample increases. Therefore, the measured first order trends for the fundamental plane indexes with mass, are seen to agree with the expectations of the model presented.

Local dwarf spheroidal galaxies

We now turn to the well studied dSph galaxies of the local group and their scaling relations. Taking this time the first order terms for $R_e \gg 1$ in equation (6.1), we obtain

$$C\tilde{\sigma}^2 = 1 + \frac{1}{R_e^3}. \tag{6.8}$$

Again we see that the first deviations from the deep MOND regime occur after skipping the R_e^{-1} and R_e^{-2} terms. This implies that near the deep MOND regime, gravitational physics remain largely unaltered. If we now solve for r_e from the above equation, we obtain

$$r_e^3 = \frac{(GMa_0)^2}{a_0^3 \left[C\sigma^2 - (GMa_0)^{1/2} \right]}. \tag{6.9}$$

Through full integration of isothermal equilibrium density profiles for the local dSph's under a force law equivalent to what we propose here in the deep MOND regime, Hernandez et al. (2010) showed that a suitable description of the problem can be found without the need to invoke the presence of any dark matter. Still, this might require the consistent inclusion of the disruptive effects of galactic tides on the local dwarfs, as pointed out by e.g. Sánchez-Salcedo & Hernandez (2007); Angus (2008); McGaugh & Wolf (2010). It is interesting to remark that Hernandez et al. (2010) also showed that there is a very clear correlation between the mass to light ratio resulting from the dynamical modelling in MOND type prescriptions (or alternatively, the dark matter fraction for these systems), and the relative ages of the stellar populations present, as directly inferred from statistical studies of their resolved stellar populations (see e.g. Hernandez et al., 2000; Dolphin, 2002; Helmi, 2008; Martin et al., 2008; Tolstoy et al., 2009). This correlation is natural in a scenario where only the stars present act as a source of gravity, shining less as they age,

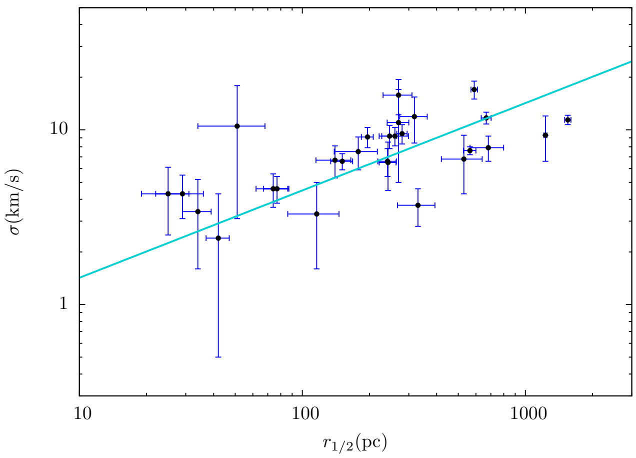

Figura II.6: Measured values of half light radii $r_{1/2}$ and internal velocity dispersions σ for the local dSph galaxies with their associated one sigma uncertainties. The solid line gives a best fit $\sigma^2 \propto r_{1/2}$ relation, the exponent of which results from the proposed force law. Data from the compilation of Walker et al. (2010). The point for the heavily distorted Sagittarius dwarf shown at the right end has been excluded from the fit.

but appears contrived under the dark matter hypothesis. Also, Hernandez et al. (2010) showed the expected scaling on average between σ^4 and total stellar mass for the objects in question, within observational errors, expected from the deep MOND regime. Introducing this scaling into equation (6.9) we obtain

$$r_e \propto \sigma^2. \tag{6.10}$$

We can now test the expectations of our proposed force law through equation (6.10), by plotting the measured values of the velocity dispersion for the local dSph galaxies, against their observed half-light radii. We assume that our r_e in equation (6.10) will be equal to some constant times the measured half light radius $r_{1/2}$ and take the compilation for these values found in (Walker et al., 2010) to plot Figure II.6. We see that in spite of the large errors present, the expected scaling of $\sigma^2 \propto r_{1/2}$ given by the line shown, clearly holds rather well. We have preferred this particular plane to test our model, as velocity dispersions and half light radii are the directly measured observables, not affected by the assumption of rather uncertain mass to light ratios.

Finally, we can try a first order extrapolation from the smallest $R_e \ll 1$ systems, the dSph galaxies, to the largest astrophysical structures in that regime, galaxy clusters, by taking values of $r = 1$kpc and $\sigma = 10\,\mathrm{km\,s^{-1}}$ from Figure II.6 and increasing the velocity dispersion by a factor of 50. The expectation of equation (6.10) then being radii of 2.5×10^3 times larger. Indeed, $2.5\,\mathrm{Mpc}$ and $500\,\mathrm{km\,s^{-1}}$ are representative values for the structural parameters of large clusters of galaxies. Explaining the dynamics of galaxy clusters in the pure MOND formalism requires some unseen dark matter (see e.g. Sanders, 2003; Angus et al., 2010), so we defer the details of this problem to a latter study. Here we only point out that the first order scaling of equation (6.10), extrapolated across ~ 6 orders of magnitude in radius, is at least not significantly in error.

Again, we see that having a well defined force law, rather than dealing with a cumbersome MOND interpolation function, facilitates the physics considerably. This allows a more direct tracing of the astrophysical consequences, and yields more transparent predictions. Notice that the general results of the whole Section §6 are not dependent on the details of $f(x)$, and will hold for any such function within the theoretical framework introduced. We note however, the good agreement with observations for the particular $f(x)$ used.

As a final summary of this section, we present Figure II.7, with typical values of the parameter x for average masses and radii for a range of astrophysical objects. This figure, when compared to Figure II.5 clearly complements the results of the previous sections. We see that solar system dynamics will be utterly unaffected by the modification proposed. Marginal modifications will appear in the dynamics of globular clusters (to within observational uncertainties, see e.g. Hernandez et al., 2010). A slight tilt is to be expected in the fundamental plane of elliptical galaxies and we see that the external regions of spiral galaxies, galaxy clusters and local dwarf spheroidal galaxies should appear as the most heavily "dark matter dominated" systems.

§7. Discussion

We have shown in this Chapter that if a modification of either the dynamical or the gravitational part of Newton's gravity is to be performed, a gravitational modification has a greater advantage for physical clarity and ease of calculations. Furthermore, it has become clear that with the addition of Milgrom's acceleration constant a_0, all physical relations must be at least functions of the parameter x defined in equation (4.2), according to Buckingham's theorem of dimensional analysis. As examples, we have shown that the acceleration felt by a particle on a centrally symmetric gravitational field satisfies this condition. Also, we show that the velocity dispersion associated to a system in the linear approximation gives rise to a generalised gravitational equilibrium relation which converges to the well known results of the fundamental plane in the description of elliptical galaxies, the Tully-Fisher relation for spiral galaxies, the Faber-Jackson relation for elliptical galaxies, the Jeans' stability criterion in the Newtonian limit and to corresponding observed relations for dSph's galaxies.

The inclusion of a_0 in the theory of gravity means that a characteristic mass-length l_M has been added. This scale approximately indicates the transition between the MONDian and Newtonian regimes. By a suitable choice of model, this transition can be built smoothly. On very general grounds, a characteristic scale l_M has been introduced into gravitation and so, gravity is no longer scale invariant.

We note also another testable prediction: as precision increases at solar system measurements the upper limits to departures from Newtonian acceleration will converge to definite values, as implied by Figure II.3.

The particular acceleration function we built is perhaps only an approximation to a more general gravitational law. The functions we built, although logical and quite precise in principle, may just be good candidates for the real gravitational theory and they should be thought of as such: good approximations which fit a wide variety of available astrophysical data to within errors. However, it has to be through an extension of the general theory of relativity that a more precise and fundamental form is to be found. This extended relativistic theory of gravity must show that the PPN approximations converge in some limit to the acceleration function built in this Chapter.

The Newtonian character of gravity has not been convincingly proved across the astrophysical scales explored in this work and as such, modifications to the theory are feasible. It is also important to note that a very wide variety of modified gravity theories of the type developed here, including MOND, are fundamentally falsifiable: a single $a > a_0$ system appearing as heavily dark matter dominated, would invalidate them all as an alternative to the dark matter hypothesis.

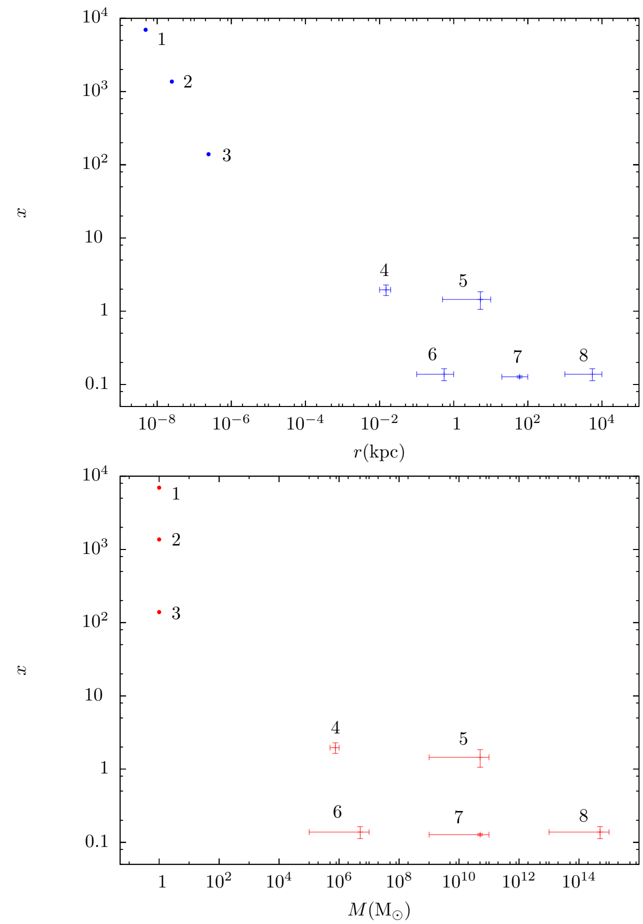

Figura II.7: The figure shows different values of $x \propto M^{1/2}/r$ for typical masses M and radii r of different astrophysical systems: (1) Solar system at Earth's orbit. (2) Solar system at Jupiter's orbit. (3) Solar system at the Kuiper belt radius. (4) Globular clusters. (5) Elliptical galaxies and bulges of spirals. (6) Dwarf spheroidal galaxies. (7) Outer regions of spiral galaxies. (8) Galaxy clusters.

Capítulo III

MOND de teorías métricas extendidas

Siguiendo con el trabajo mostrado en el Capítulo anterior, en este se muestra cómo el régimen de MOND puede recuperarse en su totalidad como el límite de campo débil de una teoría gravitacional particular formulada en la aproximación métrica. Al igual que para la gravedad Newtoniana extendida, esto es posible cuando la constante de aceleración de Milgrom se toma como una cantidad fundamental que se acopla a la teoría de manera consistente. Como consecuencia de ello, la invarianza de escala de la interacción gravitacional se rompe naturalmente. En este sentido, la gravedad Newtoniana es el límite de campo débil de la relatividad general y MOND es el límite de campo débil de la teoría de gravedad extendida particular. También demostramos que la aproximación de simetrías de Noether al problema produce una cantidad conservada coherente con esta extensión MONDiana relativista.

* * *

MOND from extended metric theories

We show that the Modified Newtonian Dynamics (MOND) regime can be fully recovered as the weak-field limit of a particular theory of gravity formulated in the metric approach. This is possible when Milgrom's acceleration constant is taken as a fundamental quantity which couples to the theory in a very consistent manner. As a consequence, the scale invariance of the gravitational interaction is naturally broken. In this sense, Newtonian gravity is the weak-field limit of general relativity and MOND is the weak-field limit of that particular extended theory of gravity. We also prove that a Noether's symmetry approach to the problem yields a conserved quantity coherent with this relativistic MONDian extension.

§8. Introduction

Milgrom (1983b,a) developed a non-relativistic theory of gravity in order to explain observed flat rotation curves of spiral galaxies. This Modified Newtonian Dynamics (MOND) theory has proved useful in explaining a great variety of astronomical phenomena without requiring the presence of a dark matter component (see e.g. Milgrom, 2010a, and references therein).

As explained by an extended Newtonian approach to gravity, Mendoza et al. (2011) showed that the key feature of MOND is the introduction of a fundamental acceleration a_0 in the theory, which on itself at the non-relativistic level, makes gravity *scale dependent*.

Through the years, finding the relativistic extension of MOND has become a big challenge. The most successful attempt was proposed by Bekenstein (2004) who formulated a Tensor-Vector-Scalar (TeVeS) relativistic theory of MOND. This approach requires tensor, vector and scalar fields to achieve a self-consistent description. However, the many cumbersome mathematical complications that TeVeS present are evident. Furthermore, it cannot reproduce crucial astrophysical phenomena (see e.g. Ferreras et al., 2009).

On the other hand, since extended metric $f(R)$ theories of gravity have proved very successful on a wide variety of cosmological scenarios (see e.g. Capozziello & Faraoni, 2011; Sotiriou & Faraoni, 2010; Nojiri & Odintsov, 2006, 2011, and references therein) it is natural to seek a relativistic generalisation of MOND in this direction, which has not deserved too much attention since Soussa (2003) and Soussa & Woodard (2003) developed a no-go theorem which prevented all metric $f(R)$ theories of gravity to become relativistic candidates for MOND. However, Mendoza & Rosas-Guevara (2007) showed counterexamples of this no-go theorem disproving its general validity. Furthermore, the works by Capozziello et al. (2006a, 2007a, 2009a) and Sobouti (2007) made clear that particular $f(R)$ models are capable of explaining phenomena usually ascribed to MOND. Up to now, the challenge has been to formulate a suitable $f(R)$ theory able to converge to standard MOND in the non-relativistic regime.

In this Chapter, we show how a particular $f(R)$ metric theory of gravity, derived from first principles, is capable of reproducing MOND when its non-relativistic regime is reached. To do so, in Section §9 we set the foundations of a metric theory of gravity with the use of correct dimensional quantities. Then, in Section §10 we solve the problem of a point mass source on a static space-time and find, at first order of approximation, the particular form of the function $f(R)$. In Section §11 the integration constants of the theory are fixed by solving the same problem in the formalism of metric perturbations. In Section §12 we show that the existence of a Noether symmetry confirms the appearance of a characteristic scale of the problem in the MONDian regime. Finally in Section §13 we comment on the obtained results.

§9. Dimensional grounding

Let us assume that a point mass M located at the origin of coordinates generates a relativistic gravitational field in the MONDian regime and that a metric formalism describes the field equations.

This problem is characterised by the following quantities: the speed of light in vacuum c, the mass M of the central object generating the gravitational field, Newton's constant of gravity G and Milgrom's acceleration constant a_0. With these parameters, two "fundamental lengths" can be built:

$$r_g := \frac{GM}{c^2}, \qquad l_M := \left(\frac{GM}{a_0}\right)^{1/2}. \tag{9.1}$$

The gravitational radius r_g is a length that appears once relativistic effects are introduced on a theory of gravity. The mass-length scale l_M, as described in (Mendoza et al., 2011), is a characteristic length which appears on a gravitational theory when MONDian effects are to be taken into account (for consistency we note here that a third length, $\lambda := l_M^2/r_g = c^2/a_0$, that does not contain the mass, can be constructed as a combination of the previous two).

In the non-relativistic regime, a test particle located at the radial coordinate r from the origin will obey the MONDian dynamics when $l_M/r \ll 1$. When $l_M/r \gg 1$ the gravitational field is Newtonian. As such, when relativistic effects are taken into account for the gravitational field, i.e. $r \gtrsim r_g$, then standard general relativity should be recovered in the limit $l_M/r \gg 1$, and a relativistic version of MOND should be obtained when $l_M/r \ll 1$. This shows that the pursue of a complete metric description leads one to consider the scale-dependence of gravity.

The length scales presented in equation (9.1) must somehow appear in a relativistic theory of gravity which accepts the fundamental nature of the constant a_0. For example, in the metric formalism, a generalised Hilbert action S_H can be written in the following way:

$$S_H = -\frac{c^3}{16\pi G L_M^2} \int f(\chi) \sqrt{-g}\, d^4x, \tag{9.2}$$

which slightly differs from its traditional form (see e.g. Capozziello & Faraoni, 2011; Sotiriou & Faraoni, 2010; Capozziello et al., 2010b)

$$S_H = -\frac{c^3}{16\pi G} \int f(R) \sqrt{-g}\, d^4x\,, \tag{9.3}$$

since we have introduced the following dimensionless quantity:

$$\chi := L_M^2 R, \tag{9.4}$$

where R is Ricci's scalar and L_M defines a length fixed by the parameters of the theory. The explicit form of the length L_M has to be obtained once a certain known limit of the theory is taken, usually a non-relativistic limit. Note that the definition of χ gives a correct dimensional character to the action (9.2), something that is not completely clear in all previous works dealing with a metric description of the gravitational field. For $f(\chi) = \chi$ the standard Einstein-Hilbert action is obtained.

On the other hand, the matter action has its usual form,

$$S_m = -\frac{1}{2c}\int \mathcal{L}_m \sqrt{-g}\, d^4x, \tag{9.5}$$

with $\mathcal{L}_m$ the Lagrangian density of the system. The null variations of the complete action, i.e. $\delta\left(S_H + S_m\right) = 0$, yield the following field equations:

$$f'(\chi)\,\chi_{\mu\nu} - \frac{1}{2}f(\chi)g_{\mu\nu} - L_M^2\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Delta\right)f'(\chi) = \frac{8\pi G L_M^2}{c^4}T_{\mu\nu}, \tag{9.6}$$

where the dimensionless Ricci tensor $\chi_{\mu\nu}$ is given by:

$$\chi_{\mu\nu} := L_M^2 R_{\mu\nu}, \tag{9.7}$$

and $R_{\mu\nu}$ is the standard Ricci tensor. The Laplace-Beltrami operator has been written as $\Delta := \nabla^\alpha\nabla_\alpha$ and the prime denotes derivative with respect to its argument. The energy-momentum tensor $T_{\mu\nu}$ is defined through the following standard relation: $\delta S_m = -\left(1/2c\right)T_{\alpha\beta}\,\delta g^{\alpha\beta}$. In here and in what follows, we choose a $(+,-,-,-)$ signature for the metric $g_{\mu\nu}$ and use Einstein's summation convention over repeated indices.

The trace of equation (9.6) is:

$$f'(\chi)\,\chi - 2f(\chi) + 3L_M^2\,\Delta f'(\chi) = \frac{8\pi G L_M^2}{c^4}T, \tag{9.8}$$

where $T := T^\alpha_\alpha$.

Since we are only interested on the gravitational field produced by a point mass source located at the origin, then the mass density ρ is given by

$$\rho = M\,\delta(\mathbf{r}), \tag{9.9}$$

where $\delta(\mathbf{r})$ represents the three dimensional Dirac delta function. With this, it follows that the only non-zero component of the energy-momentum tensor is given by

$$T_{00} = \rho c^2 = c^2 M\,\delta(\mathbf{r}). \tag{9.10}$$

A point mass distribution generates a stationary spherically symmetric space-time and so, the trace equation (9.8) contains all the relevant information relating the field equations. In what follows we assume a power law form for the function $f(\chi)$, i.e.

$$f(\chi) = \chi^b. \tag{9.11}$$

§10. Order of magnitude approach

Let us first analyse the problem described in the previous Section by performing an order of magnitude approximation of the trace equation (9.8). Under these circumstances, $d/d\chi \approx 1/\chi$, $\Delta \approx -1/r^2$ and the mass density $\rho \approx M/r^3$. This approximation implies that equation (9.8) takes the following form:

$$\chi^b\,(b-2) - 3bL_M^2 \frac{\chi^{(b-1)}}{r^2} \approx \frac{8\pi G M L_M^2}{c^2 r^3}. \tag{10.1}$$

Note that the second term on the left-hand side of equation (10.1) is much greater than the first term when the following condition is satisfied:

$$Rr^2 \lesssim \frac{3b}{2-b}. \tag{10.2}$$

At the same order of approximation, Ricci's scalar $R \approx \kappa = R_c^{-2}$, where κ is the Gaussian curvature of space and R_c its radius of curvature and so, relation (10.2) essentially means that

$$R_c \gg r. \tag{10.3}$$

In other words, the second term on the left-hand side of equation (10.1) dominates the first one when the local radius of curvature of space is much grater than the characteristic length r. This should occur in the weak-field regime, where MONDian effects are expected. For a metric description of gravity, this limit must correspond to the relativistic regime of MOND. In this Chapter we will only deal with this approximation. At the end of the current Section we show an equivalent relation to inequality (10.3) which has a more physical meaning.

Under assumption (10.3), equation (10.1) takes the following form:

$$R^{(b-1)} \approx -\frac{8\pi GM}{3bc^2 r L_M^{2(b-1)}}. \tag{10.4}$$

We now recall the well known relation followed by the Ricci scalar at second order of approximation at the non-relativistic level (Landau & Lifshitz, 1975):

$$R = -\frac{2}{c^2}\nabla^2\phi = +\frac{2}{c^2}\nabla\cdot\mathbf{a}, \tag{10.5}$$

where the negative gradients of the gravitational potential ϕ provide the acceleration $\mathbf{a} := -\nabla\phi$ felt by a test particle on a non-relativistic gravitational field. At order of magnitude, equation (10.5) can be approximated as

$$R \approx -\frac{2\phi}{c^2 r^2} \approx \frac{2a}{c^2 r}. \tag{10.6}$$

Substitution of this last equation on relation (10.4) gives

$$\begin{aligned} a &\approx -\frac{c^2 r}{2L_M^2}\left(\frac{8\pi GM}{3bc^2 r}\right)^{1/(b-1)} \\ &\approx -c^{(2b-4)/(b-1)} r^{(b-2)/(b-1)} L_M^{-2} (GM)^{1/(b-1)}. \end{aligned} \tag{10.7}$$

This last equation converges to a MOND-like acceleration $a \propto 1/r$ if $b-2 = -(b-1)$, i.e. when

$$b = 3/2. \tag{10.8}$$

Also, at the lowest order of approximation, in the extreme non-relativistic limit, the velocity of light c should not appear on equation (10.7) and so, the only possibility is that L_M depends on a power of c, i.e.

$$L_M^{-2} \propto c^{(4-2b)/(b-1)} = c^2, \quad \text{and so,} \qquad L_M \propto c^{-1}. \tag{10.9}$$

As discussed in Section §9, the length L_M must be constructed by fundamental parameters describing the theory of gravity and so, let us assume that

$$L_M = \zeta\, r_g^{\alpha} l_M^{\beta}, \qquad \text{with} \qquad \alpha + \beta = 1, \tag{10.10}$$

where the constant of proportionality ζ is a dimensionless number of order one that will be formally obtained in Section §11. Substituting equation (10.10) and the value obtained in (10.8) into relation (10.9) it then follows that

$$\alpha = \beta = 1/2, \qquad \text{i.e.} \qquad L_M \approx r_g^{1/2} l_M^{1/2}. \tag{10.11}$$

If we now substitute this last result and relation (10.8) in equation (10.7) it follows that

$$a \approx -\frac{(a_0 GM)^{1/2}}{r}, \tag{10.12}$$

which is the traditional form of MOND in spherical symmetry (see e.g. Milgrom, 2010a; Bekenstein, 2006, and references therein). Also, the results of equation (10.12) in (10.6) mean that

$$R \approx \frac{r_g}{l_M} \frac{1}{r^2}, \tag{10.13}$$

and so, inequality (10.3) is equivalent to

$$l_M \gg r_g. \tag{10.14}$$

The regime imposed by equation (10.14) is precisely the one for which MONDian effects should appear in a relativistic theory of gravity, since as discussed above $l_M \gg r \gtrsim r_g$. This is an expected generalisation of the results presented by Mendoza et al. (2011).

§11. Weak field limit approach

We now use the trace (9.8) to the lowest order of perturbation. Results of this perturbation in the Newtonian limit for other metric theories of gravity have been reported by Capozziello et al. (2007b); Capozziello & Stabile (2009). However, since we are interested in the lowest order of approximation in the MONDian regime, we expect different results.

Under the assumption of spherical symmetry for a static space-time, with a diagonal metric given by (see e.g. Landau & Lifshitz, 1975)

$$
\begin{aligned}
g_{00} &= 1 + \frac{2\phi}{c^2}, & g_{11} &= -1, \\
g_{22} &= -r^2, & g_{33} &= -r^2 \sin^2\theta,
\end{aligned}
\tag{11.1}
$$

then

$$
\begin{aligned}
\Delta f'(\chi) &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial f'(\chi)}{\partial x^\nu}\right), \\
&= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f'(\chi)}{\partial r}\right) = -\nabla^2 f'(\chi),
\end{aligned}
\tag{11.2}
$$

at the lowest order of perturbation in ϕ/c^2. In this case, when condition (10.3) or equivalently equation (10.14) is satisfied, the trace (9.8) of the field equations is given by:

$$
-3\nabla^2 f'(\chi) = \frac{8\pi G}{c^2}\rho. \tag{11.3}
$$

Note that this equation can also be obtained by performing a direct perturbation of (9.8). This is so because in the weak field limit, the field equations (9.1) are studied at orders of powers of c^{-2} (Capozziello et al., 2007b, 2009a). At the lowest zeroth-order of the perturbation, the Ricci scalar ${}^{(0)}R = 0$ everywhere and so, it describes a flat space-time. At the next second perturbation order $\mathcal{O}(\chi) = \mathcal{O}(L_M^2) + \mathcal{O}({}^{(2)}R)$, where ${}^{(2)}R$ represents Ricci's scalar at the second order of the perturbation. Since $\mathcal{O}(L_M) = 1$ according to relation (10.9), then $\mathcal{O}(\chi) = 4$. If we now use the power law relation (9.11) with (10.8) and perturb the trace (9.8), it follows that the first two terms on the right-hand side are of order $\mathcal{O}(\chi^{3/2}) = 6$, while the remaining two terms are of order $\mathcal{O}(L_M^2 \chi^{1/2}) = 4$ and so, it follows that the trace (9.8) at its lowest non-zero perturbation order is exactly relation (11.3).

Equation (11.3) can be integrated straightforward using the standard Poisson equation of Newtonian gravity:

$$
\nabla^2 \phi_N = 4\pi G \rho, \tag{11.4}
$$

for the Newtonian potential ϕ_N. Substitution of this relation on (11.3) yields

$$
\nabla^2 \left(f'(\chi) + \frac{2}{3c^2}\phi_N \right) = 0. \tag{11.5}
$$

The trivial solution of the previous Laplace-like equation occurs when the argument of the Laplacian is equal to zero, and so

$$f'(\chi) = -\frac{2}{3c^2}\phi_N. \tag{11.6}$$

Substitution of equations (9.11), (10.8), (10.10), and (10.11) in relation (11.6) leads to

$$R = \left(\frac{4}{9}\right)^2 \zeta^{-2} \left(\frac{r_g}{l_M}\right) \frac{1}{r^2}, \tag{11.7}$$

where we have used the fact that for a point mass M, the Newtonian potential is given by:

$$\phi_N = -G\frac{M}{r}. \tag{11.8}$$

We now substitute relation (10.5) into equation (11.7) to obtain:

$$\frac{2}{c^2}\nabla \cdot \mathbf{a} = \left(\frac{4}{9}\right)^2 \zeta^{-2} \left(\frac{r_g}{l_M}\right) \frac{1}{r^2}. \tag{11.9}$$

Integrating this equation over a spherical volume of radius r and using Gauss's theorem on the left-hand side, we obtain

$$a = -\left(\frac{2\sqrt{2}}{9\zeta}\right)^2 \left(\frac{c^2 r_g}{l_M}\right) \frac{1}{r}, \tag{11.10}$$

and so, by choosing

$$\zeta = \frac{2\sqrt{2}}{9}, \tag{11.11}$$

we reach the MOND acceleration limit

$$a = -\frac{(a_0 G M)^{1/2}}{r}, \tag{11.12}$$

at the lowest perturbation order of the theory.

The formulation of the theory described so far implies that, as soon as we relax the hypothesis for which the gravitational action is strictly the one described by standard general relativity, new characteristic lengths have to be considered. In the following Section, we show that such scales coherently appear as constants of motion of the problem.

§12. Noether's symmetries

The above results are based on the fact that we are assuming the power law relation (9.11) for the gravitational action. In particular, for $f(R) \propto R^{3/2}$ the MOND acceleration regime is recovered. This particular theory also admits exact cosmological solutions where a matter dominated era evolves towards the accelerated universe observed today (Capozziello et al., 2008) and recovers the observed dynamics at astrophysical scales (Capozziello et al., 2009b). Also, this solution is particularly relevant since its conformal transformation is exactly invertible, as shown in (Capozziello, 2002). Noether's symmetries give rise to conserved quantities that are directly related to characteristic length scales (Capozziello et al., 2008).

In order to develop a Noether's approach to the problem, let us consider $f(\chi)$ gravity in static spherical symmetry, following the same ideas as the ones exposed in Section §11. The spherical point–like $f(\chi)$ Lagrangian can be obtained by imposing the spherical symmetry directly into the action (9.2). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier defined by assuming the known explicit form of Ricci's scalar R (Capozziello et al., 2007c).

The static spherically symmetric metric can be expressed as

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - C(r)d\Omega^2, \tag{12.1}$$

where $d\Omega^2 := d\theta^2 + \sin^2\theta d\varphi^2$ is the angular displacement and $C(r) := r^2$. Since we are interested in the weak-field limit approach of the $f(\chi)$ theory, let us assume $B(r) = 1$.

The point–like $f(\chi)$ Lagrangian L is obtained by rewriting the action (9.2) as

$$S = -\frac{c^3}{16\pi G L_M^2}\int [f(\chi) - \lambda(\chi - \bar{\chi})]\sqrt{-g}\, d^4x, \tag{12.2}$$

where λ is a Lagrange multiplier and $\bar{\chi} = L_M^2\bar{R}$, for the known Ricci scalar $\bar{R}$ expressed in terms of the metric (12.1) with $B(r) = 1$:

$$\bar{R} = \frac{A''}{A} + \frac{2C''}{C} + \frac{A'C'}{AC} - \frac{A'^2}{2A^2} - \frac{C'^2}{2C^2} - \frac{2}{C}. \tag{12.3}$$

In the previous equation, the prime denotes the derivative with respect to the radial coordinate r. Variations of the action with respect to χ give the explicit form of the Lagrange multiplier:

$$\lambda = \frac{df(\chi)}{d\chi} := f_\chi.$$

Substituting this result in the action (12.2) and eliminating the boundary terms (cf. Capozziello et al., 2007c), the point–like Lagrangian is obtained:

$$L = -\frac{L_M^2}{\sqrt{A}}\left[\frac{Af_\chi}{2C}C'^2 + f_\chi A'C' + Cf_{\chi\chi}A'\chi' + 2Af_{\chi\chi}C'\chi'\right] - \sqrt{A}\left[(2L_M^2 + C\chi)f_\chi - Cf\right]. \tag{12.4}$$

Note that this Lagrangian is canonical since only the generalised positions, $\mathbf{q} = (A, C, \chi)$, and their generalised velocities $\mathbf{q}' = (A', C', \chi')$, appear explicitly.

In the MONDian regime, where equation (10.3) holds, the last two terms on the right-hand side of equation (12.4) are of order $Cf \approx C\chi^{3/2}$ and so, both are much smaller than $L_M^2 f_\chi$. This statement is also true at the lowest non-zero order of perturbation, since all terms on the right-hand side of equation (12.4) are of order 4, except the last two which are of order 6. With this, it follows that in the MONDian regime the Lagrangian (12.4) can be written as

$$L = -\frac{L_M^2}{\sqrt{A}}\left[\frac{Af_\chi}{2C}C'^2 + f_\chi A'C' + Cf_{\chi\chi}A'\chi' + 2Af_{\chi\chi}C'\chi' + 2A\right]. \tag{12.5}$$

We now search for symmetries related to the cyclic variables and then reduce the dynamics. According to Noether's theorem, the existence of symmetry properties for the dynamics described by the Lagrangian implies the existence of conserved quantities (Arnol'd, 1989; Marmo et al., 1985; Morandi et al., 1990). In principle, this approach allows to select particular $f(\chi)$ gravity models compatible with spherical symmetry.

A conserved quantity exists if the Lie derivative of the Lagrangian (12.5) along the vector field $\mathbf{X}$ vanishes:

$$\mathcal{L}_{\mathbf{X}} L = \alpha_i \nabla_{q_i} L + \alpha_i' \nabla_{q_i'} L = 0, \tag{12.6}$$

for $i = 1, 2, 3$, in the configuration space (A, C, χ). Solving equation (12.6) means to find out the functions α_i which constitute the Noether vector (Arnol'd, 1989; Marmo et al., 1985). However, the relation (12.6) implicitly depends on the form of $f(\chi)$ and then, by solving it, we also get $f(\chi)$ models compatible with spherical symmetry. On the other hand, by fixing the $f(\chi)$ function, we can explicitly solve (12.6). In principle, the same procedure can be worked out any time Noether's symmetries are identified (Capozziello & DeFelice, 2008).

The general form of the Noether vector is given by the solution of the Killing equations for the components α_i of the vector $\boldsymbol{\alpha}$ in a flat space-time (cf. Townsend, 1997):

$$\begin{aligned} \alpha_1 &= k_1 A + p_1, \\ \alpha_2 &= k_2 C + p_2, \\ \alpha_3 &= k_3 \chi + p_3, \end{aligned} \tag{12.7}$$

with k_i and p_i constants. Using the power law (9.11) and the general solution (12.7) in equation (12.6), it follows that

$$\boldsymbol{\alpha} = \left(2(1-b)kA,\ 0,\ k\chi\right), \tag{12.8}$$

where k is an integration constant and we have assumed in the calculations that $b \neq 1$, $\chi' \neq 0$ and $2AC' + A'C \neq 0$ in order to obtain solutions different from general relativity (cf. Section §10). For this case, the related constant of motion, Σ_0, is given by

$$\begin{aligned} \Sigma_0 &= \alpha_i \nabla_{q_i'} L, \\ &= L_M^2 b(b-1)kA^{-1/2}C\chi^{b-2}\left[2(b-1)A\chi' - A'\chi\right]. \end{aligned} \tag{12.9}$$

In general relativity, where $b = 1$, Noether's symmetry approach gives $\Sigma_0 = 2r_g$, which is exactly the Schwarzschild radius (cf. Townsend, 1997). On the other hand, in the MONDian regime, where equation (12.9) is valid for $b = 3/2$, $C(r) = r^2$ and at the lowest order of perturbation $A(r) = 1 + 2\phi/c^2$, the related constant of motion is given by

$$\Sigma_0 = \frac{3}{2} k r_g^2 l_M. \tag{12.10}$$

The key point in this relation is that it includes two characteristic lengths, namely the mass length-scale l_M and the gravitational radius r_g. This is coherent with the results discussed on the previous Sections, since both characteristic lengths must appear on a correct relativistic metric theory of MOND.

§13. Discussion

We have shown that a metric theory of gravity $f(\chi) = \chi^{3/2}$, where the dimensionless Ricci scalar χ is given by equation (9.4), converges to MOND in the non-relativistic regime. We note that a previous attempt in this direction was made in the B.Sc. thesis of Rosas–Guevara (2006), where a metric $f(R) \propto R^{3/2}$ theory was proposed to account for relativistic effects of MOND. However, the correct approximation (10.14) was never introduced on that analysis and therefore, the MONDian limit was not achieved on that work.

We have also shown that the appearance of a new characteristic length related to MOND's acceleration is coherent with Noether's symmetries related to the problem.

The metric theory of gravity presented here is by no means a complete description at all scales of gravitation. It only deals with the MONDian regime of gravity, i.e. when equation (10.14) is valid. In other words, our description breaks the scale invariance of gravity in a more general way than the one described by Mendoza et al. (2011).

The mass dependence of χ means that the mass needs to appear on Hilbert's action (9.2). This is traditionally not the case, since that action is thought to be purely a function of the geometry of space-time due to the presence of mass and energy. However, it was Sobouti (2007) who first encountered this peculiarity in the Hilbert action when dealing with a metric generalisation of MOND. Following his remarks (Sobouti, 2007), one should not be surprised if some of the commonly accepted notions, even at the fundamental level of the action, require generalisations and re-thinking. An extended metric theory of gravity goes beyond the traditional general relativity ideas and in this way, we probably need to change our standard view of its fundamental principles.

Capítulo IV

Lentes gravitacionales con gravedad $f(\chi)$

En este Capítulo se muestra el análisis de perturbaciones completo a segundo orden de la teoría métrica de gravedad $f(\chi) = \chi^{3/2}$, desarrollada en el Capítulo anterior. Demostramos que la teoría ajusta perfectamente para dos hechos observacionales: (1) la fenomenología de las curvas de rotación planas asociadas a la relación Tully–Fisher observada en galaxias espirales y (2) los detalles de las observaciones de lentes gravitacionales en galaxias y grupos de galaxias, sin necesidad de materia oscura. Mostramos cómo todas las observaciones dinámicas en curvas de rotación planas y lentes gravitacionales pueden ser resumidas en términos de los coeficientes métricos empíricos requeridos de cualquier teoría métrica de gravitación. Además construimos los componentes de la métrica correspondientes para la teoría presentada hasta segundo orden en las perturbaciones y se muestra que son compatibles perfectamente con los derivados empíricamente. También se muestra que bajo la teoría presentada, con el fin de ajustar completamente con los resultados observacionales, se debe elegir un signo específico del tensor de Riemann. Este signo corresponde al utilizado más ampliamente hoy en día en la teoría relativista. Además se presenta y se hace disponible públicamente un programa computacional, el código MEXICAS (Metric EXtended-gravity Incorporated through a Computer Algebraic System), desarrollado para su uso en el Sistema de Cómputo Algebraico (CAS) Maxima para la manipulación de las perturbaciones en una teoría métrica de gravitación.

* * *

Gravitational lensing with $f(\chi)$ gravity

In this Chapter we perform a second order perturbation analysis of the gravitational metric theory of gravity $f(\chi) = \chi^{3/2}$ developed on the last chapter. We show that the theory accounts in detail for two observational facts: (1) the phenomenology of flattened rotation curves associated to the Tully-Fisher relation observed in spiral galaxies, and (2) the details of observations of gravitational lensing in galaxies and groups of galaxies, without the need of any dark matter. We show how all dynamical observations on flat rotation curves and gravitational lensing can be synthesised in terms of the empirically required metric coefficients of any metric theory of gravity. We construct the corresponding metric components for the theory presented at second order in perturbation, which are shown to be perfectly compatible with the empirically derived ones. It is also shown that under the theory being presented, in order to obtain a complete full agreement with the observational results, a specific signature of Riemann's tensor has to be chosen. This signature corresponds to the one most widely used nowadays in relativity theory. Also, a computational program, the MEXICAS (Metric EXtended-gravity Incorporated through a Computer Algebraic System) code, developed for its usage in the Computer Algebraic System (CAS) Maxima for working out perturbations on a metric theory of gravity, is presented and made publicly available.

§14. Introduction

When Einstein introduced his theory of general relativity, an astrophysical prediction for the motion of the planet Mercury (a massive particle) through its orbit was made (Einstein, 1916). The second step was to test general relativity through the deflection of light (massless particles) coming from stars appearing near the Sun's limb during a solar eclipse (Dyson et al., 1920). Both observations constituted the first coherent steps towards the solid foundation of general relativity, a theory capable of describing gravitation through a correct relativistic description.

In this sense, any metric theory of gravity must be compatible with both kinds of observations, the dynamical ones for massive particles and the observations of the deflection of light for massless particles. The correct approach is extensively described in the monograph by Will (1993) where it is shown that when working with the weak field limit of a relativistic theory of gravity in a static spherically spacetime, the dynamics of massive particles determine the functional form of the time component of the metric, while the deflection of light determines the form of the radial one (see also Will, 2006, and references therein).

To order of magnitude and through a first perturbation analysis, Bernal et al. (2011b) have shown that it is possible to recover flat rotation curves and the Tully-Fisher relation (i.e. a MONDian-like weak field limit) from a metric theory of gravity, which includes the mass of the system in the gravitational field's action. Such limit is of high astrophysical relevance at the scales of galaxies, where MOND accurately describes the rotation curves of spiral galaxies and the Tully-Fisher relation without the need of dark matter (see e.g. Milgrom, 1983a; Famaey & McGaugh, 2011). In this Chapter we show the strength of the calculations made by Bernal et al. (2011b) by doing an extensive analysis from perturbation theory for a static spherically symmetric metric and show that in the weak field limit our results are in perfect agreement not only with the Tully-Fisher relation, but are also in exact accordance with observations of gravitational lensing at a wide range of astrophysical scales.

Extensions to Einstein's general theory of relativity have been proposed since the publication of the theory itself (see e.g. Schimming & Schmidt, 2004). However, it has not been until recent times that observations at different mass and length scales have concluded that in order to keep Einstein's field equations valid, unknown dark matter and energy entities need to be added to the theory. In this work, an equally important approach is taken where the existence of these dark unknown entities is not required. We show the theory built by Bernal et al. (2011b) to be in accordance not only with the very well established observations of the dynamics of massive particles through the Tully-Fisher relation, but also with the dynamics of massless particles through the bending of light as astrophysically observed.

Mendoza & Rosas-Guevara (2007) and Rosas–Guevara (2006) showed for the first time that metric theories of gravity are capable of producing more deflection of light that the one produced by Einstein's general relativity. This was done using the metric theory of gravity constructed by Sobouti (2007). The implications of this result invalidated the so called no-go theorem for metric $f(R)$ theories of gravity proposed by Soussa & Woodard

(2003); Soussa (2003). Furthermore, in the present work we show that it is possible to explain the observed gravitational lensing for galaxies, and groups of galaxies without the need of invoking dark matter. Developments by Capozziello et al. (see e.g. 2006b); Horváth et al. (see e.g. 2012); Nzioki et al. (see e.g. 2011) in weak and strong lensing regimes of extended metric theories of gravity have followed the work by Mendoza & Rosas-Guevara (2007) but are not completely coherent with different astrophysical observations.

Testing any metric theory of gravity against observations can be cumbersome. From an action principle one must derive field equations, which in principle, have to be solved for e.g. in spherically symmetric spacetimes. The solutions to this lead to metric coefficients which in turn yield orbits for massive and massless particles, to be then compared against astrophysical observations. These last are varied and diverse e.g., centrifugal equilibrium orbits at a variety of radii, for systems having total masses spanning several orders of magnitude, and the observed shears and caustic positions of gravitational lensing observations.

Fortunately, we have derived a much more direct and generic approach. First, dynamical observations regarding the amplitudes of galactic flat rotation curves satisfy a well known scaling with the fourth root of the total baryonic content: the Tully-Fisher relation. To second order in perturbations of the velocity measured in units of the speed of light, this can be shown to imply a definitive empirical form for the time component of any metric theory not requiring dark matter. Second, we show that all gravitational lensing observations on elliptical and spiral galaxies, as well as for groups of galaxies can be synthesised as the requirement for the same isothermal total matter distribution as needed to explain the observed spiral rotation curves and dynamics about elliptical galaxies, if one assumes Einstein's general relativity. This in turn, from studying directly the lens equation in general relativity, implies a bending angle which is independent of the impact parameter, and which scales with the square root of the total baryonic mass of a system. It can then be shown that this, in combination with the empirical time component mentioned above, leads to a definitive empirical form for the radial component, for any metric theory not requiring dark matter.

Thus, we synthesise all dynamical and gravitational lensing astrophysical observations at galactic and galaxy group scales, to second order in perturbation, into empirical time and radial metric components. It is through comparing the above to perturbed metric coefficients to the same order coming from the metric theory treated in this paper that we are able to show its full compatibility with all relevant dynamical and gravitational lensing astrophysical observations.

The Chapter is organised as follows. In section §15 the concept of weak field limit for a static spherically symmetric metric is established and we define the relevant orders of perturbation to be used throughout the Chapter.

In section §16 we perturbe the vacuum field equations for the metric theory built by Bernal et al. (2011b) and show that for a point mass source they closely resemble the ones usually adopted in $f(R)$ gravity in vacuum. However, these equations slightly differ under the approximations of the mass and length scales associated to galaxies and groups of galaxies -where gravity is expected to differ from Einstein's general relativity in the absence of any dark matter component. In section §17 we obtain the solution for the Ricci scalar up to the second order from the perturbed field equations and discuss the importance of the signature in the Riemann tensor to yield the correct results. In section §18 we obtain the coefficients of the metric up to the second order in perturbation. In section §19 we obtain the metric coefficients up to the second order in an empirical way, without reference to any specific metric theory of gravity, using the dynamical phenomenology of galaxies and groups of galaxies and the gravitational lensing produced by these objects. In that section we also compare the metric coefficients obtained in §18 with those empirically obtained and show full consistency. Finally in section §20 we discuss our results.

§15. The weak field limit

An excellent account of perturbation theory applied to metric theories of gravity (in particular general relativity) can be found in the monograph written by Will (1993); Capozziello & Stabile (2009). In this section we define the relevant properties of the perturbation theory having in mind applications to the metric theory developed by Bernal et al. (2011b).

Let us consider a fixed point mass M at the centre of coordinates generating a gravitational field. Under these considerations, the spacetime is static and its spherically symmetric metric $g_{\mu\nu}$ is generated by the interval

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00}\, c^2 dt^2 + g_{11} dr^2 - r^2 d\Omega^2. \tag{15.1}$$

In the previous equation and in what follows, Einstein's summation convention over repeated indices is used. Greek indices take values $0, 1, 2, 3$ and Latin ones $1, 2, 3$. As such, in spherical coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$, where c is the speed of light, t is the time coordinate, r the radial one, and θ and φ are the polar and azimuthal angles respectively. Also, the angular displacement $d\Omega^2 := d\theta^2 + \sin^2\theta\, d\varphi^2$. Due to the symmetry of the problem, the unknown functions g_{00} and g_{11} are functions of the radial coordinate r only. We choose a $(+, -, -, -)$ signature for the metric of the spacetime.

The radial component of the geodesic equations

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\lambda}\frac{dx^\beta}{ds}\frac{dx^\lambda}{ds} = 0\,, \tag{15.2}$$

for the metric (15.1) in the weak field limit, i.e. when the speed of light $c \to \infty$, is given by (see e.g. Will, 1993)

$$\frac{1}{c^2}\frac{d^2r}{dt^2} = \frac{1}{2}g^{11}g_{00,r}, \tag{15.3}$$

where the subscript $(\ )_{,r} := \partial/\partial r$ denotes the derivative with respect to the radial coordinate. In the above relation we have assumed that for the weak field limit $ds = c\ dt$ and since the velocity $v \ll c$ then $v^i \ll dx^0/dt$ with $v^i := (dr/dt,\ rd\theta/dt,\ r\sin\theta d\varphi/dt)$.

In this limit, a particle bounded to a circular orbit about the mass M experiences a centrifugal radial acceleration given by:

$$\frac{d^2r}{dt^2} = \frac{v^2}{r}, \tag{15.4}$$

for a circular or tangential velocity v.

When material particles are used as test particles in the weakest limit of the theory, the metric takes the form (11.1) for a Newtonian gravitational potential ϕ.

For a particle on circular motion about the mass M in the weak field limit, the lowest order of the theory is obtained when the left-hand side of equation (15.3) is of order v^2/c^2 and when the right-hand side is of order ϕ/c^2. Both are just orders $\mathcal{O}(1/c^2)$ of the theory, or simply $\mathcal{O}(2)$. As such, when lower or higher order corrections of the theory are introduced we will use the notation $\mathcal{O}(n)$ for $n = 0, 1, 2, \ldots$ meaning $\mathcal{O}(0), \mathcal{O}(c^{-1}), \mathcal{O}(c^{-2}), \ldots$ respectively.

Having in mind further astrophysical applications (of motion of material particles and bending of light -massless particles), we expand the metric $g_{\mu\nu}$ about a flat Minkowski metric $\eta_{\mu\nu} := \mathrm{diag}(1, -1, -1, -1)$ up to the second order in time and radial position in such a way that

$$\begin{aligned} g_{00} =& g^{(0)}_{00} + g^{(2)}_{00} = 1 + g^{(2)}_{00} + \mathcal{O}(4), \\ g_{11} =& g^{(0)}_{11} + g^{(2)}_{11} = -1 + g^{(2)}_{11} + \mathcal{O}(4), \\ g_{22} =& g^{(0)}_{22} = -r^2, \\ g_{33} =& g_{22} \sin^2\theta, \end{aligned} \tag{15.5}$$

where the superscript (p) denotes the order $\mathcal{O}(p)$ at which a particular quantity is approximated. From equations (15.5) it follows that the contravariant metric components are given by

$$\begin{aligned} g^{00} =& g^{00(0)} + g^{00(2)} = 1 - g^{(2)}_{00} + \mathcal{O}(4), \\ g^{11} =& g^{11(0)} + g^{11(2)} = -1 - g^{(2)}_{11} + \mathcal{O}(4), \\ g^{22} =& g^{22(0)} = -1/r^2, \\ g^{33} =& g^{22}/\sin^2\theta. \end{aligned} \tag{15.6}$$

In fact, to the lowest order of perturbation, we need to find the time $g^{(2)}_{00}$ and radial $g^{(2)}_{11}$ metric components up to the second order to compare with the astrophysical observations of material particles and bending of light (Will, 1993, 2006).

§16. Field equations

Bernal et al. (2011b) proposed an extended gravitational field's action in the metric approach given by equation (9.2) for any arbitrary dimensionless function $f(\chi)$ of the dimensionless Ricci scalar (9.4) and

$$L_M = \zeta r_g^{1/2} l_M^{1/2}, \tag{16.1}$$

is a length scale with r_g and l_M given by equation (9.1), with l_M the mass-length scale of the system defined by (Mendoza et al., 2011), and $a_0 = 1.2 \times 10^{-10}\ \mathrm{m/s^2}$ Milgrom's acceleration constant (see e.g. Famaey & McGaugh, 2011, and references therein) and ζ is a coupling constant of order one which has to be calibrated through astrophysical observations. This $f(\chi)$ theory was constructed by the inclusion of a_0 as a fundamental physical constant, which has been shown to be of astrophysical and cosmological relevance (see e.g. Bernal et al., 2011b,a; Carranza et al., 2013; Mendoza et al., 2011; Mendoza, 2012; Hernandez et al., 2010, 2012; Hernandez & Jiménez, 2012).

Following the description of Bernal et al. (2011b) the matter action takes its ordinary form (9.5), with $\mathcal{L}_m$ the Lagrangian matter density of the system. The null variation of the complete action, i.e. $\delta\left(S_f + S_m\right) = 0$, with respect to the metric $g_{\mu\nu}$ yields the field equations (9.6), where the Laplace-Beltrami operator has been written as $\Delta := \nabla^\mu \nabla_\mu$, the prime denotes the derivative with respect to the argument and the energy-momentum tensor $T_{\mu\nu}$ is defined through the standard relation $\delta S_m = -(1/2c) T_{\alpha\beta} \delta g^{\alpha\beta}$. Also, the dimensionless Ricci tensor is defined by equation (9.7). The trace of equations (9.6) is given by equation (9.8).

Bernal et al. (2011b) have shown that the function $f(\chi)$ must satisfy the following limits:

$$f(\chi) = \begin{cases} \chi, & \text{when } \chi \gg 1, \\ \chi^{3/2}, & \text{when } \chi \ll 1. \end{cases} \tag{16.2}$$

The limit $\chi \gg 1$ recovers Einstein's general relativity and the condition $\chi \ll 1$ yields a relativistic version of MOND. In this last regime it follows (see Bernal et al., 2011b) from the trace equation (9.8) that:

$$f'(\chi)\,\chi - 2f(\chi) \ll 3L_M^2\, \Delta f'(\chi), \tag{16.3}$$

to all orders of approximation and so the trace (9.8) can be written as:

$$3L_M^2 \Delta f'(\chi) = \frac{8\pi G L_M^2}{c^4} T. \tag{16.4}$$

Since we are interested in the field produced by a point mass M, then the right-hand side of equations (9.6), (9.8) and (16.4) are null and so the last relation in vacuum can be rewritten as:

$$\Delta f'(\chi) = 0. \tag{16.5}$$

As shown by Bernal et al. (2011b), $f(\chi) = \chi^{3/2}$ yields the correct MONDian non-relativistic limit. However, for the sake of generality we will assume in what follows that the function $f(\chi)$ is of power-law form (9.11). In this case, relation (16.5) is equivalent to

$$\Delta f'(R) = 0, \tag{16.6}$$

to all orders of approximation for a power-law function of the Ricci scalar

$$f(R) = R^b. \tag{16.7}$$

Substitution of the power-law function (9.11) in the null variations of the gravitational field's action (9.2) in vacuum means that

$$\delta S_f = -\frac{c^3}{16\pi G} L_M^{2(b-1)} \delta \int R^b \sqrt{-g}\, d^4x = 0, \tag{16.8}$$

and so

$$\delta \int R^b \sqrt{-g}\, d^4x = 0. \tag{16.9}$$

This equation gives the same field equations as the null variation of the action for a standard power-law metric $f(R)$ theory (16.7) in vacuum. With this in mind, we can follow the standard perturbation analysis for $f(R)$ restricted by the constraint equation (16.6) needed to yield the correct MOND-like limit. Since we are only interested in a power-law description of gravity far away from general relativity (cf. equation (16.2)), then in what follows we use the standard $f(R)$ field equations for vacuum as described by Capozziello & Faraoni (2011) for a power-law description of gravity given by equation (16.7) with $b = 3/2$, with the constraint (16.6). To follow their notation, we write the field equations (9.6) in vacuum as

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \mathcal{H}_{\mu\nu} = 0\,, \tag{16.10}$$

where the fourth-order terms are grouped into the following quantity:

$$\mathcal{H}_{\mu\nu} := -\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Delta\right) f'(R)\,. \tag{16.11}$$

The trace of equation (16.10) is thus given by

$$f'(R)R - 2f(R) + \mathcal{H} = 0\,, \tag{16.12}$$

with

$$\mathcal{H} := \mathcal{H}_{\mu\nu} g^{\mu\nu} = 3\Delta f'(R). \tag{16.13}$$

In what follows the sign convention used in the definition of the Riemann tensor becomes a relevant point in the subsequent discussion. As discussed in appendix A the solutions to the differential field equations of any $f(R)$ theory of gravity greatly depend on the signature chosen for Riemann's tensor, a bifurcation which does not appear in Einstein's general relativity. Throughout the work we select a particular branch of solutions given by the nowadays almost standard definition of Riemann's tensor in equation (0.3).

In dealing with some of the cumbersome algebraic manipulations that a perturbation to an $f(R)$ theory of gravity presents, we have used the Computer Algebra System (CAS) Maxima to facilitate the computations. The MEXICAS (Metric EXtended-gravity Incorporated through a Computer Algebraic System) code (Copyright of T. Bernal, S. Mendoza and L.A. Torres and licensed with a GNU Public License Version 3) we wrote for this is described in appendix B and can be downloaded from: http://www.mendozza.org/sergio/mexicas. Further development on the treatment of the field equations by the MEXICAS code is described in appendix C.

For the case of a static spherically symmetric spacetime (15.1) it follows that

$$\begin{aligned} \mathcal{H}_{\mu\nu} = &- f'' \left\{ R_{,\mu\nu} - \Gamma^{1}_{\mu\nu} R_{,r} - g_{\mu\nu} \left[\left(g^{11}_{\ ,r} \right.\right.\right. \\ &\left.\left.\left. + g^{11} \left(\ln \sqrt{-g} \right)_{,r} \right) R_{,r} + g^{11} R_{,rr} \right] \right\} \\ &- f''' \left\{ R_{,\mu} R_{,\nu} - g_{\mu\nu} g^{11} R_{,r}^{\ 2} \right\}, \end{aligned} \tag{16.14}$$

and

$$\begin{aligned}\mathcal{H} = 3f'' \left[\left(g^{11}_{,r} + g^{11} \left(\ln \sqrt{-g} \right)_{,r} \right) R_{,r} + g^{11} R_{,rr} \right] \\ + 3f''' g^{11} R_{,r}^{\,2}.\end{aligned} \tag{16.15}$$

We note that since

$$\sqrt{-g} = r^2 \sin\theta \left\{ 1 + \left[g^{(2)}_{00} - g^{(2)}_{11} \right] + \mathcal{O}(4) \right\}^{1/2}, \tag{16.16}$$

then, by using the fact that $\ln\left(\sqrt{-g}\right)_{,r} = \left(\sqrt{-g}\right)_{,r} / \sqrt{-g}$, it follows that

$$\ln\left(\sqrt{-g}\right)_{,r} = \frac{2}{r} + \frac{1}{2}\left[g^{(2)}_{00,r} - g^{(2)}_{11,r} \right] + \mathcal{O}(4)\,. \tag{16.17}$$

Since Ricci's scalar depends on the metric components and their derivatives up to the second order with respect to the coordinates, it follows it can only have a non-null second and higher perturbation orders, i.e.

$$R = R^{(2)} + R^{(4)} + \mathcal{O}(6)\,. \tag{16.18}$$

The fact that $R^{(0)} = 0$ is consistent with the flatness of spacetime assumption at the lowest zeroth order of perturbation. The expression for the second order component of Ricci's scalar from the metric components (15.5) is given by

$$R^{(2)} = -\frac{2}{r}\left[g^{(2)}_{11,r} + \frac{g^{(2)}_{11}}{r} \right] - g^{(2)}_{00,rr} - \frac{2}{r} g^{(2)}_{00,r}\,. \tag{16.19}$$

The global minus sign that appears on the right-hand side of equation (16.19) for Ricci's scalar $R^{(2)}$ at second perturbation order differs from that reported by Capozziello et al. (2007b); Capozziello & Stabile (2009). As mentioned above, and discussed in appendix A, this fact occurs due to the choice of signs in the definition of Riemann's tensor. The particular choice used throughout the Chapter is the one given by equation (0.3) and so, our solutions lie in a different branch as the one reported by those authors.

§17. Lowest order solution

Let us now calculate the order of the trace equation (16.12) using relations (16.7) and (16.18). On the one hand, the lowest order of the first two terms on the left-hand side of the trace equation is $\mathcal{O}(2b)$. On the other hand, direct inspection of the right-hand side of equation (16.15) results in the fact that the lowest order of $\mathcal{H}$ is $\mathcal{O}(2b-2)$. Indeed, the last term of the right-hand side of this equation is $\propto f'''g^{11}R_{,r}^{\,2}$ and so, to lowest order of perturbation of relations (15.6) and (16.18), this means that $\mathcal{H}$ contains terms of the form $R^{(2)b-3}R_{,r}^{(2)2}$ and so, $\mathcal{H}$ is of order $\mathcal{O}(2b-2)$. This analysis indicates that to lowest order the trace equation to consider is

$$\mathcal{H}^{(2b-2)} = 3\Delta f'^{(2b-2)}(R) = 0. \tag{17.1}$$

This result is consistent with relation (16.6) to lowest order of approximation and is in perfect agreement with the perturbative study performed by Bernal et al. (2011b). Note also that this is the only independent equation at this order.

Direct substitution of equations (16.7) and (16.18) into the last equation leads to

$$\begin{aligned}\mathcal{H}^{(2b-2)} &= 3b(b-1)R^{(2)b-2}g^{11(0)}\left[\left(\ln\sqrt{-g}\right)_{,r}^{(0)}R_{,r}^{(2)}\right.\\ &\left.+R_{,rr}^{(2)}\right] + 3b(b-1)(b-2)R^{(2)b-3}g^{11(0)}R_{,r}^{(2)2} = 0.\end{aligned} \tag{17.2}$$

Substitution of expressions (15.6) and (16.17) in the previous equation leads to the following differential equation for Ricci's scalar at order $\mathcal{O}(2)$:

$$R^{(2)}\left[\frac{2}{r}R_{,r}^{(2)} + R_{,rr}^{(2)}\right] + (b-2)R_{,r}^{(2)2} = 0\,, \tag{17.3}$$

which can be written in a more suitable form as

$$\left[\ln R_{,r}^{(2)}\right]_{,r} + (b-2)\left[\ln R^{(2)}\right]_{,r} = -\frac{2}{r}\,. \tag{17.4}$$

The solution of the previous equation is:

$$R^{(2)}(r) = \left[(b-1)\left(\frac{A}{r}+B\right)\right]^{1/(b-1)}, \tag{17.5}$$

where A and B are constants of integration.

Far away from the central mass, spacetime is flat and so Ricci's scalar must vanish at large distances from the origin. This means that the constant $B = 0$ and so

$$R^{(2)}(r) = \left[(b-1)\frac{A}{r}\right]^{1/(b-1)}. \tag{17.6}$$

As explained by Bernal et al. (2011b), the case $b = 3/2$ yields a MOND-like weak field limit and so, substituting $b = 3/2$ in relation (17.6) yields:

$$R^{(2)}(r) = \frac{\hat{R}}{r^2}, \tag{17.7}$$

where $\hat{R} := A^2/4$. This is exactly the same result as the one obtained by Bernal et al. (2011b). As these authors have shown, this result yields a MONDian-like behaviour for the gravitational field in the limit $r \gg l_M \gg r_g$. For this particular case, the lowest order of approximation of the theory is $\mathcal{O}(1)$, which has a higher relevance as compared to the order $\mathcal{O}(2)$ of standard general relativity for which $b = 1$. Using very general arguments, the authors also showed that the constant $\hat{R} \propto r_g/l_M$ is proportional to the square root of the mass of the central object. In order to calculate $\hat{R}$ from perturbation analysis we need to find the expressions for the metric at order $\mathcal{O}(2)$ of approximation.

§18. $f(\chi) = \chi^{3/2}$ metric components

Let us now solve the field equations at the next order $\mathcal{O}(2b)$ of approximation. At this order we expect the metric components $g^{(2)}_{00}$, $g^{(2)}_{11}$ and Ricci's scalar $R^{(4)}$ to play a relevant role in the description of the gravitational field. In fact, the field equations at this order are given by

$$bR^{(2)b-1}R^{(2)}_{\mu\nu} - \frac{1}{2}R^{(2)b}g^{(0)}_{\mu\nu} + \mathcal{H}^{(2b)}_{\mu\nu} = 0, \tag{18.1}$$

where

$$\mathcal{H}^{(2b)}_{\mu\nu} = -\left(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Delta\right) f'^{(2b)}(R). \tag{18.2}$$

The complete $\mathcal{H}^{(2b)}_{\mu\nu}$ from equation (16.14) is written in appendix C.

Now, from equation (16.6) it follows that the Laplace-Beltrami operator applied to $f'(R)$

must be zero at all perturbation orders. In particular $\Delta f'^{(2b)} = 0$. With this condition, the field equations (18.1) simplify greatly and can be written as

$$\begin{aligned} & bR^{(2)b-1}R^{(2)}_{\mu\nu} - \frac{1}{2}R^{(2)b}g^{(0)}_{\mu\nu} - b(b-1)\Big\{ R^{(2)b-2}\left[R^{(4)}_{\mu\nu} - \Gamma^{1(0)}_{\mu\nu}R^{(4)}_{,r} - \Gamma^{1(2)}_{\mu\nu}R^{(2)}_{,r}\right] \\ & +(b-2)R^{(2)b-3}R^{(4)}\left[R^{(2)}_{\mu\nu} - \Gamma^{1(0)}_{\mu\nu}R^{(2)}_{,r}\right]\Big\} - b(b-1)(b-2)\Big[2R^{(2)b-3}R^{(2)}_{,\mu}R^{(4)}_{,\nu} \\ & +(b-3)R^{(2)b-4}R^{(4)}R^{(2)}_{,\mu}R^{(2)}_{,\nu}\Big] = 0. \end{aligned} \tag{18.3}$$

Direct substitution of the following Christoffel symbols

$$\Gamma^{1(0)}_{00} = 0\,, \qquad \Gamma^{1(2)}_{00} = -\frac{1}{2}g^{11(0)}g^{(2)}_{00,r}\,, \tag{18.4}$$

and relations (15.5) and (15.6) in the 00 component of equation (18.3) leads to

$$bR^{(2)b-1}R^{(2)}_{00} - \frac{1}{2}R^{(2)b} + \frac{1}{2}b(b-1)g^{(2)}_{00,r}R^{(2)b-2}R^{(2)}_{,r} = 0. \tag{18.5}$$

If we now substitute $b = 3/2$, expression (17.7) and the value of Ricci's tensor at $\mathcal{O}(2)$ of approximation:

$$R^{(2)}_{00} = -\frac{rg^{(2)}_{00,rr} + 2g^{(2)}_{00,r}}{2r}, \tag{18.6}$$

into equation (18.5), we obtain the following differential equation for $g^{(2)}_{00}$:

$$r^2g^{(2)}_{00,rr} + 3rg^{(2)}_{00,r} + \frac{2\hat{R}}{3} = 0, \tag{18.7}$$

and so

$$g^{(2)}_{00}(r) = -\frac{\hat{R}}{3}\ln\left(\frac{r}{r_\star}\right) + \frac{k_1}{r^2}, \tag{18.8}$$

where k_1 and $r_\star$ are constants of integration. By substitution of this result in equation (16.19) and using equation (17.7) we get the following differential equation for $g^{(2)}_{11}$:

$$rg^{(2)}_{11,r} + g^{(2)}_{11} + \frac{k_1}{r^2} + \frac{\hat{R}}{3} = 0, \tag{18.9}$$

with solution:

$$g_{11}^{(2)}(r) = \frac{k_1}{r^2} + \frac{k_2}{r} - \frac{\widehat{R}}{3}, \tag{18.10}$$

where k_2 is a constant of integration.

§19. Metric coefficients from astronomical observations

In this section we derive the constraints that the well established astrophysical phenomenology of asymptotically flat galactic rotation curves satisfying the Tully-Fisher relation, and the cumulative gravitational lensing observations for elliptical and spiral galaxies and galaxy groups, imply for the metric coefficients for static, spherically symmetric spacetimes for any metric theory of gravity where dark matter is not required.

To begin with, let us take the radial component (15.3) of the geodesic equations (15.2) in the weakest limit of the theory. In this limit, the rotation curve for test particles bound to a circular orbit about a mass M with circular velocity $v(r)$ given by equation (15.4) is

$$\frac{v^2(r)}{c^2 r} = \frac{1}{2} g^{11} g_{00,r}. \tag{19.1}$$

Except for the inner regions of spiral galaxies, $v(r)$ can be well approximated by a constant which scales with the fourth root of the total baryonic mass M_b of the spiral galaxy in question, as described by the Tully-Fisher empirical relation (see e.g. Milgrom & Sanders, 2008; Famaey & McGaugh, 2011)

$$v = (G M_b a_0)^{1/4}. \tag{19.2}$$

In fact, it is from numerous observations of galactic rotation curves and total baryonic mass estimates, that the constant a_0 has been calibrated (see e.g. Famaey & McGaugh, 2011, and references therein).

We now substitute equations (15.5) and (15.6) to order $\mathcal{O}(2)$ of approximation and the relation (19.2) in equation (19.1) to obtain the following differential equation for $g_{00}^{(2)}$:

$$-g_{00,r}^{(2)} = \frac{2}{r}\left(\frac{v}{c}\right)^2 = \frac{2(G M_b a_0)^{1/2}}{c^2 r}, \tag{19.3}$$

having as solution

$$\begin{aligned} -g_{00}^{(2)}(r) &= 2\left(\frac{v}{c}\right)^2 \ln\left(\frac{r}{r_\star}\right) \\ &= \frac{2(GM_b a_0)^{1/2}}{c^2} \ln\left(\frac{r}{r_\star}\right) = \frac{2r_g}{l_M} \ln\left(\frac{r}{r_\star}\right), \end{aligned} \tag{19.4}$$

where $r_\star$ is a scale radius which, from the point of view only of the flat rotation curves of galaxies and the Tully-Fisher relation, remains arbitrary. We therefore see that a necessary and sufficient condition in any metric relativistic theory of gravity, where all observational constraints of galactic rotation curves are satisfied without invoking dark matter, is that $g_{00}^{(2)}$ must satisfy the previous empirically derived relation.

Comparing the theoretical metric coefficient $g_{00}^{(2)}$ given by (18.8) (obtained from perturbation theory for $f(\chi) = \chi^{3/2}$) and the empirical one (19.4) (obtained from the phenomenology of flat rotation curves and the Tully-Fisher relation), give the following values for the integration constants needed in equation (18.8):

$$k_1 = 0, \qquad \hat{R} = 6r_g/l_M. \tag{19.5}$$

In this case, the gravitational potential ϕ from equation (11.1) takes the form:

$$\phi = -v^2 \ln\left(\frac{r}{r_\star}\right) = -\left(GM_b a_0\right)^{1/2} \ln\left(\frac{r}{r_\star}\right), \tag{19.6}$$

which yields a radial MONDian acceleration:

$$|a| = |\nabla\phi| = \frac{\left(GM_b a_0\right)^{1/2}}{r}, \tag{19.7}$$

Thus, in the $v/c \ll 1$ limit, the $f(\chi) = \chi^{3/2}$ presented is seen to agree to the observed phenomenology of the observed galactic rotation curves in the absence of dark matter, as already shown by Bernal et al. (2011b).

The g_{11} metric coefficient will be obtained from gravitational lensing phenomenology. We begin from the general deviation angle equation (see e.g. Schneider et al., 1992; Weinberg, 1972)

$$\beta = 2\int_{r_i}^{\infty} \frac{\left[-g_{00}(r) g_{11}(r)\right]^{1/2} dr}{r\left[(r/r_i)^2 g_{00}(r_i) - g_{00}(r)\right]^{1/2}} - \pi, \tag{19.8}$$

where r_i is the closest approach to the central mass M, and it is related to the impact parameter b through the relation $r_i^2 = b^2 g_{00}(r_i)$.

Over the last few years it has become clear that the complete phenomenology of gravitational lensing, at the level of extensive massive elliptical galaxies (see e.g. Gavazzi et al., 2007; Koopmans et al., 2006; Barnabè et al., 2011), galaxy groups (see e.g. More et al., 2012), clusters of galaxies (see e.g. Newman et al., 2009; Limousin et al., 2007) and more recently spiral galaxies (see e.g. Dutton et al., 2011; Suyu et al., 2012) can be accurately modelled using total matter distributions having isothermal profiles, when treating the problem from the point of view of Einstein's general relativity. All these observations show that the dark matter halos needed to explain gravitational lensing under Einstein's general relativity obey the same Tully-Fisher scaling with total baryonic mass as the ones needed to explain the observed rotation curves of spiral galaxies. This means that for a given total baryonic mass, spiral and elliptical galaxies, as well as clusters and groups of galaxies require dark matter halos having the same physical properties to explain the observations; from kinematics of rotation curves in the former case to gravitational lensing in the latter one (Dutton et al., 2011; Suyu et al., 2012). Under Einstein's general relativity the majority of these isothermal matter distribution, particularly at large radii, must be composed of a hypothetical dark matter.

For a static spherically symmetric total matter distribution M_T, since assuming the validity of Einstein's general relativity Schwarzschild's metric holds, and therefore $g_{00S} = -1/g_{11S}$, we get:

$$g_{00S} = 1 - \frac{2r_g}{r} = 1 - \frac{2GM_T(r)}{c^2 r} = 1 - 2\left(\frac{v}{c}\right)^2. \tag{19.9}$$

The subscript S identifies the coefficients of the Schwarzschild metric, and $M_T(r) = v^2 r/G$ refers to the hypothetical isothermal total matter distribution (cf. Binney & Tremaine, 2008) needed to explain the observed lensing, when assuming general relativity. From this it follows that the dark matter hypothesis provides a self-consistent interpretation of observed phenomenology: the same dark matter halos, which are required to explain the observed rotation curves, have been solved for by analysing extensive lensing observations.

From equation (19.9) it follows that for isothermal total matter halos under Einstein's general relativity, the metric coefficient g_{00S} does not depend on the radial coordinate. We can see this by using the empirical Tully-Fisher relation (19.2) between the velocity and the total baryonic mass in the last identity above. Thus, the coefficient (19.9) can then be taken outside of the integral (19.8) of the deviation angle, where for the Schwarzschild metric and isothermal total matter halos we now obtain

$$\beta = \frac{2}{[1-2(v/c)^2]^{1/2}} \int_{r_i}^{\infty} \frac{dr}{r\left[(r/r_i)^2 - 1\right]^{1/2}} - \pi. \tag{19.10}$$

The above radial integral yields $\pi/2$ and we obtain the observed bending angle as

$$\beta = \frac{\pi}{[1-2(v/c)^2]^{1/2}} - \pi = \frac{\pi}{[1-2(GM_b a_0)^{1/2}/c^2]^{1/2}} - \pi. \tag{19.11}$$

We see that the well established empirical result of lensing observations yielding isothermal total dark matter halos under the standard theory is strictly the observation of constant bending angles which do not depend on the impact parameter, scaling with the observed baryonic total masses as indicated above.

Now, since $(v/c)^2$ is of order $\mathcal{O}(2)$ we can write equation (19.11) as

$$\beta = \pi\left(\frac{v}{c}\right)^2 = \pi\frac{(GM_b a_0)^{1/2}}{c^2} = \pi\frac{r_g}{l_M}. \tag{19.12}$$

The above equation summarises all empirical results of gravitational lensing at galactic and galaxy group scales: the bending angle does not depend on the impact parameter and scales with the square root of the total baryonic mass. This last equation gives a clear illustration of the link between the dynamics and the spacetime curvature effects induced by the presence of an observed baryonic mass.

We can now use the result of equation (19.12) to constrain the metric coefficient g_{11} for any metric theory of gravity, seeking an accurate description of the observed gravitational lensing phenomena without the introduction of any hypothetical dark matter. To do this, let us return to the general lensing equation (19.8), and ask that the results obtained under the Schwarzschild metric with isothermal total matter halos match those under any metric theory of gravity, at all closest approaches to the lens and for any total baryonic masses:

$$
\begin{aligned}
\frac{1}{2}(\beta+\pi) &= \left[1+\left(\frac{v}{c}\right)^2\right]\int_{r_i}^{\infty}\frac{dr}{r\left[(r/r_i)^2-1\right]^{1/2}} \\
&= \int_{r_i}^{\infty}\frac{\left[-g_{00}(r)g_{11}(r)\right]^{1/2}dr}{r\left[(r/r_i)^2 g_{00}(r_i)-g_{00}(r)\right]^{1/2}},
\end{aligned}
\tag{19.13}
$$

at $O(2)$ of approximation from equations (19.8) and (19.10). Let us rearrange integral (19.13) in such a way that:

$$
\int_{r_i}^{\infty}\left\{\left[1+\left(\frac{v}{c}\right)^2\right]\frac{1}{r\left[(r/r_i)^2-1\right]^{1/2}}-\frac{\left[-g_{00}(r)g_{11}(r)\right]^{1/2}}{r\left[(r/r_i)^2 g_{00}(r_i)-g_{00}(r)\right]^{1/2}}\right\}dr=0. \tag{19.14}
$$

Since the result must hold for all impact parameters, the integrand of the above equation must be equal to zero and so,

$$
\left[1+\left(\frac{v}{c}\right)^2\right]^2\frac{1}{(r/r_i)^2-1}=\frac{-g_{00}(r)g_{11}(r)}{(r/r_i)^2 g_{00}(r_i)-g_{00}(r)}, \tag{19.15}
$$

Taking again $v/c \ll 1$ it follows that that the metric coefficient g_{11} is given by:

$$
g_{11}(r)=-\left[1+2\left(\frac{v}{c}\right)^2\right]\frac{(r/r_i)^2\left[g_{00}(r_i)/g_{00}(r)\right]-1}{(r/r_i)^2-1}. \tag{19.16}
$$

From a mathematical point of view, since the contribution to the integral in the lensing equation (19.8) is fully dominated by the region $r \approx r_i$, and given the very mild radial dependence of the empirical g_{00} term, we can take $g_{00}(r_i) \approx g_{00}(r)$ in the above equation to yield:

$$
\begin{aligned}
g_{11}(r) &= -1-2\left(\frac{v}{c}\right)^2 \\
&= -1-\frac{2(GM_b a_0)^{1/2}}{c^2} = -1-2\frac{r_g}{l_M}.
\end{aligned}
\tag{19.17}
$$

Thus, any metric theory of gravity where g_{11} matches the above expression in the regime where gravitational lenses are observed will accurately reproduce all the observed lensing phenomenology, with the total baryonic mass of the object in question (galaxies or group of galaxies), and no hypothetical dark matter is assumed to exist. Equations (19.4) and (19.17) give empirical mathematical relations for the metric coefficients at perturbation order $\mathcal{O}(2)$ which reproduce all observed rotation velocity and gravitational lensing phenomenology, without the inclusion of any dark matter component.

Notice that the mass dependence of the second term on the right-hand side in expression (19.17) for the metric coefficient g_{11} is the same as the factor in expression (19.4) for g_{00}. This last was obtained for a rigorously flat rotation curve in accordance with the Tully-Fisher relation. This shows that the ratio r_g/l_M of the two important characteristic lengths of the extended metric theory of gravity proposed by Bernal et al. (2011b) is the determinant dimensionless measure of deviations from flat spacetime at galactic scales, exactly as expected from the dimensional analysis in Hernandez (2012).

The metric coefficient g_{11} in equation (19.17) can be directly compared to the results for the $f(\chi) = \chi^{3/2}$ metric theory of Bernal et al. (2011b) obtained in equation (18.10) with the inclusion of the results of equation (19.5). This means that the choice of the integration constant

$$k_2 = 0, \tag{19.18}$$

makes these expressions for the metric component g_{11} identical.

Use of the mathematical approximation $A^x \approx 1 + x \ln A$ to write the following expressions for the full empirical metric coefficients gives:

$$g_{00} \approx 1 + (2r_g/l_M) \ln (r_\star/r) \approx (r_\star/r)^{2r_g/l_M}, \tag{19.19}$$

$$g_{11} \approx -1 - (2r_g/l_M) \approx e^{2r_g/l_M}. \tag{19.20}$$

We note that all the approximations used in this section introduce an error several orders of magnitude smaller than the intrinsic observational uncertainties in the empirical relations used. Therefore, all of the expressions given can be considered as strictly equivalent in regards to the accurate modelling of astrophysical rotation curves and gravitational lensing data.

§20. Discussion

Through the use of the weak field limit of the metric $f(\chi) = \chi^{3/2}$ theory of gravity constructed by Bernal et al. (2011b), we have shown that it is possible to explain both the dynamics of massive particles and the deflection of light by observed astronomical systems such as elliptical galaxies, spiral galaxies and groups of galaxies. Recently, the same metric theory of gravity was shown to be coherent also with the expansion dynamics of the observed universe (Carranza et al., 2013; Mendoza, 2012). This is an expected result from a theory of gravity constructed through astronomical observations: it must be coherent at all scales. The regime of Einstein's general relativity is by no means violated, since the applications developed in this work ($r \gg l_M$) lie far away from the mass and length scales associated to the ones of Einstein's general relativity ($r \ll l_M$) (see e.g. Mendoza, 2012).

The results of this work were constructed using a static spherically symmetric metric with the time and radial components perturbed up to order $\mathcal{O}(2)$ of approximation. This work generalises the one of Bernal et al. (2011b) in which the radial metric component was assumed up to order $\mathcal{O}(0)$ only and so, information on the choice of signature of the Riemann tensor was lost (see appendix A). Such information is very important while working with fourth order metric theories of gravity.

We mention again the tremendous importance of a correct choice for the signature of the Riemann tensor as described in appendix A. The choice (0.3), and only that choice, used in this work yields results in agreement with astronomical observations. In other words, astronomical observations fix the correct (and unique) choice of signature for Riemann's tensor. This is an important result, since otherwise solutions from the other branch appear which are not in accordance with astronomical observations.

Table IV.1 summarises our main results. It is important to note that the empirical values of the metric components $g_{00}^{(2)}$ and $g_{11}^{(2)}$ do not depend on any gravitational theory and as such, they represent functions that any successful theory of gravity (such as the one used in this Chapter) needs to match.

An important fact arises from the usage of the $f(\chi)$ metric theory of gravity and not the $f(R)$ formalism. Although closely related to each other, the correct dimensional approach $f(\chi)$ introduces mass and length scales that, as shown by Bernal et al. (2011b), need to be incorporated into the gravitational field action. Although the field equations in vacuum for both $f(\chi)$ and $f(R)$ under a power-law representation yield the same field equations (since the mass M generating the gravitational field is a constant), $f(R)$ gravity is not capable of reproducing the crucial lensing observations as it lacks a crucial constraint equation. The gravitational theory $f(\chi) = \chi^{3/2}$ is able to do so since under this approach the

	$g_{00}^{(2)}$	$g_{11}^{(2)}$
Observations	$-\frac{2r_g}{l_M}\ln\left(\frac{r}{r_\star}\right)$ (Tully-Fisher)	$-\frac{2r_g}{l_M}$ (lensing)
Theory $f(\chi)=\chi^{3/2}$	$-\frac{\hat{R}}{3}\ln\left(\frac{r}{r_\star}\right)+\frac{k_1}{r^2}$ $\hat{R}=6r_g/l_M \quad k_1=0$	$\frac{k_1}{r^2}+\frac{k_2}{r}-\frac{\hat{R}}{3}$ $\hat{R}=6r_g/l_M \; k_2=0$

Tabla IV.1: The table shows the results obtained for the metric components $g_{00}^{(2)}$ and $g_{11}^{(2)}$ for a static spherical symmetric spacetime in scales of galaxies and galaxy groups obtained empirically from astronomical observations of these systems and the ones predicted by the metric $f(\chi)=\chi^{3/2}$ theory of gravity of Bernal et al. (2011b). A good metric theory of gravity must be such that it converges to the inferred values presented in the table. The theory $f(\chi)=\chi^{3/2}$ is in perfect agreement with the observed metric components. The dimensionless ratio formed by the quotient of the gravitational radius r_g to the mass-length scale l_M (see equation (9.1)) is the determinant dimensionless quantity of the problem. Since the metric components determine the "gravitational potential" of the system, the length $r_\star$ is undetermined. However, since the natural length scale of the system is l_M one can always asumme $r_\star = l_M$, which also ensures no sign change in the potential in equation (19.6) over the domain of applicability $r > l_M$.

correct limit where MONDian-like effects are expected yield the constraint equation (16.5) or (16.6). Notice however that both $f(R)$ and $f(\chi)$ with the appropriate choice of Riemann's tensor (0.3) are able to reproduce the flat rotation curves of galaxies and the Tully-Fisher relation.

In an effort to generalise and look for a fundamental basis to an $f(\chi)$ theory of gravity, Carranza et al. (2013) and Mendoza (2012) have shown that these metric theories are equivalent to the the $F(R,T)$ construction of Harko et al. (2011). These authors have also shown that the particular theory $f(\chi) = \chi^{3/2}$ is in excellent agreement with cosmological observations of SNIa.

An $f(\chi)$ theory of gravity satisfying the limits of equation (16.2) implies that gravity is no longer scale-invariant. In fact, precise gravity tests have been performed only at strong regimes of Einstein's gravity, where $\chi \gg 1$, and so the involved accelerations of test particles are such that $a \gg a_0$ (see e.g. Will, 2006). In exactly the opposite regime, where $\chi \ll 1$, where the involved accelerations of test particles are such that $a \lesssim a_0$, gravity differs from Einstein's general relativity. The traditional approach of assuming Einstein's general relativity to be valid at all scales means that unknown dark entities are needed to explain various astrophysical observations. This work heavily reinforces many others (Bernal et al., 2011b,a; Carranza et al., 2013; Mendoza et al., 2011; Mendoza, 2012; Hernandez et al., 2010, 2012; Hernandez & Jiménez, 2012) that show how astrophysical and cosmological observations can be accounted for without assuming the existence of dark entities and extending gravity so as to be non scale-invariant.

Capítulo V

Escala de MOND como cantidad fundamental

En este Capítulo muestro la posibilidad de que la constante de aceleración de MOND, a_0, sea una cantidad fundamental relacionada con algunas constantes universales y no únicamente una constante fenomenológica. Esto a partir de que algunas relaciones de escalamiento cósmico–cuánticas indican que la escala de aceleración de MOND podría ser una cantidad fundamental que determina las estructuras autogravitantes, desde estrellas y cúmulos globulares hasta cúmulos de galaxias y todo el universo observado. Se discuten además las relaciones de 'coincidencia' a partir de la condición de cuantización de Dirac que determina las masas de los agujeros negros primordiales.

Es a través de este trabajo que se da una justificación teórica para utilizar la escala de aceleración a_0 como fundamental en la teoría de gravitación, hecho que hemos asumido en los Capítulos anteriores para construir nuestra teoría de gravedad extendida, tanto no–relativista como relativista.

* * *

MOND's scale as a fundamental quantity

Some quantum–cosmic scaling relations indicate that the MOND acceleration parameter a_0 could be a fundamental quantity ruling the self-gravitating structures, ranging from stars and globular clusters up to superclusters of galaxies and the whole observed universe. We discuss such coincidence relations starting from the Dirac quantization condition ruling the masses of primordial black holes.

§21. Introduction

The observed scales of astrophysical self-gravitating systems, like galaxies, galaxy clusters and the universe itself, constitute a puzzle of modern physics. From a fundamental point of view, it is expected that all bounded self-gravitating structures have a common origin related to the cosmic microwave background radiation. Such structures should bring the signature of primordial quantum fluctuations which should have determined the scales of systems that we observe today.

The main goal of studies dealing with this issue is to frame the large scale structure into some unifying theory in which all the today observed structures can be treated under the same fundamental standard. In this sense, the remnants of the primordial epochs, probed by the cosmological observations, are the ideal test to constrain such theories.

Looking for a fundamental theory connecting the microscopic to the macroscopic scales, many "coincidences" have been identified among cosmic and fundamental parameters, like the cosmological constant, the gravitational constant, the speed of light and the Planck constant. The so-called Large Numbers Coincidence (LNC) (Eddington, 1931), for example, refers to some fundamental relations to pure numbers of order 10^{40}. Eddington and Dirac hypothesized that such coincidences could be dynamically generated by the cosmological parameters related to quantum ones. Furthermore, this coincidence seems to to occur at the same epoch in which other coincidences related to cosmic parameters occur. This fact suggests some underlying physical law that should be fully understood (Funkhouser, 2006).

In addition, there is the issue involving the so-called "dark" components, i.e. dark matter and dark energy, being approximately the 95 % of the total mass-energy content of the universe. In principle, the dark matter was postulated to explain the observed rotation curves of spiral galaxies (Zwicky, 1937); then it was necessary to explain the mass to light ratios of galaxies and galaxy clusters, the gravitational lensing, the structure formation and most of astrophysical phenomena and structures that escape the standard description by Einstein (Newton) gravity and luminous matter (Binney & Tremaine, 2008). On the other hand, the dark energy has been postulated in order to explain the today observed accelerated expansion rate of the Hubble cosmic fluid, firstly deduced by the supernovae Ia observations (Riess et al., 1998; Perlmutter et al., 1999).

As alternatives to the dark matter and energy problems, several theories modifying the gravity, instead of the inclusion of unknown components, have been proposed, in both non-relativistic and relativistic regimes, indicating a failure in the Newtonian and general relativistic theories of gravity at large scales (see e.g. Capozziello & Faraoni, 2011; Mendoza et al., 2011).

In the non-relativistic limit, Milgrom introduced the Modified Newtonian Dynamics (MOND) (Milgrom, 1983b, 2008, 2009a) to explain the observed rotation curves of spiral galaxies, through the inclusion of a phenomenological acceleration scale a_0 such that, for accelerations much greater than a_0, the dynamics is Newtonian and for accelerations much less than a_0, the dynamics is modified and reduces to constant circular velocities at large radii.

The MOND frame has been proved to be successful on many astrophysical situations, but difficult on others (see e.g. Mendoza et al., 2011, and references therein). Despite these facts, the key feature of this modification, is the introduction of an acceleration scale a_0 into the gravitational interaction (Binney & Tremaine, 2008). Concerning a_0, some of the mentioned coincidences can be considered involving the Hubble constant, the cosmological constants, and the the nucleon size (Funkhouser, 2006). Quantum mechanics and quantum field theory play a main role in understanding such coincidences showing that a_0 could be actually considered among the number of fundamental parameters of physics.

In this Chapter, we will face this problem. Starting from a quantum field theory approach, a quantum relation for primordial black holes has been proposed whose validity extends from the Planck scales to the today observed astrophysical structures (Capozziello et al., 2010a). It can be proven that such a quantum relation is a scaling one containing, in principle, the signature of astrophysical structures starting from their basic constituents, the nucleons (Capozziello et al., 2011). Specifically, it reproduces the so-called Eddington-

Weinberg relation (Weinberg, 1972), as well as the phenomenological statistical hypothesis for self-gravitating systems (Capozziello & Funkhouser, 2009a,b), where the characteristic sizes of astrophysical structures can be recovered assuming that gravity is the overall interaction assembling systems from fundamental microscopic constituents.

In this Chapter, we show how, starting from fundamental scaling relations involving cosmological coincidences, MOND's acceleration emerges as a characteristic scale for self-gravitating astrophysical systems. We take into account only characteristic masses, radii and the baryonic components. It is worth noting that the acceleration scale is addressed without assuming any dark component.

The Chapter is organized as follows. In Section §22, we recall the scaling hypothesis working, in principle, for all self-gravitating structures, from Planck's scales to the universe itself. From the scaling relations, a characteristic acceleration is deduced and identified as a_0 in terms of fundamental quantities.

In Section §23, the signature of scaling relations for several self-gravitating systems is discussed as a consistency check to obtain the characteristic sizes of the structures from their characteristic baryonic masses.

Section §24 is devoted to a brief summary on how the acceleration scale a_0 modifies the Newtonian dynamics. In Section §25, we discuss the coincidence relations showing that a_0 can be considered among the fundamental parameters of scaling relations.

Discussion and conclusions are reported in Section §26.

§22. Scaling hypothesis for astrophysical systems

Assuming the Dirac quantization condition (Dirac, 1931) and a quantum field theory result relating the electric and magnetic charges of primordial black holes to their masses M, the following quantization relation can be obtained (Capozziello et al., 2010a):

$$GM^2 = n\hbar c\,, \tag{22.1}$$

where G is the gravitational constant, n is the quantization number and $\hbar$ is the reduced Planck constant. For $n = 1$, the lowest mass allowed for a quantum black hole is the Planck mass.

Moreover, it can be shown that the above relation is valid for self-gravitating astrophysical structure, from globular clusters to the observed universe itself (Capozziello et al., 2010a), and numerical agreement has been found with the statistical hypothesis (based on phenomenological considerations) applied to self-gravitating systems, defined as bound states with a very large number N_{as} of constituents (Capozziello & Funkhouser, 2009a,b). This connection can be equivalently obtained either by considering the protons as the elementary constituents or, as usual in astrophysics, by considering stars as the granular components of galaxies, and the galaxies the granular components of galaxy clusters and superclusters ("granular approximation").

In this context, the working hypothesis is that the total action of the astrophysical self-gravitating system, A_{as}, can be achieved from the Planck constant as

$$A_{as} \approx \hbar N_{as}^{3/2} . \tag{22.2}$$

This means that the total action is given by $A_{as} = n_{as}\hbar$, where n_{as} is the value of the quantum level number appearing in equation (22.1). By comparing this number with the corresponding value $N_{as}^{3/2}$ from equation (22.2), a numerical scaling with characteristic sizes of globular clusters, galaxies, galaxy clusters and the observed universe can be achieved (Capozziello et al., 2000).

However, it is interesting to note that the quantization relation (22.1) scales "analytically" for any astrophysical system down to the Planck mass. In fact, the quantum relation can be rewritten in a more suitable form as (Capozziello et al., 2011):

$$GM_{as}^2 = \left(\frac{N_{as}}{N_{BH}}\right)^2 \hbar c \, , \tag{22.3}$$

where N_{BH} is the number of protons in primordial black holes.

Writing the mass of the self-gravitating system considering its number of protons as

$$M_{as} \approx N_{as}\, m_p \, , \tag{22.4}$$

one gets simply back $GM_{Pl}^2 = \hbar c$, and the scaling hypothesis (22.2) is immediately verified (Capozziello et al., 2010a).

Another interesting outcome obtained by the quantization relation (22.1), can be achieved by using the relation

$$\frac{GM_{as}}{R_{as}^2} \equiv 2\pi a_\star\,, \tag{22.5}$$

where R_{as} is the characteristic radius of the astrophysical structure and the acceleration $a_\star$ can be shown to be a universal constant.

In fact, it is possible to derive the size of the astrophysical structure in terms of the number of its fundamental constituents (protons), N_{as}, and the fundamental scale of the proton, the Compton wavelength λ_p. It is

$$R_{as} \approx 10\sqrt{N_{as}}\,\lambda_p\,. \tag{22.6}$$

Such a result reproduces the statistical hypothesis relation proposed by Capozziello & Funkhouser (2009a,b) with a correcting factor of order 10 coming from statistical uncertainties (Capozziello et al., 2011).

By using the relation (22.4) for the mass M_{as} and the size R_{as} from equation (22.6), we can rewrite the scaling relation (22.5) as

$$\frac{GM_{as}}{R_{as}^2} = \frac{G\,m_p}{(10\lambda_p)^2} \equiv 2\pi a_\star\,. \tag{22.7}$$

Also, by considering the case of the observed universe and using the Raychaudhuri equation (Raychaudhuri, 1979), the last relation can be written as

$$\frac{GM_u}{R_u^2} = \frac{c^2}{R_u} = 2\pi a_\star\,. \tag{22.8}$$

It is interesting to notice that specializing equation (22.6) to the radius of the universe and using equations (22.4) and (22.8), the Eddington-Weinberg relation for the radius of the universe is easily obtained with the correcting factor 10:

$$h = \frac{1}{10}\sqrt{GR_u\,m_p^3}\,. \tag{22.9}$$

In addition, from equation (22.7), the value of the acceleration $2\pi a_\star$ is found to be

$$2\pi a_\star = \frac{G\,m_p^3\,c^2}{(10h)^2}\,, \tag{22.10}$$

which is a constant. Evaluating $a_\star$ in the last equation, we find the value $a_\star \approx 1.01 \times 10^{-10}\,\mathrm{m/s^2}$.

It is important to stress that the equation for the constant acceleration is valid for any astrophysical system where equation (22.6) gives its characteristic radius and equation (22.4) gives its mean mass.

As we will see, the derived value for $a_\star$ is very close to the phenomenological value obtained for the constant acceleration scale introduced in the MOND theory, that is $a_\star \approx a_0$. However the value of $a_\star$ has been obtained here combining the fundamental quantities G, h, c and m_p. The meaning of this result is that the acceleration a_0 could be a fundamental quantity as we want to show in the discussion below.

§23. Astrophysical systems from scaling relations

A self-gravitating astrophysical system is a gravitationally bounded assembly of stars and gas, and other point masses that can be neglected with respect to the main stellar and gas content. These systems vary over more than fourteen orders of magnitude in size and mass, from individual stars to star clusters containing 10^2 to 10^6 stars, through galaxies containing 10^5 to 10^{12} stars, to vast clusters containing thousands of galaxies (Binney & Tremaine, 2008).

The properties of these self-gravitating structures can be deduced by assuming them to be relaxed and virialized systems where gravity is the only overall interaction (Binney & Tremaine, 2008). First, we define globular clusters, galaxies and clusters of galaxies, considering their typical lengths and masses. For the characteristic mass we will only take the baryonic one and, as the boundary ("size") of these systems is not univocally defined, we will suppose a characteristic length for the baryonic mass gravitationally bounded to the system. It is important to remark that we are not assuming a dark matter component for the systems. For example, for a spiral galaxy, we take a typical radius of ∼10 kpc, which is the typical radius for the visible disk, not ∼100 kpc for the hypothetical dark matter halo.

Now, a star is not a purely self-gravitating system since, inside it, gravity is balanced by the pressure due to electromagnetic and nuclear interactions. However, we can find a characteristic gravitational length for the gravitating objects around a star (as the size of a "planetary system" around it). Of course, this depends on the environment in which the star is embedded, but we can assume a typical one, as the solar neighborhood (Binney & Tremaine, 2008).

To obtain the characteristic interaction lengths for the systems, we are using the granular approximation, considering the stars as the granular constituents of the astrophysical structures, whose main constituents are the protons.

The globular clusters are very massive stellar systems, containing up to 10^6 stars ($M_{gc} \sim 10^6\, M_\odot$). The typical radii are of the order ~10 pc. These systems are assumed completely virialized due to collisional interactions between stars (Binney & Tremaine, 2008).

A galaxy is a collisionless gravitating system, with masses ranging from $M_{gal} \sim 10^8 \div 10^9\, M_\odot$, for dwarf galaxies, and $M_{gal} \sim 10^{10} \div 10^{12}\, M_\odot$, for giant galaxies. The sizes are not well defined since there is no effective boundary for the systems. Astronomers define operative characteristic sizes as the *effective radius,* R_e, containing half of the total luminosity or the *tidal radius,* R_t, defined as the distance from the center where the density drops to zero (Binney & Tremaine, 2008). Assuming as a typical interaction length $R_{gal} \sim 1 \div 10$ kpc is quite reasonable for systems ranging from dwarf to giant galaxies.

Groups of galaxies are systems containing $10 \div 20$ galaxies, but they are not considered self-gravitating systems because they are always part of more extended self-gravitating structures (clusters of galaxies). For example, our Local Group is a part of the Virgo Cluster.

The clusters of galaxies are the largest self-gravitating structures in the universe. The biggest clusters have masses $\sim 10^{15}\, M_\odot$ within ~1 Mpc from their centers.

The superclusters of galaxies are not considered effective self-gravitating systems because of their large sizes. For this reason, they are considered as expanding with the expansion rate of the universe. It is supposed that all the clusters of galaxies are part of a larger supercluster.

Finally, the mean number of atoms (baryons) in the *observable universe* is supposed to be $\sim 10^{80}$, corresponding to a mass of the order $M_u \sim 10^{23}\, M_\odot$. Furthermore, the estimated size for the observed universe vary between $R_u \sim 10^{25} - 10^{27}$ m.

Let us now show how the scaling relation (22.6) holds for astrophysical structures. To this purpose, we write the mean radius R_{as}, coming from equation (22.6), in terms of the mean baryonic mass M_{as}, given by equation (22.4) for the given astrophysical structure, as

$$R_{as} = 10 \left(\frac{M_{as}}{m_p^3} \right)^{1/2} \frac{h}{c} \,. \tag{23.1}$$

From this relation, we obtain, for a globular cluster with mean mass $M_{gc} \sim 10^6 M_\odot$, the characteristic radius $R_{gc} \sim 15$pc. For a typical giant galaxy with mean mass $M_{gal} \sim 10^{11}\, M_\odot$, the corresponding characteristic radius is $R_{gal} \sim 5$ kpc. For a cluster of galaxies with mean mass $M_{cg} \sim 5 \times 10^{14}\, M_\odot$, we have a characteristic radius of $R_{cg} \sim 0.5$ Mpc. For the universe, assuming the mean mass $M_u \sim 10^{23}\, M_\odot$, we obtain the mean radius $R_u \sim 10^{26}$ m.

In this sense, we can calculate the corresponding characteristic interaction radius of a star with mean mass $M_{star} \sim 1\, M_\odot$, as $R_{star} \sim 3 \times 10^3$au, where "au" is the astronomic unit.. This radius corresponds to the radius of the *Oort cloud*, which defines the typical outest gravitational boundary of our Solar System.

For the proton, we have that its characteristic interaction radius, from the scaling relation (22.6), is given by $R_p \approx 10\lambda_p$, and its characteristic acceleration scales as equation (22.7).

§24. The MOND acceleration parameter

The acceleration constant a_0 has been phenomenologically introduced by Milgrom (1983b) as a cut-off parameter in the MOND theory to discriminate between Newtonian gravity and modified dynamics. MOND is constructed to obviate the need of dark matter when applied to galactic systems, for which the standard Newtonian dynamics is a good approximation only for accelerations much larger than a_0, and the so-called deep MOND regime is valid for accelerations much less than a_0.

The best value for a_0 is obtained from the fit to the rotation curves of spiral galaxies in the deep MOND regime, in the vicinity of our Galaxy and for a value for the Hubble constant $H_0 = 75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, as $a_0 \approx 1.2 \times 10^{-10}\mathrm{m/s^2}$ (Begeman et al., 1991). This value is taken as a constant of the theory for all the applications to the different systems in the universe.

At this stage, MOND is purely phenomenological, and people has done many efforts to construct a fundamental theory from which MOND would be the correct limit for the accelerations $a \ll a_0$ (see e.g. Mendoza et al., 2011, and references therein).

In the search of such fundamental theory, Milgrom first noticed the following coincidence between the value of the acceleration scale a_0, the Hubble constant at the present epoch and the speed of light (Milgrom, 1983b, 2008):

$$2\pi a_0 \approx cH_0 \,. \tag{24.1}$$

From the Friedmann-Lemaître-Robertson-Walker cosmological equations, the last relation can be written in terms of the cosmological constant Λ as (see e.g. Milgrom, 2008)

$$2\pi a_0 \approx c \left(\frac{\Lambda}{3}\right)^{1/2} , \tag{24.2}$$

since $H_0 \approx (\Lambda/3)^{1/2}$ at the present time.

This fact can be interpreted, from one side, as the coincidence connecting MOND to cosmology and dark energy, and, on the other side, as the influence of the cosmic large scales to the local dynamics (Milgrom, 2008, 2009a).

An additional comment to this coincidence is the fact that, as the Hubble parameter H_0 varies with cosmic time, a_0 varies too. This is not necessarily the case since a_0 could be related to Λ with the latter being constant. In this case, however, the problem of the tiny value of Λ with respect to the large value of gravitational vacuum state still remains (the so-called Cosmological Constant Problem). Interestingly, variations of a_0 could induce secular evolution in galaxies and other galactic systems (Milgrom, 2008). Although the value obtained from the rotation curve of a spiral galaxy, at the redshift $z = 2.38$, is consistent with the local measured value from the best studied rotation curves of spiral galaxies in vicinity of the Milky Way, this possibility is not completely discarded because of the high uncertainties in the observations (Milgrom, 2008). If a_0 does not vary with the cosmic time, then this coincidence just occurs at the present epoch.

§25. "Coincidences" for the value of a_0

Assuming $a_\star = a_0$, the scaling relation (22.5) holds for two very different systems: the observed universe and the nucleon.

In the case of the universe, it is possible to construct a length scale l_0 from the constants

G, c and a_0, such that $l_0 \equiv c^2/a_0 \approx 10^{27}$ m, and a mass scale $M_0 \equiv \mu_0 c^4 \approx 6 \times 10^{23}\, M_\odot$ (see e.g. Milgrom, 2008). Looking for a connection with quantum theory, these scales can be seen as the Planck length and mass constructed from the fundamental constants $\hbar$, G and c. In this sense, l_0 and M_0 can give the scales where MOND effects, combined with gravity, are expected. In fact, the Hubble radius and the mass of the universe can be written as

$$R_H \equiv \frac{c}{H_0} \approx \frac{l_0}{2\pi} \sim 10^{26}\,\mathrm{m}\,, \tag{25.1}$$

$$M_u \approx \frac{c^3}{GH_0} \approx \frac{M_0}{2\pi} \sim 10^{23}\, M_\odot\,. \tag{25.2}$$

From these two relations, the coincidence (24.1) can be written as (Milgrom, 2008)

$$2\pi a_0 \approx \frac{GM_u}{R_H^2}\,. \tag{25.3}$$

We can see that this result is exactly reproduced by equation (22.8) because, despite of the different definitions for the radius of the universe, the corresponding value for the Hubble radius (25.1) and the value we obtained in Section §23 are of the same order of magnitude. This is because the same mass scale (25.2) is assumed.

Additionally, Milgrom pointed out that a_0 is also the gravitational acceleration produced by a particle of mass $\sim 100\,\mathrm{MeV}/c^2$ at a distance equal to its Compton wavelength and Funkhouser (2006) has studied the LNC for the fundamental quantities and the cosmological coincidence (24.1). He found that there exists a critical acceleration coming from these coincidences and has solved them proposing the following scaling law for the cosmological constant:

$$\Lambda \approx \left(\frac{8\pi G}{3c^2}\right)\frac{Gm_n^2}{\lambda_n^4}\,, \tag{25.4}$$

where m_n is the nucleon mass and λ_n is its Compton wavelength. This equation may be interpreted as the energy density associated with the cosmological constant, scaled to the gravitational energy density of the nucleon mass confined to a sphere with its Compton wavelength as the radius.

Evaluating the r.h.s. of the last scaling relation, a discrepancy of ~ 4 orders of magnitude is found with respect to the value of the cosmological constant, $\Lambda \approx 3.9 \times 10^{-36}\,\mathrm{s}^{-2}$. This discrepancy is approximately solved by replacing λ_n by $b\lambda_n$, where b is a constant of order

10 (Funkhouser, 2006).

Assuming the scaling relation (25.4), and using the Eddington-Weinberg relation, the MOND's acceleration scale is found to be the characteristic gravitational acceleration of the nucleon mass at its Compton wavelength, scaled by the same factor $b = 10$, that is

$$a_0 \approx \frac{G m_n}{(b\lambda_n)^2}. \tag{25.5}$$

This relation is equivalent to the Milgrom result for a particle of mass ~ 100 MeV, except for a factor 2π, recovered assuming equation (22.7).

Finally, it is possible to show how to obtain the coincidence relation (24.1) from the quantization relation (22.1). Rewriting such a quantum relation in terms of the angular momentum J of the mass M, we have

$$GM^2 = n\hbar c = Jc\,. \tag{25.6}$$

Now, the intrinsic angular momentum for a mass M and a radius R rotating at the angular velocity ω, is given by

$$J \sim \omega M R^2. \tag{25.7}$$

Writing the last two relations for the mass and radius of the observed universe, by using equation (25.7) in (25.6), and after rearranging terms, we have

$$\frac{GM_u^2}{R_u^2} \sim \omega_0 c \sim \frac{c}{T_0}\,, \tag{25.8}$$

where ω_0 represents the 'intrinsic angular velocity' of the universe and T_0 its 'rotation period', that can be approximated by the age of the universe. Then, from the Hubble constant, $H_0 \sim 1/T_0$, and from the scaling relation (22.8), where $a_\star = a_0$, we recover the coincidence given by equation (24.1).

An intrinsic angular momentum for the entire universe from which this coincidence can be derived, may have some implication for the so-called "axis of evil" (Land & Magueijo, 2007).

§26. Discussion

In this chapter we have discussed some quantum-cosmic coincidence relations involving the phenomenological parameter a_0 which sets the so-called MOND scale.

In this perspective, such an acceleration seems more than only a phenomenological parameter, but a fundamental quantity related to some universal constants (c.f. equation (22.10)) and coming from a quantization condition (22.1) for quantum-gravitational systems.

From the identification $a_\star = a_0$, the scaling relations (22.6) and (22.5) hold for any self-gravitating astrophysical structure where gravity is the overall interaction that bounds the system. Such relations connect the microscopic constituents (protons) with the macroscopic features of the astrophysical systems (radius and mass); the acceleration a_0 gives the natural cut-off where dynamics changes regime without invoking any dark matter. Being connected also with the Hubble parameter, H_0, equation (24.1), and the cosmological constant, Λ (equation (24.2)), a_0 could give rise also to a natural explanation for dark energy phenomena.

As a final comment, we have to say that the relations presented here should be seriously considered in view to explain the universe content just with observable quantities. However, stating that a_0 is a fundamental parameter, we do not pretend to be conclusive since we need a self-consistent relativistic quantum field theory where MOND is fully recovered in the weak field limit.

Conclusiones

A lo largo del trabajo de tesis se ha mostrado que es posible construir una teoría de gravedad modificada a partir de considerar la escala de aceleración de MOND a_0, como una constante fundamental involucrada en los fenómenos gravitacionales. Esta es la hipótesis clave de todo el trabajo. De este modo construimos la teoría de gravedad extendida no–relativista, a partir de argumentos dimensionales, modificando la fuerza Newtoniana a partir de una función únicamente de la variable x, que depende de la escala de masa l_M y el radio r del sistema.

Esta teoría es sólo una aproximación a una ley de gravedad más general. Sin embargo, mostramos que el considerar a a_0 como constante fundamental en la gravitación nos arroja resultados muy interesantes, al poder ajustar diversos fenómenos astrofísicos dentro de los errores observacionales, por ejemplo, las curvas de rotación planas de galaxias espirales y la relación Tully–Fisher asociada, las relaciones de equilibrio de las galaxias esferoidales enanas y la relación Faber–Jackson y el plano fundamental de las galaxias elípticas. Además, nuestro modelo $n = 3$ es consistente al límite máximo de las desviaciones a la gravedad Newtoniana dentro del Sistema Solar.

Trabajos posteriores han mostrado que esta teoría de gravedad extendida no–relativista ajusta las dispersiones de velocidades observadas en cúmulos globulares (Hernandez & Jiménez, 2012; Hernandez et al., 2013), las velocidades relativas en sistemas binarios extendidos (Hernandez et al., 2012) y las dispersiones de velocidades en galaxias elípticas (Jimenez et al., 2012), para un valor del parámetro $n \geq 8$. Además, nuestro modelo $n = 3$ genera desviaciones de la fuerza Newtoniana menores a las incertidumbres experimentales medidas en el laboratorio (Meyer et al., 2012) y en el sistema Tierra–Luna (Exirifard, 2011), lo cual implica que los modelos con $n \geq 3$ también pasan este test de consistencia.

En base a este trabajo nos propusimos construir una teoría de gravedad extendida relativista, que en su límite de campo débil convergiera a nuestra propuesta de gravedad Newtoniana extendida. Para ello nos basamos en las funciones $f(R)$ en su aproximación métrica y consideramos también a la escala a_0 como constante fundamental en la teoría gravitacional. En el Capítulo III, proponemos una función de la variable adimensional $\chi = L_M^2 R$, en la que la masa del sistema en estudio es incluida a través del parámetro L_M. En este mismo Capítulo encontramos que esta nueva longitud característica es consistente con las simetrías de Noether del problema.

En los Capítulos III y IV se muestra que el régimen de aceleración MONDiano es justo el límite de campo débil para la teoría $f(\chi) = \chi^{3/2}$. Esto lo mostramos de dos maneras: primero en la sección §10, a partir de un análisis a orden de magnitud de las ecuaciones de campo resultantes para la teoría $\chi^{3/2}$, para una métrica estática esféricamente simétrica, se perturbó la métrica en su componente temporal hasta segundo orden en potencias de v/c y hasta orden cero en las componentes radial y angulares. Posteriormente, en las secciones §17 y §18, se muestra el análisis completo de perturbaciones hasta segundo orden en términos de v/c en las componentes temporal y radial de dicha métrica. Luego, en la sección §19 se muestra a partir de los componentes de la métrica obtenidos, que la teoría es consistente con dos observaciones astronómicas cruciales: en el límite de campo débil es capaz de ajustar la fenomenología de las curvas de rotación planas y la relación Tully–Fisher asociada, así como también ajusta los lentes gravitacionales observados en galaxias y grupos de galaxias, ambas sin la necesidad de materia oscura.

Adicionalmente, encontramos que hay una degeneración en la elección de signo del tensor de Riemann cuando se trabaja con teorías de orden mayor en la acción gravitacional. Esta degeneración desaparece en relatividad general de manera natural, pero en teorías métricas $f(R)$ de cuarto orden, y en particular en nuestras teorías métricas $f(\chi)$, concluimos que el signo debe ser elegido de la comparación con las observaciones. Es aquí donde es muy importante la introducción del escalar de Ricci adimensional χ en nuestra teoría, pues esta aproximación, correcta dimensionalmente, es capaz de reproducir los lentes gravitacionales requeridos para galaxias y grupos de galaxias, mediante la constricción dada por la ecuación (16.5). De la comparación con las observaciones de lentes gravitacionales, elegimos el signo correcto del tensor de Riemann que reprodujera la métrica a segundo orden requerida. Este resultado hace una distinción de nuestras teorías métricas $f(chi)$ con las teorías métricas $f(R)$, pues ambas teorías son capaces de reproducir el límite MONDiano para recuperar las curvas de rotación planas y la relación Tully–Fisher, pero únicamente nuestra teoría $f(\chi) = \chi^{3/2}$ reproduce los lentes gravitacionales esperados.

En trabajos posteriores, Carranza et al. (2013) y Mendoza (2012) han mostrado que estas teorías $f(\chi)$ son equivalentes a las teorías $f(R, T)$ (Harko et al., 2011) que discutimos en la sección §2. Además estos trabajos han mostrado que la teoría particular que utilizamos, $f(\chi) = \chi^{3/2}$, genera una expansión acelerada a épocas tardías. Carranza et al. (2013) han mostrado que dicha expansión es consistente con la expansión acelerada deducida de las supernovas tipo Ia.

Por tanto, nuestra propuesta además es consistente al nivel de las observaciones cosmológicas. Esto es muy importante, pues la misma teoría $\chi^{3/2}$ es capaz de reproducir las observaciones astrofísicas relativistas y no–relativistas, mostrando consistencia en un rango de escalas desde los pársecs hasta los cientos de megapársecs.

Hasta aquí sólo hemos mostrado que es posible obtener el límite MONDiano y ajustar con las observaciones para sistemas en las regiones en que la gravitación entra en este régimen. Sin embargo, hemos visto que la transición entre los regímenes Newtoniano y MONDiano se da de manera muy abrupta, por tanto, la región de transición es muy pequeña (una función 'escalón'; cf. Figura II.1). En analogía con el régimen no–relativista, Mendoza (2012) ha propuesto una función de transición entre el régimen Newtoniano y el régimen profundo de MOND:

$$f(\chi) = \chi^{3/2} \, \frac{1 \pm \chi^{p+1}}{1 \pm \chi^{3/2+p}} \rightarrow \begin{cases} \chi^{3/2}, & \text{para } \chi \ll 1, \\ \chi, & \text{para } \chi \gg 1, \end{cases} \tag{26.1}$$

que cumple con los límites correctos: relatividad general se recupera para $\chi \gg 1$ y la versión relativista de MOND, con $\chi^{3/2}$, se recupera para $\chi \ll 1$. Al igual que con el parámetro n para la función de transición dada por la ecuación (5.7), el parámetro $p \geq -1$ debe calibrarse del ajuste con las observaciones astronómicas y cosmológicas, un problema para investigación futura.

Una consecuencia de ambas teorías de gravedad extendida, relativista y no–relativista, es que la invariancia de la interacción gravitacional con la escala y con la masa se pierde. De este modo, la gravedad de Newton es el límite de campo débil de relatividad general o $f(\chi) = \chi$ y MOND es el límite de campo débil de nuestra gravedad extendida $f(\chi) = \chi^{3/2}$.

La dependencia de χ en la masa implica que ésta entra en la acción gravitacional. En teoría lagrangiana esto no ha sido propuesto anteriormente, en la cual se asume que la acción es función únicamente de la geometría del espacio–tiempo debido a la presencia de las masas. Sin embargo, fue Sobouti (2007) quien se enfrentó primero con esta peculiaridad en la acción gravitacional cuando trabajaba en una generalización métrica de MOND. Una teoría de gravedad extendida va más allá de las ideas tradicionales de la relatividad general y, en este sentido, probablemente necesitamos cambiar las nociones comúnmente aceptadas, incluso al nivel más fundamental en la acción.

Nuestra teoría de gravedad extendida está formulada en base a la fenomenología observada. Este es el valor principal de las teorías presentadas. Aún cuando la teoría última de gravedad modificada sea distinta a la presentada en esta tesis, podemos asegurar que las componentes métricas deberán coincidir con nuestros resultados si es que pretenden reemplazar la materia oscura como componente dominante del universo.

Finalmente, también exploré en mi trabajo de tesis la posibilidad de que la constante de aceleración de MOND sea una cantidad fundamental relacionada con algunas constantes universales y no únicamente una constante fenomenológica (cf. ecuación 22.10). A partir de una condición de cuantización para sistemas autogravitantes, obtuvimos que es posible recuperar relaciones que inicialmente parecían 'coincidencias' numéricas, pero que pudieran ser manifestaciones de relaciones fundamentales en la física. Por ejemplo, la relación existente entre a_0, la velocidad de la luz c y la constante de Hubble H_0 (cf. ecuación 24.1) aparece naturalmente. Esta misma relación fue obtenida a partir de nuestra teoría $f(\chi) = \chi^{3/2}$ por Carranza et al. (2013), para la época cósmica actual. Por tanto, se puede postular la escala a_0 como una constante fundamental de la naturaleza, hecho que hemos supuesto para construir nuestra teoría de gravedad extendida.

Conclusions

Throughout this thesis it has been shown that it is possible to construct a modified theory of gravity from considering that MOND's acceleration scale a_0, is a fundamental constant that enters into the gravitational phenomena. This is the key hypothesis of all the work. In this way, we have constructed the extended non–relativistic theory of gravity, from dimensional arguments, modifying the Newtonian force through a function only of the variable x, which depends on the mass length scale l_M and the radius r of the system.

This theory is only one approximation to a more general law of gravity. However, we have showed that considering a_0 as a fundamental constant yields very interesting results, being able to adjust diverse astrophysical phenomena within the observational errors, e.g. the flat rotation curves of spiral galaxies and the associated Tully–Fisher relation, the equilibrium relations for dwarf spheroidal galaxies and the Faber–Jackson relation and the Fundamental Plane of elliptical galaxies. Also, our model $n = 3$ is consistent with the upper limits of the deviations to Newtonian gravity in the Solar System.

Subsequent works have shown that this non–relativistic extended theory of gravity adjusts the velocity dispersions observed in globular clusters (Hernandez & Jiménez, 2012; Hernandez et al., 2013), the relative velocities in wide binary systems (Hernandez et al., 2012) and the velocity dispersions in elliptical galaxies (Jimenez et al., 2012), for a value of the parameter $n \geq 8$. In addition, our model $n = 3$ generates deviations from the Newtonian force smaller than the experimental uncertainties measured in the laboratory (Meyer et al., 2012) and in the Earth–Moon system (Exirifard, 2011), which means that models with $n \geq 3$ also agree with these tests of consistency.

Based on this work, we aim to construct an extended relativistic theory of gravity, that in its weak field limit converge to our proposed extended Newtonian gravity. To do this we used the $f(R)$ functions in the metric approach and also considered the scale a_0 as a fundamental constant in the gravitational theory. In Chapter III, we propose a function on the dimensionless variable $\chi = L_M^2 R$, in which the mass of the system is included through the parameter L_M. In the same Chapter, we find that this new characteristic length is consistent with the Noether symmetries of the problem.

In Chapters III and IV it is shown that the weak field limit of the theory $f(\chi) = \chi^{3/2}$ gives us the MONDian acceleration regime. We obtained this result in two ways: first, in section §10, from an order of magnitude approach of the resulting field equations for the theory $\chi^{3/2}$, for a static spherically symmetric metric, the perturbation was performed in the time component up to the second order in powers of v/c and up to zeroth order in the radial and angular components. Then, in sections §17 and §18, it is shown the complete perturbation analysis up to the second order in terms of v/c in the time and radial components of the metric. Later, in section §19, from the metric components obtained before, it is shown that our theory is consistent with two crucial astronomical observations: in its weak field limit adjusts the phenomenology of the flat rotation curves for disc galaxies and the associated Tully–Fisher relation, also the gravitational lenses observed in galaxies and groups of galaxies, both without the need of dark matter.

Additionally, we find that there is a degeneration in the election of sign in the Riemann tensor when working with higher order theories in the gravitational action. This degeneration disappears naturally in general relativity, but in $f(R)$ fourth order metric theories, and particularly in our $f(\chi)$ metric theories, we conclude that the sign must be chosen from the comparison with the observations. Here, the introduction in our theory of the dimensionless Ricci scalar χ is very important, since this approximation, dimensionally correct, is capable of reproducing required gravitational lenses in galaxies and groups of galaxies, through the constriction given by equation (16.5). Through the comparison with the gravitational lensing observations, we choose the correct sign for the Riemann tensor to reproduce the metric up to the second order required. This result makes a distinction of our $f(\chi)$ metric theories with the $f(R)$ ones, because both theories are able to reproduce the MONDian limit to recover the flat rotation curves and the Tully–Fisher relation, but only our theory $f(\chi) = \chi^{3/2}$ reproduces the expected gravitational lensing.

In subsequent works, Carranza et al. (2013) and Mendoza (2012) have shown that these $f(\chi)$ theories are equivalent to the $f(R, T)$ theories (Harko et al., 2011), discussed in section §2. Moreover, these studies have shown that the particular theory $f(\chi) = \chi^{3/2}$

generates an accelerated expansion at late epochs. Carranza et al. (2013) have shown that such expansion is consistent with the accelerated expansion signalled by type Ia supernovae.

Therefore, our proposal is also consistent at the cosmological observational level. This is very important because the same theory $\chi^{3/2}$ is be able to reproduce non–relativistic and relativistic astrophysical observations, showing consistency in a range of scales from the parsecs to hundreds of megaparsecs.

So far, we have only shown that it is possible to obtain the MONDian limit and fit the observations for the systems in the regions where the gravitation is in this regime. However, we have seen that the transition between the Newtonian and MONDian regimes occurs very abruptly, therefore, the transition region is very small (a step–like function; cf. Figure II.1). In analogy to the non–relativistic regime, Mendoza (2012) has proposed a transition function from the Newtonian regime to the deep MOND one:

$$f(\chi) = \chi^{3/2} \frac{1 \pm \chi^{p+1}}{1 \pm \chi^{3/2+p}} \rightarrow \begin{cases} \chi^{3/2}, & \text{for } \chi \ll 1, \\ \chi, & \text{for } \chi \gg 1, \end{cases} \tag{26.2}$$

that holds the correct limits: general relativity is recovered for $\chi \gg 1$ and the relativistic version of MOND, with $\chi^{3/2}$, is recovered for $\chi \ll 1$. In the same form as the parameter n for the transition function given by equation (5.7) was handled, the parameter $p \geq -1$ must be calibrated with astronomical and cosmological observations, matter of future investigation.

A consequence of the both relativistic and non–relativistic extended theories of gravity, is that the scale and mass invariance of the gravitational interaction is broken. Thus, Newton's gravity is the weak field limit of general relativity or $f(\chi) = \chi$ and MOND is the corresponding weak field limit of our extended gravity $f(\chi) = \chi^{3/2}$. The dependence of χ on the mass implies that the mass enters into the gravitational action. This has not been proposed previously in Lagrangian theory, in which it is assumed that the action is a function only of the spacetime geometry, due to the presence of the masses. However, it was Sobouti (2007) who first encountered this peculiarity in the Hilbert action when dealing with a metric generalisation of MOND. An extended metric theory of gravity goes beyond the traditional general relativity ideas and in this way, we probably need to change the commonly accepted notions, even at the fundamental level of the action.

Our extended theory of gravity is formulated on the basis of the observed phenomenology. This is the most valuable characteristic of the theories presented in this work. Even if the final theory of gravity is different from the theory presented in this thesis, we can assert that the metric components must coincide with our results if they intend to replace the dark matter as the dominant component of the universe.

Finally in this thesis, I also explored the possibility of MOND's acceleration constant to be a fundamental quantity related to some universal constants and not only a phenomenological constant (cf . equation 22.10). From a quantisation condition for autogravitating systems, we obtained that it is possible to recover some relations initially thought as numerical 'coincidences', but that may be manifestations of some fundamental relations in physics. For example, a relation between a_0, the speed of light c and the Hubble constant H_0 (cf. equation 24.1) appears naturally. This same relation was obtained from our theory $f(\chi) = \chi^{3/2}$ by Carranza et al. (2013) for the actual cosmic epoch. Therefore, one can postulate the scale a_0 as a fundamental constant of nature, the basic foundation in the developments presented in this thesis.

Apéndice A

About the sign in Riemann's tensor

In the study of the gravitational field equations, the link between the curvature of spacetime and the matter content is a key fact. All the information regarding the curvature of spacetime is contained in the Riemann curvature tensor $R^{\alpha}{}_{\beta\eta\theta}$, which is a function of the first and second derivatives of the metric. From a purely mathematical point of view, the Riemann tensor can be obtained from the commutator of covariant derivatives (Carroll, 2004):

$$[\nabla_{\mu}, \nabla_{\nu}]V^{\rho} = R^{\rho}{}_{\sigma\mu\nu}V^{\sigma}, \tag{0.1}$$

for any vector field V^{α}. From a geometrodynamical point of view, the curvature tensor is constructed through the change ΔA_{μ} in a vector A_{μ} after being displaced about any infinitesimal closed contour (Landau & Lifshitz, 1975): $\Delta A_{\mu} = \oint \Gamma^{\lambda}_{\mu\nu} A_{\lambda} dx^{\nu}$. By the use of Stokes' theorem it then follows that for a sufficiently small closed contour:

$$\Delta A_{\mu} \approx \frac{1}{2} R^{\lambda}{}_{\mu\nu\theta} A_{\lambda} \Delta f^{\nu\theta}, \tag{0.2}$$

where $\Delta f^{\nu\theta}$ represents the infinitesimal area enclosed by the contour of the line integral. In this respect, it follows that the Riemann tensor measures the curvature of spacetime (cf. Landau & Lifshitz, 1975).

In equations (0.1) and (0.2), the Riemann tensor has been defined as:

$$R^{\beta}{}_{\mu\nu\alpha} := \Gamma^{\beta}{}_{\mu\alpha,\nu} - \Gamma^{\beta}{}_{\mu\nu,\alpha} + \Gamma^{\beta}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\alpha} - \Gamma^{\beta}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\mu\nu}. \tag{0.3}$$

If Riemann's tensor is defined by equation (0.3), then Ricci's tensor is $R_{\nu\alpha} := g^{\beta\mu} R_{\beta\mu\nu\alpha}$ and Ricci's scalar is $R^{\alpha}{}_{\alpha}$. Since these are the most used definitions in relativity theory nowadays, we will refer to these quantities as "*standard*".

However, there is another way in which Riemann's tensor (and Ricci's tensor) can be defined, usually adopted by mathematicians and by Computer Algebra Systems (CAS) such as Maxima (http://maxima.sourceforge.net). In these cases, the syntaxis is such that (see e.g. Toth, 2005)

$$\begin{aligned} R[\mu, \nu, \alpha, \beta] &:= R^{\beta}{}_{\mu\nu\alpha} \\ &= \Gamma^{\beta}{}_{\mu\nu,\alpha} - \Gamma^{\beta}{}_{\mu\alpha,\nu} + \Gamma^{\beta}{}_{\lambda\alpha}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\beta}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\alpha}. \end{aligned} \tag{0.4}$$

If Riemann's tensor is defined by equation (0.4), then Ricci's tensor is $R_{\nu\alpha} := g^{\beta\mu} R_{\beta\mu\nu\alpha}$ and Ricci's scalar is $R^{\alpha}{}_{\alpha}$. Although this choice of signs for the Riemann and Ricci tensors is not very much in use these days, some well-known textbooks use them (see e.g. the Table of Sign Conventions at the beginning of reference Misner et al., 1973). The CAS Maxima uses the definition (0.4) and is such that:

$$\mathbf{R}_{\text{maxima}} = -\mathbf{R}_{\text{standard}}, \tag{0.5}$$

in free-index notation.

As discussed in the Table of Sign Conventions of Misner et al. (1973), general relativity can use any of the above definitions (and a few more) simply because of the linearity with which Ricci's scalar and Ricci's tensor appear in Einstein's field equations. This is however not the case in metric $f(R)$ theories of gravity, since for example in those theories, the trace of the field equations is given by (see e.g. Capozziello & Faraoni, 2011):

$$f'(R)R - 2f(R) + 3\Delta f'(R) = \frac{8\pi G}{c^4} T. \tag{0.6}$$

To highlight the point, let us substitute the power-law function (16.7) in the previous equation to obtain:

$$(b-2)R^{b} + 3b\Delta R^{b-1} = \frac{8\pi G}{c^4} T. \tag{0.7}$$

This equation reflects a crucial fact about the choice of sign in Riemann's tensor. Due to the presence of the derivative term $f'(R) = bR^{b-1}$, depending on the sign convention of the definition of the Ricci scalar, there appears a sign factor $(\pm)^{b-1}$ which is not global to all the terms in the equation. This establishes a *bifurcation* in this class of solutions of the theory. Indeed, for a situation where $f(R) = R^a + R^b$ or any more complicated function of R, there is not (*a priori*) any indication of which convention in the definition of Riemann's tensor should be used to describe a particular physical phenomena. In this article we show that the convention can be settled through the use of astrophysical observations. For example, the results presented in this article were obtained with the standard definition of Riemann's tensor in equation (0.3). That choice (and only that one) can account for both observed dynamics of massive particles in spiral galaxies through the Tully-Fisher relation, and for the deflection of light observed in gravitational lenses. An important aspect to point out is that the case $f(R) = R$ of Einstein's general relativity is free from the above ambiguity. This is so because it is possible to redefine the signature for the energy-momentum tensor to recover the same field equations (see e.g. Misner et al., 1973; Hobson et al., 2006).

We see from this result that previous works by Capozziello et al. (2007b); Capozziello & Stabile (2009) have selected the convention used by the CAS Maxima in order to compute their results. In that respect, their results lie in another branch of the solutions of the field equations. If we would have taken for example, the definition of Riemann's tensor by Maxima, then the metric coefficients would have been: $g^{(2)}_{00} = 2\hat{R}\ln^2(r)/9 + A\ln(r) + B$ and $g^{(2)}_{11} = -2\hat{R}\ln(r)/9 + D/r + (\hat{R} - A)/2$ (where A, B and D are constants). These are very different from the ones obtained in equations (18.8) and (18.10) and would have never reproduced the astrophysical observations treated in this article. It is only through the correct choice of signs in the definition of Riemann's tensor, such as the ones used in the present article and represented in equation (0.3), that the good agreement with the Tully-Fisher relation and lensing observations can be correctly obtained.

Apéndice B

Comments about the Maxima code

In this section we give a brief introduction to the code we wrote in the Computer Algebra System (CAS) Maxima (http://maxima.sourceforge.net) to obtain the field equations. Specifically, we work with the module `ctensor` (cf. Toth, 2005). The syntax of such module is that, when invoked, it runs an input interface to design the form of the covariant metric.

The Maxima code MEXICAS (Metric EXtended-gravity Incorporated through a Computer Algebraic System) is Copyright of T. Bernal, S. Mendoza and L.A. Torres, licensed under a GNU Public General License (GPL), version 3 (see http://www.gnu.org/licenses) can be obtained from: http://www.mendozza.org/sergio/mexicas (see the section about copyright and usage in that webpage).

For the implementation of the code, we consider a perturbative approach in the parameter $\epsilon := 1/c$, such that the covariant components of the metric are given by

$$\begin{aligned} g_{00} &= 1 + \epsilon^2 g_{00}^{(2)} + \mathcal{O}(4), \\ g_{11} &= -1 + \epsilon^2 g_{11}^{(2)} + \mathcal{O}(4), \end{aligned} \tag{0.1}$$

where the angular components are given by the standard expressions for spherical coordinates as shown in equation (15.5). With these equations, it is simple to construct the contravariant components of the metric:

$$\begin{aligned} g^{00} &= 1 - \epsilon^2 g_{00}^{(2)} + \mathcal{O}(4), \\ g^{11} &= -1 - \epsilon^2 g_{11}^{(2)} + \mathcal{O}(4). \end{aligned} \tag{0.2}$$

With these considerations, the metric is recorded in the `ctensor` module. From this fact, it is simple to invoke all the quantities required to construct the field equations, either in general relativity or for any extended metric theory of gravity. For example, in a descriptive way concerning the syntaxis of maxima it follows that:

$$\texttt{christof(mcs)} \longrightarrow \Gamma^{\lambda}{}_{\mu\nu}, \tag{0.3}$$

and with similar syntaxis for the Riemann tensor, the Ricci tensor and the Ricci scalar.

Due to the fact that the metric has an order parameter ϵ, all the tensorial quantities involved in the construction of the field equations will gain this dependence. In the formalism of the code, it is a crucial fact to extract the perturbation order of every metric quantity to construct the field equations at the desired perturbation order. For example, for a generic quantity q calculated from the manipulation of the metric, if we consider that $q^{(n)}$ represents such quantity at order n, we have:

$$q^{(0)} = \lim_{\epsilon \to 0} q, \tag{0.4}$$

which reproduces the flat spacetime limit. For the second order we have

$$q^{(2)} = \lim_{\epsilon \to 0} \frac{q - q^{(0)} - \cancelto{0}{\epsilon q^{(1)}}}{\epsilon^2}, \tag{0.5}$$

and consequently the fourth order is obtained by

$$q^{(4)} = \lim_{\epsilon \to 0} \frac{q - q^{(0)} - \cancelto{0}{\epsilon q^{(1)}} - \epsilon^2 q^{(2)} - \cancelto{0}{\epsilon^3 q^{(3)}}}{\epsilon^4}. \tag{0.6}$$

Similarly, higher perturbation orders can be obtained by the obvious generalisation of the previous relation.

In equations (0.5) and (0.6), it is implied that the first order quantities vanish, as is also the case for the Christoffel symbols. This computational procedure gives as an output a key result used in the article corresponding to Ricci's scalar at second perturbation order, given by equation (16.19).

Apéndice C

Extended field equations using Maxima

By using the Computer Algebra System (CAS) Maxima and the MEXICAS code (see appendix B), we obtained the field equations up to the second order.

The trace (16.12) of the field equations (18.1) to the order $\mathcal{O}(2b)$ of approximation can be simplified with the aid of the solutions found at the lowest order of approximation in Section §17 to obtain

$$\begin{aligned} &(b-2)R^{(2)4} - 3b(b-1)R^{(2)}\left\{R^{(2)}\left[R^{(4)}_{,rr} + \frac{2}{r}R^{(4)}_{,r}\right.\right. \\ &\left.\left.+\frac{1}{2}R^{(2)}_{,r}\left(g^{(2)}_{00,r} + g^{(2)}_{11,r}\right)\right] + 2(b-2)R^{(2)}_{,r}R^{(4)}_{,r}\right\} \\ &+ 3b(b-1)(b-2)R^{(2)2}_{,r}R^{(4)} = 0. \end{aligned} \tag{0.1}$$

The components $\mathcal{H}^{(2b)}_{\mu\nu}$ of the field equations (16.10) at order $\mathcal{O}(2b)$ are given by:

$$\begin{aligned} \mathcal{H}^{(2b)}_{\mu\nu} &= -b(b-1)\left\{R^{(2)b-2}\left[R^{(4)}_{,\mu\nu} - \Gamma^{1(0)}_{\mu\nu}R^{(4)}_{,r} - \Gamma^{1(2)}_{\mu\nu}R^{(2)}_{,r}\right.\right. \\ &- g^{(0)}_{\mu\nu}\left(R^{(2)}_{,r}\left[g^{11(2)}_{,r} + g^{11(0)}(\ln\sqrt{-g})^{(2)}_{,r} + g^{11(2)}\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\times(\ln\sqrt{-g})^{(0)}_{,r}\Big] + g^{11(0)}\left[(\ln\sqrt{-g})^{(0)}_{,r}R^{(4)}_{,r} + R^{(4)}_{,rr}\right] \\
&+ g^{11(2)}R^{(2)}_{,rr}\Bigg) - g^{(2)}_{\mu\nu}g^{11(0)}\left(R^{(2)}_{,r}(\ln\sqrt{-g})^{(0)}_{,r} + R^{(2)}_{,rr}\right)\Bigg] \\
&+ (b-2)R^{(2)b-3}R^{(4)}\Bigg[R^{(2)}_{,\mu\nu} - \Gamma^{1(0)}_{\mu\nu}R^{(2)}_{,r} - g^{(0)}_{\mu\nu}g^{11(0)} \\
&\times\left((\ln\sqrt{-g})^{(0)}_{,r}R^{(2)}_{,r} + R^{(2)}_{,rr}\right)\Bigg]\Bigg\} - b(b-1)(b-2) \\
&\times\Bigg\{R^{(2)b-3}\Bigg[2R^{(2)}_{,\mu}R^{(4)}_{,\nu} - g^{(0)}_{\mu\nu}\Bigg(2g^{11(0)}R^{(2)}_{,r}R^{(4)}_{,r} + g^{11(2)} \\
&\times R^{(2)2}_{,r}\Bigg) - g^{(2)}_{\mu\nu}g^{11(0)}R^{(2)2}_{,r}\Bigg] + (b-3)R^{(2)b-4}R^{(4)} \\
&\times\left[R^{(2)}_{,\mu}R^{(2)}_{,\nu} - g^{(0)}_{\mu\nu}g^{11(0)}R^{(2)2}_{,r}\right]\Bigg\}.
\end{aligned}
\tag{0.2}
$$

Dividing the field equations (18.1) by $R^{(2)b-4}$ and using the trace (0.1) and the last equation, a reduced expression for the field equations is found:

$$
\begin{aligned}
&-\frac{2b-1}{6}R^{(2)4} + bR^{(2)3}R^{(2)}_{\mu\nu} - b(b-1)R^{(2)2}\Big[R^{(4)}_{,\mu\nu} \\
&-\Gamma^{1(0)}_{\mu\nu}R^{(4)}_{,r} - \Gamma^{1(2)}_{\mu\nu}R^{(2)}_{,r}\Big] - b(b-1)(b-2)R^{(2)} \\
&\times\left[R^{(4)}\left(R^{(2)}_{,\mu\nu} - \Gamma^{1(0)}_{\mu\nu}R^{(2)}_{,r}\right) + 2R^{(2)}_{,\mu}R^{(4)}_{,\nu}\right] \\
&- b(b-1)(b-2)(b-3)R^{(2)}_{,\mu}R^{(2)}_{,\nu}R^{(4)} = 0\,,
\end{aligned}
\tag{0.3}
$$

which can also be regarded as the traceless component of the field equations.

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