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Contents

Chapter 1

Introduction

During the development of physics, there has been one fundamental question in its history. The question is what are the laws that control physical systems' behavior?

In the atomic domain, experiments based on the physical phenomenon known as scattering have been performed and have given a variety of answers. To mention some of these we have spectroscopy, diffraction and collision in particle accelerator experiments.

In this introductory part we speak of the scattering phenomenon. Scattering theory is a tool which helps explain some of the phenomena in the atomic world, it forms an important part of a great physical theory known as quantum mechanics (it is also important in classical mechanics). There are a variety of scattering experiments, but in general those experiments consist of four essential parts:

(*a*) The source, which is an apparatus that will produce particles that will interact with those in the target. Here, it is important to mention that the source must produce particles in a recurrent, continuous way and under practically the same conditions, that is because all scattering experiments involve recurrent measurements for identical systems.

(*b*) The preparing apparatus (for example it can be a "colimator", the beam of an spectrometer, or a polarizer), it serves to define initial conditions of the incident particles.

(*c*) The target, it contains particles that will interact with incident particles. Conditions on the state of the target will largely affect measurements, that is why they must be taken into consideration to be able to give a correct interpretation. For example, if the target is thick then it will be possible to observe multiple scattering, and if for example, the target has crystalline structure then it will be possible to observe a diffraction pattern. On the other hand, if we deal with a moving target (for example a gas) then that effect will be present in the measurements made. The easiest interpretation of the results is found when the target is very thin and it contains a random distribution of particles at rest.

(*d*) The detector, which is a device that is responsible of recording results and is usually found in a place where it is possible to only detect scattered particles, in other words, if the target is taken off from the experimental arrangement then the detector does not register anything. In practice this condition is not so easy to have since it is not always possible to produce a sufficiently collimated beam, or because there exists scattering remains in the material due to other interactions (This is the reason why good calibration of the detector is important). It is also important to locate the detector sufficiently far from the target not to detect interactions between scattered particles and particles in the target. In almost every case the detector has a finite resolution angle, the better we can do is to make this angle small enough to have better measurements, but it is also true that there exists certain physical limitations that do not allow to make this resolution very precise, and the same problems happen with the preparing apparatus.

I The physical characteristics of scattering systems

The essential physical characteristics of a scattering system are the following:

In a scattering process we have to distinguish three moments in the temporal evolution of the system.

At the beginning, the system's state is found in the remote past. At this time, the incident particle and the target particle are located far enough so their mutual interaction is negligible. Thus, it is expected that the system's state evolves obeying the laws that govern the behavior of free particles.

During the second moment the particles will mutually interact and the evolution of the system is ruled by a movement equation in which the interaction term plays an important role. It is in the interaction moment that scattering occurs.

For the third moment one is placed in an analogous situation to that at the first moment. In fact, when the scattering phenomenon has happened, the particles are far from each other such that mutual interaction is again negligible and it has no effect in the future of the system's evolution. At this stage, the detector, observes the new state of the particles created by the scattering process.

The states describing scattering phenomena must be characterized in time in the remote past $t = (-\infty)$ and in the remote future $t = (+\infty)$ by quantities concerning the dynamics of free particles is known as the asymptotic condition. To be able to describe such states in mathematical terms, it is necessary to study the description of the evolution in time of quantum mechanical systems (it can also be done for classical mechanical systems) and to introduce a topology (a convergence notion) that will help express the difference between the perturbed system and the free system (in the remote past and in the remote future).

II Different types of scattering

So far we have not made any distinction between different types of scattering. The simplest process in scattering is the elastic scattering between different particles. This process can be symbolically represented as follows:

$$a + b \to a + b. \tag{I.2.1}$$

Expression (I.2.1) indicates that particles a and b in some initial state are scattered to the particles a and b in some final state. We say that the scattering is elastic because it refers to the fact that the total kinetic energy of the particles is the same before and after the scattering process. A special case of elastic scattering is the scattering between identical particles.

$$a + a \to a + a. \tag{I.2.2}$$

When the incident or target particles have internal degrees of freedom, it is possible that during the scattering process one of the particles undergoes a change on its internal state. If this occurs then the kinetic energy of the particles before and after the scattering is no longer the same and we speak of inelastic scattering. The most frequent case is the excitation of an internal state, the final kinetic energy is less than the initial energy in a quantity equals the excitation energy. This is called hypoelastic or endoergic scattering. However, it is also possible that some excited metastable states of one of the particles gets unexcited during the scattering process. In this case, the final kinetic energy is grater than the initial one and we may speak of hyperelastic or exoergy scattering.

Another type of scattering associated to the internal degrees of freedom is observed when those degree are degenerated in energy. This is the case for particles with spin, the scattering process would be still described by (I.2.1) but one has to take into account that the effect in dispersion over the internal degrees of freedom of the incident particles. This gives place to a variety of interesting phenomena of polarization which can provide significant information on the dependence-spin of the interaction.

An entirely different type of scattering is observed if the outgoing particles differ in number and class compared with the incoming ones. For example, the constituents of the incident beam and the scattering centers in the target cannot be elementary particles but complex structures such as α particles, hydrogen atoms, etc., and through the scattering process the interaction can decompose such composite systems into some of the constituent parts or reorganize these parts into new composite systems. We speak then of scattering reordering and we can schematically write

$$a + b \to c + d + e + \dots \tag{I.2.3}$$

Scattering Theory can be developed in such a way elastic, inelastic, hypoelastic, hyperelastic scattering and scattering with reordering can be treated in a unified way. This unified theory is called multichannel scattering theory.

A typical example of multichannel scattering is the scattering of a deuteron d by a fixed center of force. A deuteron can be elastically scattered or can be decomposed into its constituent parts (a proton p and a neutron n), in such a way that we have two possibilities.

$$d \begin{array}{l} \nearrow\; d \\ \searrow\; p + n \end{array}$$

Each different collection possible of particles and composed systems after the collision determines the so called scattering channel. In the example above, there are only two scattering channels.

Another reason for a change in the type and number of particles is the creation of new types of particles during the scattering process. These creation processes are frequently observed in scattering at high energies; at highly enough energies this creation processes are always present. A collision between two nucleons N_1 and N_2, for example, can be carried as in the following schemes:

$$N_1 + N_2 \begin{array}{ll} \nearrow\; N_1 + N_2 & \text{elastic channel} \\ \to\; N_1 + N_2 + \pi & \\ \to\; N_1 + N_2 + K + \bar{K} & \left.\right\} \text{ many inelastic channels} \\ \searrow\; \cdots\cdots & \end{array}$$

Here new particles, like mesons π and K, can be created, and some final channels clearly represent more than a simple reordering of the constituents of N_1 and N_2. The usual theory of multichannel scattering does not include such collisions and its theoretical descriptions goes beyond the scope of this work.

In all these cases we have considered, so far, that the initial number of particles or composite systems, which participate in a individual scattering is always two. It is logically possible to consider scattering process with more than two initial particles. Such situations seldom are found in the laboratory due to experimental difficulties to prepare such states. Those experiments play certain role in dense gas theory. But in elementary particle physics they are not important.

Another case of importance, however, is that of one incident free particle only. Evolution of the state of only one free stable particle does not give information on any interaction. If it is unstable, though, it will decay in some stables and unstables fragments and it is perfectly possible to look a decay like a certain type of scattering process degenerated in concordance to the scheme (I.2.4):

$$a \to b + d + \dots \tag{I.2.4}$$

In this case one does not need any target to observe scattering. It is possible to consider such decay of a unstable particle as part of an ordinary scattering process, because the unstable particle must have been created in some point in the past thanks to another ordinary scattering process. This way to see unstable systems is specially useful if the life time of the unstable particle is very short. We have then a case of resonant scattering (formation and decay of a resonance). This is often observed and is known in nuclear physics where resonances are associated with some excited state of an intermediate nucleus of short duration.

If the unstable particle has a life time largely enough, it is possible to produce ordinary scattering effects with this particle and to develop a theory where the particle is treated as a stable particle with a sufficient good approximation.

III Observable quantities

Every physical theory must have an interpretation in terms of certain effects physically observable. These effects are usually expressed giving numerical values of some quantities. The main quantities that appear in scattering theory are the scattering cross section, life time of an unstable particle or the resonance width and the bifurcation rate of multichannel processes. By the reasons given in sections above the life time is in some sense a secondary quantity which can be related to the behavior of the cross section in a appropriate scattering process.

The bifurcation rate (i.e. the probability of the scattering in a particular channel) in reality is not a independent physical quantity, because one can obtain it if individual cross sections for different channels are known. Often it is possible on symmetry considerations, without a detailed theory, to obtain expressions for the bifurcation rates without the knowledge of the cross sections' values.

There exists another theoretical approach, which is known as inverse scattering. In this Theory, it is assumed that some experimental data, i.e., results from a scattering process are given then one finds an interaction (usually a potential) that produces exactly the given data when one considers the direct scattering problem.

In the following sections, we will discuss the concepts on inverse problems, first in general and then in the context of quantum mechanics; these sections will be mainly based on the book of Chadan and Sabatier [63] and the contents of the course "Inverse scattering in Quantum Mechanics" lectured by professor Ricardo Weder at the spring school in classical and quantum mechanics in IIMAS-UNAM in March, 2008.

IV Inverse problems in physics

The normal work of physicists can be schematically thought with a movement prediction of the particles over the base of known forces, or the propagation of radiation on the ground of the knowledge on the constitution of matter. The inverse problem is to conclude what forces or components are on the basis of the observed movement. A great part of our sensorial contact with the world that surrounds us depends on an intuitive solutions of an inverse problem: We infer the shape, size and texture of a surface of strange objects from scattering and absorption of light, which is detected by our eyes. When one uses scattering experiments to know the size or the shape of particles, or the forces that some particles exert on others, the nature of the problem is similar, or more refined. Kinematics, movements equations, usually are supposed to be known. We research how forces are and how they change on time.

The mathematical expression of a physics law is a rule that defines a mapping $\mathcal{M}$ of a set of functions $\mathcal{C}$, called parameters, a set of functions $\mathcal{E}$ called results. This rule is usually a set E of equations, in terms of the parameters $\mathcal{C}$, solutions to these equations are the corresponding elements of $\mathcal{E}$, therefore, the definition of $\mathcal{M}$ is given implicitly. Nevertheless, from the sole definition of a mapping, the solution of E must exist and to be unique in $\mathcal{E}$ for any element of $\mathcal{C}$. This is the only constraint that must be asked to E. We call computed results to elements in $\mathcal{E}$ which are thus obtained from those obtained from $\mathcal{C}$. Deriving the computed results for a given element of $\mathcal{C}$ is called to solve the direct problem. Conversely, to obtain the subset of $\mathcal{C}$ that corresponds to a given element of $\mathcal{E}$ is called to solve the inverse problem.

To give physical meaning to $\mathcal{M}$, $\mathcal{E}$ must be such that its elements can be compared with experimental results. From now on, we assume that the result of any relevant measure is an element of a subset, $\mathcal{E}_e$ of $\mathcal{E}$, called the subset of experimental results. $\mathcal{E}$ therefore it contains the union of experimental results and computed results. Also we assume that $\mathcal{E}$ can be given the structure of a metric space. The comparison of a given computed result e_i, and an experimental result, e_j, is then measured by the distance $d(e_i, e_j)$.

The set $\mathcal{C}$ was defined as the set of functions such that E can be solved. Greater limitations often appear when physical properties are taken into consideration. In other words, $\mathcal{C}$ could be the set $\mathcal{S}$ of all functions such that E can be solved and that are consistent with all the "physical information" that comes either of general principles or from previous measures. However, the definition of $\mathcal{S}$ is, in most of the cases, indirect or difficult to be made precise, and since that, one is left the choice of $\mathcal{C}$ as a convenient subset of $\mathcal{S}$, with a clear definition. On the other hand, the aim of incrementing the definition of $\mathcal{C}$ many times gives acces to a new class of parameters for those the direct and inverse problem can be solved.

With these definition, it could seem that all physical problems are inverse problems. Really, one generally reserves this name for problems with precise mathematical expressions for the generalized inverse transformations from $\mathcal{E}$ into $\mathcal{C}$. This excludes the so called data fit process in which models that depend on some parameters and that give a good data fitting of the experimental results are obtained by trial and error or by another technique.

The first person that studied inverse problem of the class that we consider was Lord Rayleigh (1877) [76], who discussed the possibility of infering the density of a string by means of their vibration frequencies. More recently, a generalization was exposed by Marc Kac (1966) [75] in his famous paper: "Can one hear the shape of a drum ?".

V Inverse scattering in quantum mechanics

With the invention of Schrödinger equation, the applicability of spectral problems in partial differential equations to physics problems increased: The type of equations that had only applications to problems of mechanical vibrations in the past, now, they will be used to describe atoms and molecules.

Some experimental results in physics are quantities measured in scattering experiments, e.g., cross sections o related quantities. Since this quantities are associated to an asymptotical behavior of wave functions, we will always consider problems where the set $\mathcal{E}$ consists of "theoretical measurements" of this asymptotical behaviour, e.g., the scattering amplitude or the phase shift. This leads, in a natural way, to the particular problem of building the scattering amplitude from the cross section. Leaving this discussion, the equations E that define the transformation $\mathcal{M}$ consist of a wave equation (e.g., Schrödinger equation, Klein-Gordon equation, Dirac equation with the proper conditions). The sets $\mathcal{C}$ of "parameters" are local or non local potential sets, from which it is possible to predict scattering results.

Inverse scattering problems in quantum mechanics have been extensively studied since the seminal work of W. Heisenberg in the theory of scattering theory in 1943 and 1944, [71–73]. In precise terms, there are three problems of inverse scattering problems in quantum mechanics:

- Uniqueness. To prove that scattering operators uniquely determines potentials.
- Reconstruction. To give methods to reconstruct potentials from a scattering operator.
- Characterization. To give necessary and sufficient conditions for an operator to be the scattering operator associated to a potential that belongs, to a certain given class.

There are different ways to provide scattering information. For example, one can give the scattering operator for all energies, the limit of high energies of the scattering operator, or the scattering operator at fixed energy.

Because all the information that can be obtained on nuclei, physical particles and sub-particles is gotten from scattering experiments, these problems are of obvious physical importance. Moreover, there exists the very related problem of inverse scattering of acoustic, electromagnetic and elastic waves, that has a lot of technological applications, for example, tomography.

Inverse problems are related to the following mathematical tools: advanced results in differential and integral equations theory, harmonic analysis, spectral operator theory, holomorphic functions, asymptotic expansions, numerical analysis, etc.

The majority of contributions to inverse problems in quantum physics use stationary methods. On the contrary, in this work we use a method that depends on time. More information on this subject is found in Chapter 2.

VI Stark Effect

In 1913 Johannes Stark observed that spectral lines in the Balmer series split and shift in the presence of a uniform electric field. This phenomenon was called Stark effect and it is the electric counterpart to the Zeeman effect, where, by the spin of the electron and the presence of a magnetic field, spectral lines also split and shift. One of the first applications of Schrödinger's quantum theory was the explanation of the Stark Effect made by Epstein in 1926 [68]. Currently the Stark Effect is referred to physical phenomena where a uniform electric field is present.

VII Structure of this thesis

The structure of this thesis is the following: This introduction gives the physical motivation and locates the work inside the specific area of inverse problems in non-relativistic quantum mechanics scattering theory. Chapter 2 is an exact transcription of the paper published in the Journal of Mathematical Physics [84].

Assuming measure theory and some notions of general topology then we study Hilbert spaces (chapter 3) necessary to postulate quantum mechanics and to understand our work. We expand our understanding of Fourier transform by studying Fourier transform on groups and we close that chapter with unbounded operators and the construction of their evolution groups, all this material being used in chapter 2.

Finally, in the appendix we give more details on assertions made in chapter 2.

Chapter 2

High-Velocity Estimates and Inverse Scattering for Quantum N-Body Systems with Stark Effect

Abstract

In an N–body quantum system with a constant electric field, by inverse scattering, we uniquely reconstruct pair potentials, belonging to the optimal class of short-range potentials and long-range potentials, from the high-velocity limit of the Dollard scattering operator. We give a reconstruction formula with an error term.

I Introduction

We study the direct and inverse scattering problems for an N–body quantum mechanical system in an $n \geq 2$ dimensional space under Stark effect, i.e. in a constant electric field, with interactions given by pair potentials (multiplication operators).

When we speak of scattering by a potential V, it is common that V is classified as being short-range if the canonical wave operators $W_{\pm}(H_0 + V, H_0)$ exist, where H_0 is the unperturbed Hamiltonian. On the other hand, if they do not exist, we say that we have a long-range potential; in this case we have to modify the free evolution and thus, to define modified wave operators.

As it is well known, the Coulomb potential $V_c(x) = q/|x|$ is long-range when $H_0 = -\triangle$. It is also well known, that V_c is short-range in the case of the Stark effect, where $H_0 = -\triangle - E \cdot x$, and E is a constant electric field. More generally, potentials V that decay at infinity as $V(x) \approx |x|^{-\gamma}$, $\gamma \leq 1$ are long-range when $H_0 = -\triangle$ and on the contrary, when there is a constant electric field, they are short-range if $1/2 < \gamma \leq 1$ and long-range if $0 < \gamma \leq 1/2$.

This feature of the Stark effect is particularly interesting in inverse scattering. For example, because it allows to prove that the Coulomb potential is uniquely determined by the scattering matrix, defined from canonical wave operators, without having to modify the free dynamics, as first proved in [40].

We denote by $m_j \in \mathbb{R}^+, q_j \in \mathbb{R}$ and $\tilde{\mathbf{x}}_j \in \mathbb{R}^n, j = 1, 2, \ldots, N$, respectively, the masses, the charges and the positions of the particles. The free Hamiltonian generates the free time evolution,

$$\tilde{H}_0 = \sum_{j=1}^{N} (2m_j)^{-1} \; \tilde{\mathbf{p}}_j^2 \; + \sum_{j=1}^{N} q_j \mathbf{E} \cdot \tilde{\mathbf{x}}_j, \quad \tilde{\mathbf{p}}_j = -i \nabla_{\tilde{\mathbf{x}}_j}, \tag{II.1.1}$$

where the electric field $\mathbf{E} = (-E, 0, \ldots, 0)$, $E = |\mathbf{E}| > 0$ is directed along minus the first coordinate direction.

We study the system in the center of mass frame and we separate off the motion of the center of mass

$$H_{CM} = (2M)^{-1} (\mathbf{P}_{CM})^2 + Q\mathbf{E} \cdot \mathbf{X}_{CM},$$

where $M = \sum_{j=1}^{N} m_j$, is the total mass, $\mathbf{X}_{CM} = (1/M) \sum_{j=1}^{N} m_j \tilde{\mathbf{x}}_j$, is the center of mass, $\mathbf{P}_{CM} = \sum_{j=1}^{N} \tilde{\mathbf{p}}_j$, is the momentum of the center of mass, $Q = \sum_{j=1}^{N} q_j$, is the total charge.

The free Hamiltonian in the center of mass frame is $H_0 := \tilde{H}_0 - H_{CM}$,

$$H_0 = \sum_{j=1}^{N} (2m_j)^{-1} \, \tilde{\mathbf{p}}_j^2 \, - (2M)^{-1} (\mathbf{P}_{CM})^2 + \sum_{j=1}^{N} (q_j - m_j Q/M) \mathbf{E} \cdot \tilde{\mathbf{x}}_j.$$

H_0 is essentially self-adjoint in the space of Schwartz. We also denote by H_0 the unique self-adjoint extension.

In the center of mass frame the space of states is the Hilbert space, $\mathcal{H}$, represented in configuration space by wave functions ϕ in

$$L^2(\mathbf{X}), \quad \mathbf{X} = \left\{ \tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N) \middle| \sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j = 0 \right\} \cong \mathbb{R}^{n(N-1)} \tag{II.1.2}$$

with the measure induced on $\mathbf{X}$ by the following norm on $\mathbb{R}^{nN}$: $|||\tilde{\mathbf{x}}||| = \left[\sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j^2 \right]^{1/2}$. The space

$$L^2(\hat{\mathbf{X}}), \quad \hat{\mathbf{X}} = \left\{ \tilde{\mathbf{p}} = (\tilde{\mathbf{p}}_1, \ldots, \tilde{\mathbf{p}}_N) \middle| \sum_{j=1}^{N} \tilde{\mathbf{p}}_j = 0 \right\} \cong \mathbb{R}^{n(N-1)}, \tag{II.1.3}$$

where $\hat{\mathbf{X}}$ is equipped with the dual metric induced by $\left[\sum_{j=1}^{N} (m_j)^{-1} \, \tilde{\mathbf{p}}_j^2 \right]^{1/2}$ on $\mathbb{R}^{nN}$, is the set of momentum space wave functions $\hat{\phi}$. Fourier transform maps unitarily $L^2(\mathbf{X})$ onto $L^2(\hat{\mathbf{X}})$. The measures on $\mathbf{X}$ and $\hat{\mathbf{X}}$ are equivalent to Lebesgue measure. Given an (abstract) state $\Phi \in \mathcal{H}$ we use both its configuration or momentum space wave functions where appropriate.

As a general reference for multiparticle scattering see e.g. [36], where Jacobi coordinates are defined

$$\xi_j := \tilde{\mathbf{x}}_{j+1} - \left(\sum_{k=1}^{j} m_k \right)^{-1} \left(\sum_{k=1}^{j} m_k \tilde{\mathbf{x}}_k \right), \quad j = 1, \ldots, N-1. \tag{II.1.4}$$

These coordinates are obtained by first changing variables from $(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2)$ to $\xi_1 = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1$ and the center of mass of particles (1) and (2), $\tilde{\mathbf{R}}_{12} = (m_1 + m_2)^{-1}(m_1 \tilde{\mathbf{x}}_1 + m_2 \tilde{\mathbf{x}}_2)$. Then we change from $(\tilde{\mathbf{R}}_{12}, \tilde{\mathbf{x}}_3)$ to $\xi_2 = \tilde{\mathbf{x}}_3 - \tilde{\mathbf{R}}_{12}$ and the center of mass of particles (1), (2), and (3), and so on. In the end we obtain the Jacobi coordinates $\xi_j, 1 \leq j \leq N-1$, on $\mathbf{X}$ and the center of mass coordinate $\mathbf{X}_{CM}$. In these coordinates H_0 is expressed as

$$H_0 = \sum_{j=1}^{N-1} ((2\nu_j)^{-1} \hat{\mathbf{p}}_j^2 + q_j^R \mathbf{E} \cdot \xi_j), \quad \hat{\mathbf{p}}_j = -i \nabla_{\xi_j}, \tag{II.1.5}$$

where

$$\nu_j^{-1} = m_{j+1}^{-1} + \left(\sum_{k=1}^{j} m_k \right)^{-1}, \quad 1 \leq j \leq N-1,$$

$$q_j^R = (q_{j+1} M_j - m_{j+1} Q_j)/(m_{j+1} + M_j), \quad M_j = \sum_{k=1}^{j} m_k, \tag{II.1.6}$$

$$Q_j = \sum_{k=1}^{j} q_k, \quad 1 \le j \le N-1,$$

ν_j and $q_j^R, 1 \le j \le N-1$, are, respectively, the reduced mass and the relative charge of the particle $(j+1)$ with respect to the masses and the charges of the first j particles. Formula (II.1.5) shows that the proof that H_0 is essentially self-adjoint in the space of Schwartz reduces to the one in the two-body case. The Jacobi coordinates above are based in the pair of particles (1, 2) in the sense that we have taken as the first coordinate $\xi_1 = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1$ the relative distance of the particles (1) and (2). Of course, we can base Jacobi coordinates in any pair of particles $(j,k), j,k = 1,2,...,N$.

In order to determine the potential for a given pair we number the particles in such a way that the given pair consists of particles one and two. By (II.1.5) we write

$$H_0 = \left[(2\nu_1)^{-1} \hat{\mathbf{p}}_1^2 + \frac{(q_2 m_1 - m_2 q_1)}{m_1 + m_2} \mathbf{E} \cdot \xi_1 \right] \otimes I + I \otimes \hat{H}_0, \tag{II.1.7}$$

under the decomposition of $L^2(\mathbf{X})$ as

$$L^2(\mathbf{X}) = L^2(\mathbb{R}^n_{\xi_1}) \otimes \left[\otimes \prod_{j=2}^{N-1} L^2(\mathbb{R}^n_{\xi_j}) \right],$$

where

$$\hat{H}_0 = \sum_{j=2}^{N-1} ((2v_j)^{-1} \hat{\mathbf{p}}_j^2 + q_j^R \mathbf{E} \cdot \xi_j). \tag{II.1.8}$$

This shows that if the relative charge of the pair (1, 2), $(q_2 m_1 - m_2 q_1)/(m_1 + m_2)$, is different from zero the relative distance of the pair (1, 2) is accelerated by the electric field as in the two-body case. However, if the relative charge is zero both particles are accelerated in the same way by the electric field and the relative distance is not accelerated, and then, with respect to the pair (1, 2), the relative scattering is as in the case when the external constant electric field is zero. This shows that, for any given pair of particles, the inverse scattering problem has to be formulated as in the two-body case with no electric field if the relative charge of the pair is zero and, as in the two-body case with an electric field, if the relative charge of the pair is different from zero.

For any given pair of particles we construct as in Enss and Weder [20] appropriate states where all particles have high-velocity relative to each other in order to reconstruct the corresponding pair potential. For this purpose we first introduce some kinematical notation. We use a numbering of the particles such that the pair of interest consists of particles 1 and 2. As usual we take as one n-dimensional variable the relative distance $\mathbf{x}$ and momentum $\mathbf{p}$ of the chosen pair $(1,2)$.

$$\mathbf{x} = \xi_1 := \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \qquad \mathbf{p} = \hat{\mathbf{p}}_1 = -i\nabla_{\mathbf{x}} = \mu_{12}[(-i\nabla_{\tilde{\mathbf{x}}_2}/m_2) - (-i\nabla_{\tilde{\mathbf{x}}_1})/m_1)], \tag{II.1.9}$$

where μ_{12} is the reduced mass of the pair $(1,2)$, $\mu_{12} = m_1 m_2/(m_1+m_2)$. We also use the position $\mathbf{x}_j$ and the momentum $\mathbf{p}_j$ of the jth particle, $j = 1,...,N$, relative to the center of mass of the pair (1,2),

$$\mathbf{x}_j := \tilde{\mathbf{x}}_j - (m_1 \tilde{\mathbf{x}}_1 + m_2 \tilde{\mathbf{x}}_2)/(m_1 + m_2), \quad j = 1, \ldots, N, \tag{II.1.10}$$

$$\mathbf{p}_j = \mu_j (\tilde{\mathbf{p}}_j/m_j - (\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2)/(m_1 + m_2)), \quad j = 1, \ldots, N, \tag{II.1.11}$$

where μ_j is the reduced mass of the jth particle with respect to the center of mass of the pair (1,2),

$$\mu_j = m_j (m_1 + m_2)/(m_j + m_1 + m_2), \quad j = 1, \ldots, N,$$

and $\tilde{\mathbf{p}}_j = -i\nabla_{\tilde{\mathbf{x}}_j}$ is the momentum relative to some origin (see (II.1.1)). Note that $\mathbf{x}$ is the first Jacobi coordinate ξ_1, $\mathbf{p} = \mu_{12}(\tilde{\mathbf{p}}_2/m_2 - \tilde{\mathbf{p}}_1/m_1)$ and $\mathbf{p}_j/\mu_j$ are the relative velocities with respect the center of mass of the distinguished pair $(1,2)$. $\{\mathbf{x}, \mathbf{x}_3, \ldots \mathbf{x}_N\}$ and $\{\mathbf{p}, \mathbf{p}_3, \ldots \mathbf{p}_N\}$ are sets of $N-1$ independent n-dimensional variables in the configuration and momentum space, respectively, relative to the center of mass frame.

Let $\Phi_0 \in \mathcal{H}$ be an asymptotic configuration with the product wave function of the form in momentum space,

$$\Phi_0 \sim \hat{\phi}_{12}(\mathbf{p})\hat{\phi}_3(\mathbf{p}_3, \ldots, \mathbf{p}_N), \tag{II.1.12}$$

where $\hat{\phi}_{12} \in C_0^\infty(\mathbb{R}^n)$ varies while $\hat{\phi}_3 \in C_0^\infty(\mathbb{R}^{n(N-2)})$ is a fixed normalized function with support in $\{(\mathbf{p}_3, \ldots, \mathbf{p}_N) : |\mathbf{p}_j| < \mu_j\}$; i.e., the particles 3 to N have speed smaller than one relative to the pair (1,2). We take an $\eta > 0$ such that $\hat{\phi}_{12} \in C_0^\infty(B_{\mu_{12}\eta})$, where $B_{\mu_{12}\eta}$ denotes the open ball of center zero and radius $\mu_{12}\eta$ in $\mathbb{R}^n$.

The high-velocity state is defined as (see Enss and Weder [20])

$$\Phi_{\mathbf{v}} \sim \hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3, \ldots, \mathbf{p}_N - \mu_N\mathbf{v}_N), \tag{II.1.13}$$

where $\mathbf{v} = v\hat{\mathbf{v}}, |\hat{\mathbf{v}}| = 1, \mathbf{v}_j = v^2\mathbf{d}_j$, with $\mathbf{d}_j \neq 0$, for $j = 3, \ldots, N$ and where we assume that $\mathbf{d}_j - \mathbf{d}_k \neq 0$ for $j, k = 3, \ldots N$. We, moreover, define $\mathbf{v}_1 = -\mathbf{v}\mu_{12}/m_1, \mathbf{v}_2 = \mathbf{v}\mu_{12}/m_2$.

We denote the relative velocities by

$$\mathbf{v}_{jk} = \mathbf{v}_k - \mathbf{v}_j, \quad v_{jk} = |\mathbf{v}_{jk}|, \quad j, k = 1, \ldots, N.$$

Then with $d_j = |\mathbf{d}_j|$,

$$\begin{aligned} \mathbf{v}_{1,j} &= v^2(\mathbf{d}_j + \mu_{12}\hat{\mathbf{v}}/(m_1 v)) \neq 0 \quad \text{if } v > \mu_{12}/(m_1 d_j),\ j = 3, \ldots, N, \\ \mathbf{v}_{2,j} &= v^2(\mathbf{d}_j - \mu_{12}\hat{\mathbf{v}}/(m_2 v)) \neq 0 \quad \text{if } v > \mu_{12}/(m_2 d_j),\ j = 3, \ldots, N, \\ \mathbf{v}_{j,k} &= v^2(\mathbf{d}_k - \mathbf{d}_j) \neq 0 \qquad j, k = 3, \ldots, N. \end{aligned} \tag{II.1.14}$$

We denote $\hat{\mathbf{v}}_{jk} = \mathbf{v}_{jk}/|\mathbf{v}_{jk}|$. We assume for all pairs (j,k) with $q_{j,k} \neq 0$ that $|\hat{\mathbf{v}}_{jk} \cdot \hat{\mathbf{E}}| \leq \delta$ for some $0 \leq \delta < 1$. It follows that in our high-velocity states the relative average velocity of the pair $(1,2)$ is v while all other particles travel with minimal velocity proportional to v^2 relative to each other as well as with respect to particles 1 and 2.

The relative momentum of particles j and k is

$$\mathbf{p}_{jk} = -i\nabla_{(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)}, \tag{II.1.15}$$

where in the derivative the positions of all other particles, as well as of the center of mass, are kept fixed. The relative velocity of the pair (j,k) is

$$\mathbf{p}_{jk}/\mu_{jk} = \tilde{\mathbf{p}}_k/m_k - \tilde{\mathbf{p}}_j/m_j = \mathbf{p}_k/\mu_k - \mathbf{p}_j/\mu_j. \tag{II.1.16}$$

It follows from the definition that $\phi_0 \in \mathcal{S}(\mathbb{R}^{n(N-1)})$ and that

$$\Phi_{\mathbf{v}} = e^{i\mu_{12}\mathbf{v}\cdot\mathbf{x}} \prod_{j=3}^{N} e^{i\mu_j \mathbf{v}_j \cdot \mathbf{x}_j} \Phi_0. \tag{II.1.17}$$

Moreover, by (II.1.10)

$$|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j| = |\mathbf{x}_k - \mathbf{x}_j| \leq |\mathbf{x}_k| + |\mathbf{x}_j|,\ j, k = 1, \cdots, N,\ |\mathbf{x}_1| \leq |\mathbf{x}|, |\mathbf{x}_2| \leq |\mathbf{x}|.$$

Hence, we have good initial localization uniformly in $\mathbf{v}$,

$$\|(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^2 \Phi_{\mathbf{v}}\| \leq C, \quad j, k = 1, \ldots, N. \tag{II.1.18}$$

Additionally, by (1.15) there are functions $f_{jk} \in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$ such that

$$\Phi_{\mathbf{v}} = f_{jk}(\mathbf{p}_{jk} - \mu_{jk}\mathbf{v}_{jk})\Phi_{\mathbf{v}}, \tag{II.1.19}$$

where μ_{jk} is the reduced mass of the pair (j,k),

$$\mu_{jk} = \frac{m_j m_k}{m_j + m_k}. \tag{II.1.20}$$

Furthermore, $\eta_{12} = \eta, \eta_{1j} = 2(1 + \eta\mu_{12}/m_1), \eta_{2j} = 2(1 + \eta\mu_{12}/m_2), j = 3, \dots, N$, and $\eta_{jk} = 4$, for $j,k = 3, \dots, N$.

Note that by (II.1.10)

$$\mathbf{x} = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1 = i\frac{\partial}{\partial \mathbf{p}}, \quad \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j = i\frac{\partial}{\partial \mathbf{p}_k} - i\frac{\partial}{\partial \mathbf{p}_j}, \quad j,k = 3, \dots, N, \tag{II.1.21}$$

$$\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_1 = i\frac{\partial}{\partial \mathbf{p}_k} + \frac{\mu_{12}}{m_1} i\frac{\partial}{\partial \mathbf{p}}, \quad \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_2 = i\frac{\partial}{\partial \mathbf{p}_k} - \frac{\mu_{12}}{m_2} i\frac{\partial}{\partial \mathbf{p}}, \quad k = 3, \dots, N. \tag{II.1.22}$$

As in Enss and Weder [20] and Weder [40], (II.1.18), (II.1.19), (II.1.21) and (II.1.22) allow us to reduce the proofs in the N-body case to the ones for two bodies. We introduce below an appropriate class of potentials where D^β, D^{α_0} denotes the derivative with the usual multi-index notation.

DEFINITION II.1.1. *We denote by $\mathcal{V}_0$ the class of real-valued potentials, $V^0(\mathbf{x})$, defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ such that $V^0(\mathbf{x}) = V^{0,\,vs}(\mathbf{x}) + V^{0,\,l}(\mathbf{x})$ with $V^{0,\,vs}(\mathbf{x}) \in \mathcal{V}_{0,\,vs}$, $V^{0,\,l}(\mathbf{x}) \in \mathcal{V}_{0,\,l}$, where $\mathcal{V}_{0,\,vs}$ is the class of real-valued potentials, $V^{0,\,vs}$, that are relatively bounded with respect to the Laplacian with relative bound zero and*

$$\int_0^\infty dR \left\| V^{0,\,vs}(\mathbf{x})\,(-\Delta + I)^{-1} F(|\mathbf{x}| \geq R) \right\| < \infty. \tag{II.1.23}$$

$\mathcal{V}_{0,\,l}$ is the class of real-valued potentials $V^{0,\,l}$ that satisfy $V^{0,\,l}(\mathbf{x}) \in C^1_\infty(\mathbb{R}^n)$, the space of all continuously differentiable functions that tend to zero at infinity, and that

$$|D^\beta V^{0,\,l}(\mathbf{x})| \quad \leq \quad C(1 + |\mathbf{x}|)^{-\gamma_1}, \quad |\beta| = 1, \gamma_1 > 3/2, \tag{II.1.24}$$

where without loss of generality we assume that $\gamma_1 \leq 2$, otherwise $V^{0,\,l}$ would be of short range.

Let ϵ_0 satisfy: $0 < \epsilon_0 < \gamma_1 - \frac{3}{2}$. After Hörmander [25], we can write, without loss of generality that, for all $V^0(\mathbf{x}) \in \mathcal{V}_0$, $V^0(\mathbf{x}) = V^{0,\,vs}(\mathbf{x}) + V^{0,\,l}(\mathbf{x})$ with $V^{0,\,vs} \in \mathcal{V}_{0,\,vs}$, $V^{0,\,l} \in C^4(\mathbb{R}^n)$ and

$$|D^{\alpha_0} V^{0,\,l}(\mathbf{x})| \leq C(1 + |\mathbf{x}|)^{-1-|\alpha_0|(\epsilon_0 + 1/2)}, \text{ for } 2 \leq |\alpha_0| \leq 4. \tag{II.1.25}$$

The more intuitive condition

$$\int_0^\infty dR \left\| F(|\mathbf{x}| \geq R)\, V^{0,\,vs}(\mathbf{x})\,(-\Delta + I)^{-1} \right\| < \infty,$$

by Reed and Simon [36], is equivalent to the decay property (II.1.23).

DEFINITION II.1.2. *[2]. We denote by $\mathcal{V}_E$ the class of potentials, $V^E(\mathbf{x})$, defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ such that $V^E(\mathbf{x}) = V^{E,\,vs}(\mathbf{x}) + V^{E,\,s}(\mathbf{x}) + V^{E,\,l}(\mathbf{x})$ with $V^{E,\,vs}(\mathbf{x}) \in \mathcal{V}_{E,\,vs}$, $V^{E,\,s}(\mathbf{x}) \in \mathcal{V}_{E,\,s}$, and $V^{E,\,l}(\mathbf{x}) \in \mathcal{V}_{E,\,l}$, where $\mathcal{V}_{E,\,vs}$ is the class of real-valued potentials, $V^{E,\,vs}$, that satisfy $V^{E,\,vs} = V_1^{E,\,vs} + V_2^{E,\,vs}$ with $(1 + |x_1|)V_1^{E,\,vs}$ relatively bounded with respect to the Laplacian with relative bound zero and $V_2^{E,\,vs}$ bounded and that*

$$\int_0^\infty dR \left\| V^{E,\,vs}(\mathbf{x})\,(-\Delta + I)^{-1} F(|\mathbf{x}| \geq R) \right\| < \infty. \tag{II.1.26}$$

$\mathcal{V}_{E,\,s}$ is the class of real-valued potentials $V^{E,\,s}$ that satisfy $V^{E,\,s}(\mathbf{x}) \in C^1(\mathbb{R}^n)$ and that

$$|V^{E,\,s}(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-\gamma}, \tag{II.1.27}$$
$$|D^\beta V^{E,\,s}(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-1-\alpha}, \quad |\beta|=1, \tag{II.1.28}$$

with some $1/2 < \alpha \leq \gamma \leq 1$. $\mathcal{V}_{E,l}$ is the class of real-valued potentials $V^{E,l}$ that satisfy $V^{E,l}(\mathbf{x}) \in C^2(\mathbb{R}^n)$ and that

$$|D^\beta V^{E,\,l}(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-\gamma_D-\mu|\beta|}, \quad |\beta| \leq 2, \tag{II.1.29}$$

with $0 < \gamma_D \leq 1/2$ and $1-\gamma_D < \mu \leq 1$.

The class of potentials $\mathcal{V}_E$ in Definition II.1.2 is the same as in Adachi and Maehara [2]. Again we can assume, without loss of generality by [25], that for all $V^E(\mathbf{x}) \in \mathcal{V}_E$, $V^E(\mathbf{x}) = V^{E,\,vs}(\mathbf{x}) + V^{E,\,s}(\mathbf{x}) + V^{E,\,l}(\mathbf{x})$ with $V^{0,\,vs} \in \mathcal{V}_{0,\,vs}$, $V^{E,\,s}(\mathbf{x}) \in \mathcal{V}_{E,\,s}$, $V^{E,\,l} \in C^4(\mathbb{R}^n)$ with $V^{E,\,l}$ satisfying (II.1.29) and

$$|D^\beta V^{E,\,l}(\mathbf{x})| \leq C(1+|\mathbf{x}|)^{-\gamma_D-\mu(2+|\beta|)/2}, \quad 3 \leq |\beta| \leq 4. \tag{II.1.30}$$

We call the potentials $V^{0,\,vs}$ and $V^{E,\,vs}$ very short-range, the potential $V^{E,\,s}$ short-range and the potentials $V^{0,\,l}$ and $V^{E,\,l}$ long-range.

For a particle with mass m and charge q, there is a formula for the free time evolution, it was proven simultaneously by Avron and Herbst [8] and by Veselić and Weidmann [39]. There is also a generalization for the time-dependent case considered by Kitada and Yajima [31],

$$e^{-it(\mathbf{p}^2/(2m)-qE\mathbf{x}_1)} = e^{iqE\mathbf{x}_1 t} e^{-it^3q^2E^2/(6m)} e^{-i\mathbf{p}_1 qEt^2/(2m)} e^{-it\mathbf{p}^2/(2m)}. \tag{II.1.31}$$

We will also make frequent use of the following relations that are obtained under translation in configuration or momentum space generated by $\mathbf{x}$ or $\mathbf{p}$, respectively,

$$e^{i\mathbf{p}\cdot\mathbf{v}t} f(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{v}t} = f(\mathbf{x}+\mathbf{v}t), \tag{II.1.32}$$

$$e^{-im\mathbf{v}\cdot\mathbf{x}} f(\mathbf{p}) e^{im\mathbf{v}\cdot\mathbf{x}} = f(\mathbf{p}+m\mathbf{v}), \tag{II.1.33}$$

for any measurable and bounded function f. In particular, (II.1.33) implies that

$$e^{-im\mathbf{v}\cdot\mathbf{x}} e^{-it\mathbf{p}^2/(2m)} e^{im\mathbf{v}\cdot\mathbf{x}} = e^{-i\mathbf{p}\cdot\mathbf{v}t} e^{-it\mathbf{p}^2/(2m)} e^{-imv^2t/2}, \tag{II.1.34}$$

where $v = |\mathbf{v}|$. Since $e^{it\mathbf{p}^2/(2m)}\,\mathbf{x}\,e^{-it\mathbf{p}^2/(2m)} = \mathbf{x} + t\,\mathbf{p}/m$ and functional calculus,

$$e^{it\mathbf{p}^2/(2m)}\, f(\mathbf{x})\, e^{-it\mathbf{p}^2/(2m)} = f(\mathbf{x}+t\,\mathbf{p}/m). \tag{II.1.35}$$

We denote by $\mathbf{e}_1 = (1,0,\ldots,0)$ the unit vector along the x_1 direction and $\hat{\mathbf{E}} = \mathbf{E}/|\mathbf{E}|$. We designate by $q_{jk} = (q_k m_j - q_j m_k)/(m_j + m_k)$ the relative charge of the pair (j,k) and we denote by $\sum_{j<k}^0$ and $\sum_{j<k}^E$, respectively, the sum over all indices, $j<k, j,k = 1,\ldots,N$, with $q_{jk} = 0$, and $q_{jk} \neq 0$.

We assume that the potential of the N-body system is a multiplication operator that is a sum of pair potentials,

$$V = \sum_{j<k}^{0} V_{jk}^0(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + \sum_{j<k}^{E} V_{jk}^E(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j), \tag{II.1.36}$$

with $V_{jk}^0 \in \mathcal{V}_0$ (see Definition II.1.1), and $V_{jk}^E \in \mathcal{V}_E$ (see Definition II.1.2). By using a decomposition of H_0 as in (II.1.7) for each pair (j,k) we see that each of the pair potentials V_{jk}^0 and V_{jk}^E are relatively bounded with respect to H_0 with relative bound zero. Note that for a given pair the corresponding pair potential belongs to $\mathcal{V}_0$ if the relative charge of the pair is zero and that it belongs to $\mathcal{V}_E$ if the relative charge is different from zero. Then V is relatively bounded with respect to H_0 with relative bound zero and the interacting Hamiltonian,

$$H = H_0 + V, \tag{II.1.37}$$

is self-adjoint on $D(H) = D(H_0)$.

It is convenient to split the potential into the very short-, short- and long-range potentials. For this purpose we define

$$\mathcal{V}_{VSR} = \left\{ V^{VS} = \sum_{j<k}^{0} V_{jk}^{0,\,vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + \sum_{j<k}^{E} V_{jk}^{E,\,vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \;\Big|\; V_{jk}^{0,\,vs} \in \mathcal{V}_{0,\,vs},\; V_{jk}^{E,\,vs} \in \mathcal{V}_{E,\,vs} \right\}, \tag{II.1.38}$$

$$\mathcal{V}_{SR} = \left\{ V^{S} = \sum_{j<k}^{E} V_{jk}^{E,\,s}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \;\Big|\; V_{jk}^{E,\,s} \in \mathcal{V}_{E,\,s} \right\}, \tag{II.1.39}$$

$$\mathcal{V}_{LR} = \left\{ V^{L} = \sum_{j<k}^{0} V_{jk}^{0,\,l}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + \sum_{j<k}^{E} V_{jk}^{E,\,l}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \;\Big|\; V_{jk}^{0,\,l} \in \mathcal{V}_{0,\,l}, V_{jk}^{E,\,l} \in \mathcal{V}_{E,\,l} \right\}. \tag{II.1.40}$$

Then

$$V = V^{VS} + V^{S} + V^{L}, \qquad H = H_0 + V = H_0 + V^{VS} + V^{S} + V^{L}. \tag{II.1.41}$$

Let $S^D = S^D(V^L; V^{VS} + V^S)$ be the Dollard modified scattering operator defined in equation (II.2.10) below.

Our main results are the reconstruction formulae given in Theorems II.2.8 and II.2.10 that we prove in Section II. The uniqueness result given in Theorem II.1.3 follows from Theorem II.2.8.

THEOREM II.1.3. *Let γ_1 be as in Definition II.1.1 and, γ_D and μ as in Definition II.1.2. If there are two pairs $1 \leq j < k \leq N$, $1 \leq j' < k' \leq N$, with $q_{jk} \neq 0$ and $q_{j'k'} = 0$ we assume that $\gamma_1 > 3 - 4(\gamma_D + \mu)/3$. Then,*

1. *Suppose that $V^i = V^{VS,\,i} + V^{S,\,i} + V^{L,\,i} \in \mathcal{V}_{VSR} + \mathcal{V}_{SR} + \mathcal{V}_{LR}$, $i = 1, 2$, and that $S^D(V^{L,\,1}; V^{VS,\,1} + V^{S,\,1}) = S^D(V^{L,\,2}; V^{VS,\,2} + V^{S,\,2})$. Then, $V^1 = V^2$.*

2. *Furthermore, it is possible to uniquely reconstruct the total potential V from any Dollard scattering operator S^D.*

REMARK II.1.4. Note that in item 1 of Theorem II.1.3 it is enough to assume that the high-velocity limits of $S^D(V^{L,\,1}; V^{VS,\,1} + V^{S,\,1})$ and $S^D(V^{L,\,2}; V^{VS,\,2} + V^{S,\,2})$ are the same. Furthermore, we prove item 2 of Theorem II.1.3 giving a method for the unique reconstruction of V from the high-velocity limit of any Dollard scattering operator. See the reconstruction formulae (II.2.44), (II.2.75) and the proof of Theorem II.1.3.

REMARK II.1.5. For a given $V^L \in \mathcal{V}_{LR}$ let us define, as in [2] and [40], the scattering map $S_1 := S^D(V^L; \cdot)$, $S_1(Q) = S^D(V^L; Q), Q \in \mathcal{V}_{VSR} + \mathcal{V}_{SR}$, an operator from $\mathcal{V}_{VSR} + \mathcal{V}_{SR}$ into the Banach space $\mathcal{L}(\mathcal{H})$ of all bounded operators in $\mathcal{H}$. Clearly, Theorem II.1.3 implies that $S_1 = S^D(V^L; \cdot)$ is injective.

REMARK II.1.6. For a given $V^L \in \mathcal{V}_{LR}$ and a given $V^S \in \mathcal{V}_{SR}$ we define the scattering map $S_2 := S^D(V^L; \cdot + V^S)$, $S_2(V^{VS}) = S_2(V^L; V^{VS} + V^S)$, an operator from $\mathcal{V}_{VSR}$ into $\mathcal{L}(\mathcal{H})$. It is immediate that Theorem II.1.3 implies that $S_2 = S^D(V^L; \cdot + V^S)$ is injective. However, as we show in Remark II.2.11 this result can also be proven using the reconstruction formula (II.2.75) given in Theorem II.2.10, that is simpler than the formula (II.2.44) in Theorem II.2.8, because in (II.2.75) it is not necessary to take the commutator of S^D with a component of the momentum operator. This is important in applications where the tail at infinity of the potential is already known and one wishes to uniquely reconstruct V^{VS} assuming that V^S and V^L are known.

REMARK II.1.7. Under the Stark effect, for a pair potential where the relative charge is not zero, the short-range decay rate at infinity of this potential depends on γ given in our equation (II.1.27). Theorem II.1.3 is proved by the first time by Weder [40], where he considers $\gamma > 3/4$ and N-Body pair potentials which are short-range if the corresponding relative charge is not zero and long-range if the corresponding relative charge is zero. Then, for two body short-range potentials, Nicoleau [34] proves this Theorem with $\gamma > 1/2$, the dimension of the space $n \geq 3$ and the regularity and decay of the potential:

$$V: \quad |\partial_x^\beta V(x)| \leq C_\beta (1+|x|)^{-\gamma-|\beta|}, \tag{II.1.42}$$

for all multi-index β. Later, in the two-body case, Adachi and Maehara [2] improve the results of Nicoleau [34] because, besides $\gamma > 1/2$, they relax the conditions on the derivatives on the potential and use dimension $n \geq 2$. Furthermore, Adachi and Maehara [2] consider long-range potentials whereas Nicoleau [34] does not.

We improve the N-body results of Weder [40]. Our potential V is given, by

$$V = \sum_{j<k}^{0} \Big(V_{jk}^{0,\,vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V_{jk}^{0,\,l}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \Big) + \sum_{j<k}^{E} \Big(V_{jk}^{E,\,vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V_{jk}^{E,\,s}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V_{jk}^{E,\,l}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) \Big)$$

where, for all $1 \leq j < k \leq N$, $V_{jk}^{0,\,vs} \in \mathcal{V}_{0,\,vs}$, $V_{jk}^{0,\,l} \in \mathcal{V}_{0,\,l}$, $V_{jk}^{E,\,vs} \in \mathcal{V}_{E,\,vs}$, $V_{jk}^{E,\,s} \in \mathcal{V}_{E,\,s}$, $V_{jk}^{E,\,l} \in \mathcal{V}_{E,\,l}$. Our potential $V_{jk}^{E,\,l}$ has no counterpart in [40], i.e. potentials that are long-range with respect to the Stark effect, when the relative charge $q_{jk} \neq 0$, are not allowed in [40] whereas, here, we do. This is our first improvement over [40]. Secondly, in equation (1.4) of [40] $\gamma > 3/4$ and in our equation (II.1.27) we have $\gamma > 1/2$, thus we improve the results of [40] because our potential $V_{jk}^{E,\,s}$ is allowed to have the optimal short-range decay rate at infinity.

We give a reconstruction formula with an error term that goes to zero as an inverse power of the velocity, that depends on the decay rate of the potentials, see Theorems II.2.8 and II.2.10. If we only assume (II.1.26), our results coincide with those of Adachi and Maehara [2], in the case $N = 2$ and $q_{12} \neq 0$. If, instead of (II.1.26), we assume (II.2.21), we give a sharper error term than theirs. In this sense, we can say that we obtain a new result, even in the two body case. ■

In this paper, we prove Theorem II.1.3 by extending to the N-body case the results obtained, in the two-body case, by Adachi and Maehara [2] using the the findings published in 1993 [18], 1994 [19], 1995 [20] by V. Enss and R. Weder where a new time-dependent method was developed. Here, physical propagation properties of finite energy wave functions are used to estimate the high-velocity behavior of solutions of the Schrödinger equation and solve inverse scattering problems in quantum mechanics. It is intuitive from the point of view of the physics related to the problem. Contrary to the stationary approach, this method can be applied to study non-linear equations [37, 42–48, 50, 52–54]. Lately, this time-dependent approach [51] has been exploited to study: Hamiltonians with electric and magnetic fields [5–7, 30], N-body systems [19–21, 40, 41], the Stark effect [1, 2, 34, 35, 40], the Aharanov-Bohm effect [9–11, 33, 49, 55, 56], time-dependent potentials [35, 41], Dirac equation [16, 26–29], Klein-Gordon equation [16, 17, 45, 50], mass and charge of black holes [12, 13], amongst others.

II Reconstruction Formulae

Let us define

$$V_{jk}^{vs} = \begin{cases} V_{jk}^{0,\,vs}, & \text{if } q_{jk} = 0, \\ V_{jk}^{E,\,vs}, & \text{if } q_{jk} \neq 0, \end{cases} \quad V_{jk}^{s} = \begin{cases} 0, & \text{if } q_{jk} = 0, \\ V_{jk}^{E,\,s}, & \text{if } q_{jk} \neq 0, \end{cases} \quad V_{jk}^{l} = \begin{cases} V_{jk}^{0,\,l}, & \text{if } q_{jk} = 0, \\ V_{jk}^{E,\,l}, & \text{if } q_{jk} \neq 0. \end{cases} \tag{II.2.1}$$

where $V_{jk}^{0,\,vs}$ and $V_{jk}^{0,\,l}$ are defined in Definition II.1.1, and $V_{jk}^{E,\,vs}, V_{jk}^{E,\,s}$ and $V_{jk}^{E,\,l}$ are defined in Definition II.1.2. Moreover, the decays of $V_{jk}^{0,\,l}$ and $V_{jk}^{E,\,l}$ are to be taken as in (II.1.25), (II.1.29) and (II.1.30), respectively.

We introduce the Graf-type modifier (II.2.2) Graf [22] and Zorbas [57], to define auxiliary wave operators, whose existence and completeness were proven in the N body case for long-range Stark Hamiltonians by Adachi and Tamura [3]. We note that the $\mathbf{v}$ dependence of the Graf-type modifier (II.2.2) is first introduced in Adachi and Maehara [2] by taking into account the $\mathbf{v}$ dependence of $\Phi_{\mathbf{v}}$. In Graf [22] and Zorbas [57], $\mathbf{v}$ is taken 0 in the definiton of the Graf-type modified wave operators. We define the Graf-type modifier [2], [22], [57] and the Dollard-type modifier by (II.2.2), and (II.2.3), respectively

$$\tilde{U}_{G,v}(t) = \exp\left(-i\sum_{j<k}^{E}\int_0^t ds\, V_{jk}^{E,\,s}(\mathbf{v}_{jk}\, s + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk}))\right), \tag{II.2.2}$$

$$\tilde{U}_{D}(t) = \exp\left(-i\sum_{j<k}\int_0^t ds\, V_{jk}^{l}(s\mathbf{p}_{jk}/\mu_{jk} + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk}))\right). \tag{II.2.3}$$

For completeness we mention, for the short-range case, the modified Graf propagator, the modified Graf wave operators [22], [57] and the wave operators for the free channel which are defined, respectively, by (II.2.4) (II.2.5) and (II.2.6):

$$U^{G,v}(t) = e^{-itH_0}\tilde{U}_{G,v}(t), \tag{II.2.4}$$

$$\Omega_{\pm}^{G,v} = \mathrm{s}-\lim_{t\to\pm\infty} e^{itH}\, U^{G,v}(t), \tag{II.2.5}$$

$$W_{\pm} = \mathrm{s}-\lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}. \tag{II.2.6}$$

The modified Dollard-Graf propagator, the modified Dollard-Graf wave operators [22], [57] and the modified Dollard wave operators for the free channel are defined, respectively, by (II.2.7) (II.2.8) and (II.2.9):

$$U^{D,G,v}(t) = e^{-itH_0}\tilde{U}_{D}(t)\tilde{U}_{G,v}(t), \tag{II.2.7}$$

$$\Omega_{\pm}^{D,G,v} = s-\lim_{t\to\pm\infty} e^{itH}\, U^{D,G,v}(t), \tag{II.2.8}$$

$$W_{\pm}^{D} = s-\lim_{t\to\pm\infty} e^{itH}\, e^{-itH_0}\tilde{U}_{D}(t). \tag{II.2.9}$$

Tamura proved the existence of the $W_{\pm}$ for short range N-Body Stark systems [38], Korotyaev [32] did it for the case N=3. Adachi and Tamura [4], and, Herbst, Møller and Skibsted [24] proved the existence of $W_{\pm}^{D}$ given by (II.2.9) for the N-Body long-range case. Actually, the existence of the $W_{\pm}^{D}$ and $W_{\pm}$ also follows from our estimates. We give the simple proof of the existence of $\Omega_{\pm}^{D,G,v}$ and $\Omega_{\pm}^{G,v}$ in Proposition II.2.6. The Dollard scattering operator S^D from the free channel to the free channel is defined as

$$S^D = S^D(V^L; V^{VS} + V^S) := (W_{+}^{D})^* W_{-}^{D}, \tag{II.2.10}$$

S^D is not unique because there is more than one short- and long-range splitting of the potential. We also mention the scattering operator S from the free channel to the free channel defined for the short-range case as

$$S = (W_+)^* W_-. \tag{II.2.11}$$

Proposition II.2.1, below, shall be frequently used in this text. Its proof is given in the Proposition 2.10 in Enss [15].

PROPOSITION II.2.1. *For any* $f \in C_0^{\infty}(\mathbb{R}^n)$ *with* $\operatorname{supp} f \subset B_{m\eta_0}$, *for some* $m, \eta_0 > 0$ *and any* $l = 1, 2, 3, \ldots$ *there is a constant* C_l *such that the following estimate is true:*

$$\left\| F(\mathbf{x}\in\mathcal{M}')\, e^{-it\mathbf{p}^2/(2m)} f(\mathbf{p}-m\mathbf{v}) F(\mathbf{x}\in\mathcal{M}) \right\| \le C_l(1+r+|t|)^{-l},$$

for every $\mathbf{v}\in\mathbb{R}^n$, $t\in\mathbb{R}$ *and any measurable sets* $\mathcal{M}'$, $\mathcal{M}$ *such that* $r := \mathit{dist}(\mathcal{M}',\, \mathcal{M}+\mathbf{v}t) - \eta_0|t| > 0$.

To treat the case whether or not the relative charge q_{jk} is zero, we define

$$\delta_{jk} := \begin{cases} \delta, & \text{if } q_{jk} \neq 0, \\ 0, & \text{if } q_{jk} = 0. \end{cases} \tag{II.2.12}$$

where δ is such that $|\hat{\mathbf{v}}_{jk} \cdot \hat{\mathbf{E}}| \leq \delta < 1$, for all integers $1 \leq j < k \leq N$ with $q_{jk} \neq 0$.

A cornerstone throughout this work is the existence of $0 < \delta_1, \delta_2 \leq 1$ such that

$$|\mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})| \geq \sqrt{\delta_1 |vt|^2 + \delta_2 (q_{12}E/(2\mu_{12}))^2 t^4} \geq \sqrt{\delta_1}|vt|. \tag{II.2.13}$$

When $q_{12} = 0$, we can take $\delta_1 = \delta_2 = 1$, and if $q_{12} \neq 0$, we use $\delta_1 = \delta_2 = 1 - \delta$, Moreover, if $0 \leq \tilde{\sigma} \leq 1$, $q_{12} \neq 0$, $|\mathbf{p}| \leq \mu_{12}\eta$, and $\eta/v < \sqrt{1-\delta}/4$, from a simple computation, there exist two positive constants c_1 and c_2 such that

$$\begin{aligned} |t\,\mathbf{p}/\mu_{12} + \mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})| &\geq c_1 |vt|, \\ |t\,\mathbf{p}/\mu_{12} + \mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})| &\geq c_2 t^2, \\ |t\,\mathbf{p}/\mu_{12} + \mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})| &\geq c_1^{\tilde{\sigma}} c_2^{1-\tilde{\sigma}} |vt|^{\tilde{\sigma}} t^{2(1-\tilde{\sigma})}. \end{aligned} \tag{II.2.14}$$

For any pair (j,k), we establish three conditions: ζ^a_{jk} as "$\gamma_1 < 2$ and there is, at least, one pair (j',k') with $q_{j'k'} = 0$, $V^l_{j',k'} \neq 0$, and either $j' = j$ or $j' = k$ or $k' = j$ or $k' = k$ or $j' + j = 3$", ζ^b_{jk} as "$\gamma_1 = 2$ and there is, at least, one pair (j',k') with $q_{j'k'} = 0$, $V^l_{j',k'} \neq 0$, and either $j' = j$ or $j' = k$ or $k' = j$ or $k' = k$ or $j' + j = 3$", and ζ^c_{jk} as "there is no pair (j',k') with $q_{j'k'} = 0$, $V^l_{j',k'} \neq 0$, and either $j' = j$ or $j' = k$ or $k' = j$ or $k' = k$ or $j' + j = 3$". We define the following constant, for any $\epsilon > 0$:

$$\theta_{jk} := \begin{cases} 2 - \gamma_1, & \text{if } \zeta^a_{jk}, \\ \epsilon, & \text{if } \zeta^b_{jk}, \\ 0, & \text{if } \zeta^c_{jk}. \end{cases} \tag{II.2.15}$$

LEMMA II.2.2. *Let $\tilde{U}_D(t)$ be given as in* (II.2.3)*. Then there exists a constant C, such that for all $t \in \mathbb{R}$, for every $\mathbf{v}_{jk} \in \mathbb{R}^n$, as in* (II.1.14)*, with $v_{jk} \geq 4\eta_{jk}/\sqrt{1-\delta_{jk}}$ and $v = v_{12}$, for all $f_{jk} \in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$, for all integers $1 \leq j < k \leq N$, for all $\kappa > 0$ and $0 < \tilde{\epsilon} < \min\{4\epsilon_0,\, 2\gamma_D + 5\mu + \epsilon_0 - 5/2,\, 2\gamma_D + 6\mu - 3\}$, one has that*

$$\begin{aligned} A_{jk} &:= \left\| (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\ \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-1/2} \right\| \\ &\leq C \begin{cases} 1 + v_{jk}^{-(2-\gamma_1/2)} |v_{jk}t|^{2-\gamma_1}, & \text{if } \zeta^a_{jk}, \\ 1 + v_{jk}^{-1} \ln(1 + v_{jk}^{-1/2}|v_{jk}t|), & \text{if } \zeta^b_{jk}, \\ 1, & \text{if } \zeta^c_{jk}, \end{cases} \leq C\left(1 + v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{\theta_{jk}}\right), \end{aligned} \tag{II.2.16}$$

$$\begin{aligned} B_{jk} &:= \left\| F(|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j| > \kappa |v_{jk}t|)\, \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-2} \right\| \\ &\leq C\,(1 + |v_{jk}t|)^{-2-\tilde{\epsilon}}. \end{aligned} \tag{II.2.17}$$

PROOF.

By (II.1.21) and (II.1.22), for $1 \leq a \leq 4$, multiplication by $(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a$ becomes derivatives in the $\mathbf{p}, \mathbf{p}_k, k = 3, \ldots, N$ variables.

The norm

$$\left\| (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a\ \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-a/2} \right\|$$

is bounded by a finite sum of terms of the form $C\prod_{b=1}^{a} I_{\beta_b}$, with $I_{\beta_b}=1$ if the multi-index $\beta_b=0$ and if $|\beta_b|>0$,

$$I_{\beta_b}=\left\|\int_0^t |s|^{|\beta_b|}\left(D^{\beta_b}V^l_{j'k'}\right)\left(s(\mathbf{p}_{j'k'}/\mu_{j'k'}+\mathbf{v}_{j'k'})+\mathbf{e}_1 q_{j'k'}Es^2/(2\mu_{j'k'})\right)ds\, g_{j'k'}(\mathbf{p}_{j'k'})\right\|, \tag{II.2.18}$$

where (j',k') is a pair of integers $1\le j'<k'\le N$ such that $j'=j$ or $j'=k$ or $k'=j$ or $k'=k$ or $j'+j=3$, $g_{j'k'}\in C_0^\infty(B_{\mu_{j'k'}\eta_{j'k'}})$ and $g_{j'k'}=1$ in the support of $f_{j'k'}$. Note that $\sum_{b=1}^{a}|\beta_b|\le a$.

Below, we take $\tilde\sigma=0$, if $q_{j'k'}\neq 0$ and $\tilde\sigma=1$, if $q_{j'k'}=0$. We define

$$i_{\beta_b,v_{j'k'}}(s):=\begin{cases}(1+|v_{j'k'}s|)^{-\gamma_1}, & \text{if } q_{j'k'}=0 \text{ and } |\beta_b|=1,\\ (1+|v_{j'k'}s|)^{-1-|\beta_b|(\epsilon_0+1/2)}, & \text{if } q_{j'k'}=0 \text{ and } 2\le|\beta_b|\le 4,\\ (1+|s|^2)^{-\gamma_D-\mu|\beta_b|}, & \text{if } q_{j'k'}\neq 0 \text{ and } 1\le|\beta_b|\le 2,\\ (1+|s|^2)^{-\gamma_D-\mu(2+|\beta_b|)/2}, & \text{if } q_{j'k'}\neq 0 \text{ and } 3\le|\beta_b|\le 4,\end{cases} \tag{II.2.19}$$

It follows from (II.1.24), (II.1.25), (II.1.29), (II.1.30), (II.2.13), (II.2.14), (II.2.18) and (II.2.19) that

$$I_{\beta_b}\quad\le\quad C\int_0^{|t|} s^{|\beta_b|} i_{\beta_b,v_{j'k'}}(s)\,ds\le Cv_{j'k'}^{-(|\beta_b|+1)\tilde\sigma/(2-\tilde\sigma)}\int_0^{v_{j'k'}^{\tilde\sigma/(2-\tilde\sigma)}|t|}\tau^{|\beta_b|}i_{\beta_b,1}(\tau)\,d\tau. \tag{II.2.20}$$

Let us assume $|\beta_b|=1$ in (II.2.20). If $q_{j'k'}\neq 0$,

$$I_{\beta_b}\le C\int_0^{|t|}\tau(1+\tau)^{2(-\gamma_D-\mu)}\,d\tau\le C.$$

If $q_{j'k'}=0$, we have that

$$I_{\beta_b}\quad\le\quad Cv_{j'k'}^{-2}\int_0^{v_{j'k'}|t|}\tau(1+\tau)^{-\gamma_1}\,d\tau\le Cv_{j'k'}^{-2}\begin{cases}(1+|v_{j'k'}t|)^{2-\gamma_1}, & \text{if } \gamma_1<2,\\ \ln(1+|v_{j'k'}t|), & \text{if } \gamma_1=2.\end{cases}$$

$$\le\quad C\begin{cases}1+v_{jk}^{-2}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2 \text{ and either } (j',k')=(j,k)=(1,2),\\ & \text{or } (j',k')\neq(1,2) \text{ and } (j,k)\neq(1,2),\\ 1+v_{jk}^{-(2+\gamma_1)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2, (j',k')\neq(1,2) \text{ and } (j,k)=(1,2),\\ 1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2, (j',k')=(1,2) \text{ and } (j,k)\neq(1,2)\\ v_{jk}^{-2}\ln(1+|v_{jk}t|), & \text{if } \gamma_1=2 \text{ and either } (j',k')=(j,k)=(1,2),\\ & \text{or } (j',k')\neq(1,2) \text{ and } (j,k)\neq(1,2),\\ v_{jk}^{-4}\ln(1+|v_{jk}^2t|), & \text{if } \gamma_1=2, (j',k')\neq(1,2) \text{ and } (j,k)=(1,2),\\ v_{jk}^{-1}\ln(1+|v_{jk}^{1/2}t|), & \text{if } \gamma_1=2, (j',k')=(1,2) \text{ and } (j,k)\neq(1,2).\end{cases}$$

This implies that (II.2.16) is true.

In the other hand, similarly to (II.2.20), we have that,

$$\begin{aligned}I_{\beta_b}\quad&\le\quad C\int_0^{|t|}s^{|\beta_b|}i_{\beta_b,v}(s)\,ds\le Cv^{-(|\beta_b|+1)\tilde\sigma/(2-\tilde\sigma)}\int_0^{v^{\tilde\sigma/(2-\tilde\sigma)}|t|}\tau^{|\beta_b|}i_{\beta_b,1}(\tau)\,d\tau\\ &\le\quad C\begin{cases}1+|vt|^{|\beta_b|(-\epsilon_0+1/2)}, & \text{if } q_{j'k'}=0 \text{ and } 1\le|\beta_b|\le 4,\\ 1, & \text{if } q_{j'k'}\neq 0 \text{ and } 1\le|\beta_b|\le 2,\\ 1+|vt|^{\max\{|\beta_b|+1-2\gamma_D-(|\beta_b|+2)\mu,\,0\}}, & \text{if } q_{j'k'}\neq 0 \text{ and } 3\le|\beta_b|\le 4.\end{cases}\end{aligned}$$

Then, it follows that

$$C\prod_{b=1}^{4} I_{\beta_b} \le C(1+|vt|)^{2-\tilde{\epsilon}},$$

hence

$$\begin{aligned}\kappa^4|v_{jk}t|^4\,B_{jk} &\le \left\| |\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^4 F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|>\kappa|v_{jk}t|)\,\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\right\| \\ &\le C(1+|vt|)^{2-\tilde{\epsilon}} \le C(1+|v_{jk}t|)^{2-\tilde{\epsilon}}.\end{aligned}$$

This proves Equation (II.2.17). ∎

Lemma (II.2.3), below, is a generalization of equations (3.8) and (3.17) in Weder [40]. Note that conditions (II.1.23) and (II.1.26) imply that

$$\|V^{vs}(\mathbf{x})g(\mathbf{p})F(|\mathbf{x}|\ge R)\|$$

is an integrable function of R for all $g\in C_0^\infty(\mathbb{R}^n)$ (see Corollary 2.4 in Enss [15]). It follows that potentials in $\mathcal{V}_{E,vs}$ and $\mathcal{V}_{0,vs}$, satisfy condition (II.2.21) below with $\rho=0$. Of course, larger ρ means faster decay.

LEMMA II.2.3. *Suppose that V_{jk}^{vs} is given as in* (II.2.1) *and satisfies*

$$(1+R)^\rho\|V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)g(\mathbf{p}_{jk})F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|\ge R)\|\in L^1((0,\infty),dR), \tag{II.2.21}$$

for some $0\le\rho\le 1$ and all $g\in C_0^\infty(\mathbb{R}^n)$, $\tilde{U}_D(t)$ is given as in (II.2.3). *Then, for all functions $f_{j'k'}\in C_0^\infty(B_{\mu_{j'k'}\eta_{j'k'}})$ with $1\le j'<k'\le N$, there is a function h_{jk} with $(1+\tau)^\rho h_{jk}(\tau)\in L^1((0,\infty))$ such that for every $\mathbf{v}_{jk}\in\mathbb{R}^n$ with $v_{jk}>c$ for some constant $0<c$, we have the following estimate, for all integers $1\le j<k\le N$:*

$$D_{jk} := \left\|V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\,e^{-itH_0}\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\right\| \le h_{jk}(|v_{jk}t|). \tag{II.2.22}$$

Proof. Let us take $g_{jk}\in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$ that satisfies $g_{jk}\equiv 1$ on the support of f_{jk}.

$$D_{jk} \le I_1+I_2+I_3, \tag{II.2.23}$$

where, for any positive constant λ,

$$\begin{aligned} I_1 = {} & \left\|V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)g_{jk}(\mathbf{p}_{jk}-\mu_{jk}\mathbf{v}_{jk})\right\|\,\big\|F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j-\mathbf{v}_{jk}t-\mathbf{e}_1q_{jk}Et^2/(2\mu_{jk})|\ge\lambda|v_{jk}t|5/8)\,e^{-itH_0} \\ & \times g_{jk}(\mathbf{p}_{jk}-\mu_{jk}\mathbf{v}_{jk})F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|<\lambda|v_{jk}t|/8)\big\| \\ & \times\left\|\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\right\|,\end{aligned}$$

$$\begin{aligned} I_2 = {} & \left\|V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)g_{jk}(\mathbf{p}_{jk}-\mu_{jk}\mathbf{v}_{jk})F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j-\mathbf{v}_{jk}t-\mathbf{e}_1q_{jk}Et^2/(2\mu_{jk})|\ge\lambda|v_{jk}t|5/8)\,e^{-itH_0}\right\| \\ & \times\left\|F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|\ge\lambda|v_{jk}t|/8)\,\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\right\|,\end{aligned}$$

$$\begin{aligned} I_3 = {} & \left\|V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)g_{jk}(\mathbf{p}_{jk}-\mu_{jk}\mathbf{v}_{jk})F(|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j-\mathbf{v}_{jk}t-\mathbf{e}_1q_{jk}Et^2/(2\mu_{jk})|<\lambda|v_{jk}t|5/8)\right\| \\ & \times\left\|e^{-itH_0}\,\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\right\|.\end{aligned}$$

We give the proof for the pair $(1, 2)$, the other cases are similar, using Jacobi coordinates based in the pair (j, k). Let us set $\mathbf{x} = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \mathbf{p} = -i\nabla_{\mathbf{x}}$ we obtain as in (II.1.7) that

$$H_0 = \left[(2\nu_1)^{-1}\mathbf{p}^2 + q_{12}\mathbf{E}\cdot\mathbf{x}\right] \otimes I + I \otimes \hat{H}_0,$$

where $I \otimes \hat{H}_0$ conmutes with $\mathbf{x}$, by virtue of $\hat{H}_0$'s independence from $\mathbf{x}$. Note that $\nu_1 = \mu_{12}$. Let us write $v = v_{12} = |\mathbf{v}|$. Therefore, thanks to commutativity

$$e^{-itH_0} = e^{-it\left[(2\nu_1)^{-1}\mathbf{p}^2+q_{12}\mathbf{E}\cdot\mathbf{x}\right]\otimes I - itI\otimes\hat{H}_0} = e^{-it\left[(2\nu_1)^{-1}\mathbf{p}^2+q_{12}\mathbf{E}\cdot\mathbf{x}\right]} \otimes e^{-it\hat{H}_0}.$$

We observe that the second factor in the tensorial product above conmutes with any operator depending on $\mathbf{x}$ and $\mathbf{p}$. It is also unitary, thus it disappears from the following norm estimations. We define $\mathcal{M}' = \{\mathbf{x} \in \mathbb{R}^n \,\big|\, |\mathbf{x} - \mathbf{v}t| \geq \lambda|vt|5/8\}$ and $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n \,\big|\, |\mathbf{x}| < \lambda|vt|/8\}$. We proceed as in Weder [40] using (II.1.31)-(II.1.34).

$$\begin{aligned}
I_1 &\leq C\left\|F(|\mathbf{x} - \mathbf{v}t - \mathbf{e}_1 q_{12}Et^2/(2\mu_{12})| \geq \lambda|vt|5/8)\, e^{-itH_0} g_{12}(\mathbf{p} - \mu_{12}\mathbf{v})F(|\mathbf{x}| < \lambda|vt|/8)\right\| \\
&= C\Big\|F(|\mathbf{x} - \mathbf{v}t - \mathbf{e}_1 q_{12}Et^2/(2\mu_{12})| \geq \lambda|vt|5/8)\, e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}Et^2/(2\mu_{12})} \\
&\qquad \times e^{-it\mathbf{p}^2/(2\mu_{12})} g_{12}(\mathbf{p} - \mu_{12}\mathbf{v})F(|\mathbf{x}| < \lambda|vt|/8)\Big\| \\
&= C\left\|F(|\mathbf{x} - \mathbf{v}t| \geq \lambda|vt|5/8)\, e^{-it\mathbf{p}^2/(2\mu_{12})} g_{12}(\mathbf{p} - \mu_{12}\mathbf{v})F(|\mathbf{x}| < \lambda|vt|/8)\right\| \\
&= C\|F(\mathbf{x} \in \mathcal{M}')\, e^{-it\mathbf{p}^2/(2\mu_{12})} g_{12}(\mathbf{p} - \mu_{12}\mathbf{v})F(\mathbf{x} \in \mathcal{M})\| \\
&\leq C(1 + \lambda|vt|/4 + |t|)^{-3} \leq C(1 + |vt|)^{-3}. \qquad\qquad \text{(II.2.24)}
\end{aligned}$$

To justify (II.2.24), we will prove that $r \geq \lambda|vt|/4$ in Proposition II.2.1, provided $v > 4\eta/\lambda$. Let us take $x \in \mathcal{M}'$ and $y \in \mathcal{M} + \mathbf{v}t$, then $|\mathbf{x} - \mathbf{y}| = |\,(\mathbf{x} - \mathbf{v}t) - (\mathbf{y} - \mathbf{v}t)\,| \geq \lambda|vt|5/8 - \lambda|vt|/8 = \lambda|vt|/2$. Thus, $r \geq \lambda|vt|/2 - \eta|t| \geq \lambda|vt|/2 - \lambda|vt|/4$.

Application of Lemma II.2.2, equation (II.2.17), yields for an $\epsilon > 0$,

$$I_2 \leq C(1 + |vt|)^{-2-\epsilon}. \qquad\qquad \text{(II.2.25)}$$

Then, by (II.2.13),
$$\begin{aligned}
I_3 &\leq C\|V_{12}^{vs}(\mathbf{x})g(\mathbf{p})F(|\mathbf{x} - \mathbf{v}t - \mathbf{e}_1 q_{12}Et^2/(2\mu_{12})| < \lambda|vt|5/8)\| \\
&\leq C\|V_{12}^{vs}(\mathbf{x})g(\mathbf{p})F(|\mathbf{x}| \geq |vt|(\sqrt{\delta_1} - 5\lambda/8))\| \\
&:= h_{12}(|vt|), \qquad\qquad \text{(II.2.26)}
\end{aligned}$$

where, by (II.2.21), $h_{12}(\tau) \in L^1((0, \infty))$, provided $\lambda < 8\sqrt{\delta_1}/5$.

Inequalities (II.2.23), (II.2.24), (II.2.25) and (II.2.26) prove the Lemma. ■

LEMMA II.2.4. *Given $V_{jk}^{E,\,s} \in \mathcal{V}_{E,\,s}$, where $1 \leq j < k \leq N$, α as in Definition II.1.2, $\tilde{U}_D(t)$ be given as in* (II.2.3)*. Then for all functions $f_{j'k'} \in C_0^\infty(B_{\mu_{j'k'}\eta_{j'k'}})$ with $1 \leq j' < k' \leq N$, there is a constant $0 < c$ such that for every $\mathbf{v}_{jk} \in \mathbb{R}^n$ with $v_{jk} > c$, the following estimate is true for all $0 < \epsilon_1 < 1$:*

$$\begin{aligned}
&\int_{-\infty}^{\infty} dt \Big\| \left(V_{jk}^{E,\,s}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^{E,\,s}(\mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) \\
&\times e^{-itH_0}\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-2}\Big\| = \begin{cases} O(v_{jk}^{-\alpha}), & \text{if } \alpha < 1, \\ O(v_{jk}^{-1+\epsilon_1}), & \text{if } \alpha = 1. \end{cases} \qquad \text{(II.2.27)}
\end{aligned}$$

Proof. The proof is quite similar to that of Lemma 2.2 in Adachi and Maehara [2]. To simplify the notation let us assume, in this Lemma, that $q_{12} \neq 0$ and consider the pair (1,2), i.e. $\mathbf{x} = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \mathbf{p} = -i\nabla_{\mathbf{x}}$. Let us take $g_{12} \in C_0^\infty(B_{\mu_{12}\eta})$ that satisfies $g_{12} \equiv 1$ on the support of f_{12}.

We simplify as follows, noting that $V_{12}^{E,\,s}$ is bounded:

$$\begin{aligned} I &= \left\| \left(V_{12}^{E,\,s}(\mathbf{x}) - V_{12}^{E,\,s}(\mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})) \right) e^{-itH_0} \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2} \right\| \\ &\leq C(I_1 + I_2 + I_3), \end{aligned}$$

where, for $0 < \tilde{\alpha} < 1$,

$$\begin{aligned} I_1 &= \left\| F(|\mathbf{x} - \mathbf{v}t - \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})| \geq 3|v^{\tilde{\alpha}}t|)\, e^{-itH_0} g_{12}(\mathbf{p} - \mu_{12}\mathbf{v}) F(|\mathbf{x}| < |v^{\tilde{\alpha}}t|) \right\| \\ &= \left\| F(|\mathbf{x} - \mathbf{v}t| \geq 3|v^{\tilde{\alpha}}t|)\, e^{-it\mathbf{p}^2/(2\mu_{12})} g_{12}(\mathbf{p} - \mu_{12}\mathbf{v}) F(|\mathbf{x}| < |v^{\tilde{\alpha}}t|) \right\|, \\ I_2 &= \left\| F(|\mathbf{x}| \geq |v^{\tilde{\alpha}}t|) \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2} \right\|, \\ I_3 &= \left\| \left(V_{12}^{E,\,s}(\mathbf{x}) - V_{12}^{E,\,s}(\mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})) \right) F(|\mathbf{x} - \mathbf{v}t - \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})| < 3|v^{\tilde{\alpha}}t|) \right\| \\ &= \left\| \left(V_{12}^{E,\,s}(\mathbf{x} + \mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})) - V_{12}^{E,\,s}(\mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})) \right) F(|\mathbf{x}| < 3|v^{\tilde{\alpha}}t|) \right\|. \end{aligned}$$

I_1 and I_2 are estimated as in the proof of Lemma II.2.3, by Proposition II.2.1 and equation II.2.17, respectively:

$$\int_{-\infty}^{\infty} (I_1 + I_2)\, dt = O(v^{-\tilde{\alpha}}).$$

By lemma 2.2 of Adachi and Maehara [2] (see also page 042101-5, equation 2.10 of [2]), we get, for all $0 < \tilde{\alpha} < 1$ and v sufficiently large that

$$\int_{-\infty}^{\infty} I_3\, dt = \begin{cases} O(v^{\tilde{\alpha}-2\alpha}), & \text{if } \alpha < 1, \\ O(v^{\tilde{\alpha}-2}|\ln v|), & \text{if } \alpha = 1. \end{cases}$$

We finish the proof by setting $\tilde{\alpha} = \begin{cases} \alpha, & \text{if } \alpha < 1, \\ 1 - \epsilon_1, & \text{if } \alpha = 1,\ 0 < \epsilon_1 < 1. \end{cases}$ ■

LEMMA II.2.5. *Let V_{jk}^l and $\tilde{U}_D(t)$ be given as in* (II.2.1) *and* (II.2.3)*, respectively. Let γ_1, ϵ_0 be as in Definition II.1.1, γ_D, μ be as in Definition II.1.2, θ_{jk} as in* (II.2.15)*. Let us define two constants σ_{jk} and $\tilde{\sigma}_{jk}$; if $q_{jk} \neq 0$ and $V_{jk}^l \neq 0$, then $\sigma_{jk} = \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}}$ and $0 < \tilde{\sigma}_{jk} < 2 - \max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\}$, else, if $q_{jk} = 0$ or $V_{jk}^l = 0$, then $\sigma_{jk} := \tilde{\sigma}_{jk} := 1$. Then for all functions $f_{j'k'} \in C_0^\infty(B_{\mu_{j'k'}\eta_{j'k'}})$ with $1 \leq j' < k' \leq N$, and for all integers $1 \leq j < k \leq N$, there is a constant $v_0 > 0$ such that for every $\mathbf{v}_{jk} \in \mathbb{R}^n$ with $v_{jk} > v_0^{1/\sigma_{jk}}$, we have the following estimate:*

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \left\| \left(V_{jk}^l(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^l(t\,\mathbf{p}_{jk}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk})) \right) \right. \\ &\left. \times e^{-itH_0} \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-2} \right\| \leq O(v_{jk}^{-\sigma_{jk}}). \qquad \text{(II.2.28)} \end{aligned}$$

PROOF. The proof in the case where $q_{jk} = 0$ is quite similar to that of Lemma 3.3 in Enss and Weder [20], and the proof in the case where $q_{jk} \neq 0$ is quite similar to that of Lemma 3.4 in Adachi and Maehara [2]. In this Lemma, let us denote $\mathbf{x} = \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j, \mathbf{p} = -i\nabla_{\mathbf{x}}$. From (II.2.14), a constant is defined as follows

$$c := \begin{cases} c_1^{\tilde{\sigma}_{jk}} c_2^{1-\tilde{\sigma}_{jk}}, & \text{if } q_{jk} \neq 0, \\ 1/2, & \text{if } q_{jk} = 0. \end{cases}$$

Let's split the long-range potential V_{jk}^l into two parts with controllable decay properties. Let $\chi \in C^\infty(\mathbb{R}^n)$ satisfy, $0 \leq \chi \leq 1, \chi(\mathbf{u}) = 1$, for $|\mathbf{u}| \geq c$ and $\chi(\mathbf{u}) = 0$, for $|\mathbf{u}| \leq c/2$; and $V_{jk,v_{jk}t}^l(\mathbf{u}) = V_{jk}^l(\mathbf{u})\chi(\mathbf{u}/(v_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}))$. In consequence, $\operatorname{supp}\left(V_{jk,v_{jk}t}^l - V_{jk}^l\right) \subset B_{cv_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}}$, and $\|V_{jk,v_{jk}t}^l - V_{jk}^l\| \leq \|V_{jk}^l\|$.

Choosing again $g \in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$ such that $g \equiv 1$ on the support of f_{jk}, it follows that

$$\begin{aligned} &\left\|\left(V_{jk}^l(\mathbf{x}) - V_{jk}^l(t\,\mathbf{p}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\right) e^{-itH_0} \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2}\right\| \\ \leq\ & I_1 + I_2 + I_3, \end{aligned} \tag{II.2.29}$$

where

$$\begin{aligned} I_1 = &\left\|\left(V_{jk,v_{jk}t}^l(\mathbf{x}) - V_{jk}^l(t\,\mathbf{p}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\right) e^{-itH_0} g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk})\right. \\ &\left.\times \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2}\right\|, \end{aligned} \tag{II.2.30}$$

$$\begin{aligned} I_2 = &\left\|\left(V_{jk}^l - V_{jk,v_{jk}t}^l\right)(\mathbf{x})\, e^{-itH_0} g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) F(|\mathbf{x}| < v_{jk}^{\sigma_{jk}}|t|/8)\right\| \\ &\times \left\|\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2}\right\|, \end{aligned} \tag{II.2.31}$$

$$\begin{aligned} I_3 = &\left\|\left(V_{jk}^l - V_{jk,v_{jk}t}^l\right)(\mathbf{x})\, e^{-itH_0} g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk})\right\| \\ &\times \left\|F(|\mathbf{x}| \geq v_{jk}^{\sigma_{jk}}|t|/8)\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2}\right\|. \end{aligned} \tag{II.2.32}$$

If $q_{jk} \neq 0$, for $\mathbf{q} \in B_{\mu_{jk}\eta_{jk}}$, $v_0 \geq 4\eta_{jk}/\sqrt{1-\delta_{jk}}$, by (II.2.14), we have

$$|t\,\mathbf{q}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk})| \geq c v_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}. \tag{II.2.33}$$

If $q_{jk} \neq 0$ with $\mathbf{p}$ in the support of g, we note, by (II.2.33), that $V_{jk,v_{jk}t}^l(t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk})) = V_{jk}^l(t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))$. If $q_{jk} = 0$, $\mathbf{p}$ belongs to the support of $g(\cdot - \mu_{jk}\mathbf{v}_{jk})$, and $v_0 > 2\eta_{jk}$ then $V_{jk,v_{jk}t}^l(t\,\mathbf{p}/\mu_{jk}) = V_{jk}^l(t\,\mathbf{p}/\mu_{jk})$.

As in Enss and Weder [20] and Adachi and Maehara [2], by (II.1.31)-(II.1.35) and the Baker-Campbell-Hausdorff formula [14],

$$\begin{aligned} I_1 \quad \leq \quad & \int_0^1 ds \left\| \left[\left(\nabla V^l_{jk,v_{jk}t} \right) (s\mathbf{x} + t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk})) \cdot \mathbf{x} \right.\right. \\ & + \frac{it}{(2\mu_{jk})} \left(\Delta V^l_{jk,v_{jk}t} \right) (s\mathbf{x} + t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk})) \Big] g(\mathbf{p})\, e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}} \\ & \left. \times \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2} \right\| . \end{aligned} \tag{II.2.34}$$

For $q_{jk} = 0$, $\epsilon_0 < \gamma_1 - 3/2$, having in consideration that in the support of $V^l_{jk,v_{jk}t}$ we must have $|\mathbf{x}| \geq (c/2)|v_{jk}t|$, (II.2.16) and (II.2.34) imply that

$$I_1 \quad \leq \quad C \left[|v_{jk}t|^{-\gamma_1} \left(1 + v_{jk}^{-(2-\gamma_1/2)} |v_{jk}t|^{\theta_{jk}} \right) + |v_{jk}t|^{-1-2\epsilon_0} \right] \;\leq\; C|v_{jk}t|^{-1-2\epsilon_0}.$$

Then, by the fact that I_1 is uniformly bounded, for all $t \in \mathbb{R}$ and all v_{jk}, $\int dt\, I_1 = O(v_{jk}^{-1})$.

We consider now the case when $q_{jk} \neq 0$. Recall that in the support of $V^l_{jk,v_{jk}t}$ we must have $|\mathbf{x}| \geq (c/2)v_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}$ for $0 < \tilde{\sigma}_{jk} < 1$. For $0 < b$, (II.2.16) and (II.2.34) imply:

$$I_1 \leq I_{11} + I_{12}, \text{ where, } I_{12} \leq C\|V^l_{jk}\|,$$

$$\begin{aligned} I_{11} \quad \leq \quad & C \left(\left(v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)} + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+1)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)} \right) \right. \\ & \left. \times \left(1 + v_{jk}^{-(2-\gamma_1/2)} |v_{jk}t|^{\theta_{jk}} \right) F(|t| > v_{jk}^{-b}) + \|V^l_{jk}\|\, F(|t| \leq v_{jk}^{-b}) \right) \text{ and} \\ I_{12} \quad \leq \quad & C \left(v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+1} + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu+1)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)+1} \right. \\ & \left. + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+2)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+2)+1} \right). \end{aligned}$$

By a straightforward calculation, provided $\tilde{\sigma}_{jk} < 2 - (1+\theta_{jk})/(\gamma_D+\mu)$,

$$\int dt\, I_{11} = O(v_{jk}^{-b}),$$

having taken

$$b = \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}} = \min\left\{ \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}}, \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+1)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+1)}, \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+\mu)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+\mu)} \right\}.$$

Using Adachi and Maehara's computations [2] of the last three terms of the integral of I_3 in the proof of their Lemma 3.4, assuming that $\tilde{\sigma}_{jk} < 2 - 2/(\gamma_D + 2\mu)$, we obtain:

$$\int_{-\infty}^{+\infty} dt\, I_{12} \quad = \quad O(v_{jk}^{-\tilde{\sigma}_{jk}/[(2-\tilde{\sigma}_{jk})-1/(\gamma_D+2)]}).$$

Thus we have, in general:

$$\int_{-\infty}^{+\infty} dt\, I_1 \quad = \quad O(v_{jk}^{-\sigma_{jk}}). \tag{II.2.35}$$

If $|\mathbf{x}| \leq (5/8)\, v_{jk}^{\sigma_{jk}}|t|$ and $v_{jk}^{\sigma_{jk}-1} \leq (2/5)\sqrt{1-\delta_{jk}}$, for $q_{jk} \neq 0$, we obtain as in (II.2.33)

$$|\mathbf{x} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk})| \quad \geq \quad c v_{jk}^{\tilde{\sigma}_{jk}} |t|^{2-\tilde{\sigma}_{jk}}, \tag{II.2.36}$$

by equations (II.1.31)-(II.1.34) and (II.2.36) we can invoke Proposition 2.10 from Enss [15] in (II.2.31) to estimate I_2 with $v_0 > 4\eta_{jk}$,

$$\begin{aligned} I_2 &\leq C \left\| \left(V^l_{jk} - V^l_{jk,v_{jk}t} \right) (\mathbf{x} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\, e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p}) F(|\mathbf{x}| < v_{jk}^{\sigma_{jk}}|t|/8) \right\| \\ &\leq C \left\| F \left(|\mathbf{x} - \mathbf{v}_{jk}t| \geq \begin{cases} 5v_{jk}^{\sigma_{jk}}|t|/8, & \text{if } q_{jk} \neq 0, \\ v_{jk}|t|/2, & \text{if } q_{jk} = 0, \end{cases} \right) e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) F(|\mathbf{x}| < v_{jk}^{\sigma_{jk}}|t|/8) \right\| \\ &\leq C(1 + v_{jk}^{\sigma_{jk}}|t|)^{-2}. \end{aligned} \tag{II.2.37}$$

Again, by Lemma II.2.2, equation (II.2.17), we estimate I_3,

$$I_3 \leq C(1 + v_{jk}^{\sigma_{jk}}|t|)^{-2}. \tag{II.2.38}$$

By (II.2.35), (II.2.37) and (II.2.38) we finish the proof. ■

Let us denote,

$$I_{G,v,a,b} = \exp\left(-i \sum_{j<k}^{E} \int_a^b ds\; V_{jk}^{E,\,s}(\mathbf{v}_{jk}s + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk})) \right) \text{ and } I_{G,v} = I_{G,v,-\infty,\infty}. \tag{II.2.39}$$

Observe that $\tilde{U}_{G,v}(t) = I_{G,v,0,t}$.

PROPOSITION II.2.6. *The wave operators $\Omega_\pm^{D,G,v}$ and $\Omega_\pm^{G,v}$ exist and, moreover,*

$$\Omega_\pm^{D,G,v} = W_\pm^D I_{G,v,0,\pm\infty}, \quad \Omega_\pm^{G,v} = W_\pm I_{G,v,0,\pm\infty}. \tag{II.2.40}$$

Proof. We give only the proof for $\Omega_\pm^{D,G,v}$, the other is similar. Note that:

$$\text{s} - \lim_{t\to\pm\infty} e^{itH} U^{D,G,v}(t) = \text{s} - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} \tilde{U}_D(t) I_{G,v,0,t} = W_\pm^D \ \text{s} - \lim_{t\to\pm\infty} I_{G,v,0,t}.$$

Furthermore, for any $\Phi \in L^2$ we have that:

$$\|\left(I_{G,v,0,t} - I_{G,v,0,\pm\infty}\right)\Phi\|^2 = \int_{\mathbb{R}^n} |I_{G,v,0,t} - I_{G,v,0,\pm\infty}|^2 |\Phi|^2 \longrightarrow_{t\to\pm\infty} 0,$$

by the Lebesgue dominated convergence theorem, taking into account that the integrand is dominated by $4|\Phi|^2$ for all t. This proves the proposition. ■

Now we focus in the wave operator estimates. We use Jacobi Coordinates based on the pair (1,2), where $v = |\mathbf{v}| = |\mathbf{v}_2 - \mathbf{v}_1|$ and $v_{jk} = O(v^2)$, for $(j,k) \neq (1,2)$. Lemma II.2.7, below, is a N-body generalization of Lemma 3.5 in Adachi and Maehara [2]. See also Lemma 4.6 of Adachi, Kamada, Kazuno and Toratani [1], for a generalization of Lemma 3.5 of Adachi and Maehara [2] to the case where the external electric field is asymptotically zero in time, in the two-body case.

LEMMA II.2.7. *Let α be as in Definition II.1.2, where, without loss of generality, $\alpha = 1$ if $q_{jk} = 0$ for all $1 \leq j < k \leq N$. For all $1 \leq j < k \leq N$, let $0 < \sigma_{jk} \leq 1$ be as in Lemma II.2.5. Let us take $V^{VS} \in \mathcal{V}_{VSR}, V^S \in \mathcal{V}_{SR}, V^L \in \mathcal{V}_{LR}$. Then, for all $\Phi_\mathbf{v}$ as in* (II.1.13) *with a fixed normalized $\hat{\phi}_3$, where, with δ_{jk} being defined as in* (II.2.12)*, the relative velocities satisfy $|\hat{\mathbf{v}}_{jk} \cdot \hat{\mathbf{E}}| \leq \delta_{jk}$ for all integers $1 \leq j < k \leq N$ with $q_{j,k} \neq 0$, and $v_{jk} > v_0^{1/\sigma_{jk}}$ for some $v_0 > 0$ and all integers $1 \leq j < k \leq N$:*

$$\sup_{t\in\mathbb{R}} \left\| (\Omega_\pm^{D,G,v} - e^{itH} U^{D,G,v}(t))\Phi_\mathbf{v} \right\| = O(v^{-\min\{\alpha,\, \sigma_{jk}\,|\,1\leq j<k\leq N\}}). \tag{II.2.41}$$

In the short-range case, where $V^l_{jk}=0$ (see (II.2.1)*) for all $1\le j<k\le N$, we obtain the following result*

$$\sup_{t\in\mathbb{R}}\left\|(\Omega^{G,v}_{\pm}-e^{itH}U^{G,v}(t))\Phi_{\mathbf{v}}\right\| \;=\; \begin{cases} O(v^{-\alpha}), & \text{if } \alpha<1 \text{ and } \sum_{j<k}|q_{jk}|>0,\\ O(v^{-(1-\epsilon_1)}), & \text{if } \alpha=1 \text{ and } \sum_{j<k}|q_{jk}|>0,\\ O(v^{-1}), & \text{if } \sum_{j<k}|q_{jk}|=0,\end{cases} \tag{II.2.42}$$

for all $0<\epsilon_1<1$.

Proof. We give the proof for $\Omega^{D,G,v}_{+}$. By Duhamel's formula, (II.1.31) and (II.1.33):

$$\begin{aligned}\Omega^{D,G,v}_{+}-e^{itH}U^{D,G,v}(t) &= \lim_{t'\to+\infty}e^{it'H}U^{D,G,v}(t')-e^{itH}U^{D,G,v}(t)=\lim_{t'\to+\infty}\int_t^{t'}ds\,\frac{d}{ds}\left(e^{isH}U^{D,G,v}(s)\right)\\ &= i\int_t^{\infty}ds\,e^{isH}\Bigg(\sum_{j<k}\big[V^{vs}_{jk}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)+V^{l}_{jk}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\\ &\quad -V^l_{jk}(s\mathbf{p}_{jk}/\mu_{jk}-\mathbf{e}_1q_{jk}Es^2/(2\mu_{jk}))\big]\\ &\quad +\sum_{j<k}^{E}\Big[V^{E,\,s}_{jk}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)-V^{E,\,s}_{jk}(\mathbf{v}_{jk}s+\mathbf{e}_1q_{jk}Es^2/(2\mu_{jk}))\Big]\Bigg)U^{D,G,v}(s).\end{aligned}$$

Using that the Graf-type modifier $\tilde{U}_{G,v}(t)$ (II.2.2) conmutes with any operator, and Lemmata II.2.3, II.2.4, II.2.5, it follows for any $0<\epsilon_1<1$, $t\in\mathbb{R}$:

$$\begin{aligned}\left\|\left(e^{-itH}\Omega^{D,G,v}_{+}-U^{D,G,v}(t)\right)\Phi_{\mathbf{v}}\right\| &\le C\sum_{j<k}\int_{-\infty}^{\infty}\Big\|V^{vs}_{jk}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\\ &\qquad\times e^{-isH_0}\tilde{U}_D(s)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\Big\|\,ds\\ &\quad +C\sum_{j<k}\int_{-\infty}^{\infty}\Big\|\left(V^l_{jk}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)-V^l_{jk}(s\mathbf{p}_{jk}/\mu_{jk}-\mathbf{e}_1q_{jk}Es^2/(2\mu_{jk}))\right)\\ &\qquad\times e^{-isH_0}\tilde{U}_D(s)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\Big\|\,ds\\ &\quad +C\sum_{j<k}^{E}\int_{-\infty}^{\infty}\Big\|\left(V^s_{jk}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)-V^s_{jk}(\mathbf{v}_{jk}s+\mathbf{e}_1q_{jk}Es^2/(2\mu_{jk}))\right)\\ &\qquad\times e^{-isH_0}\tilde{U}_D(s)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\Big\|\,ds\\ &\le \sum_{j<k}\left(O(v_{jk}^{-1})+O(v_{jk}^{-\sigma_{jk}})\right)+\sum_{j<k}^{E}\begin{cases}O(v_{jk}^{-\alpha}), & \text{if } \alpha<1,\\ O(v_{jk}^{-(1-\epsilon_1)}), & \text{if } \alpha=1.\end{cases}\end{aligned}$$

The proof is finished by the use of the following arguments: $\alpha<1$ implies for $v\ge 1$ that $v^{-1}\le v^{-\alpha}$, $0<\sigma_{jk}<1$ implies, for ϵ_1 sufficiently small that $O(v_{jk}^{-(1-\epsilon_1)})\le O(v_{jk}^{-\sigma_{jk}})$, and noting that $v_{12}=v$ and v_{jk} is $O(v^2)$ for $j<k=3,\dots,N$. ■

Lemma II.2.5 above defines two sets of exponents σ_{jk} and $\tilde{\sigma}_{jk}$. Theorem II.2.8 below needs $\sigma_{jk}>1/2$. For this purpose we have to ask, for all $1\le j<k\le N$ with $q_{jk}\ne 0$ and $V^l_{jk}\ne 0$, that θ_{jk}, being as in

(II.2.15), must hold:

$$2 - \max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\} > \frac{2}{3} \Longleftrightarrow \theta_{jk} < \frac{4}{3}(\gamma_D+\mu) - 1. \quad \text{(II.2.43)}$$

In particular, inequality (II.2.43) is always true if $\theta_{jk} \leq 1/3$ because $1/3 < 4(\gamma_D+\mu)/3 - 1$ for all γ_D and μ as in Definition II.1.2. Inequality (II.2.43) is always met in conditions ζ^b_{jk} and ζ^c_{jk}, see (II.2.15), because in the former, θ_{jk} can be taken arbitrarily small, and in the later, θ_{jk} is zero. If there is a pair (j,k) with $q_{jk} \neq 0$ and the condition ζ^a_{jk} is true, (II.2.43) is equivalent to $\max\{3/2,\, 3 - 4(\gamma_D+\mu)/3\} < \gamma_1 < 2$. If $\sum |q_{jk}| = 0$ we just need $3/2 < \gamma_1 \leq 2$. Theorem II.1.3 is stated considering long-range potentials, in this case, ζ^a_{jk} is true for some pair (j,k) with $q_{jk} \neq 0$, if and only if, there are two pairs (j^*,k^*) and (j',k') such that $1 \leq j^* < k^* \leq N$, $1 \leq j' < k' \leq N$, $q_{j^*k^*} \neq 0$ and $q_{j'k'} = 0$. We can also use Theorem II.1.3 with short-range potentials: the condition $3 - 4(\gamma_D+\mu)/3 < \gamma_1$ is always true because, without loss of generality, we can take $\gamma_1 = 2$ in this situation.

THEOREM II.2.8. *(Reconstruction Formula) Let γ_1 be as in Definition II.1.1, α, γ_D, μ be as in Definition II.1.2, where, without loss of generality, $\alpha = 1$ if $q_{jk} = 0$ for all $1 \leq j < k \leq N$. If there exists two pairs $1 \leq j < k \leq N$, $1 \leq j' < k' \leq N$ such that $q_{jk} \neq 0$, $q_{j'k'} = 0$, $V^l_{j'k'} \neq 0$, and either $j' = j$ or $j' = k$ or $k' = j$ or $k' = k$ or $j' + j = 3$, we additionally assume $\gamma_1 > 3 - 4(\gamma_D+\mu)/3$. For all $1 \leq j < k \leq N$, let $0 < \sigma_{jk} \leq 1$ be as in Lemma II.2.5. Let us take $V^{VS} \in \mathcal{V}_{VSR}, V^S \in \mathcal{V}_{SR}, V^L \in \mathcal{V}_{LR}$, where V^{vs}_{12} satisfies* (II.2.21) *for all $g \in C^\infty_0(\mathbb{R}^n)$, with $0 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$. Let us set $\mathbf{p}_l = \mathbf{p} \cdot \mathbf{e}_l$ for any $l = 1, \ldots, N$. Then, for all $\Phi_\mathbf{v}, \Psi_\mathbf{v}$ as in* (II.1.13) *with the same fixed normalized $\hat{\phi}_3$, with δ_{jk} being defined as in* (II.2.12)*, the relative velocities satisfy $|\hat{\mathbf{v}}_{jk} \cdot \hat{\mathbf{E}}| \leq \delta_{jk}$ for all integers $1 \leq j < k \leq N$ with $q_{j,k} \neq 0$, and $v_{jk} > v_0^{1/\sigma_{jk}}$ for some $v_0 > 0$, as in Lemma II.2.5, and all integers $1 \leq j < k \leq N$:*

$$\begin{aligned} v(i[S^D, \mathbf{p}_l]\Phi_\mathbf{v}, \Psi_\mathbf{v}) \quad = \quad & \int_{-\infty}^{\infty} d\tau \Big[(V^{vs}_{12}(\mathbf{x}+\tau\hat{\mathbf{v}})\mathbf{p}_l\Phi_{12}, \Psi_{12}) - (V^{vs}_{12}(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, \mathbf{p}_l\Psi_{12}) \\ & + i\left(\left(\frac{\partial V^s_{12}}{\partial x_l}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, \Psi_{12}\right) + i\left(\left(\frac{\partial V^l_{12}}{\partial x_l}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, \Psi_{12}\right)\Big] \\ & + \begin{cases} o(v^{-\rho_l}), & \text{if } \gamma_2 - 1 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1, \\ & \quad \text{for any } \rho_l,\, 0 \leq \rho_l < \gamma_2 - 1, \\ o(v^{-\rho}), & \text{if } 0 \leq \rho < \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < \gamma_2 - 1, \\ o(v^{-\rho_l}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} \neq 0, \\ & \quad \text{for any } \rho_l,\, 0 \leq \rho_l < \gamma_1 - 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} = 0. \end{cases} \end{aligned} \quad \text{(II.2.44)}$$

where $\gamma_2 := \begin{cases} \gamma_1, & \text{if } q_{12} = 0 \text{ and } V^l_{12} \neq 0, \\ \gamma_D + \mu, & \text{if } q_{12} \neq 0, \text{ and } V^l_{12} \neq 0, \\ 2, & \text{if } V^l_{12} = 0. \end{cases}$

REMARK II.2.9. Note that the first term in the right-hand side of (II.2.44) can be written as

$$i\int_{-\infty}^{\infty} d\tau \left(\left(\frac{\partial V_{12}}{\partial x_l}\right)(x+\tau\hat{v})\Phi_{12},\, \Psi_{12}\right),$$

where $V_{12} = V^{vs}_{12} + V^s_{12} + V^l_{12}$, and the derivative $\frac{\partial V_{12}}{\partial x_l}$ is taken in distribution sense. This shows that the high-velocity limit of $v(i[S^D, \mathbf{p}_l]\Phi_\mathbf{v}, \Psi_\mathbf{v})$ is independent of the decomposition of the potential V into the part $V^{VS} + V^S$, that is of short range under the constant electric field $\mathbf{E}$ and the part V^L that is long-range; that is used for the definition of the Dollard scattering operator (II.2.10).

Proof.

The scattering operator can be expressed as $S^D = (\Omega_+^{D,G,v} I^{-1}_{G,v,0,+\infty})^* (\Omega_-^{D,G,v} I^{-1}_{G,v,0,-\infty}) = I_{G,v}(\Omega_+^{D,G,v})^* \Omega_-^{D,G,v}$, by (II.2.10) and (II.2.40). Noting that $[S^D, \mathbf{p}_l] = [S^D, \mathbf{p}_l - \mu_{12} v_l] = [S^D - I_{G,v}, \mathbf{p}_l - \mu_{12} v_l]$ and $(\mathbf{p}_l - \mu_{12} v_l)\Phi_{\mathbf{v}} = (\mathbf{p}_l \Phi_0)_{\mathbf{v}}$ where $\mathbf{p}_l$ and v_l are the l-th components of the relative momentum and the velocity $\mathbf{v}$ of the chosen pair $(1,2)$, respectively. By the fact that $\Omega_\pm^{D,G,v}$ are partially isometric and the application of Duhamel formula, (II.1.31) and (II.1.33), as in the proof of Lemma II.2.7, we obtain

$$i(S^D - I_{G,v})\Phi_{\mathbf{v}} = I_{G,v} i \left(\Omega_+^{D,G,v} - \Omega_-^{D,G,v}\right)^* \Omega_-^{D,G,v}\Phi_{\mathbf{v}} = I_{G,v}\int_{-\infty}^{+\infty} dt \ \left(U^{D,G,v}(t)\right)^* V_t(\tilde{\mathbf{x}}) e^{-iHt}\Omega_-^{D,G,v}\Phi_{\mathbf{v}},$$

with $\tilde{\mathbf{x}}$ defined as (II.1.2) and $V_t = V_{3,t} + V_{12,t}$ where

$$\begin{aligned} V_{3,t}(\tilde{\mathbf{x}}) \;=\; & \sum_{j<k,3\le k\le N} \left[V_{jk}^{vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) + V_{jk}^l(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^l(t\,\mathbf{p}_{jk}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\right] \\ & + \sum_{j<k,3\le k\le N}^{E} \left[V_{jk}^s(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^s(\mathbf{v}_{jk} t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\right] \end{aligned}$$

and

$$\begin{aligned} V_{12,t} \;=\; & V_{12}^{vs}(\mathbf{x}) + V_{12}^l(\mathbf{x}) - V_{12}^l(t\,\mathbf{p}/\mu_{12} - \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})) \\ & + V_{12}^s(\mathbf{x}) - V_{12}^s(\mathbf{v}t + \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})). \end{aligned} \tag{II.2.45}$$

Thus we have

$$v\left(i[S^D, \mathbf{p}_l]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) = I_{G,v}\left(I(v) + R(v)\right) \tag{II.2.46}$$

with

$$I(v) \;=\; \int_{-\infty}^{+\infty} d\tau\, l_v(\tau), \tag{II.2.47}$$

where

$$\begin{aligned} l_v(vt) \;=\; & \left(V_{12,t}(\mathbf{x}) e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Phi_0\right)_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ & - \left(V_{12,t}(\mathbf{x}) e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Psi_0\right)_{\mathbf{v}}\right) \end{aligned} \tag{II.2.48}$$

and

$$\begin{aligned} R(v)/v \;=\; & \int_{-\infty}^{+\infty} dt \left[\left(V_{3,t} e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Phi_0\right)_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)\right. \\ & - \left(V_{3,t} e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Psi_0\right)_{\mathbf{v}}\right) + \left(\left(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t)\right)\left(\mathbf{p}_l\Phi_0\right)_{\mathbf{v}},\right. \\ & \left.\left. V_t U^{D,G,v}(t)\Psi_{\mathbf{v}}\right) - \left(\left(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t)\right)\Phi_{\mathbf{v}}, V_t U^{D,G,v}(t)\left(\mathbf{p}_l\Psi_0\right)_{\mathbf{v}}\right)\right]. \end{aligned} \tag{II.2.49}$$

In the derivation of (II.2.48) and (II.2.49) we used that $\tilde{U}_{G,v}(t)$ commutes with any operator.

We are going to need to translate the Dollard-type modifier (II.2.3)

$$\tilde{U}_D(\mathbf{v}, t) = e^{-i\mu_{12}\mathbf{v}\cdot\mathbf{x}} \prod_{j=3}^{N} e^{-i\mu_j \mathbf{v}_j\cdot\mathbf{x}_j}\ \tilde{U}_D(t)\ e^{i\mu_{12}\mathbf{v}\cdot\mathbf{x}} \prod_{j=3}^{N} e^{i\mu_j\mathbf{v}_j\cdot\mathbf{x}_j}. \tag{II.2.50}$$

Using equations (II.1.31)-(II.1.35) and substituting (II.2.45) and (II.2.50) in (II.2.48), it follows that

$$I(v) = J_1(v) - J_2(v) + iJ_3(v) + iJ_4(v), \tag{II.2.51}$$

where

$$J_1(v) = \int \Big(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}}+\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12}))\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\tilde{U}_D(\mathbf{v},\tau/v)\mathbf{p}_l\Phi_0, \quad e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Psi_0\Big)\, d\tau, \tag{II.2.52}$$

$$J_2(v) = \int \Big(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}}+\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12}))\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Phi_0, \quad e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\mathbf{p}_l\Psi_0\Big)\, d\tau, \tag{II.2.53}$$

$$J_3(v) = \int \Big(\Big(\partial V_{12}^{E,s}/\partial x_l\Big)\,(\mathbf{x}+\tau\hat{\mathbf{v}}+\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12}))\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Phi_0, \quad e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Psi_0\Big)\, d\tau. \tag{II.2.54}$$

$$J_4(v) = \int \Big(\big(\partial V_{12}^{l}/\partial x_l\big)\,(\mathbf{x}+\tau\hat{\mathbf{v}}-\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12}))\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Phi_0, \quad e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\mathbf{p}_l\Psi_0\Big)\, d\tau. \tag{II.2.55}$$

There exists $C>0$ that uniformly bounds the following expression, for all $j<k$:

$$\|\Phi_{\mathbf{v}}\| + \|(\mathbf{p}_l\Phi_0)_{\mathbf{v}}\| + \big\|(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^2\Phi_{\mathbf{v}}\big\| + \big\|(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^2\,(\mathbf{p}_l\Phi_0)_{\mathbf{v}}\big\| \le C.$$

Then,

$$\begin{aligned}
\frac{|R(v)|}{v} \le\; & C \sum_{j<k,3\le k\le N} \int dt \left\| V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)e^{-itH_0}\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2} \right\| \\
& + C \sum_{j<k,3\le k\le N} \int dt \Big\| \big(V_{jk}^{l}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j) - V_{jk}^{l}(t\,\mathbf{p}_{jk}/\mu_{jk}-\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\big) \\
& \quad \times e^{-itH_0}\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2} \Big\| \\
& + C \sum_{j<k,3\le k\le N}^{E} \int dt \Big\| \big(V_{jk}^{s}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j) - V_{jk}^{s}(\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\big) \\
& \quad \times e^{-itH_0}\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2} \Big\| \\
& + C \left[\sup_{t\in\mathbb{R}} \Big\| \Big(e^{-iHt}\Omega_{-}^{D,G,v} - U^{D,G,v}(t)\Big)\,(\mathbf{p}_l\Phi_0)_{\mathbf{v}} \Big\| + \sup_{t\in\mathbb{R}} \Big\| \Big(e^{-iHt}\Omega_{-}^{D,G,v} - U^{D,G,v}(t)\Big)\,\Phi_{\mathbf{v}} \Big\| \right] \\
& \times \left[\sum_{j<k} \int dt \left\| V_{jk}^{vs}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)e^{-itH_0}\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2} \right\| \right. \\
& \quad + \sum_{j<k} \int dt \Big\| \big(V_{jk}^{l}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j) - V_{jk}^{l}(t\,\mathbf{p}_{jk}/\mu_{jk}-\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\big) \\
& \quad \times e^{-itH_0}\tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2} \Big\|
\end{aligned}$$

$$+\sum_{j<k}^{E}\int dt\left\|\left(V_{jk}^{s}(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)-V_{jk}^{s}(\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right)\right.$$
$$\left.\times\ e^{-itH_0}\tilde{U}_D(t)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-2}\right\|\Bigg].$$

Thus, by Lemmata II.2.3, II.2.4 and II.2.7, if $V_{jk}^l=0$ for all $1\le j<k\le N$:

$$R(v)\ =\ \begin{cases} o(v^{-\rho}), & \text{if } 0\le\rho<2\alpha-1,\\ O(v^{-\rho}), & \text{if } \rho=2\alpha-1<1,\ \sum_{j<k}|q_{jk}|>0,\\ O(v^{-1}), & \text{if } \rho=1,\ \sum_{j<k}|q_{jk}|=0,\end{cases} \tag{II.2.56}$$

Similarly, by Lemmata II.2.3, II.2.4, II.2.5 and II.2.7, if $V_{jk}^l\ne 0$ for some $1\le j<k\le N$:

$$\begin{aligned} R(v)\ &=\ \begin{cases} O\left(v^{1-2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}}\right), & \text{if } \sum_{j<k}|q_{jk}|>0,\\ O(v^{-1}), & \text{if } \sum_{j<k}|q_{jk}|=0,\end{cases}\\ &=\ \begin{cases} o(v^{-\rho}), & \text{if } 0\le\rho<2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1,\\ O(v^{-\rho}), & \text{if } \rho=2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<1,\\ O(v^{-1}), & \text{if } \rho=1,\ \sum_{j<k}|q_{jk}|=0.\end{cases} \end{aligned}\tag{II.2.57}$$

Now, let us compute the following: $\lim_{v\to\infty} v\left(i[S^D,\mathbf{p}_l]\Phi_{\mathbf{v}},\Psi_{\mathbf{v}}\right)$, using (II.2.46):

$$\lim_{v\to\infty} I_{G,v}=\exp\left(-i\sum_{j<k}^{E}\lim_{v\to\infty}\int_{-\infty}^{\infty}ds\ V_{jk}^{s}(\mathbf{v}_{jk}s+\mathbf{e}_1 q_{jk}Es^2/(2\mu_{jk}))\right)=1.$$

We have used the Lebesgue dominated convergence theorem: There exist $0<\delta_1,\delta_2\le 1$ such that $|\mathbf{v}_{jk}s+\mathbf{e}_1 q_{jk}Es^2/(2\mu_{jk})|\ \ge\sqrt{\delta_1|\mathbf{v}_{jk}t|^2+\delta_2(q_{jk}E/(2\mu_{jk}))^2t^4}$, since, $|\hat{\mathbf{v}}_{jk}\cdot\hat{\mathbf{E}}|\le\delta<1$, by (II.2.13), when $q_{12}=0$, we can take $\delta_1=\delta_2=1$, and if $q_{12}\ne 0$, we use $\delta_1=\delta_2=1-\delta$. We can estimate $V_{jk}^s(\mathbf{v}_{jk}s+\mathbf{e}_1 q_{jk}Es^2/(2\mu_{jk}))$ as follows:

$$\begin{aligned} \left|V_{jk}^{s}(\mathbf{v}_{jk}s+\mathbf{e}_1 q_{jk}Es^2/(2\mu_{jk}))\right| &\le C\left(1+\left|\mathbf{v}_{jk}s+\mathbf{e}_1 q_{jk}Es^2/(2\mu_{jk})\right|\right)^{-\gamma}\\ &\le C\left(1+\delta_1\mathbf{v}_{jk}^2s^2+\delta_2|q_{jk}E/(2\mu_{jk})|^2s^4\right)^{-\gamma/2}\\ &\le C(1+s^{-2\gamma}). \end{aligned}$$

This last term is integrable in $\mathbb{R}$ because $1/2<\gamma\le 1$.

Note that pointwise in τ,

$$\begin{aligned} \lim_{v\to\infty} l_v(\tau)\ &=\ (V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\,(\mathbf{p}_l\Phi_{12}),\ \Psi_{12})-(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12},\ (\mathbf{p}_l\Psi_{12}))\\ &+i\left(\left(\frac{\partial V_{12}^{s}}{\partial x_l}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12},\Psi_{12}\right)+i\left(\left(\frac{\partial V_{12}^{l}}{\partial x_l}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12},\Psi_{12}\right). \end{aligned}\tag{II.2.58}$$

We want to compute $\lim_{v\to\infty} I(v)$, by (II.2.47), (II.2.58) and the Lebesgue dominated convergence theorem, thus showing the rate of convergence when $\rho=0$ in (II.2.21):

$$\begin{aligned} \lim_{v\to\infty} I(v)\ &=\ \int_{-\infty}^{\infty}\Big[(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\,(\mathbf{p}_l\Phi_{12}),\ \Psi_{12})-(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12},\ (\mathbf{p}_l\Psi_{12}))\\ &+i\left(\left(\frac{\partial V_{12}^{s}}{\partial x_l}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12},\Psi_{12}\right)+i\left(\left(\frac{\partial V_{12}^{l}}{\partial x_l}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12},\Psi_{12}\right)\Big]\,d\tau, \end{aligned}\tag{II.2.59}$$

this means in terms of the J_1, J_2, J_3, J_4 functions that

$$\lim_{v\to\infty} J_1(v) = \int (V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})(\mathbf{p}_l\Phi_{12}), \Psi_{12})\, d\tau, \tag{II.2.60}$$

$$\lim_{v\to\infty} J_2(v) = \int (V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12}, (\mathbf{p}_l\Psi_{12}))\, d\tau, \tag{II.2.61}$$

$$\lim_{v\to\infty} J_3(v) = \int \left(\left(\partial V_{12}^{E,s}/\partial x_l\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12}, \Psi_{12}\right) d\tau, \tag{II.2.62}$$

$$\lim_{v\to\infty} J_4(v) = \int \left((\partial V_{12}^{l}/\partial x_l)\,(\mathbf{x}+\tau\hat{\mathbf{v}})\,\Phi_{12}, \Psi_{12}\right) d\tau. \tag{II.2.63}$$

To justify the use of the Lebesgue dominated convergence theorem observe that $\frac{\partial V_{12}^s}{\partial x_l}$ and $\frac{\partial V_{12}^l}{\partial x_l}$ are very short-range. By (II.2.48) and Lemma II.2.3:

$$\begin{aligned}
|l_v(\tau)| \leq\ & C\left\| V_{12}^{vs}(\mathbf{x})e^{-i(\tau/v)H_0}\tilde{U}_D(\tau/v)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\mathbf{x}|^2)^{-2}\right\| \\
& +C\left\| \frac{\partial V_{12}^s}{\partial x_l}(\mathbf{x})e^{-i(\tau/v)H_0}\tilde{U}_D(\tau/v)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\mathbf{x}|^2)^{-2}\right\| \\
& +C\left\| \frac{\partial V_{12}^l}{\partial x_l}(\mathbf{x})e^{-i(\tau/v)H_0}\tilde{U}_D(\tau/v)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\mathbf{x}|^2)^{-2}\right\| \\
\leq\ & Ch_{12}(|\tau|),
\end{aligned}$$

where $h_{12} \in L^1((0,\infty))$.

Let us find the rate of convergence of (II.2.59) when $\rho > 0$ in (II.2.21). We estimate the rate of convergence of, J_1, the first term in the right-hand side of (II.2.51) (i.e. (II.2.52)) to its limit. From (II.2.52) and (II.2.60) we have:

$$\begin{aligned}
J_1(v) - \lim_{v\to\infty} J_1(v) =\ & \int_{-\infty}^{-\infty} d\tau \Big(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}}+\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12}))\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)(\mathbf{p}_l\Phi_0), \\
& e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Psi_0\Big) - \int_{-\infty}^{-\infty} d\tau\,(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})(\mathbf{p}_l\Phi_{12}),\Psi_{12})
\end{aligned}$$

$$\begin{aligned}
=\ & \int d\tau \Big[\Big(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)(\mathbf{p}_l\Phi_0), \\
& e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)\Psi_0\Big) \\
& -\Big(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)(\mathbf{p}_l\Phi_0), \Psi_0\Big)\Big] \\
& + \int d\tau\,[-((\mathbf{p}_l\Phi_0), V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\Psi_0) \\
& + \Big(e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)(\mathbf{p}_l\Phi_0), V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\Psi_0\Big)\Big].
\end{aligned}$$

The latter calculations suggest us to define:

$$\begin{aligned}
h_{\mathbf{v}}^{(1)} =\ & \Big(V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v)(\mathbf{p}_l\Phi_0), \\
& \Big(e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v) - I\Big)\,\Psi_0\Big),
\end{aligned} \tag{II.2.64}$$

$$h_{\mathbf{v}}^{(2)} = \Big(\Big(e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{12}E\tau^2/(2v^2\mu_{12})}\, e^{-i\tau\mathbf{p}^2/(2v\mu_{12})}\,\tilde{U}_D(\mathbf{v},\tau/v) - I\Big)(\mathbf{p}_l\Phi_0), V_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\Psi_0\Big). \tag{II.2.65}$$

With this notation

$$J_1(v) - \lim_{v\to\infty} J_1(v) = \int d\tau \left(h_{\mathbf{v}}^{(1)} + h_{\mathbf{v}}^{(2)} \right). \tag{II.2.66}$$

Let us analyze the rate of convergence of $h_{\mathbf{v}}^{(1)}$. On one hand, with $t = \tau/v$:

$$\left\|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)} \tilde{U}_D(\mathbf{v}, \tau/v) - I)\Psi_0\right\|^2 \quad \leq \quad \left[C\Big|\tau/v\Big| \left(1 + \Big|\tau/v\Big|\right)\right]^2,$$

On the other hand:

$$\left\|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\, \tilde{U}_D(\mathbf{v}, \tau/v) - I)\Psi_0\right\| \quad \leq \quad 2\,\|\Psi_{12}\|\,.$$

Consider two cases, with $0 < a \leq 1$ in mind:

(a) $|\tau/v| < 1$: Clearly we have that $|\tau/v|^a \geq |\tau/v|$, therefore:

$$\left\|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\, \tilde{U}_D(\mathbf{v}, \tau/v) - I)\Psi_0\right\| \leq C\Big|\tau/v\Big| \left(1 + \Big|\tau/v\Big|\right) \leq C\Big|\tau/v\Big|^a. \tag{II.2.67}$$

(b) $|\tau/v| \geq 1$: In this case $|\tau/v|^a \geq 1$, thus:

$$\left\|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\, \tilde{U}_D(\mathbf{v}, \tau/v) - I)\Psi_0\right\| \leq C \leq C\Big|\tau/v\Big|^a. \tag{II.2.68}$$

Now we study $|h_{\mathbf{v}}^{(1)}(\tau)|$'s decay as $v \to \infty$ applying Lemma II.2.3, and (II.2.67), (II.2.68) with $a = \rho$:

$$\begin{aligned} |h_{\mathbf{v}}^{(1)}(\tau)| \quad \leq \quad & C\Big|\tau/v\Big|^\rho \Bigg\| V_{12}^{vs}\left(\mathbf{x} + \tau\,\mathbf{p}/(\mu_{12}v) + \mathbf{e}_1 q_{12} E\tau^2/(2v^2\mu_{12})\right)\, \tilde{U}_D(\tau/v) \\ & \times \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\mathbf{x}|^2)^{-2} \Bigg\|. \end{aligned}$$

Then

$$v^\rho |h_{\mathbf{v}}^{(1)}(\tau)| \leq C|\tau|^\rho h_{12}(|\tau|) \in L^1(-\infty, \infty). \tag{II.2.69}$$

Hence, for $\rho = 1$

$$v \int |h_{\mathbf{v}}^{(1)}(\tau)| d\tau \leq C.$$

For $0 \leq \rho < 1$, by Lebesgue dominated convergence theorem

$$\lim_{v\to\infty} v^\rho \int h_{\mathbf{v}}^{(1)}(\tau) d\tau = \int \lim_{v\to\infty} v^\rho h_{\mathbf{v}}^{(1)}(\tau) d\tau = 0,$$

where we used that $\lim_{v\to\infty} v^\rho h_{\mathbf{v}}^{(1)}(\tau) = 0$, since by (II.2.67) and (II.2.68) with $a = 1$ we have $v^\rho |h_{\mathbf{v}}^{(1)}(\tau)| \leq C|\tau| v^{\rho-1}$.

As a result

$$\int_{-\infty}^{+\infty} d\tau h_{\mathbf{v}}^{(1)}(\tau) = \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1. \end{cases} \tag{II.2.70}$$

At this moment, we turn our attention to the rate of convergence of $h_{\mathbf{v}}^{(2)}$. When $|\mathbf{x} + \tau\hat{\mathbf{v}}| \leq |\tau|/2$, we have $|\mathbf{x}| \geq |\tau| - |\mathbf{x} + \tau\hat{\mathbf{v}}| \geq |\tau|/2$. With the last inequality we can estimate the second factor in the scalar product of (II.2.65). Let g be a $C_0^\infty(\mathbb{R}^n)$ such that $g(\mathbf{p})\hat{\psi}_{12} = \hat{\psi}_{12}$. By (II.2.67) and (II.2.68):

$$\begin{aligned} v^\rho \int_{-\infty}^{\infty} d\tau\, |h_{\mathbf{v}}^{(2)}(\tau)| \quad \leq \quad & C \int_{-\infty}^{+\infty} d\tau |\tau|^\rho \left(\|V_{12}^{vs}(\mathbf{x} + \hat{\mathbf{v}}\tau) g(\mathbf{p}) F(|\mathbf{x} + \hat{\mathbf{v}}\tau| \geq |\tau|/2)\| \right. \\ & \left. + \|V_{12}^{vs}(\mathbf{x} + \hat{\mathbf{v}}\tau) g(\mathbf{p})\|\, \|F(|\mathbf{x}| \geq |\tau|/2)\Psi_{12}\| \right). \end{aligned} \tag{II.2.71}$$

Due to the short-range condition (II.2.21), the first integral in (II.2.71) is finite; the fast decay in configuration space of Ψ_{12} makes the second integral in (II.2.71) be bounded:

$$\int_{-\infty}^{\infty} d\tau\, |\tau|^{\rho}\, \|F(|\mathbf{x}| \geq |\tau|/2)\Psi_{12}\| \;=\; \int_{-\infty}^{\infty} d\tau\, |\tau|^{\rho}(1+|\tau|)^{-3} \left\|(1+|\tau|)^3 F(|\mathbf{x}| \geq \frac{|\tau|}{2})\Psi_{12}\right\| < \infty.$$

Hence, for $\rho = 1$

$$v\int |h_{\mathbf{v}}^{(2)}(\tau)| d\tau \leq C,$$

and for $0 \leq \rho < 1$, by Lebesgue dominated convergence theorem

$$\lim_{v\to\infty} v^{\rho} \int h_{\mathbf{v}}^{(2)}(\tau) d\tau = \int \lim_{v\to\infty} v^{\rho} h_{\mathbf{v}}^{(2)}(\tau) d\tau = 0,$$

where we used that $\lim_{v\to\infty} v^{\rho} h_{\mathbf{v}}^{(2)}(\tau) = 0$, since by (II.2.67) and (II.2.68) with $a = 1$ we have $v^{\rho}|h_{\mathbf{v}}^{(2)}(\tau)| \leq C|\tau| v^{\rho-1}$.

As a result

$$\int_{-\infty}^{+\infty} d\tau h_{\mathbf{v}}^{(2)}(\tau) = \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1. \end{cases} \tag{II.2.72}$$

We have just estimated the rate of convergence of J_1, the first term in the right-hand side of (II.2.51). Since $\hat{\phi}_{12} \in C_0^{\infty}(B_{\mu_{12}\eta})$, we have that $\left(p_l \hat{\phi}_{12}\right) \in C_0^{\infty}(B_{\mu_{12}\eta})$, therefore, we can apply the same treatment to J_2, the second term in the right-hand side of (II.2.51). For J_3, in the right-hand side of (II.2.51), when we estimate the term with $\partial/\partial x_l V_{12}^{E,s}$ we have that

$$(1+|x|)^{\rho_s} |\partial/\partial x_l V_{12}^{E,s}(x)| \leq C(1+|x|)^{-1-\alpha+\rho_s}$$

satisfies the very short-range condition if $\rho_s < \alpha \leq 1$. Nevertheless, when $q_{12} \neq 0$, we do not have an extra error term of the form $o(v^{-\rho_s})$ because in (II.2.56) and (II.2.57) $\rho < \alpha$, for that reason one can always choose ρ_s such that $\rho < \rho_s < \alpha \leq 1$. Regarding J_4 in (II.2.51), we estimate the term with $\partial/\partial x_l V_{12}^{l}$, one sees that

$$(1+|x|)^{\rho_l} |\partial/\partial x_l V_{12}^{l}(x)| \leq C \begin{cases} (1+|x|)^{-\gamma_D-\mu+\rho_l}, & \text{if } q_{12} \neq 0, \\ (1+|x|)^{-\gamma_1+\rho_l}, & \text{if } q_{12} = 0, \end{cases}$$

satisfies the very short-range condition if $\rho_l < \begin{cases} \gamma_D + \mu - 1, & \text{if } q_{12} \neq 0, \\ \gamma_1 - 1, & \text{if } q_{12} = 0. \end{cases}$ Therefore, we have another error term of the form

$$o(v^{-\rho_l}). \tag{II.2.73}$$

Moreover, when there is at least one pair with non-zero relative charge, we have to estimate the following error, see (II.2.39) and (II.2.46). In this case, $\rho < 1$, and $-(2\gamma-1) \leq -(2\alpha-1) \leq -\rho$, where γ is as in Definition II.1.2. By equation (II.2.13):

$$\begin{aligned} |I_{G,v} - 1| &\leq \sum_{j<k}^{E} \int_{-\infty}^{\infty} ds\, |V_{jk}^{s}(\mathbf{v}_{jk}s + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk}))| \leq C \begin{cases} v^{-(2\gamma-1)}, & \text{if } 1/2 < \gamma < 1, \\ \frac{\ln v}{v}, & \text{if } \gamma = 1, \end{cases} \\ &= \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1. \end{cases} \end{aligned} \tag{II.2.74}$$

Finally, to prove the convergence rate given in (II.2.44) we sum the terms, corresponding to $I(v)$, $R(v)$, $I_{G,v}$, respectively, in (II.2.46), recalling (II.2.56), (II.2.57), (II.2.66), (II.2.70), (II.2.72), (II.2.74) and taking in consideration (II.2.73) with the highest possible values of ρ_l in order to have the optimal error rate in all the cases enounced in Theorem II.2.8. ■

The following reconstruction formula is of independent interest.

THEOREM II.2.10. *Assume the same hypothesis as in Theorem II.2.8, Then*

$$v(i[S^D - I_{G,v}]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}) - I_{G,v}\int_{-\infty}^{\infty} vdt\,\Big(\big(V_{12}^s(\mathbf{x}) - V_{12}^s(\mathbf{v}t + \mathbf{e}_1 q_{12} Et^2/(2\mu_{12})) + V_{12}^l(\mathbf{x})$$

$$-V_{12}^l(t\,\mathbf{p}/\mu_{12} - \mathbf{e}_1 q_{12} Et^2/(2\mu_{12}))\big)\, e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\Big) = \int_{-\infty}^{\infty} d\tau (V_{12}^{E,\,vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi_{12}, \Psi_{12})$$

$$+\begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 2\min\{\alpha,\, \sigma_{jk}\,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk}\,|\, 1 \le j < k \le N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1, \text{ and } \sum |q_{jk}| = 0. \end{cases} \tag{II.2.75}$$

Proof. The left hand side of (II.2.75) can be written as equal to the right hand side of (II.2.46) exactly with the same $I_{G,v}$ but with

$$I(v) = v\int_{-\infty}^{+\infty} dt\,\Big(V_{12}^{vs}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\Big),$$

and, with the same $V_{3,t}$ and V_t as in the proof of Theorem II.2.8,

$$\begin{aligned} R(v)/v \;\; = \;\; & \int_{-\infty}^{+\infty} dt\,\Big[\Big(V_{3,t}e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\Big) \\ & + \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t)\Big)\Phi_{\mathbf{v}}, V_t U^{D,G,v}(t)\Psi_{\mathbf{v}}\Big)\Big]. \end{aligned}$$

The convergence rate of $I(v)$ is computed like that one of J_1, see equations (II.2.52), (II.2.64), (II.2.65), (II.2.70), (II.2.72). $R(v)$ and $I_{G,v}$ are estimated like in (II.2.57) and (II.2.74), respectively. ■

Proof of Theorem II.1.3:

Let us consider the states $\Phi \sim \hat{\phi}_{12}(p)\hat{\phi}_3(p_3,\ldots,p_N)$, $\Psi \sim \hat{\psi}_{12}(p)\hat{\phi}_3(p_3,\ldots,p_N)$, such that $\hat{\phi}_{12}, \hat{\psi}_{12} \in C_0^\infty(\mathbb{R}^n)$ and $\hat{\phi}_3$ is like in (II.1.12). Let $\mathbf{y}$ be an element of a two dimensional subspace of $\mathbb{R}^n$, for instance, we associate each $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$ with the vector $y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \in \mathbb{R}^n$. We express by

$$\Phi^{\mathbf{y}} = e^{-i\mathbf{p}\cdot\mathbf{y}}\Phi \Leftrightarrow \phi^{\mathbf{y}} = \phi_{12}(\mathbf{x}-\mathbf{y})\phi_3(\mathbf{x}_3,\ldots,\mathbf{x}_N), \quad \Psi^{\mathbf{y}} = e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi \Leftrightarrow \psi^{\mathbf{y}} = \psi_{12}(\mathbf{x}-\mathbf{y})\phi_3(\mathbf{x}_3,\ldots,\mathbf{x}_N), \tag{II.2.76}$$

the states, translated in the configuration space by $\mathbf{y}$, considered as an vector in $\mathbb{R}^n$.

Suppose that $V^i = V^{VS,\,i} + V^{S,\,i} + V^{L,\,i} \in \mathcal{V}_{VSR} + \mathcal{V}_{SR} + \mathcal{V}_{LR}$, $i = 1,2$, and that $S^D(V^{L,\,1}; V^{VS,\,1} + V^{S,\,1}) = S^D(V^{L,\,2}; V^{VS,\,2} + V^{S,\,2})$. Then, we can write the potentials V^i, $i = 1,2$,

$$V^i = \sum_{1\le j<k\le N} V_{jk}^i(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j), \qquad V_{jk}^i = V_{jk}^{vs,\,i} + V_{jk}^{s,\,i} + V_{jk}^{l,\,i},$$

with, for all $1 \le j < k \le N$, $V_{jk}^{vs,\,i} \in \mathcal{V}_{VSR}$, $V_{jk}^{s,\,i} \in \mathcal{V}_{SR}$, and $V_{jk}^{l,\,i} \in \mathcal{V}_{LR}$.

It is enough to prove uniqueness for the pair $(1,2)$. Let us assume $q_{12} \neq 0$, the other case is similar and simpler. Note that as $q_{12} \neq 0$, $V_{12}^{vs,\,i} \in \mathcal{V}_{E,\,vs}$, $V_{12}^{s,\,i} \in \mathcal{V}_{E,\,s}$, and $V_{12}^{l,\,i} \in \mathcal{V}_{E,\,l}$. We define

$$\begin{cases} Q_{12}^{vs}(\mathbf{x}) & = V_{12}^{vs,2}(\mathbf{x}) - V_{12}^{vs,1}(\mathbf{x}), \\ Q_{12}^{s}(\mathbf{x}) & = V_{12}^{s,2}(\mathbf{x}) - V_{12}^{s,1}(\mathbf{x}), \\ Q_{12}^{l}(\mathbf{x}) & = V_{12}^{l,2}(\mathbf{x}) - V_{12}^{l,1}(\mathbf{x}), \\ Q_{12}(\mathbf{x}) & = Q_{12}^{vs}(\mathbf{x}) + Q_{12}^{s}(\mathbf{x}) + Q_{12}^{l}(\mathbf{x}). \end{cases} \tag{II.2.77}$$

With $\Phi^{\mathbf{y}}$ and $\Psi^{\mathbf{y}}$ as in (II.2.76), and $\mathbf{p}_1 = \mathbf{p}\cdot\mathbf{e}_1$, the function $f : \mathbb{R}^2 \to \mathbb{C}$ is defined as

$$f(\mathbf{y}) := f_1(\mathbf{y}) + f_2(\mathbf{y}) + f_3(\mathbf{y}), \tag{II.2.78}$$

where

$$\begin{aligned} f_1(\mathbf{y}) &:= (Q_{12}^{vs}(\mathbf{x})\mathbf{p}_1\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}), \\ f_2(\mathbf{y}) &:= -(Q_{12}^{vs}(\mathbf{x})\Phi^{\mathbf{y}}, \mathbf{p}_1\Psi^{\mathbf{y}}), \\ f_3(\mathbf{y}) &:= i\left(\left(\frac{\partial Q_{12}^s}{\partial x_1} + \frac{\partial Q_{12}^l}{\partial x_1}\right)(\mathbf{x})\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}\right). \end{aligned}$$

Let us focus on f_1. Let g_1 be a $C_0^\infty(\mathbb{R}^n)$ function such that $g_1(p)\hat{\phi}_{12}(p) = \hat{\phi}_{12}(p)$,

$$|f_1(\mathbf{y})| \leq C\|Q_{12}^{vs}(\mathbf{x})g_1(\mathbf{p})\| \tag{II.2.79}$$

$$\begin{aligned} |f_1(\mathbf{y})| &\leq C\left(\|Q_{12}^{vs}(\mathbf{x})g_1(\mathbf{p})F(|\mathbf{x}| \geq |\mathbf{y}|/2)\| \right.\\ &\quad \left. + \|Q_{12}^{vs}(\mathbf{x})\mathbf{p}_1 g_1(\mathbf{p})\| \, \|F(|\mathbf{x}| < |\mathbf{y}|/2)\phi_{12}(\mathbf{x}-\mathbf{y})\|\right). \end{aligned} \tag{II.2.80}$$

Inequality (II.2.79) shows that f_1 is bounded. By the very short range condition (II.2.21):

$$\|Q_{12}^{vs}(\mathbf{x})g_1(\mathbf{p})F(|\mathbf{x}| \geq |\mathbf{y}|/2)\| \in L^2(\mathbb{R}^2).$$

Additionally

$$\begin{aligned} \|F(|\mathbf{x}| < |\mathbf{y}|/2)\phi_{12}(\mathbf{x}-\mathbf{y})\| &= \left\|\frac{1}{1+|\mathbf{x}-\mathbf{y}|^2}F(|\mathbf{x}| < |\mathbf{y}|/2)(1+|\mathbf{x}-\mathbf{y}|^2)\phi_{12}(\mathbf{x}-\mathbf{y})\right\| \\ &\leq \frac{C}{(1+|\mathbf{y}|/2)^2} \in L^2(\mathbb{R}^2). \end{aligned}$$

Then, $f_1(\mathbf{y}) \in L^2(\mathbb{R}^2)$. Moreover, $f_1(\mathbf{y})$ is continuous because the operator $e^{-i\mathbf{p}\cdot\mathbf{y}}$ is strongly continuous on $L^2(\mathbb{R}^2)$.

Working with f_2 and f_3 is analogous to the case of f_1, remarking that (II.1.28) and (II.1.29) imply that $\frac{\partial Q_{12}^s}{\partial x_1} + \frac{\partial Q_{12}^l}{\partial x_1}$ belongs to our very short-range class $\mathcal{V}_{E,\,vs}$. Thus $f(\mathbf{y}) \in L^2(\mathbb{R}^2)$ and it is a bounded continuous function.

The Radon transform of $f(\mathbf{y})$, for any $\mathbf{v}$ in the $\mathbf{y}$-plane satisfying $|\hat{\mathbf{v}}\cdot\hat{\mathbf{E}}| < 1$, is given by

$$\begin{aligned} \tilde{f}(\hat{\mathbf{v}};\mathbf{y}) &:= \int_{-\infty}^{\infty} f(\mathbf{y}+\tau\hat{\mathbf{v}})d\tau = \int_{-\infty}^{\infty} [(Q_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\mathbf{p}_1\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}) \\ &\quad -(Q_{12}^{vs}(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi^{\mathbf{y}}, \mathbf{p}_1\Psi^{\mathbf{y}}) \\ &\quad +i\left(\left(\frac{\partial Q_{12}^s}{\partial x_1} + \frac{\partial Q_{12}^l}{\partial x_1}\right)(\mathbf{x}+\tau\hat{\mathbf{v}})\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}\right)\Bigg]. \end{aligned} \tag{II.2.81}$$

By Theorem II.2.8 applied to the pair $(1,2)$, we have that

$$\begin{aligned} \tilde{f}(\hat{\mathbf{v}};\mathbf{y}) &= \lim_{v\to\infty}\left[v(i[S^D(V^{L,1}; V^{VS,1}+V^{S,1}), \mathbf{p}_1]\Phi_{\mathbf{v}}^{\mathbf{y}}, \Psi_{\mathbf{v}}^{\mathbf{y}}) \right.\\ &\quad \left. -v(i[S^D(V^{L,2}; V^{VS,2}+V^{S,2}), \mathbf{p}_1]\Phi_{\mathbf{v}}^{\mathbf{y}}, \Psi_{\mathbf{v}}^{\mathbf{y}})\right] \\ &\equiv 0. \end{aligned}$$

Then, the Plancherel formula associated with the Radon transform [23] implies that $f(\mathbf{y}) = 0$. From (II.2.78) we have that

$$\frac{\partial}{\partial y_1}(Q_{12}\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}) = -if(\mathbf{y}).$$

This implies that $(Q_{12}\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}})$ does not depend on y_1. Moreover, $\lim_{|y_1|\to\infty}(Q_{12}\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}) = 0$ by (II.1.26), (II.1.27) and (II.1.29). Therefore, $(Q_{12}\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}) \equiv 0$. In particular $(Q_{12}\Phi^0, \Psi^0) = (Q_{12}\phi_{12}, \psi_{12}) = 0$, what implies by the density of the states ϕ_{12}, ψ_{12} that $Q_{12}(\mathbf{x}) \equiv 0$ a.e. We conclude that the total potential V

is uniquely determined by the high-velocity limit of the commutator of any Dollard scattering operator S^D and some component of the momentum.

We consider the reconstruction problem of the total potential V as in (II.1.36), by means of Theorem II.2.8. We assume $q_{12} \neq 0$ because the case $q_{12} = 0$ is easier. Let us compute $V_{12} := V_{12}^{vs} + V_{12}^{s} + V_{12}^{l} \in \mathcal{V}_{VSR} + \mathcal{V}_{SR} + \mathcal{V}_{LR}$ from the high-velocity limit of $[S^D, \mathbf{p}_1]$. We substitute Q_{12}^{vs} by V_{12}^{vs}, Q_{12}^{s} by V_{12}^{s} and Q_{12}^{l} by V_{12}^{l} in (II.2.78). We know $\lim_{v\to\infty} v(i[S^D, \mathbf{p}_1]\Phi_{\mathbf{v}}^{\mathbf{y}}, \Psi_{\mathbf{v}}^{\mathbf{y}})$ for all $\Phi^{\mathbf{y}}$ and $\Psi^{\mathbf{y}}$ as in (II.2.76). Then, by Theorem II.2.8 and (II.2.81) we reconstruct $\widetilde{f}(\hat{\mathbf{v}}; \mathbf{y})$ and by the inversion of the Radon transform [23], we uniquely reconstruct $f(\mathbf{y})$. From (II.1.26), (II.1.27), (II.1.28) and (II.1.29) f is integrable along any line and $\lim_{\mathbf{y}\to\infty}((V_{12})\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}) = 0$. Then we have

$$(V_{12}\phi_{12}, \psi_{12}) = i \int_0^\infty f(y_1, 0) dy_1,$$

in a dense set in L^2. Hence V_{12} is obtained almost everywhere as a function. Repeating this process for all pairs we reconstruct V. ■

REMARK II.2.11. As we have already mentioned in Remark II.1.6 the reconstruction formula (II.2.75) from Theorem II.2.10 is simpler than the formula (II.2.44) in Theorem II.2.8. Let us show how (II.2.75) can be used. Let us suppose that $q_{12} \neq 0$. The case $q_{12} = 0$ follows in the same way. The potentials $V_{12}^{E,\,vs} \in \mathcal{V}_{E,\,vs}$, $V_{12}^{E,\,s} \in \mathcal{V}_{E,\,s}$, $V_{12}^{E,\,l} \in \mathcal{V}_{E,\,l}$ are the very short-, short- and long-range potentials, respectively, for the pair $(1, 2)$. Let us assume that we want to recover $V_{12}^{E,\,vs}$ knowing $V_{12}^{E,\,s}$, $V_{12}^{E,\,l}$ and the high-velocity limit of S^D for each $\Phi^{\mathbf{y}}$ and $\Psi^{\mathbf{y}}$ as in (II.2.76). Defining

$$h(\mathbf{y}) = (V_{12}^{E,vs}(\mathbf{x})\Phi^{\mathbf{y}}, \Psi^{\mathbf{y}}), \tag{II.2.82}$$

using Theorem II.2.10 and inverting the Radon transform we obtain $h(\mathbf{y})$. Then, we can compute $(V_{12}^{E,\,vs}\phi_{12}, \psi_{12}) = h(0)$ in a dense set in L^2. This implies that we recover $V_{12}^{E,\,vs}$ almost everywhere as a function.

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Chapter 3

Linear operators in Hilbert space

This chapter contains the mathematical material used in chapters 2 and the appendix. A brief description of L^p spaces is given in section I. In section II we introduce Hilbert space, where we discuss some of its elementary properties, which are illustrated in the case of L^2 spaces. Hilbert space is a basic mathematical object, which is necessary to describe particles in quantum mechanics. In section III many simple results concerning linear operators in Hilber space are presented. We develop Fourier transform in sections IV and V, we have chosen the approach taken by Rudin to generalize Fourier transforms to Groups (where a manifold is a group).

In sections VI, VII, IX and X we deal with bounded and unbounded operators, the spectral theorem applied to both types of operators and tensorial products of them. We present, in section, VIII the Stone theorem used to prove the existence of evolution group of self-adjoint operators. For the writting of this chapter we have used Amrein books such as [58] and [59].

I L^p Spaces

Briefly we collect here a few definitions and results from the theory of L^p spaces. Let $(M;\mu)$ be a measure space, in other words let μ be a measure defined on a σ-algebra $\mathcal{R}$ of subsets of the set M. If Δ is a measurable subset of M (i.e. an element of $\mathcal{R}$), we denote by χ_Δ the characteristic function of Δ, which is defined as follows:

$$\chi_\Delta(s) = \begin{cases} 1 & s \in \Delta \\ 0 & s \notin \Delta. \end{cases} \tag{III.1.1}$$

For $p \in [1,\infty]$, $L^p(M;d\mu)$ is the set of all equivalence classes of measurable functions $f : M \to \mathbb{C}$ satisfying $\|f\|_p < \infty$, where two functions are said equivalent if they are equal μ-almost everywhere, and where $\|f\|_p$ is defined as follows:

$$\|f\|_p := \left[\int_M |f(s)|^p d\mu(s)\right]^{1/p} \quad \text{if } p < \infty \tag{III.1.2}$$

and

$$\|f\|_\infty := \operatorname*{ess\,sup}_{s\in M} |f(s)|. \tag{III.1.3}$$

Here $\operatorname{ess\,sup} f(s)$ is the infimum of $\sup h(s)$ as h varies over all functions that are equal to f almost everywhere. In other words $\operatorname{ess\,sup} f(s)$ is the infimum of all m such that the measure $\mu(\Delta_m)$ of the set $\Delta_m = \{s \in M \mid f(s) > m\}$ is zero. If for example, M is the closed interval $M = [a,b]$, μ Lebesgue measure and f is continuous in $[a,b]$, then $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$.

$L^p(M;d\mu)$ is a complete normed linear space with respect to the norm $\|\cdot\|_p$. If $f \in L^p(M;d\mu), g \in L^p(M;d\mu)$ and $1/r = 1/p + 1/q$, entonces $f(\cdot)g(\cdot) \in L^r(M;d\mu)$ and

$$\|fg\|_r \leq \|f\|_p \|g\|_q \tag{III.1.4}$$

Inequality (III.1.4) is known as the Hölder inequality. The following facts about L^p spaces are often useful:

LEMMA III.1.1. *(a) If Δ is a measurable subset of M with finite measure, $p \in [1,\infty]$ and $f \in L^p(M;d\mu)$, then $\chi_\Delta(\cdot)f(\cdot) \in L^r(M;d\mu)$ for each $r \in [1,p]$. (b) If $1 \leq p < q \leq \infty$, then $L^p(M;d\mu) \cap L^q(M;d\mu) \subset L^r(M;d\mu)$ for each $r \in [p,q]$.*

An important theorem, which allows one to interchange a limit with an integral, is the Lebesgue Dominated convergence Theorem. We use it only for $p = 1$:

THEOREM III.1.2. *Lebesgue Dominated convergence Theorem*

i $g, f_t \in L^1(M;d\mu)(t \in \mathbb{R})$,

ii $|f_t(s)| \leq g(s)$ *for almost all* $s \in M$ *and all* t,

iii $\lim_{t\to t_0} f_t(s) = f(s)$ *for almost all* $s \in M$.

Then $f \in L^1(M;d\mu)$ *and*

$$\lim_{t\to t_0} \int_M f_t(s)d\mu(s) = \int_M f_{t_0}(s)d\mu(s).$$

Instead of using functions with values in $\mathbb{C}$, one could also consider functions from M into some complete normed space $\mathcal{H}_0$ and define the spaces $L^p(M,\mathcal{H}_0;d\mu)$.

II Hilbert Space

DEFINITION III.2.1. *A Hilbert space is a set $\mathcal{H}$ of elements $f, g, h, \ldots$ called vectors satisfying the following three axioms:*

I *$\mathcal{H}$ is a linear vector space over the field $\mathbb{C}$ of complex numbers: Whenever $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ is an element of $\mathcal{H}$, and*

(*a*) $f + g = g + f, \quad (f+g)+h = f+(g+h),$
(*b*) $\alpha(f+g) = \alpha f + \alpha g, \quad (\alpha+\beta)f = \alpha f + \beta f,$
(*c*) $\alpha(\beta f) = (\alpha\beta)f, \quad 1\cdot f = f,$

and there exists a vector θ, called the zero vector and often written as 0, such that

(*d*) $f + \theta = f$, *and* $0 \cdot f = \theta$ *for all* $f \in \mathcal{H}$.

II *There exists on $\mathcal{H}$ a positive definite scalar product, i.e. a mapping from $\mathcal{H}\times\mathcal{H}$ to $\mathbb{C}$, denoted by $(\cdot,\cdot)$, such that for all $f, g, h \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$:*

$$(f,g) = \overline{(g,f)}, \tag{III.2.5}$$

$$(f, g+\alpha h) = (f,g) + \alpha(f,h) \text{ for all complex } \alpha, \tag{III.2.6}$$

$$(f,f)^{1/2} \geq 0, \text{ and } (f,f)^{1/2} = 0 \text{ only for } f = \theta. \tag{III.2.7}$$

The scalar product induces a metric on $\mathcal{H}$. The distance $d(f,g)$ between two vectors f and g is $d(f,g) = \|f-g\|$, where the norm $\|\cdot\|$ is defined as

$$\|f\| = (f,f)^{1/2}. \tag{III.2.8}$$

III *$\mathcal{H}$ is complete with respect to the norm* (III.2.8)*: If $\{f_n\}$ is a Cauchy sequence in $\mathcal{H}$, i.e such that $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, there exists a vector $f \in \mathcal{H}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.*

DEFINITION III.2.2. *Let $\mathcal{H}$ be a Hilbert space:*

IV *$\mathcal{H}$ is separable if there exists a dense numerable subset in $\mathcal{H}$. A set D is dense in $\mathcal{H}$ if for all $f \in \mathcal{H}$ and any $\eta > 0$ there exists an element f_η in D such that $\|f - f_\eta\| < \eta$.*

Throughout this thesis, $\mathcal{H}$ is assumed to be a separable complex Hilbert space.

Before introducing the L^2 spaces as explicit examples of Hilbert spaces, we give some additional definitions and some basic results that follows from the previous axioms.

PROPOSITION III.2.3. *Schwarz Inequality. For all f, g in $\mathcal{H}$*

$$|(f, g)| \leq \|f\| \, \|g\|. \tag{III.2.9}$$

A consequence of (III.2.9) is the triangle inequality:

PROPOSITION III.2.4. *Triangle Inequality. For all f, g in $\mathcal{H}$*

$$\|f + g| \leq \|f\| + \|g\|. \tag{III.2.10}$$

The triangle inequality with the Axioms I and II imply that a Hilbert space be a normed space.

Since the vectors of $\mathcal{H}$ will be interpreted as the pure states of some physical system, two states f and g are practically indistinguishable if $\|f - g\|$ is very small. By this reason we will examine the properties of convergence of sequences $\{f_n\}$ of elements of $\mathcal{H}$.

DEFINITION III.2.5. *The convergence of a sequence of vectors in the norm $\| \cdot \|$ has already been used in Axiom III. In Hilbert space theory this is called strong convergence. In this case, we write $s\text{-}\lim f_n = f$ as $n \to \infty$. The strong limit f es unique.*

A necessary and sufficient condition for strong convergence is that the sequence be Cauchy in the sense defined in Axiom III.

The convergence in $\mathcal{H}$ obtained by means of the scalar product is called weak convergence.

DEFINITION III.2.6. *A sequence $\{f_n\}$ converges weakly to a limit f if for every $g \in \mathcal{H}$ the sequence of scalar products $\{(f_n, g)\}$ converges to $\{(f, g)\}$. If this is the case, we write $w\text{-}\lim f_n = f$ as $n \to \infty$. The weak limit f is unique.*

Strong convergence implies weak convergence, but the converse is not true. If fact one has the following relation which is often very useful:

PROPOSITION III.2.7. *$s\text{-}\lim f_n = f$ if and only if $w\text{-}\lim f_n = f$ and $\lim \|f_n\| = \|f\|$.*

We have to introduce the notion of orthogonality and mutually orthogonal subsets.

DEFINITION III.2.8. *Two vectors f and g are said to be orthogonal to each other if $(f, g) = 0$.*

DEFINITION III.2.9. *Two subsets M_1 and M_2 of $\mathcal{H}$ are mutually orthogonal if $(f_1, f_2) = 0$ for all $f_1 \in M_1$ and $f_2 \in M_2$.*

An important relation concerning mutually orthogonal vectors is the following:

THEOREM III.2.10. *Pythagoras Theorem.*

$$\Big\| \sum_{i=1}^{n} f_i \Big\|^2 = \sum_{i=1}^{n} \|f_i\|^2 \ \text{if } (f_i, f_j) = 0 \text{ for all } i \neq j. \tag{III.2.11}$$

DEFINITION III.2.11. *An orthonormal sequence of vectors* $\{h_i\}$ *is characterized by the property that* $(h_i, h_j) = \delta_{i,j}$ *where* $\delta_{ij} = 1$ *if* $i = j$ *and* $\delta_{ij} = 0$ *if* $i \neq j$.

PROPOSITION III.2.12. *Bessel Inequality. Let* $\{h_i\}_{i=1}^{\infty}$ *be a orthonormal sequence of vectors in* $\mathcal{H}$ *then for all* $n \in \mathbb{N}$, $f \in \mathcal{H}$:

$$\sum_{i=1}^{n} |(h_i, f)|^2 \leq \|f\|^2 \tag{III.2.12}$$

DEFINITION III.2.13. *An orthonormal set of vectors* $\{e_i\}$ *is called an orthonormal basis of* $\mathcal{H}$ *if the set of all linear combinations of vectors belonging to* $\{e_i\}$ *is dense in* $\mathcal{H}$.

In a separable Hilbert space an orthonormal basis is always a countable set. The existence of an orthonormal set basis can be established by choosing a subset of linearly independent vectors of a countable dense set D and applying to it the Schmidt orthogonalization process.

DEFINITION III.2.14. *The dimension of a separable Hilbert space is equal to the number* N *of vectors of an orthonormal basis if this basis is finite, otherwise, if this basis is an infinite countable set, the dimension of this separable Hilbert space is equal to* $\aleph_0$.

One can prove that any two orthonormal bases of the same Hilbert space have the same cardinal, therefore, the dimension of a Hilbert space is well defined. Hilbert space Axioms apply to finite or infinite dimensional spaces. Nevertheless, in the finite dimension case, Axioms III and IV are consequence of Axiomas I and II; furthermore, in this case, strong convergence coincides with weak convergence.

THEOREM III.2.15. *Parseval Relation. If* $\{e_i\}$ *is an orthonormal basis of vectores in* $\mathcal{H}$ *and* $f \in \mathcal{H}$, *then*

$$\|f\|^2 = \sum_{i=1}^{\infty} |(e_i, f)|^2. \tag{III.2.13}$$

Parseval relation implies that each $f \in \mathcal{H}$ can be expressed as the strong limit of the sequence $\{f_n\}$, where $f_n = \sum_{i=1}^{n} (e_i, f) e_i$.

PROPOSITION III.2.16. *Let* D *be a dense set in* $\mathcal{H}$ *and* $f \in \mathcal{H}$. *If* $(f, g) = 0$ *for all* $g \in D$, *then* $f = \theta$.

DEFINITION III.2.17. *A linear manifold is a subset* M *of* $\mathcal{H}$ *that satisfies Axiom I but not necessarily Axiom III (*M *will always verify Axioms II and IV, since it is a subset of* $\mathcal{H}$*). A subset of* $\mathcal{H}$ *that satisfies all four Axioms will be called a subspace.*

An important example of a closed linear manifold (i.e. a subspace) is given in Definition III.2.18.

DEFINITION III.2.18. *The orthogonal complement* $N^{\perp}$ *of a subset* N *of* $\mathcal{H}$ *is the set of all vectors* $f \in \mathcal{H}$ *such that* $(f, g) = 0$ *for all* $g \in N$.

It is worth noticing the following fact known as the Projection Theorem.

THEOREM III.2.19. *Projection Theorem. If* M *is a subspace and* $M^{\perp}$ *is its orthogonal complement, then each vector* f *in* $\mathcal{H}$ *has a unique decomposition* $f = f_1 + f_2$ *with* $f_1 \in M$ *and* $f_2 \in M^{\perp}$.

A simple but very important consequence is the following:

PROPOSITION III.2.20. *Density criterion. If* M *is a linear manifold such that the unique vector of* $\mathcal{H}$ *that is orthogonal to* M *is the vector* θ, *then* M *is dense in* $\mathcal{H}$.

DEFINITION III.2.21. *A linear bounded functional in a Hilbert space* $\mathcal{H}$, *is a linear function* Φ *from* $\mathcal{H}$ *into* $\mathbb{C}$, *which is bounded with respect to the norm in* $\mathcal{H}$, *i.e.,*

$$\|\Phi\| = \sup_{f \neq \theta} \frac{|\Phi(f)|}{\|f\|} < \infty.$$

If g is a fixed vector in $\mathcal{H}$, one may associate with it a bounded linear functional Φ_g in $\mathcal{H}$ by $\Phi_g(f) = (g, f)$. The converse is true and known as:

THEOREM III.2.22. *Riesz representation Theorem. Let $\Phi : \mathcal{H} \to \mathbb{C}$ be a bounded linear functional. Then there exists a uniquely determined vector $g \in \mathcal{H}$ such that $\Phi(f) = (g, f)$ for all $f \in \mathcal{H}$, and $\|\Phi\| = \|g\|$.*

We present one of the most concrete and useful examples of Hilbert spaces: The set of all functions in $L^2(\mathbb{R}^n)$ form a linear vectorial space if we define the sum and multiplication by scalar as follows:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (\alpha f)(x) = \alpha f(x).$$

The scalar product between two function is defined by

$$(f, g) = \int_{\mathbb{R}^n} \overline{f(x)} g(x)\, d^n x.$$

This integral can be shown to be finite by Hölder Inequality.

The Hilbert space $L^2(\mathbb{R}^n)$ does not consist of individual functions by themselves but rather of classes of equivalent functions. Two functions are defined to be equivalent if they differ only on a set of measure zero. It is in most cases possible to transfer all operations in the Hilbert space $L^2(\mathbb{R}^n)$ to individual functions (a practice which we shall frequently follow in the traditional manner). There are occasional situations where the above remark is essential and must be borne in mind.

Completeness of $L^2(\mathbb{R}^n)$ is a classic result of analysis known as the Riesz-Fischer Theorem. Separability of $L^2(\mathbb{R}^n)$ can also be proved.

III Linear operators in Hilbert space

DEFINITION III.3.1. *A linear operator in a Hilbert space $\mathcal{H}$ is a linear mapping between vectors of $\mathcal{H}$.*

A linear operator is defined giving its domain, i.e. a linear manifold $D(A)$ in $\mathcal{H}$, and a linear mapping A of $D(A)$ in $\mathcal{H}$. The following notation is widely used: If M is a subset of $D(A)$, then AM is the subset of all vectors f in $\mathcal{H}$ such that $f = Ag$ for a g in M. The subset $AD(A)$ is known as the range of the operator A.

Two linear operators A and B are equal if and only if $D(A) = D(B)$ and $Af = Bf$ for all $f \in D(A)$.

DEFINITION III.3.2. *A linear operator A' is called an extension of A if $D(A) \subset D(A')$ and $A'f = Af$ for all $f \in D(A)$. In such case we write $A \subset A'$. One can say that A is the restriction of A' in $D(A)$. A linear operator is usually called an operator.*

DEFINITION III.3.3. *Let A be an operator in $\mathcal{H}$. We say that A is closable if the following condition holds: Whenever $\{f_n\}$ and $\{f'_n\}$ are two Cauchy sequences in $D(A)$ that strongly converge to the same limit f, and both $\{Af_n\}$ and $\{Af'_n\}$ are also Cauchy, then $s\text{-}\lim Af_n$ and $s\text{-}\lim A'f_n$ are equal.*

Since A is linear, we have the following equivalence:

PROPOSITION III.3.4. *An operator A is closable if and only if $\{f_n\}$ is a sequence in $D(A)$, such that $f_n \to \theta$ and Af_n is strongly Cauchy implies $Af_n \to \theta$.*

A very natural way to define an extension $\bar{A}$ of an operator A is the following:

DEFINITION III.3.5. *If an operator A is closable we define la closure $\bar{A}$ of the operator A whose domain is $D(\bar{A})$. We say that $f \in D(\bar{A})$ if f is the strong limit of a Cauchy sequence $\{f_n\}$ of elements in $D(A)$ such that $\{Af_n\}$ is also Cauchy and strongly converges to g. We define $\bar{A}f = g$. The closure is well defined because A is closable.*

If an operator A is closable, then its closure $\bar{A}$ is its smallest closed extension, i.e. if A' is an arbitrary closed extension of A, then $\bar{A} \subset A'$.

A very important class of closable operators is the class of bounded operators.

DEFINITION III.3.6. *We say that a linear operator A is bounded if there exists a number $M < \infty$ such that $\|Af\| \leq M\|f\|$ for all $f \in D(A)$. If there does not exist such M, A is called unbounded. For A bounded one defines its norm $\|A\|$ as*

$$\|A\| = \sup_{f\in D(A), f\neq\theta} \frac{\|Af\|}{\|f\|}. \tag{III.3.14}$$

We denote by $\mathcal{B}(\mathcal{H})$ the set of all bounded operators A in $\mathcal{H}$ such that $D(A) = \mathcal{H}$.

In consequence, one has that for $f \in D(A)$:

$$\|Af\| \leq \|A\|\|f\|, \tag{III.3.15}$$

this implies the following result very important in scattering theory.

PROPOSITION III.3.7. *If A is an bounded linear operator in a Hilbert space $\mathcal{H}$, then it has a unique bounded extension $\bar{A}$ in the subspace generated by $D(A)$ (i.e. the closure $\overline{D(A)}$ of $D(A)$). $\bar{A}$ is closed, and $\|\bar{A}\| = \|A\|$. In particular, if $D(A)$ is dense in $\mathcal{H}$, then $D(\bar{A}) = \mathcal{H}$.*

We will define the concept of adjoint operator A^* of a linear operator A.

DEFINITION III.3.8. *Assume that $D(A)$ is dense in $\mathcal{H}$. First, we define the domain $D(A^*)$: A vector $g \in \mathcal{H}$ belongs to $D(A^*)$ if there exists a vector $g^* \in \mathcal{H}$ such that*

$$(g, Af) = (g^*, f) \qquad \forall f \in D(A). \tag{III.3.16}$$

The mapping A^* is then defined as $A^*g = g^*$.

Equation (III.3.16) can be rewritten in the following way:

$$(g, Af) = (A^*g, f) \qquad \forall f \in D(A), g \in D(A^*). \tag{III.3.17}$$

One can show that A^* is well defined, i.e. the vector g^* in (III.3.16) is unique. Clearly, A^* is linear. Some of the properties of the adjoint operator of a linear operator are the following:

(a) The adjoint of a linear operator A is always a closed operator.

(b) If A is closable and $D(A)$ is dense, then

$$A^* = (\bar{A})^* \equiv \bar{A}^*. \tag{III.3.18}$$

If $D(A^*)$ is also dense in $\mathcal{H}$, then $A^{**} \equiv (A^*)^*$ exists. We have the following result:

PROPOSITION III.3.9. *Let A be a linear operator such that $D(A)$ and $D(A^*)$ are dense in $\mathcal{H}$. Then A is closable and $\bar{A} = A^{**}$.*

DEFINITION III.3.10. *A is symmetric if $D(A)$ is dense in $\mathcal{H}$ and $A \subset A^*$ (i.e. if $D(A) \subset D(A^*)$ and $A^*f = Af$ for each $f \in D(A)$).*

Condición $A \subset A^*$ can also be written as follows:

$$(Af, g) = (f, Ag) \text{ for all } f, g \in D(A). \tag{III.3.19}$$

DEFINITION III.3.11. *Self-adjointness. A is self-adjoint if $D(A)$ is dense in $\mathcal{H}$ and $A = A^*$ (i.e. if $D(A) = D(A^*)$ and $A^*f = Af$ for all $f \in D(A)$).*

Clearly every self-adjoint operator is symmetric. If A is bounded and $D(A) = \mathcal{H}$, then A is symmetric if and only if it is self-adjoint. If A is not bounded, the condition A to be self-adjoint is a very strong condition, because it requires that $D(A^*)$ be exactly equal to $D(A)$. Condition (III.3.19) which is easy to verify in applications, it is not sufficient for the self-adjointness of A.

DEFINITION III.3.12. *A symmetric operator A is called essentially self-adjoint if $\bar{A}$ is self-adjoint.*

An equivalent definition of essential self-adjointness is that $A^* = A^{**}$. An essentially self-adjoint operator has one and only one self-adjoint extension. The notion of essentially self-adjointness is important because in applications one often has a non closed symmetric operator. If it is shown that such operator is essentially self-adjoint it follows that it determines a unique self-adjoint operator.

Each self-adjoint operator in a naturally way induces a decomposition of the underlying Hilbert space $\mathcal{H}$ in a direct sum of two orthogonal subspaces.

DEFINITION III.3.13. *Let A be an operator on $\mathcal{H}$. We define $\mathcal{H}_p(A)$ as the subspace generated by all the eigenvectors of A, i.e., the closure of the linear manifold of all linear combinations of eigenvectors of A. Alternatively, $\mathcal{H}_p(A)$ is the direct sum of all eigenspaces of $A : \mathcal{H}_p(A) = \bigoplus \mathcal{M}_i = \bigoplus N(A - \lambda_i)$, where $\{\lambda_i\}$ are the eigenvectors of A.*

DEFINITION III.3.14. *We define $\mathcal{H}_c(A)$ as the orthogonal complement of $\mathcal{H}_p(A)$.*

We see that $\mathcal{H}$ is the orthogonal direct sum of $\mathcal{H}_p(A)$ and $\mathcal{H}_c(A)$:

$$\mathcal{H} = \mathcal{H}_p(A) \oplus \mathcal{H}_c(A). \tag{III.3.20}$$

Thus, each vector f in $\mathcal{H}$ has a unique decomposition as

$$f = f_p \oplus f_c \tag{III.3.21}$$

with $f_p \in \mathcal{H}_p(A), f_c \in \mathcal{H}_c(A)$ and $(f_p, f_c) = 0$. The indexes p and c are abbreviations of "puntual spectrum" and "continuous spectrum". If $\mathcal{H}_p(A) = \mathcal{H}, \mathcal{H}_c(A) = \{0\}$, we say that A has a pure puntual spectrum. An example is the Hamiltonian of the harmonic oscillator $A = \bar{P}^2 + \bar{Q}^2$ in $L^2(\mathbb{R}^n)$. If, on the other hand, $\mathcal{H}_p(A) = \{0\}, \mathcal{H}_c(A) = \mathcal{H}$, we say that A has a pure continuous spectrum. An example is the free Hamiltonian $H_0 = \bar{P}^2$ in non-relativistic quantum mechanics.

PROPOSITION III.3.15. *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Let A_p and A_c be the restrictions of A in $D(A) \cap \mathcal{H}_p(A)$ and $D(A) \cap \mathcal{H}_c(A)$, respectively. Then A_p leaves $\mathcal{H}_p(A)$ invariant and A_c leaves $\mathcal{H}_c(A)$ invariant. Therefore, we can see A_p as an operator in $\mathcal{H}_p(A)$ and A_c as an operator in $\mathcal{H}_c(A)$. With this convention, A_p and A_c are self-adjoint operators in $\mathcal{H}_p(A)$ and $\mathcal{H}_c(A)$, respectively, and we can write, in the decomposition* (III.3.20) *of $\mathcal{H}$:*

$$A = A_p \oplus A_c. \tag{III.3.22}$$

DEFINITION III.3.16. *Let A be a closed linear operator. The complex number z is called a regular point of A if*

(i) $(A - zI)$ *is invertible,*

(ii) $D((A - zI)^{-1}) = \mathcal{H}$,

(iii) $(A - zI)^{-1}$ *is bounded.*

In other words if $(A - zI)^{-1}$ exists and belongs to $\mathcal{B}(\mathcal{H})$. The set of all regular points is called the resolvent set of A and is denoted by $\rho(A)$.

The complement $\sigma(A)$ of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of A :

$$\sigma(A) := \mathbb{C}\backslash\rho(A)$$

The spectrum of the operator A_p is called the puntual spectrum $\sigma_p(A)$ of A and the spectrum of A_c the continuous spectrum $\sigma_c(A)$ of A. Thus, by definition,

$$\sigma_p(A) = \sigma(A_p), \qquad \sigma_c(A) = \sigma(A_c).$$

PROPOSITION III.3.17. *Let $A = A^*$ and $z = x + iy$ with $x, y \in \mathbb{R}$ and $y \neq 0$ then $z \in \rho(A)$.*

If λ is an eigenvalue of an operator A from $\mathcal{H}$ (into itself), $N(A - \lambda)$ is a nonempty subspace of $\mathcal{H}$, in consequence $A - \lambda I$ is not invertible. If λ is not an eigenvalue but it belongs to the continuous spectrum of A, then $A - \lambda I$ is invertible but either $D((A - \lambda I)^{-1})$ is only a proper subset of $\mathcal{H}$ or $(A - \lambda I)^{-1}$ is not bounded.

DEFINITION III.3.18. *Let A be a closed operator. Then the operator-valued function: $z \mapsto (A - zI)^{-1}$ from $\rho(A)$ into $\mathcal{B}(\mathcal{H})$ is called the resolvent of A.*

PROPOSITION III.3.19. *Let A be a closed operator and $z, z_1, z_2 \in \rho(A)$. Then*

(a) $(A - zI)^{-1}$ *maps $\mathcal{H}$ onto $D(A)$ and*

$$A(A - zI)^{-1}f = (A - zI)^{-1}Af, \ \forall f \in D(A).$$

(b) The following identity, called the first resolvent equation:

$$(A - z_1 I)^{-1} - (A - z_2 I)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1}$$

(c) $(A - z_1 I)^{-1}(A - z_2 I)^{-1} = (A - z_2 I)^{-1}(A - z_1 I)^{-1}$, *i.e. the resolvent in the point $z_1 \in \rho(A)$ commutes with the resolvent in any other point $z_2 \in \rho(A)$.*

PROPOSITION III.3.20. *Let A be a closed operator. Then*

(a) The mapping $z \mapsto (A - zI)^{-1}$ is continuous in the operator norm on $\rho(A) = \{z : (A - zI)^{-1}$ exists and lies in $\mathcal{B}(\mathcal{H})\}$ i.e. $u\text{-}\lim_{z_1 \to z,\, z, z_1 \in \rho(A)} \|(A - z)^{-1} - (A - z_1)^{-1}\| = 0$

(b) The resolvent is differentiable in the operator norm and

$$\begin{aligned} \frac{d}{dz}(A - zI)^{-1} &:= u - \lim_{z_1 \to z} (z_1 - z)^{-1}[(A - z_1)^{-1} - (A - z)^{-1}] \\ &= (A - zI)^{-2} \end{aligned}$$

DEFINITION III.3.21. *Let Δ be an open set in $\mathbb{R}^n$ and $a : \Delta \to \mathbb{C}$ be a measurable function. The multiplication operator A associated with a is the following linear operator in $L^2(\Delta)$:*

$$D(A) = \{f \in L^2(\Delta) : \int_\Delta |a(x)|^2 |f(x)|^2 d^n x < \infty\} \ y$$

$$(Af)(\bar{x}) = a(\bar{x})f(\bar{x}) \text{ for each } f \in D(A).$$

Clearly $D(A)$ is the maximal domain whereby multiplication by $a(\bar{x})$ makes sense.

PROPOSITION III.3.22. *Let $a : \Delta \to \mathbb{R}$ be measurable and $|a(\bar{x})| < \infty$ almost everywhere. Then the multiplication operator associated is a self-adjoint operator in $L^2(\Delta)$.*

PROPOSITION III.3.23. *Let A be the multiplication operator associated with a function $a : \Delta \to \mathbb{C}$. Then A is in $\mathcal{B}(L^2(\Delta))$ if and only if $\|a\|_\infty < \infty$ in which case*

$$\|A\| = \|a\|_\infty. \tag{III.3.23}$$

DEFINITION III.3.24. *An orthogonal projection (a projection, briefly) is a linear operator E that satisfies $D(E) = \mathcal{H}$ and*

$$E^2 = E = E^*. \tag{III.3.24}$$

We establish

$$M(E) = \{f \in \mathcal{H} | Ef = f\}. \tag{III.3.25}$$

It is easy to see that $M(E)$ is a subspace. Moreover, if $g \perp M(E)$, we have that for any $h \in \mathcal{H}$

$$(Eg, h) = (g, E^*h) = (g, Eh). \tag{III.3.26}$$

Now $E^2h = Eh$, in consequence $Eh \in M(E)$, in such a way that (III.3.26) implies that $(Eg, h) = 0$. Therefore $Eg = 0$ by proposition III.2.16. This shows that E is not nothing else that the orthogonal projection operation of $\mathcal{H}$ over $M(E)$.

Below, we define the concept of isometry. This approach was taken from Amrein [59]:

An isometry (or isometric operation) is a linear operator Ω in $\mathcal{B}(\mathcal{H})$ that satisfies

$$\Omega^*\Omega = I. \tag{III.3.27}$$

In, Reed and Simon [77], an isometry is defined as in (III.3.29).

PROPOSITION III.3.25. *Let Ω be an isometry. Then*

(a) *Ω preserves scalar products:*

$$(\Omega f, \Omega g) = (f, g) \text{ for all } f, g \in \mathcal{H}. \tag{III.3.28}$$

In particular

$$\|\Omega f\| = \|f\| \text{ for all } f \in \mathcal{H}. \tag{III.3.29}$$

(b) $\|\Omega\| = 1$.

(c) *$\Omega\Omega^*$ is a projection, and $M(\Omega\Omega^*) = R(\Omega)$.*

(d) *Ω is invertible.*

(e) *$\Omega^* = \Omega^{-1}f$ if $f \in R(\Omega)$, and $\Omega^* f = 0$ if $f \perp R(\Omega)$.*

Proposition III.3.25 shows that every isometric operator Ω maps the Hilbert space $\mathcal{H}$ onto a subspace $M(\Omega\Omega^*)$ while it preserves the length of vectors and the angles between vectors. A special case is the unitary operators U such that $M(UU^*) = \mathcal{H}$. Thus, U is unitary if it is isometric and $F \equiv UU^* = I$; in other words U is unitary if

$$U^*U = I \quad \text{and} \quad UU^* = I. \tag{III.3.30}$$

In this case $U^* = U^{-1}$ in $\mathcal{H}$

A generalization of the notion of isometry is the partial isometry. An operator $\Omega \in \mathcal{B}(\mathcal{H})$ is called a partial isometry if

$$\Omega^*\Omega = E, \tag{III.3.31}$$

where E is a projection. Some properties of partial isometries are given in the following proposition:

PROPOSITION III.3.26. *Let Ω a partial isometry. Then*

(a)

$$\Omega E = \Omega, \tag{III.3.32}$$

(*b*)

$$(\Omega f, \Omega g) = (Ef, Eg) \quad \forall f, g \in \mathcal{H}, \tag{III.3.33}$$

(*c*) $\|\Omega\| = 1$ *unless that* $E = 0$,

(*d*) $\Omega\Omega^*$ *is a projection, and* $M(\Omega\Omega^*) = R(\Omega)$.

IV Fourier Transform

The Fourier transform is an element ubiquitous in quantum mechanics.

DEFINITION III.4.1. *We denote by* $C_0^\infty(\mathbb{R}^n)$ *the set of all functions infinitely differentiable* $f : \mathbb{R}^n \to \mathbb{C}$ *each of them being identically zero in the complement of a compact subset of* $\mathbb{R}^n$. *Last, if* $\Gamma \subset \mathbb{R}^n$ *is closed and with zero Lebesgue measure, we denote by* $C_0^\infty(\mathbb{R}^n \backslash \Gamma)$ *the set of all functions in* $C_0^\infty(\mathbb{R}^n)$ *whose support lies in* $\mathbb{R}^n \backslash \Gamma$.

To define Fourier transform, besides infinitely differentiable functions, we need to define functions of rapidly decrease.

DEFINITION III.4.2. *A function* f *belongs to* $S(\mathbb{R}^n)$ *if it is infinitely differentiable and if for each tuple of* $2n$ *coordinates of non-negative integers* $\{j_1, \ldots, j_n, m_1, \ldots, m_n\}$ *one has that*

$$\sup_{\bar{x} \in \mathbb{R}^n} \left| x_1^{j_1} \cdots x_n^{j_n} \frac{\partial^{|m_1 + \ldots + m_n|}}{\partial x_1^{m_1} \cdots \partial x_n^{m_n}} f(x_1, \ldots, x_n) \right| < \infty.$$

Such functions are also called rapidly decreasing functions. One example is the function e^{-x^2}.

DEFINITION III.4.3. *If* $f \in S(\mathbb{R}^n)$, *we can define a new function* $\tilde{f} : \mathbb{R}^n \to \mathbb{C}$ *by means of the formula*

$$\tilde{f}(\bar{k}) = (2\pi)^{-n/2} \int d^n x e^{-i\bar{k}\cdot\bar{x}} f(\bar{x}), \quad (\bar{k} \in \mathbb{R}^n). \tag{III.4.34}$$

We have the following properties of $C_0^\infty(\mathbb{R}^n \setminus \Gamma)$ and $S(\mathbb{R}^n)$:

LEMMA III.4.4. (*a*) $S(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n), 1 \le p < \infty$.

(*b*) $S(\mathbb{R}^n)$ *is invariant under Fourier transforms.*

LEMMA III.4.5. *Let* $\Gamma \subset \mathbb{R}^n$ *be closed and with Lebesgue measure zero, then the set* $C_0^\infty(\mathbb{R}^n \setminus \Gamma)$ *is dense in* $L^p(\mathbb{R}^n), 1 \le p < \infty$. *In particular* $C_0^\infty(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n), 1 \le p < \infty$.

The result of lemma III.4.4 can still be strengthened. In fact, Fourier transform is a map of $S(\mathbb{R}^n)$ onto $S(\mathbb{R}^n)$. This can be seen defining the inverse Fourier transform in $S(\mathbb{R}^n)$ by

$$\hat{f}(\bar{x}) = (2\pi)^{-n/2} \int d^n k \, e^{i\bar{k}\cdot\bar{x}} f(\bar{k}), \quad f \in S(\mathbb{R}^n) \tag{III.4.35}$$

Equation (III.4.35) defines the inverse of (III.4.34). From this point on, we will denote the map $f \mapsto \tilde{f}$ by $\mathcal{F}$. Then we have that $f = \mathcal{F}^{-1}\tilde{f} = \hat{\tilde{f}}$. In the following sections $\mathcal{F}$ and $\mathcal{F}^{-1}$ will be extended over the whole space $L^2(\mathbb{R}^n)$.

We have that both $\mathcal{F}$ are $\mathcal{F}^{-1}$ isometric in $S(\mathbb{R}^n)$, i.e.

$$\|\tilde{f}\| = \|f\| = \|\hat{f}\| \qquad f \in S(\mathbb{R}^n), \tag{III.4.36}$$

$$(\tilde{f}, \tilde{g}) = (f, g) = (\hat{f}, \hat{g}) \qquad f, g \in S(\mathbb{R}^n), \tag{III.4.37}$$

Before enouncing Proposition III.4.6 which is a partial version of Theorem 7.2 from Rudin's book [80], we define the following:

(a) Por each $y \in \mathbb{R}^n$, the character e_y is the function defined by

$$e_y(x) = e^{iy\cdot x} = e^{i\sum_{j=1}^{n} y_j x_j}, \quad x \in \mathbb{R}^n.$$

(b) The translation operators τ_x are defined by

$$(\tau_x f)(y) = f(y-x), \quad x, y \in \mathbb{R}^n.$$

PROPOSITION III.4.6. *Suppose $f, g \in L^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. Then, denoting $\hat{}$ as the Fourier transform,*

(a) $(\tau_x f)\hat{} = e_{-x}\hat{f}$,

(b) $(e_x f)\hat{} = \tau_x \hat{f}$,

(c) $(f * g)\hat{} = \hat{f}\hat{g}$.

Proof. The proof of (a) and (b) are detailed in Theorem 7.2 [80], (c) is obtained by Fubini's Theorem and one explicit proof of (c) can be found in Reed and Simon [77] Theorem IX.3. ■

Even though the set $\{\tilde{f}|f \in L^2(\mathbb{R}^n)\}$ is again $L^2(\mathbb{R}^n)$, it is convenient to distinguish between these two representations of $L^2(\mathbb{R}^n)$, because variables $\bar{x}$ and $\bar{k}$ have different interpretations in quantum mechanics. Multiplication of $f(\bar{x})$ by x_i corresponds to the i-th component of the position operator, and multiplication of $\tilde{f}(\bar{k})$ by k_i corresponds to the i-th component of the momentum operator. Therefore, we will denote the set of functions $\{f|f \in L^2(\mathbb{R}^n)\}$ by $L^2(\mathbb{R}^n)$ and the set $\{\tilde{f}|f \in L^2(\mathbb{R}^n)\}$ of their Fourier transforms by $\tilde{L}^2(\mathbb{R}^n)$. In other word, we do not considerate $L^2(\mathbb{R}^n)$ as an abstract space but as the set of all square integrable quantum mechanical wave functions defined in the n dimensional configuration space.

One can apply Proposition III.3.7 to show that Fourier transform $\mathcal{F}$ defined in $D(\mathcal{F}) = S(\mathbb{R}^n)$ with norm equals 1 can be extended to a bounded operator with norm 1 defined in all $L^2(\mathbb{R}^n)$. For functions also in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ Fourier transform is also defined by (III.4.34). For any f in $L^2(\mathbb{R}^n)$ one has to define $\mathcal{F}f$ as $\mathcal{F}f = \text{s-lim}\,\mathcal{F}f_m$ as $m \to \infty$, where $f_m \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $f_m \to f$. Similarly $\mathcal{F}^{-1}$ can be extended to whole $L^2(\mathbb{R}^n)$ and again this extension will be denoted by $\mathcal{F}^{-1}$. One application of Proposition III.3.7 implies that $\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = I$ in the whole $L^2(\mathbb{R}^n)$.

Now let us see some very important examples in quantum mechanics:

Example III.4.7. *$Q_m, (m = 1, \ldots, n)$ the multiplication operator by x_m in $L^2(\mathbb{R}^n)$:*

$$(Q_m f)(\bar{x}) = x_m f(\bar{x}) \tag{III.4.38}$$

It is called the m-th component of the position operator in quantum mechanics.

Example III.4.8. *$P_m, (m = 1, \ldots, n)$ the multiplication operator by k_m in $\tilde{L}^2(\mathbb{R}^n)$:*

$$(\mathcal{F}P_m f)(\bar{k}) = k_m \tilde{f}(\bar{k}) \tag{III.4.39}$$

P_m is called the m-th component of the momentum operator.

Example III.4.9. *Let H_0 be the multiplication operator by $|\bar{k}|^2$ in $\tilde{L}^2(\mathbb{R}^n)$:*

$$(\mathcal{F}H_0 f)(\bar{k}) = |k|^2 \tilde{f}(\bar{k}). \tag{III.4.40}$$

This operator is called the free Hamiltonian of Schrödinger in quantum mechanics.

$$H_0 = \bar{P}^2 = \sum_{m=1}^{n} P_m^2. \tag{III.4.41}$$

Example III.4.10. *If $v : \mathbb{R}^n \to \mathbb{R}$ is any measurable function then it determines a multiplication operator V in $L^2(\mathbb{R}^n)$. We will use letters v and V for such operators when we have in mind the interaction operator of a non-relativistic quantum particle, and function v will be called a potential.*

The Hamiltonian for a particle that moves under the influence of a potential v formally is given by $H = H_0 + V$.

PROPOSITION III.4.11. *(a) H_0 is a non-bounded positive operator, its spectrum is $[0, \infty)$ and is purely continuous. In particular $\mathcal{H}_p(H_0) = \{0\}$, $\mathcal{H}_c(H_0) = \mathcal{H}$.*

(b) $D(H_0)$ lies in $D(P_m)$, and $P_m(H_0 - zI)^{-1}$ belongs to $\mathcal{B}(\mathcal{H})$ for each complex z outside of $[0, \infty)$.

(c) The resolvent of H_0 is the multiplication operator by $(\bar{k}^2 - z)^{-1}$ in $\tilde{L}^2(\mathbb{R}^n)$.

PROPOSITION III.4.12. *(a) If $f \in S(\mathbb{R}^n)$, then $f \in D(H_0)$ and*

$$(H_0 f)(\bar{x}) = -(\triangle f)(\bar{x}), \tag{III.4.42}$$

where $\triangle := \sum_{m=1}^{n} \partial^2/\partial x_m^2$ is the Laplacian.

(b) $(H_0 + I)$ transforms onto $S(\mathbb{R}^n)$ in $S(\mathbb{R}^n)$.

(c) The restriction $\hat{H}_0$ of H_0 in $S(\mathbb{R}^n)$ is essentially self-adjoint, and $\hat{H}_0^* = H_0$.

REMARK III.4.13. *There exist other linear sub-manifolds of $D(H_0)$ on which, H_0 is essentially self-adjoint. We mention two of these sub-manifolds:*

(a) The set $C_0^\infty(\mathbb{R}^n)$ of all infinitely differentiable compactly supported functions.

(b) The set $\tilde{C}_0^\infty(\mathbb{R}^n)$ of all functions $f : \mathbb{R}^n \to \mathbb{C}$ whose Fourier transform $\tilde{f}$ are infinitely differentiable and compact supported.

V Fourier Analysis on Groups

When we map the space $L^2(\mathbf{X})$ in (II.1.2) onto the space $L^2(\hat{\mathbf{X}})$ in (II.1.3) we use, in fact, a more general notion of Fourier transform than the one we have described in Section IV. To understand more this, we include the material from Rudin [79].

V.1 Topological Groups

DEFINITION III.5.1. *An Abelian Group is a set G in which a binary operation, $+$, is defined with the following properties:*

(a) $x + y = y + x$ for all $x, y \in G$.

(b) $x + (y + z) = (x + y) + z$ for all $x, y, z \in G$.

(c) G contains an element 0 such that $x + 0 = x$ for all $x \in G$.

(d) To each $x \in G$ corresponds an element $-x$ such that $x + (-x) = 0$. We write $x - x$ in place of $x + (-x)$.

A homomorphism of a group G into a group G_1 is a map φ from G into G_1 such that

$$\varphi(x + y) = \varphi(x) + \varphi(y) \quad x, y \in G.$$

A homomorphism which is one to one is an isomorphism. If there is an isomorphism of a group G into a group G_1, then G and G_1 are isomorphic groups, and for many purposes one need not to distinguish between them.

The kernel of a homomorphism φ is the set $\varphi^{-1}(0)$; the kernel is always a subgroup.

If H is a subgroup of G, the sets $H + x$, $x \in G$ are the cosets of H. Two cosets $H + x$ and $H + y$ are identical if and only if $x - y \in H$; otherwise $H + x$ and $H + y$ are disjoint. The set of all cosets of H is denoted by G/H and G/H becomes an Abelian group (the quotient group of G modulo H) if we define

$$(H + x) + (H + y) = H + (x + y), \quad x, y \in G.$$

The map $x \to H + x$ is a homomorphism of G into G/H, with kernel H.

DEFINITION III.5.2. *A Topological Abelian Group is a Hausdorff space G which is also an Abelian group, provided the map $(x, y) \to x - y$ is a continuous map of the product space $G \times G$ onto G. If, in addition, the topology of G is locally compact, then G is a locally compact Abelian (LCA) group.*

THEOREM III.5.3. *Suppose that G is LCA, φ is the natural homomorphism of G onto G/H, where H is a closed group of G, and a subset of G/H is declared open if and only if it is the image under φ of an open subset of G. Then G/H is an LCA group.*

If $\{G_\alpha\}$ is a collection of Abelian groups, their complete direct sum is the group G defined as follows: G, as a set, is the Cartesian product of the sets G_α, and addition is performed coordinatewise: If x and y belong to G, then $x + y$ is the element of G whose αth coordinate is $x(\alpha) + y(\alpha) \in G_\alpha$.

The direct sum of the groups G_α is the subgroup of their complete direct sum which consists of all x which have $x(\alpha) \neq 0$ for only finitely many α.

By the Tychonoff Theorem: The direct sum of any finite collection of LCA groups is a LCA group. The complete direct sum of any collection of compact Abelian groups is a compact Abelian group.

If $G = H_1 + H_2$, where H_1 and H_2 are subgroups of G, the G is (isomorphic to) the direct sum $H_1 \oplus H_2$ of these two subgroups if and only if $H_1 \cap H_2 = \{0\}$.

V.2 Weak topology and continuous functions

We commence with recalling some definitions from topology.

If τ_1 and τ_2 are two topologies on a set S and if $\tau_1 \subset \tau_2$, then τ_1 is said to be weaker than τ_2.

If F is a family of maps of S into a topological space Y, the collection of all finite intersections of sets of the form $f^{-1}(V)$, with $f \in F$ and V open in Y, forms a base for a topology τ_F on S. Each $f \in F$ is evidently continuous with respect to τ_F, and τ_F is the weakest topology on S with this property; τ_F is called the weak topology induced in S by F.

F is said to separate points on S if to every pair of points p_1 and p_2 in S there corresponds an $f \in F$ such that $f(p_1) \neq f(p_2)$. If F separates points on S and if Y is a Hausdorff space then S with the weak topology induced by F is also a Hausdorff space.

DEFINITION III.5.4. *If S is a topological space, $C(S)$ denotes the set of all bounded continuous complex-valued functions in S. The set of all $f \in C(S)$ whose support is compact is denoted by $C_c(S)$. If, for each $\epsilon > 0$, the inequality $|f(p)| < \epsilon$ holds for all p in the complement of some compact set in S, then f is said to vanish at infinity. The set of all f in $C(S)$ such that f vanishes at infinity is denoted by $C_0(S)$.*

The spaces $C(S), C_0(S)$ and $C_c(S)$ are closed under pointwise addition, multiplication, and scalar multiplication: $(f + g)(p) = f(p) + g(p)$; $(fg)(p) = f(p)g(p)$; $(\alpha f)(p) = \alpha(f(p))$. Since the usual commutative, associative, and distributive laws holds, these spaces are algebras over the complex field.

If we introduce a norm in $C(s)$ by setting

$$\|f\|_\infty = \sup_{p \in S} |f(p)|, \qquad f \in C(S),$$

The metric $\|f - g\|_\infty$ turns $C(S)$ and $C_0(S)$ into complete metric spaces, since they are closed under the formation of limits of uniformly convergent sequences.

THEOREM III.5.5. *Stone-Weierstrass Theorem. Let S be a locally compact Hausdorff space and let A be a subalgebra of $C_0(A)(S)$ which separates points on S, which is self-adjoint (i.e., $f \in A$ implies $\overline{f} \in A$, where $\overline{f}$ is the complex conjugate of f) and which contains, for each $p_0 \in S$, a function f such that $f(p_0) \neq 0$. Then A is dense in $C_0(A)(S)$.*

V.3 Haar measure

Let X be a locally compact Hausdorff space, B be the family of Borel subsets of X and μ a measure defined on (X, B). With each measure μ on X there is associated a set function $|\mu|$, the total variation of μ, defined by

$$|\mu|(E) = \sup \sum |\mu(E_j)|,$$

the supremum being taken over all finite collections of pairwise disjoint Borel sets E_j whose union is E. Then $|\mu|$ is also a measure on X. If

$$|\mu|(E) = \sup |\mu|(K) = \inf |\mu|(V)$$

for every Borel set E, where K ranges over all compact subsets of E and V ranges over all open supersets of E, the μ is called regular.

DEFINITION III.5.6. *We put*

$$\|\mu\| = |\mu|(X)$$

and define $M(X)$ to be the set of all complex-valued regular measures μ on X for which $\|\mu\|$ is finite.

On every LCA group G there exists a non-negative regular measure m, the so called Haar measure of G, which is not identically zero and which is translation invariant. That is to say

$$m(E + x) = m(E)$$

for every $x \in G$ and every Borel set E in G. For the construction of that measure see [79] section 1.1.1 and the references given therein.

There is a uniqueness theorem for the Haar measure.

THEOREM III.5.7. *If m and m' are two Haar measures on G, then $m' = \lambda m$ where λ is a positive constant.*

As usual, for $1 \leq p \leq \infty$, $L^p(G) = L^p(G, B, m)$ is the space of complex valued functions f defined on G such that

$$\|f\|_p = \left(\int_G |f(x)|^p \, dm(x) \right)^{1/p} < \infty.$$

$L^2(G)$ is a Hilbert space with the customary scalar product $(f, g) = \int_G f(x)\overline{g(x)} \, dm(x)$.

DEFINITION III.5.8. *Convolution. For any pair of Borel functions f and g on the LCA group G we define their convolution $f * g$ by the formula*

$$(f * g)(x) = \int_G f(x - y) g(y) \, dm(y) = \int_G (\tau_y f)(x) g(y) \, dm(y),$$

provided $f(x)g(y) \in L^1(y \in G)$.

We have the following theorem

THEOREM III.5.9. *For any LCA group G, $L^1(G)$ is a commutative Banach algebra, if multiplication is defined by convolution.*

PROOF. Look at Theorem 1.1.7 in Rudin [79]. ■

V.4 The Dual Group and the Fourier Transform

DEFINITION III.5.10. *Characters. A complex function γ on a LCA group G is called a character of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the functional equation,*

$$\gamma(x+y) = \gamma(x)\gamma(y) \quad x, y \in G,$$

is satisfied. The set of all continuous characters of G forms a group Γ, the dual group of G, if addition is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad x \in G;\ \gamma_1, \gamma_2 \in \Gamma.$$

Throughout this section, the letter Γ will denote the dual group of the LCA group G. In view of the duality between G and Γ which will be established in Subsection V.6 below, it is customary to write

$$(x, \gamma)$$

in place of $\gamma(x)$.

We identify the relation that exists between Γ and $L^1(G)$.

THEOREM III.5.11. *If $\gamma \in \Gamma$ and if*

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)\, dm(x), \quad f \in L^1(G), \tag{III.5.43}$$

then the map $f \to \hat{f}(\gamma)$ is a complex homomorphism of $L^1(G)$, and is not identically zero. Conversely, every non-zero complex homomorphism of $L^1(G)$ is obtained in this way, and distinct characters induce distinct homomorphisms. Proof. *See Theorem 1.2.2 in Rudin [79].* ■

DEFINITION III.5.12. *The Fourier Transform. For all $f \in L^1(G)$, the function $\hat{f}$ defined on Γ by* (III.5.43) *with $\gamma \in \Gamma$ is called the Fourier transform of f. The set of all functions $\hat{f}$ so obtained will be denoted throughout by $A(\Gamma)$.*

Since $\hat{f} : \Gamma \to \mathbb{C}$, we give Γ the weak topology induced by $A(\Gamma)$.

We limit ourselves to enounce some results from Theorem 1.2.4 in Rudin [79].

THEOREM III.5.13. (*a*) *$A(\Gamma)$ is a separating self-adjoint subalgebra of $C_0(T)$, so that $A(\Gamma)$ is dense in $C_0(T)$, by the Stone Weierstrass Theorem.*

(*b*) *The Fourier transform, considered as a map of $L^1(G)$ into $C_0(T)$, is norm-decreasing and therefore continuous: $\|\hat{f}\|_\infty \le \|f\|_1$.*

At present, we have that Γ is a group and a locally compact Hausdorff space

By an alternative description of the topology of Γ one can prove the following:

THEOREM III.5.14. (*a*) *(x, γ) is a continuous function on $G \times \Gamma$.*

(*b*) *Let K and C be compact subsets of G and Γ, respectively, let U_r be the set of all complex numbers z with $|1 - z| < r$, and put*

$$\begin{aligned} N(K, r) &= \{\gamma : (x, \gamma) \in U_r\, \forall\, x \in K\}, \\ N(C, r) &= \{x : (x, \gamma) \in U_r\, \forall\, \gamma \in C\}. \end{aligned}$$

Then $N(K, r)$ and $N(K, r)$ are open subsets of Γ and G, respectively.

(*c*) *The family of all sets $N(K, r)$ and their translates is a base for the topology of Γ.*

(*d*) *Γ is a LCA group.*

V.5 The inversion theorem

DEFINITION III.5.15. *Let $B(G)$ be the set of all functions f on G which are representable in the form*

$$f(x) = \int_\Gamma (x,\gamma)\, d\mu(\gamma) \quad x \in G,$$

for some $\mu \in M(X)$.

Let us denote $d\gamma$ the Haar measure in the LCA group Γ.

THEOREM III.5.16. (*a*) *If $f \in L^1(G) \cap B(G)$, then $\hat{f} \in L^1(G)$.*

(*b*) *If the Haar measure of G is fixed, the Haar measure of Γ can be so normalized that the inversion formula,*

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x,\gamma)\, d\gamma, \quad x \in G,$$

is valid for every $f \in L^1(G) \cap B(G)$.

If the Haar measure of G given, the inversion theorem singles out a specific Haar measure of Γ, adjusted so that the inversion theorem holds.

From now on, it will always be tacitly assumed that the Haar measure of G and Γ are so adjusted that the inversion theorem holds.

THEOREM III.5.17. *Plancherel Theorem. The Fourier transform, restricted to $L^1(G) \cap L^2(G)$, is an isometry (with respect to the L^2-norms) onto a dense linear subspace of $L^2(\Gamma)$. Hence it may be extended, in a unique manner, to an isometry of $L^2(G)$ onto $L^2(\Gamma)$.*

The above extension of the Fourier transform to $L^2(G)$ is sometimes referred to as the Plancherel transform; the symbol $\hat{f}$ will be used in this context as well.

V.6 The Pontryagin duality Theorem

Since Γ is a LCA group, it has a dual group, say $\hat{\Gamma}$, and everything we have proved so far for the ordered pair (G,Γ) holds equally well for the pair $(\Gamma,\hat{\Gamma})$. The value of a character $\hat{\gamma} \in \hat{\Gamma}$ at the point $\gamma \in \Gamma$ will be temporarily written $(\gamma,\hat{\gamma})$. By Theorem III.5.14 (*a*) every $x \in G$ may be regarded as a continuous character on Γ, and thus there is a natural map α of G into $\hat{\Gamma}$, defined by

$$(x,\gamma) = (\gamma, \alpha(x)) \quad x \in G,\ \gamma \in \Gamma.$$

THEOREM III.5.18. *The Pontryagin duality Theorem. The above map α is an isomorphism and a homeomorphism of G onto $\hat{\Gamma}$.*

Thus: Every LCA group is the dual of its dual group.

V.7 Duality between subgroups and quotient groups

Suppose H is a closed subgroup of the LCA group G, and Λ is the set of all $\gamma \in \Gamma$ (the dual group of G) such that $(x,\gamma) = 1$, for all $x \in H$. We call Λ the annihilator of H.

For any fixed $x \in H$, the continuity of (x,γ) shows that the set of all γ with $(x,\gamma) = 1$ is closed, so that Λ is an intersection of closed sets. Since Λ is evidently a group, we conclude that Λ is a closed group of Γ.

THEOREM III.5.19. *With the above notation Λ and Γ/Λ are (isomorphically homeomorphic to) the dual groups of G/H and H, respectively.*

V.8 Direct sums

In subsection V.1 above we have already defined the direct sum and complete direct sum of LCA groups. The direct sum of G_1 and G_2 will be written $G_1 \oplus G_2$, and the direct sum of n copies of G will be denoted by G^n.

THEOREM III.5.20. *If $G = G_1 \oplus \ldots \oplus G_n$ and Γ_i is the dual group of G_i, $1 \le i \le n$, then $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$.*

COROLLARY III.5.21. *$\mathbb{R}^n$ is its own dual.*

V.9 Fourier transforms on subgroups and on quotient groups

Throughout this section, H will be a closed group of G, and Λ will be the annihilator of H.

THEOREM III.5.22. *The functions belonging to $B(\Lambda)$ are precisely the restrictions to Λ of the functions belonging to $B(\Gamma)$.*

Suppose m_G, m_H, and $m_{G/H}$ are the Haar measures of the indicated groups.

THEOREM III.5.23. *The functions belonging to $A(\Lambda)$ are precisely the restrictions to Λ of the functions belonging to $A(\Gamma)$. For $f \in L^1(G)$, $\hat{f}$ vanishes on Λ if and only if*

$$\int_H f(x+y)\, dm_H(y) = 0$$

for almost all $x \in G$.

In the last theorem the Haar measures can be adjusted so that

$$\int_G f\, dm_G = \int_{G/H} dm_{G/H}(\xi) \int_H f(x+y) dm_H(y),$$

where $f \in C_c(G)$, and $\xi = \xi(x)$ is the coset of H (an element in G/H) which contains x and $x \in G$.

V.10 Normalization of the Haar measure

Now, we are going to develop in more detail the case where $G = \mathbb{R}$ with the Haar measure.

Theorem 1.18 in Folland [69] shows that every Lebesgue-Stieltjes measure in $\mathbb{R}$ is regular, in particular Lebesgue measure is Lebesgue-Stieltjes. Since Lebesgue measure is translation invariant we can conclude that:

THEOREM III.5.24. *The Haar measure in the Real line is the Lebesge measure up to a positive factor.*

Take $G = \mathbb{R}$. Let Γ the dual group of the group G. Fix $\gamma \in \Gamma$. Because γ is not identically zero, by means of the fundamental Theorem of calculus, there exists an $\delta > 0$ such that

$$\int_0^\delta \gamma(t)\, dt = \alpha,$$

for some $\alpha \neq 0$.

By the functional equation

$$\gamma(x+t) = \gamma(x)\gamma(t) \quad x, t \in \mathbb{R}, \tag{III.5.44}$$

then it implies that

$$\alpha\gamma(x) = \int_0^\delta \gamma(t)\gamma(x)\, dt = \int_0^\delta \gamma(t+x)\, dt = \int_x^{x+\delta} \gamma(t)\, dt.$$

Since γ is continuous, the last expression is differentiable, and so γ has a continuous derivative γ'. We differentiate (III.5.44):

$$\gamma'(x+t) = \gamma(x)\gamma'(t),$$

setting $t = 0$, we obtain

$$\gamma'(x) = \gamma(x)\gamma'(0), \tag{III.5.45}$$

Since $|\gamma(x)| = 1$, there exists a differentiable function $\theta : G = \mathbb{R} \to \mathbb{R}$ such that $\gamma(x) = e^{i\theta(x)}$, moreover $\theta(0) = 0$ because $\gamma(0) = 1$. Equation (III.5.45) becomes

$$\begin{aligned} e^{i\theta(x)} i\theta'(x) &= e^{i\theta(x)}\, e^{i\theta(0)} i\theta'(0) \\ \theta'(x) &= \theta'(0) \\ \theta(x) &= \theta'(0)x \\ \theta(x) &= xy \end{aligned} \tag{III.5.46}$$

where $y \in \mathbb{R}$ is such that $\theta'(0) = y$. Thus, we have a map $\zeta : \Gamma \to \mathbb{R}$, such that $\gamma \mapsto y$. Let $\gamma, \gamma_1, \gamma_2 \in \Gamma$, so $\gamma(x) = e^{i\theta(x)}$, $\gamma_1(x) = e^{i\theta_1(x)}$ and $\gamma_2(x) = e^{i\theta_2(x)}$ with $\theta_1, \theta_2 : G = \mathbb{R} \to \mathbb{R}$ being differentiable and $\gamma = \gamma_1 + \gamma_2$. We know that for all $x \in \mathbb{R}$ the sum in Γ goes as follows:

$$\gamma(x) = (\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x),$$

this implies that

$$\theta(x) = (\theta_1 + \theta_2)(x).$$

Hence ζ is a homomorphism between Γ and $\mathbb{R}$. Let us see that it is one to one. Suppose that $\zeta(\gamma_1) = \zeta(\gamma_2)$ then $\theta_1'(0) = \theta_2'(0)$, by (III.5.46) $\theta_1(x) = \theta_2(x)$, for all $x \in \mathbb{R}$, finally $\gamma_1 = \gamma_2$. Then, for all $\gamma \in \Gamma$ there exist a unique $y \in \mathbb{R}$ such that, for all $x \in G = \mathbb{R}$,

$$\gamma(x) = e^{iyx}. \tag{III.5.47}$$

Therefore, Γ are $\mathbb{R}$ are isomorphic.

We use the topology of Γ described in Theorem III.5.14, known also as the Gelfand topology, to give a topology (the Gelfand topology) to the real line $\mathbb{R}$. That is a set O is open in $\mathbb{R}$ if and only if $\zeta^{-1}(O)$ is open in Γ. Then the collection of images $\zeta(N(K,r))$ and their translates is a base for $\mathbb{R}$. Let us take any open set, in the Gelfand topology of $\mathbb{R}$, V that contains 0, then it exists K compact in $\mathbb{R}$ with the usual topology and an $r > 0$ such that $0 \in \zeta(N(K,r)) \subset V$. There exists $n \in \mathcal{N}$ such that $|x| \leq n$ for all $x \in K$. If we denote $V(n,r) = \{y \in \mathbb{R} : |1 - e^{iyx}| < r, \forall |x| \leq n\}$ and B_r denotes the open ball of center zero and radius r in $\mathbb{R}^n$. We have that $0 \in V(n,r) = \zeta(N(\overline{B_n}, r)) \subset \zeta(N(K,r)) \subset V$. This proves that the sets $V(n,r)$ form a neighborhood base at zero with respect to the Gelfand topology in $\mathbb{R}$.

At this moment, we want to show for $0 < r \leq 2$ that $y \in V(n,r) \Leftrightarrow |y| < \frac{2}{n} \arcsin \frac{r}{2}$. This means that

$$V(n,r) = \begin{cases} (-\frac{2}{n}\arcsin\frac{r}{2}, \frac{2}{n}\arcsin\frac{r}{2}), & \text{if } r \leq 2, \\ \mathbb{R} & \text{if } r > 2. \end{cases}$$

So, the Gelfand and the usual topologies are the same. Meaning also that Γ with the Gelfand topology and $\mathbb{R}$ with the usual are isomorphic and homeomorphic.

First we do the following calculations with $0 < |x| \leq n$ and we take $\arcsin : [0,1] \to [0,\pi]$:

$$\begin{aligned} |1 - e^{iyx}| &< r \Leftrightarrow \\ (1 - \cos(yx))^2 + \sin^2(yx) &< r^2 \Leftrightarrow \\ 2 - 2\cos(yx) &< r^2 \Leftrightarrow \\ \sin^2(yx/2) &< (r/2)^2 \Leftrightarrow \\ |\sin(yx/2)| &< r/2 \Leftrightarrow \\ |yx/2| &< \arcsin(r/2) \Leftrightarrow \\ |y| &< \frac{2}{x}\arcsin(r/2) \end{aligned}$$

Assume $y \in V(n,r)$, then, for all $|x| \leq n$ we have that $|1 - e^{iyx}| < r$, in particular for $x = n$ this implies that $|y| < \frac{2}{n}\arcsin(r/2)$. Conversely, take $|y| < \frac{2}{n}\arcsin(r/2)$, because $\frac{2}{n}\arcsin(r/2) < \frac{2}{x}\arcsin(r/2)$ with $0 < |x| \leq n$ we have that $|1 - e^{iyx}| < r$, for all $|x| \leq n$.

Now we have established that the dual group of $\mathbb{R}$ is $\Gamma \simeq \mathbb{R}$. Let αdx, βdp the Haar measures in G and in Γ, where $\alpha, \beta > 0$ and dx and dp denote the ordinary Lebesgue Measure on the real line.

By a straightforward computation

$$\frac{2\beta}{1+x^2} = \int_{-\infty}^{\infty} e^{-|p|}e^{ixp}\,\beta dp,$$

Because $e^{-|p|}$ and $\frac{2\beta}{1+x^2}$ are L^1 functions (see theorem 7.7 in Rudin [80]) and the inversion Theorem for Fourier transform in $\mathbb{R}$, we have that

$$e^{-|p|} = 2\alpha\beta \int_{-\infty}^{\infty} \frac{1}{1+x^2} e^{-ixp}\,dx,$$

Setting $p = 0$,

$$1 = 2\alpha\beta \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\alpha\beta \arctan x\Big|_{-\infty}^{\infty} = 2\pi\alpha\beta. \tag{III.5.48}$$

Two of the possible choices that are frequently used: $\alpha = 1/(2\pi)$, $\beta = 1$ or $\alpha = \beta = (2\pi)^{-1/2}$.

We can generalize to the case $G = \Gamma = \mathbb{R}^n$. Following the same idea, let $\alpha dx_1 \cdots dx_n$, $\beta dp_1 \cdots dp_n$ the Haar measures in G and in Γ, where $\alpha, \beta > 0$

$$\begin{aligned} \int_{\mathbb{R}^n} e^{-\sum_{j=1}^n |p_j|} e^{i\sum_{j=1}^n x_j p_j}\,\beta dp_1 \cdots dp_n &= \beta \prod_{j=1}^n \left(\int_{-\infty}^{\infty} e^{-|p_j|}e^{ix_jp_j}\,dp_j \right) \\ &= 2^n \beta \prod_{j=1}^n \frac{1}{1+x_j^2} \end{aligned}$$

By the inversion theorem

$$e^{-\sum_{j=1}^n |p_j|} = \int_{\mathbb{R}^n} 2^n\beta \prod_{j=1}^n \frac{1}{1+x_j^2}\, e^{-i\sum_{j=1}^n x_jp_j}\,\alpha dx_1 \cdots dx_n$$

Setting $p_1 = \cdots = p_n = 0$:

$$\begin{aligned} 1 &= 2^n\alpha\beta \prod_{j=1}^n \int_{\mathbb{R}} \frac{1}{1+x_j^2}\,dx_j \\ 1 &= (2\pi)^n\alpha\beta. \end{aligned} \tag{III.5.49}$$

VI Unbounded operators

Many of the most important operators which occur in mathematical physics are not bounded. The Hellinger-Toeplitz Theorem (see Theorem III.6.8) says that an everywhere-defined operator A which satisfies $(A\phi, \psi) = (\phi, A\psi)$ is necesarily a bounded operator suggesting that a general unbounded operator T will only be defined on a dense linear subset of the Hilbert space $\mathcal{H}$. To identify an unbounded operator on a Hilbert space one must give the domain on which it acts and the specify how it acts on that space. Before (the Hellinger-Toeplitz) Theorem III.6.8 is enounced we give a Definition and a Theorem to help us to understand how Reed and Simon prove Hellinger-Toeplitz's Theorem.

DEFINITION III.6.1. *Let T be a mapping of a normed linear space X into a normed linear space Y. The graph of T, denoted by $\Gamma(T)$, is defined as*

$$\Gamma(T) = \{\langle x, y\rangle \in X \times Y \,\big|\, y = Tx\}.$$

If T is unbounded operator in X then we modify the later definition

$$\Gamma(T) = \{\langle x, y\rangle \,\big|\, x \in D(T) \subset X,\, y = Tx\}.$$

T is a closed operator if $\Gamma(T)$ is a closed subset in the Hilbert space $\mathcal{H} \times \mathcal{H}$ with the scalar product: $(\langle \phi_1, \psi_1\rangle, \langle \phi_2, \psi_2\rangle) = (\phi_1, \phi_2) + (\psi_1, \psi_2)$.

DEFINITION III.6.2. *An Alternative way to define closable operators. An operator T is closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure, which we denote by $\overline{T}$.*

PROPOSITION III.6.3. *If T is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

THEOREM III.6.4. *Let T be a densely defined operator on a Hilbert space $\mathcal{H}$. Then:*

(*a*) *T^* is closed.*

(*b*) *T is closable if and only if $D(T^*)$ is dense in which case $\overline{T} = T^{**}$.*

(*c*) *If T is closable, then $(\overline{T})^* = T^*$.*

As a remainder: A symmetric operator is always closable, since $D(T^*) \supset D(T)$ is dense. If T is symmetric, T^* is a closed extension of T, so the smallest closed extension T^{**} of T must be contained in T^*. Thus for symmetric operators, we have

$$T \subset T^{**} \subset T^*. \tag{III.6.50}$$

For closed symmetric operators,

$$T = T^{**} \subset T^*. \tag{III.6.51}$$

For self-adjoint operators,

$$T = T^{**} = T^*. \tag{III.6.52}$$

From this we can easily see that a closed symmetric operator is self-adjoint if and only if T^* is symmetric. The distinction between closed symmetric operators and self-adjoint operators is very important. It is only for self-adjoint operators that the spectral theorem holds and it is only self-adjoint operators that may be exponentiated to give the one-parameter unitary groups which give the dynamics in quantum mechanics.

If T is essentially self-adjoint, then it has one and only one self-adjoint extension, for that suppose that S is a self-adjoint extension of T. Then S is closed, and thereby, since $S \supset T$, $S \supset T^{**}$. By Definitions III.3.10 and III.3.12, T and T^* are symmetric and applying (III.6.50) to both T and T^* we have that $(T^{**})^* = T^{**}$. Thus, $S = S^* \subset (T^{**})^* = T^{**}$. By the results in Section X.1 of Reed and Simon [78] the converse is also true; namely, if T has one and only one self-adjoint extension, then T is essentially self-adjoint. This follows by the corollary of Theorem VIII.3 in Reed and Simon [77] and the corollary of Theorem X.2 in Reed and Simon [78].

Since $T^* = \overline{T}^* = T^{***}$, T is essentially self-adjoint if and only if

$$T \subset T^{**} = T^*.$$

The importance of essential self-adjointness is that one is often given a non-closed symmetric operator T. If T can be shown to be essentially self-adjoint, then there is uniquely associated to T a self-adjoint operator $\overline{T} = T^{**}$. Another way of saying this is that if A is a self-adjoint operator, then to specify A uniquely one need not give the exact domain of A (which is often difficult) but just some core (see Definition III.6.9) for A.

THEOREM III.6.5. *The basic criterium for self-adjointness. Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following three statements are equivalent:*

(a) *T is self-adjoint.*

(b) *T is closed and* $\ker(T^* \pm i) = \{0\}$.

(c) *Ran* $(T \pm i) = \mathcal{H}$.

We give the simple proof of Corollary III.6.6 below, in view that Reed and Simon omitted it.

COROLLARY III.6.6. *The basic criterium for essential self-adjointness. Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following three statements are equivalent:*

(a) *T is essentially self-adjoint.*

(b) $\ker(T^* \pm i) = \{0\}$.

(c) *Ran* $(T \pm i)$ *is dense.*

PROOF. Assume T is essentially self-adjoint. Then $\overline{T}$ is self-adjoint. By Theorem III.6.4, $T^* = \overline{T}^*$ and then, Theorem III.6.5 implies $\ker(T^* \pm i) = \{0\}$.

Let us suppose that $\ker(T^* \pm i) = \{0\}$, then $\overline{T}$ is closed and $\ker(\overline{T}^* \pm i) = \{0\}$. (c) in Theorem III.6.5 tell us that Ran $(\overline{T} \pm i) = \mathcal{H}$. Let g be any element in $\mathcal{H}$, then there exists a $f \in D(\overline{T} \pm i)$ such that $g = (\overline{T} \pm i)f$. Because $\overline{T} \pm i$ is the closure of $T \pm i$ there exists a sequence in $f_n \in D(T)$ such that $f_n \to f$ and $(T \pm i)f_n \to g$. This concludes that Ran $(T \pm i)$ is dense.

Conversely, If Ran $(T \pm i)$ is dense, then Ran $(\overline{T} \pm i) = \mathcal{H}$. Using Theorem III.6.5 we obtain that $\overline{T}$ is self-adjoint. ∎

THEOREM III.6.7. *Closed graph Theorem. Let X and Y be Banach spaces and T a linear map of X into Y. Then T is bounded if and only if the graph of T is closed.*

THEOREM III.6.8. *Hellinger-Toeplitz Theorem. Let A be an everywhere-defined linear operator on a Hilbert space $\mathcal{H}$ with $(x, Ay) = (Ax, y)$ for all x and $y \in \mathcal{H}$. Then A is bounded.*

PROOF. Reed and Simon [77] prove that the graph of A is closed. ∎

We summarize the concept of adjointness. The distinction between closed symmetric operators and self-adjoint operators is very important. It is only for self-adjoint operators that the spectral theorem holds and it is only self-adjoint operators that may be exponentiated to give the one parameter unitary groups which give the dynamics in quantum mechanics.

We give the notion of a core for an operator.

DEFINITION III.6.9. *If T is a closed symmetric operator, a subset $D \subset D(T)$ is called a core for T if $\overline{T \restriction D} = T$.*

VII The spectral Theorem for unbounded operators

We will transcript some theorems from Kato [70] and Reed and Simon [77]. Kato's book is used because he shows directly how we can define by functional calculus $\phi(H)$ where ϕ is any complex continuous function (in Kato's book [70], page 356 mentions that more general functions ϕ can be allowed) and H is a self-adjoint operator. Theorems from [77] section VIII.3 are helpful because they are used throughout this entire thesis and because in the case of functional calculus we arrive to define $h(A)$ by a strong limit of $h_n(A)$ with h_n a sequence of bounded measurable functions.

VII.1 Approach in Kato's book

Let T an operator in $\mathcal{H}$. The numerical range of $\Theta(T)$ of T is the set of all complex numbers (Tu, u) where u changes over all $u \in D(T)$ with $\|u\| = 1$. (We assume $dim\mathcal{H} > 0$.)

A symmetric operator T is said to be bounded from below if its numerical range (which is a subset of the real axis) is bounded from below, that is, if

$$(Tu, u) \geq \gamma(u, u), \quad u \in D(T). \tag{III.7.53}$$

In this case we simply write $T \leq \gamma$. The largest number γ with this property is the lower bound of T.

An operator is said to be accretive if the numerical range $\Theta(T)$ is a subset of the right half-plane, that is, if $\Re e(Tu, u) \geq 0$ for all $u \in D(T)$.

An operator T satisfying (III.7.54) will be said to be m-accretive:

$$(T + \lambda)^{-1} \in B(H), \quad \|(T + \lambda)^{-1}\| \leq (\Re e\lambda)^{-1}, \quad \text{for} \Re e\lambda > 0. \tag{III.7.54}$$

An m-accretive operator T is maximal accretive, in the sense that T is accretive and has no proper accretive extension. An m-accretive operator T is necessarily densely defined.

We shall say that T is quasi-accretive if $T + \alpha$ is accretive for some scalar α. This is equivalent to the condition that $\Theta(T)$ is contained in a half-plane of the form $\Re e\zeta \geq const$. In the same way we say that T is quasi-m-accretive if $T + \alpha$ is m-accretive for some α.

For some quasi-accretive operators T, the numerical range $\Theta(T)$ is not only a subset of the half-plane $\Re e\zeta \geq const$. but a subset of a sector $|\arg(\zeta - \gamma)| \leq \theta < \pi/2$. In such a case T is said to be sectorially-valued or simply sectorial; γ and θ will be called a vertex and a semi-angle of the sectorial operator (these are not uniquely determined). T is said to be m-sectorial if it is sectorial and quasi-m-accretive.

DEFINITION III.7.1. *Let $\mathcal{H}$ be a Hilbert space, we define a form t defined for u and v both belonging to a linear manifold D of $\mathcal{H}$ by the following: $t[u, v]$ is complex valued and linear in $u \in D$ for each fixed $v \in D$ and semilinear in $v \in D$ for each fixed $u \in D$. D will be called the domain of t and is denoted by $D(t)$. t is densely defined if $D(t)$ is dense in $\mathcal{H}$.*

A form t is said to be symmetric if

$$t[u, v] = \overline{t[v, u]}. \tag{III.7.55}$$

We write $t[u]$ instead of $t[u, u]$. We call $t[u]$ the quadratic form associated with $t[u, v]$.

DEFINITION III.7.2. *A symmetric form h is said to be bounded from below if the set of (real) values $h[u]$ for $\|u\| = 1$ is bounded from below or, equivalently,*

$$h[u] \geq \gamma\|u\|^2, \quad u \in D(h). \tag{III.7.56}$$

This will be simply written $h \geq \gamma$.

Let us now consider a nonsymmetric form t. The set of values of $t[u]$ for $u \in D(t)$ with $\|u\| = 1$ is called the numerical range of t and will be denoted by $\Theta(t)$.

t will be said to be sectorial if $\Theta(t)$ is a subset of a sector of the form

$$\zeta \in \mathbb{C} \text{ such that } |arg(\zeta - \gamma)| \leq \theta, \quad 0 \leq \theta < \pi/2, \quad \gamma \text{ real}. \tag{III.7.57}$$

The number γ and θ are not uniquely determined by t.

Let t be a sectorial form. A sequence $\{u_n\}$ of vector will be said to be t-convergent (to $u \in \mathcal{H}$), in symbol: $u_n \to_t u$, $n \to \infty$, if $u_n \in D(t)$, $u_n \to u$ and $t[u_n - u_m] \to 0$ for $n, m \to \infty$.

A sectorial form is said to be closed if $u_n \to_t u$ implies that $u \in D(t)$ and $t[u_n - n] \to 0$. A sectorial form is said to be closable if it has a closed extension. The closure of a closable sectorial form t is the smallest closed extension of t. When t is a closed sectorial form, a linear submanifold D' if D(t) is called a core of t if the restriction t' of t with domain D' has the closure t.

THEOREM III.7.3. *The first Representation Theorem Let $t[u,v]$ be a densely defined, closed, sectorial sesquilinear form in $\mathcal{H}$. There exists a m-sectorial operator T such that*

(*i*) *$D(T) \subset D(t)$ and*

$$t[u,v] = (Tu, v) \tag{III.7.58}$$

for every $u \in D(T)$ and $v \in D(t)$,

(*ii*) *$D(T)$ is a core of t.*

(*iii*) *if $u \in D(T)$ and $w \in \mathcal{H}$ and*

$$t[u,v] = (w, v)$$

holds for every v belonging to a core of t, then $u \in D(T)$ and $w = Tu$. The m-sectorial operator T is uniquely determined by the condition (i).

We define the order relation $h_1 \geq h_2$ for any two symmetric forms h_1 and h_2 bounded from below by

$$D(h_1) \subset D(h_2) \quad \text{and} \quad h_1[u] \geq h_2[u] \quad \text{for} u \in D(h_1). \tag{III.7.59}$$

Let H_1, H_2 be the selfadjoint operators bounded from below associated respectively with closed symmetric forms h_1, h_2 bounded from below. We write $H_1 \geq H_2$ if $h_1 \geq h_2$ in the sense defined above.

Let $\mathcal{H}$ be a Hilbert space, and suppose there is a nondecreasing family $\{M(\lambda)\}$ of closed subspaces of $\mathcal{H}$ depending on a real parameter λ, $-\infty < \lambda < \infty$, such that the intersection of all the $M(\lambda)$ is 0 and their union is dense in $\mathcal{H}$. By "nondecreasing" we mean that $M(\lambda') \subset M(\lambda'')$ for $\lambda' < \lambda''$.

For any fixed λ, then, the intersection $M(\lambda + 0)$ of all $M(\lambda')$ with $\lambda' > \lambda$ contains $M(\lambda)$. Similarly, we have $M(\lambda) \supset M(\lambda - 0)$, where $M(\lambda - 0)$ is the closure of the union of all $M(\lambda')$ with $\lambda' < \lambda$. We shall say that the family $\{M(\lambda)\}$ is right continuous at λ if $M(\lambda+0) = M(\lambda)$, left continuous if $M(\lambda-0) = M(\lambda)$ and continuous if it is right as well as left continuous. As it is easily seen, $\{M(\lambda + 0)\}$ has the same properties as those required of $\{M(\lambda)\}$ above and, moreover, it is everywhere right continuous.

These properties can be translated into properties of the associated family $\{E(\lambda)\}$ of orthogonal projections on $M(\lambda)$. We have:

$$E(\lambda') \leq E(\lambda'') \text{ for } \lambda' < \lambda''. \tag{III.7.60}$$

$$s - \lim_{\lambda \to -\infty} E(\lambda) = 0, \quad s - \lim_{\lambda \to \infty} E(\lambda) = 1. \tag{III.7.61}$$

Equation III.7.60 is equivalent to

$$E(\mu)E(\lambda) = E(\lambda)E(\mu) = E(\min\{\mu, \lambda\}). \tag{III.7.62}$$

A family $\{E(\lambda)\}$ of orthogonal projections with the properties (III.7.60) and (III.7.61) is called a spectral family or a resolution of the identity.

The projections $E(\lambda \pm 0)$ on $M(\lambda \pm 0)$ are given by

$$E(\lambda \pm 0) = s - \lim_{\epsilon \to 0^+} E(\lambda \pm \epsilon).$$

For any semiclosed interval $I = (\lambda', \lambda'']$ of the real line we set

$$E(I) = E(\lambda'') - E(\lambda'),$$

If S is the union of a finite number if intervals (open, closed or semiclosed) on the real line, S can be expressed as the union of disjoint sets of the form I stated below. If we define $E(S)$ as the sum of the corresponding $E(I)$, it is easily seen that $E(S)$ has the property that $E(S')E(S'') = E(S' \cap S'')$. $E(S)$ is called a spectral measure on the class of all sets S of the kind described. This measure $E(S)$ can then be extended to the class of all Borel sets S of the real line by a standard measure-theoretic construction.

For any $u \in \mathcal{H}$, $(E(\lambda)u, u)$ is a nonnegative, nondecreasing function of λ and tends to zero for $\lambda \to -\infty$ and to $\|u\|^2$ for $\lambda \to +\infty$. For any $u, v \in \mathcal{H}$, the polar form $(E(\lambda)u, v)$ is a linear combination of functions of the form $(E(\lambda)w, w)$. Hence the complex-valued function $(E(\lambda)u, v)$ of λ is of bounded variation. See Kato [70] for more details.

The selfadjoint operator associated with a spectral family

To any spectral family $E(\lambda)$, there is associated a selfadjoint operator H expressed by

$$H = \int_{-\infty}^{+\infty} \lambda \, dE(\lambda). \tag{III.7.63}$$

$D(H)$ is the set of all $u \in \mathcal{H}$ such that

$$\int_{-\infty}^{+\infty} \lambda^2 d(E(\lambda)u, u) < \infty.$$

For such u, (Hu, v) is given by

$$(Hu, v) = \int_{-\infty}^{+\infty} \lambda d(E(\lambda)u, v),$$

These two last integrals are Stieltjes integrals. This type of integrals, measure the length of intervals by using a increasing function and uses a similar approach to the Riemann integral. For the proof please check Kato [70].

More generally, we can define operators

$$\phi(H) = \int_{-\infty}^{+\infty} \phi(\lambda) \, dE(\lambda). \tag{III.7.64}$$

Where ϕ may be any complex-valued, continuous function. More general functions ϕ can be allowed, but then the integral $(\phi(H)u, v) = \int \phi(\lambda) \, d(E(\lambda)u, v)$ must be taken in the sense of the Radon-Stieltjes integral, where by the use of increasing functions a measure can be defined in the real line and applied to the Lebesgue approach of integrals.

An operation calculus can be developed for the operators $\phi(H)$. $\phi(H)$ is in general unbounded if $\phi(\lambda)$ is unbounded; $D(\phi(H))$ is the set of all $u \in \mathcal{H}$ such that $\int |\phi(\lambda)|^2 d(E(\lambda)u, u) < \infty$.

THEOREM III.7.4. *The Spectral Theorem Every spectral family* $\{E(\lambda)\}$ *determines a selfadjoint operator by* (III.7.63). *The spectral theorem asserts that every selfadjoint operator* H *admits an expression* (III.7.63) *by means of a spectral family* $\{E(\lambda)\}$ *which is uniquely determined by* H.

VII.2 Approach in Reed and Simon's book

PROPOSITION III.7.5. *Let* $\langle M, \mu \rangle$ *be a measure space with* μ *a finite measure. Suppose that* f *is a measurable, real-valued function on* M *which is finite a.e* $[\mu]$. *Then the operator* $\phi \xrightarrow{T_f} f\phi$ *on* $L^2(M, d\mu)$ *with domain*

$$D(T_f) = \{\phi \mid f\phi \in L^2(M, \mu)\}$$

is self-adjoint.

PROPOSITION III.7.6. *Let* f *and* T_f *obey the conditions in Proposition III.7.5 above. Suppose in addition that* $f \in L^p(M, d\mu)$ *for* $2 < p < \infty$. *Let* D *be any dense set in* $L^q(M, d\mu)$ *where* $q^{-1} + p^{-1} = 1/2$. *Then* D *is a core for* T_f.

Unless $f \in L^\infty(M, \mu)$ the operator T_f described in Propositions 1 et 2 will be unbounded. Thus, we have found a large class of unbounded self-adjoint operators. In fact, we have found them all.

THEOREM III.7.7. *Spectral Theorem-multiplication operator form. Let* A *be a self-adjoint operator on a separable Hilbert space* $\mathcal{H}$ *with domain* $D(A)$. *Then there is a measure space* $\langle M, \mu \rangle$ *with* μ *a finite measure, a unitary operator* $U : \mathcal{H} \to L^2(M, \mu)$, *and a real-valued function* f *on* M *which is finite a.e. so that* (*a*) *$\psi \in D(A)$ if and only if $f(\cdot)U(\psi)(\cdot) \in L^2(M,\mu)$.*

(*b*) *If $\phi \in U[D(A)]$, then $(UAU^{-1}\phi)(m) = f(m)\phi(m)$.*

There is a natural way to define functions of a self-adjoint operator by using the above theorem. Given a bounded Borel function h on $\mathbb{R}$ we define $h(A) = U^{-1}T_{h(f)}U$ where $T_{h(f)}$ is the operator on $L^2(M,\mu)$ which acts by multiplication by the function $h(f(m))$. Using this definition, Theorem III.7.8 below follows easily from Theorem III.7.7. In Theorem III.7.8 below $\hat{\phi}$ does not mean Fourier transform apart from the rest of this thesis.

THEOREM III.7.8. *Spectral Theorem-functional calculus form. Let A be a self-adjoint operator on $\mathcal{H}$. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions on $\mathbb{R}$ into $\mathcal{L}(\mathcal{H})$ so that*

(*a*) *$\hat{\phi}$ is an algebraic *-homeomorphism.*

(*b*) *$\hat{\phi}$ is norm continuous, that is, $\|\hat{\phi}(h)\|_{\mathcal{L}(\mathcal{H})} \leq \|h\|_\infty$.*

(*c*) *Let $h_n(x)$ be a sequence of bounded Borel functions with $h_n(x) \to x$ as $n \to \infty$, for each x and $|h_n(x)| \leq |x|$ for all x and n. Then, for any $\psi \in D(A)$, $\lim_{n\to\infty} \hat{\phi}(h_n)\psi = A\psi$.*

(*d*) *If $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\hat{\phi}(h_n) \to \hat{\phi}(h)$ strongly.*

In addition

(*e*) *If $A\psi = \lambda\psi$, $\hat{\phi}(h)\psi = h(\lambda)\psi$.*

(*f*) *if $h \geq 0$, then $\hat{\phi}(h) \geq 0$.*

The functional calculus is very useful. For example, it allows us to define the exponential e^{itA} and prove easily many of its properties as a function of t.

Finally, the spectral Theorem in its projection-valued measure form follows easily from the functional calculus. Let P_Ω be the operator $\chi_\Omega(A)$ where χ_Ω is the characterized function of the measurable set $\Omega \subset \mathbb{R}$. The family of operators $\{P_\Omega\}$ has the following properties:

(*a*) Each P_Ω is an orthogonal projection.

(*b*) $P_\emptyset = 0$, $P_{(-\infty,\infty)} = I$.

(*c*) If $\Omega = \bigcup_{n=1} \Omega_n$ with $\Omega_n \cap \Omega_m =$ if $n \neq m$, then $P_\Omega = \text{s-}\lim_{N\to\infty} \sum_{n=1}^N P_{\Omega_n}$.

(*d*) $P_{\Omega_1} \cap P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

THEOREM III.7.9. *Spectral Theorem-projection valued measure form. There is a one-to-one correspondence between self-adjoint operators A and projection valued measures $\{P_\Omega\}$ on $\mathcal{H}$, the correspondence given by*

$$A = \int_{-\infty}^{\infty} \lambda \, dP_\lambda.$$

If $g(\cdot)$ is a real-valued Borel function on $\mathbb{R}$, then

$$g(A) = \int_{-\infty}^{\infty} g(\lambda) \, dP_\lambda,$$

defined on D_g (Theorem III.7.8) is self-adjoint. If g is bounded, $g(A)$ coincides with $\hat{\phi}(g)$ in Theorem III.7.8.

We have the following remark: The spectrum of an unbounded self-adjoint operator is an unbounded set of the real axis. We note that the measure space of Theorem III.7.7 can always be chosen so that Proposition III.7.6 is applicable:

PROPOSITION III.7.10. *Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Then the measure space $\langle M,\mu\rangle$ and the function f of Theorem III.7.7 can be chosen so that $f \in L^p(M,\mu)$ for all p with $1 \leq p < \infty$.*

VIII Stone's Theorem

In this section we prove a theorem due to Stone which, like the spectral Theorem, is fundamental for quantum mechanics. Suppose that A is a self-adjoint operator on $\mathcal{H}$. If A is bounded, we can define the exponential of A by

$$e^{itA} = \sum_{n=0}^{\infty} \frac{(it)^n A^n}{n!}.$$

since the series converges in norm. If A is unbounded and self-adjoint, we cannot use the power series directly, but we can use the functional calculus developed in the last section to define e^{itA}.

THEOREM III.8.1. *Let A be a self-adjoint operator and define e^{itA}. Then*

(*a*) *For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.*

(*b*) *If $\phi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\phi \to U(t_0)\phi$.*

(*c*) *For $\phi \in D(A)$, $\frac{U(t)\psi - \psi}{t} \to iA\psi$ as $t \to 0$.*

(*d*) *If $\lim_{t\to 0} \frac{U(t)\psi-\psi}{t}$ exists then $\psi \in D(A)$.*

DEFINITION III.8.2. *An operator-valued function $U(t)$ satisfying (a) and (b) is called a strongly continuous one-parameter unitary group.*

The following theorem says that every strongly continuous unitary group arises as the exponential of a self-adjoint operator.

THEOREM III.8.3. *Stone's theorem. Let $U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$. Then, there is a self-adjoint operator A on $\mathcal{H}$ so that $U(t) = e^{itA}$.*

DEFINITION III.8.4. *If $U(t)$ is a strongly continuous one-parameter unitary group, then the self-adjoint operator A with $U(t) = e^{itA}$ is called the infinitesimal generator of $U(t)$.*

THEOREM III.8.5. *von Neumann. Let $U(t)$ be a one-parameter unitary group on a separable Hilbert space $\mathcal{H}$. Suppose that for all $\phi, \psi \in \mathcal{H}$, $(U(t)\psi, \phi)$ is measurable. Then $U(t)$ is strongly continuous.*

Now we have the following self-adjointness criterion:

THEOREM III.8.6. *Let A be a self-adjoint operator on $\mathcal{H}$ and D be a dense linear set contained in $D(A)$. If for all t, $e^{itA} : D \to D$, then D is a core for A.*

Finally, we have the following generalization of Stone's Theorem that is helpful in the case of time dependent Hamiltonians.

THEOREM III.8.7. *Let $t \to U(t) = U(t_1, \ldots, t_n)$ be a strongly continuous map of $\mathbb{R}^n$ into the unitary operators on a separable Hilbert space $\mathcal{H}$ satisfying $U(t+s) = U(t)U(s)$ and $U(0) = I$. Let D be the set of finite linear combinations of vectors of the form*

$$\phi_f = \int_{\mathbb{R}^n} f(t)U(t)\phi \, dt \quad \phi \in \mathcal{H}, f \in C_0^\infty(\mathbb{R}^n).$$

Then D is a domain of essential self-adjointness for each of the generators A_j of the one parameter subgroups $U(0, 0, \ldots, t_j, \ldots, 0)$, each $A_j : D \to D$ and the A_j commute, $j = 1, \ldots, n$. Furthermore, there is a projection-valued measure P_Ω on $\mathbb{R}^n$ so that

$$(\phi, U(t)\psi) = \int_{\mathbb{R}^n} e^{it\cdot\lambda} \, d(\phi, P_\lambda \psi)$$

for all $\phi, \psi \in \mathcal{H}$.

IX Tensor products of Hilbert spaces

Here, we present some material taken from Reed and Simon [77]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with scalar products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$, respectively. Both scalar products are linear in the first entry and antilinear in the second one. When there is no possibility of confusion, we simply denote them as $(\cdot,\cdot)$. For each $\phi_1 \in \mathcal{H}_1$ and $\phi_2 \in \mathcal{H}_2$, let $\phi_1 \otimes \phi_2$ denote the conjugate (bilinear) form which acts on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$(\phi_1 \otimes \phi_2)\langle \psi_1, \psi_2 \rangle = (\psi_1, \phi_1)_1 (\psi_2, \phi_2)_2 . \tag{III.9.65}$$

PROPOSITION III.9.1. *The form $\phi_1 \otimes \phi_2$ is bilinear.*

PROOF. Let $\alpha \in \mathbb{R}$, $\psi_1, \psi_{11}, \psi_{12} \in \mathcal{H}_1$ and $\psi_2, \psi_{21}, \psi_{22} \in \mathcal{H}_2$.

$$\begin{aligned}
(\phi_1 \otimes \phi_2)\langle \alpha\psi_{11} + \psi_{12}, \psi_2 \rangle &= (\alpha\psi_{11} + \alpha_{12}, \phi_1)(\psi_2, \phi_2) \\
&= (\alpha(\psi_{11}, \phi_1) + (\psi_{12}, \phi_1))(\psi_2, \phi_2) \\
&= \alpha(\psi_{11}, \phi_1)(\psi_2, \phi_2) + (\psi_{12}, \phi_1)(\psi_2, \phi_2) \\
&= \alpha(\phi_1 \otimes \phi_2)\langle \psi_{11}, \psi_2 \rangle + (\phi_1 \otimes \phi_2)\langle \psi_{12}, \psi_2 \rangle.
\end{aligned}$$

Then $\phi_1 \otimes \phi_2$ is linear in the first entry. To prove linearity in the second entry we proceed as follows:

$$\begin{aligned}
(\phi_1 \otimes \phi_2)\langle \psi_1, \alpha\psi_{21} + \psi_{22} \rangle &= (\psi_1, \phi_1)(\alpha\psi_{21} + \psi_{22}, \phi_2) \\
&= (\psi_1, \phi_1)(\alpha(\psi_{21}, \phi_2) + (\psi_{22}, \phi_2)) \\
&= \alpha(\psi_1, \phi_1)(\psi_{21}, \phi_2) + (\psi_1, \phi_1)(\psi_{22}, \phi_2) \\
&= \alpha(\phi_1 \otimes \phi_2)\langle \psi_1, \psi_{21} \rangle + (\phi_1 \otimes \phi_2)\langle \psi_1, \psi_{22} \rangle.
\end{aligned}$$

■

Let $\mathcal{E}$ be the set of finite linear combinations of such conjugate linear forms, see (III.9.65). We define an inner product $(\cdot,\cdot)$ on $\mathcal{E}$ by defining

$$(\phi \otimes \psi, \eta \otimes \mu) = (\phi, \eta)(\psi, \mu) \quad \phi, \eta \in \mathcal{H}_1,\ \psi, \mu \in \mathcal{H}_2. \tag{III.9.66}$$

and extending it by linearity to $\mathcal{E}$.

LEMMA III.9.2. *Suppose that μ is a finite sum which is the zero form, then $(\eta, \mu) = 0$ for all $\eta \in \mathcal{E}$.*

PROOF. Let $\eta = \sum_{i=1}^{N} c_i(\phi_i \otimes \psi_i)$, then

$$\begin{aligned}
(\eta, \mu) &= \left(\sum_{i=1}^{N} c_i(\phi_i \otimes \psi_i),\ \mu \right) \\
&= \sum_{i=1}^{N} c_i\left((\phi_i \otimes \psi_i),\ \mu\right) \\
&= \sum_{i=1}^{N} c_i \mu\langle \phi_i, \psi_i \rangle \\
&= 0,
\end{aligned}$$

because μ is the zero form. ■

PROPOSITION III.9.3. *The scalar product $(\cdot,\cdot)$ in $\mathcal{E}$ is well defined and positive definite.*

PROOF. Let us prove that (λ, λ') does not depend on which finite linear combinations are used to express λ and λ'. Let λ, λ' be written as

$$\begin{aligned}\lambda &= \sum c_i \phi_{1i} \otimes \phi_{2i} = \sum a_k \psi_{1k} \otimes \psi_{2k}, \\ \lambda' &= \sum b_j \eta_{1j} \otimes \eta_{2j}.\end{aligned}$$

By Lemma III.9.2, it follows that

$$\begin{aligned}\left(\sum c_i \phi_{1i} \otimes \phi_{2i},\, \lambda'\right) - \left(\sum a_k \psi_{1k} \otimes \psi_{2k},\, \lambda'\right) &= \sum c_i \left(\phi_{1i} \otimes \phi_{2i}, \lambda'\right) - \sum a_k \left(\psi_{1k} \otimes \psi_{2k},\, \lambda'\right) \\ &= \sum \sum c_i b_j \left(\phi_{1i} \otimes \phi_{2i}, \eta_{1j} \otimes \eta_{2j}\right) \\ &\quad - \sum \sum a_k b_j \left(\psi_{1k} \otimes \psi_{2k},\, \eta_{1j} \otimes \eta_{2j}\right) \\ &= \sum_j \left(\sum c_i \phi_{1i} \otimes \phi_{2i} - \sum a_k \psi_{1k} \otimes \psi_{2k},\, \eta_{1j} \otimes \eta_{2j}\right) \\ &= \left(\sum c_i \phi_{1i} \otimes \phi_{2i} - \sum a_k \psi_{1k} \otimes \psi_{2k},\, \lambda'\right) \\ &= 0.\end{aligned}$$

It is enough to consider λ as having two different expressions as a linear combination. If λ' had two of them then we would have to use property (III.2.5) to the scalar products in $\mathcal{H}_1$ and $\mathcal{H}_2$.

Now suppose $\lambda = \sum_{k=1}^{M} d_k(\eta_k \otimes \mu_k)$. Then $\{\eta_k\}_{k=1}^{M}$ and $\{\mu_k\}_{k=1}^{N}$ span subspaces $M_1 \subset \mathcal{H}_1$ and $M_2 \subset \mathcal{H}_2$, respectively. If we let $\{\phi_j\}_{j=1}^{N_1}$ and $\{\psi_l\}_{l=1}^{N_2}$ be orthogonal bases for M_1 and M_2, we can express each η_k in terms of the $\phi_j' s$ and each μ_k in terms of the $\psi_l' s$ obtaining:

$$\lambda = \sum_{j=1,l=1}^{M_1,M_2} c_{jl}(\phi_j \otimes \psi_l).$$

But

$$\begin{aligned}(\lambda, \lambda) &= \left(\sum c_{jl}(\phi_j \otimes \psi_l), \sum c_{im}(\phi_i \otimes \psi_m)\right) \\ &= \sum \overline{c_{jl}} c_{im} \left(\phi_j \otimes \psi_l,\, \phi_i \otimes \psi_m\right) \\ &= \sum \overline{c_{jl}} c_{im} (\phi_j, \phi_i)(\psi_l, \psi_m) \\ &= \sum |c_{jl}|^2.\end{aligned}$$

So, if $(\lambda, \lambda) = 0$, then all the $c_{jl} = 0$ and λ is the zero form. ■

DEFINITION III.9.4. *We define $\mathcal{H}_1 \otimes \mathcal{H}_2$ to be the completion of $\mathcal{E}$ under the scalar product $(\cdot, \cdot)$ defined above* (III.9.66). *$\mathcal{H}_1 \otimes \mathcal{H}_2$ is called the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$.*

PROPOSITION III.9.5. *If $\{\phi_k\}$ and $\{\psi_l\}$ are orthonormal basis for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then $\{\phi_k \otimes \psi_l\}$ is an orthonalmal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.*

PROOF. See Reed and Simon [77] page 50. ■

To show how the tensor product arises naturally, we will show how it is related to Hilbert spaces with which the reader is already familiar. First, let $\langle M_1, \mu_1 \rangle$ and $\langle M_2, \mu_2 \rangle$ be measure spaces. We consider the Hilbert spaces $L^2(M_1, \mu_1)$ and $L^2(M_2, \mu_2)$. Let $\{\phi_k\}$ and $\{\psi_l\}$ be orthogonal basis in $L^2(M_1, \mu_1)$ and

$L^2(M_2, \mu_2)$, respectively. We want to show that $\{\phi_k(x), \psi_l(y)\}$ is an orthonormal set in $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$, where $\mu_1 \otimes \mu_2$ is the measure product of μ_1 and μ_2. Suppose that $f(x, y) \in L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$, and

$$\int\int_{M_1\times M_2} \overline{f(x,y)}\phi_k(x)\psi_l(y)d\mu_1(x)d\mu_2(y) = 0 \tag{III.9.67}$$

for all k and l. Because

$$\begin{aligned}
\int\int_{M_1\times M_2} |f(x,y)\phi_k(x)\psi_l(y)|d\mu_1(x)d\mu_2(y) &= \int_{M_1} |\phi_k(x)| \left[\int_{M_2} |f(x,y)\psi_l(y)|d\mu_2(y)\right] d\mu_1(x) \\
&\leq \int_{M_1} |\phi_k(x)| \, \|f(x,\cdot)\|_{L^2(M_2,\mu_2)} d\mu_1(x) \\
&\leq \|\phi_k(x)\|^2_{L^2(M_1,\mu_1)} \int_{M_1} \|f(x,\cdot)\|^2_{L^2(M_2,\mu_2)} d\mu_1(x) \\
&= \int_{M_1}\int_{M_2} |f(x,y)|^2 d\mu_2(y)d\mu_1(x) \\
&= \|f\|^2_{L^2(M_1\times M_2,\mu_1\otimes\mu_2)} < \infty.
\end{aligned}$$

Fubini's Theorem can be used, thus (III.9.67) implies that, for all k and l

$$\int_{M_2} \left(\int_{M_1} |f(x,y)|^2 \phi_k(x) d\mu_1(x)\right) \psi_l(y)\, d\mu_2(y) = 0,$$

since $\{\psi_l\}$ is basis for $L^2(M_2, \mu_2)$, we have that

$$\int_{M_1} \overline{f(x,y)}\phi_k(x)d\mu_1(x) = 0$$

for all y except on a set $S_k \subset M_2$ with $\mu_2(S_k) = 0$. Thus, for $y \in M_2 \setminus \bigcup S_k$,

$$\int_{M_1} \overline{f(x,y)}\phi_k(x)d\mu_1(x) = 0$$

for all k, which implies that $f(x, y) = 0$, a.e. $[\mu_1]$. Thus, f $= 0$ a.e. $[\mu_1 \otimes \mu_2]$. Therefore, $\{\phi_k(x), \psi_l(y)\}$ is an orthonormal set in $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$.

Next, we present a abbreviated version of Theorem II.10 in Reed and Simon [77].

THEOREM III.9.6. *Let $\langle M_1, \mu_1\rangle$ and $\langle M_2, \mu_2\rangle$ be measure spaces so that $L^2(M_1, \mu_1)$ and $L^2(M_2, \mu_2)$ are separable. Then, there is a unique isomorphism from $L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2)$ to $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$, so that $f \otimes g \mapsto fg$.*

Proof. Let

$$U : \phi_k \otimes \psi_l \mapsto \phi_k(x)\psi_l(y).$$

Then U takes an orthonormal basis for $L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2)$ onto an orthonormal basis $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$, and extends uniquely to a unitary mapping of $L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2)$ onto $L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)$.

Notice that if $f \in L^2(M_1, \mu_1)$ and $g \in L^2(M_2, \mu_2)$, then

$$\begin{aligned}
U\,(f\otimes g) &= U\left(\sum c_k\phi_k \otimes \sum d_l\psi_l\right) = U\left(\sum_{k,l} c_k d_l\, \phi_k\otimes\psi_l\right) \\
&= \sum_{k,l} c_k d_l\, U\,(\phi_k\otimes\psi_l) = \sum_{k,l} c_k d_l\, \phi_k(x)\psi_l(y) \\
&= f(x)g(x).
\end{aligned}$$

■

Because of this property, we often say that $L^2(M_1,\mu_1)\otimes L^2(M_2,\mu_2)$ and $L^2(M_1\times M_2,\mu_1\otimes\mu_2)$ are "naturally" isomorphic. Let $M_i=\mathbb{R}$ and μ_i the Lebesgue measure, then we have shown that $L^2(\mathbb{R}^2)$ is naturally isomorphic to $L^2(\mathbb{R})\otimes L^2(\mathbb{R})$. Comment: $L^2(\mathbb{R}^{nN})\cong L^2(\mathbb{R}^n)\otimes\ldots\otimes L^2(\mathbb{R}^n)$.

X Tensor products of bounded and unbounded operators

This section is mainly based on section VIII.10 of Reed and Simon [77]. Here we describe some aspects of tensor products of operators in Hilbert spaces. Let A and B be densely defined operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. We represent by $D(A)$ and $D(B)$ the domains of A and B, respectively. We will denote $D(A)\otimes D(B)$ the set of finite linear combinations of vectors of the form $\phi\otimes\psi$ where $\phi\in D(A)$ and $\psi\in D(B)$. We define $A\otimes B$ on $D(A)\otimes D(B)$ by

$$(A\otimes B)(\phi\otimes\psi)=(A\phi\otimes B\psi)$$

and extend it by linearity.

Let us prove that $D(A)\otimes D(B)$ is dense in $\mathcal{H}_1\otimes\mathcal{H}_2$. Let us take f a bilinear form in $\mathcal{H}_1\otimes\mathcal{H}_2$ such that $f\langle\phi,\psi\rangle=(\phi\otimes\psi,f)_{\mathcal{H}_1\otimes\mathcal{H}_2}=0$, for all $\phi\in D(A)$ and $\psi\in D(B)$. Then f = 0 in the Cartesian product $D(A)\times D(B)$ which is a dense set in $\mathcal{H}_1\times\mathcal{H}_2$. If $f=\sum c_{kl}\phi_k\otimes\psi_l$ then

$$\begin{aligned} f\langle\phi,\psi\rangle &= \sum c_{kl}(\phi\otimes\psi,\phi_k\otimes\psi_l)\\ &= \sum c_{kl}(\phi,\phi_k)(\psi,\psi_l)\\ |f\langle\phi,\psi\rangle| &\le \left(\sum |c_{kl}|\,\|\phi_k\|\,\|\psi_l\|\right)\|\phi\|\,\|\psi\|. \end{aligned}$$

It follows that f is a bounded bilinear form. By Theorem III.10.2 below f is continuous, that is why $f=0$ in $\mathcal{H}_1\times\mathcal{H}_2$. So, $D(A)\otimes D(B)$ is dense in $\mathcal{H}_1\otimes\mathcal{H}_2$.

Before we enounce a simplified version of Theorem 2.17 from Rudin [80], we define the following, extracted from definition 1.8 of the same book:

DEFINITION III.10.1. *X is an F-space if its topology is induced by a complete invariant metric d.*

THEOREM III.10.2. *Suppose $B:X\times Y\to Z$ is bilinear and separately continuous, X is an F-space, Y is a metric space and Z a topological vector space, it follows that B is continuous.*

We define $A\otimes B$ on $D(A)\otimes D(B)$ by

$$(A\otimes B)(\phi\otimes\psi)=(A\phi\otimes B\psi)\,,\quad \phi\in D(A),\psi\in D(B),$$

and extend by linearity.

PROPOSITION III.10.3. *The operator $A\otimes B$ is well defined. Further, if A and B are closable, so are $A\otimes B$ and $A\otimes I+I\otimes B$.*

PROOF. See Reed and Simon [77]. ■

DEFINITION III.10.4. *Let A and B be operators in $\mathcal{H}_1,\mathcal{H}_2$, respectively. The tensor product of A and B is the closure of the operator $A\otimes B$ defined in $D(A)\otimes D(B)$. We will denote the closure by $A\otimes B$ also. Usually $A+B$ will denote the closure of $A\otimes I+I\otimes B$ on $D(A)\otimes D(B)$.*

PROPOSITION III.10.5. *Let A and B be bounded operators on $\mathcal{H}_1,\mathcal{H}_2$, respectively. Then $\|A\otimes B\|=\|A\|\,\|B\|$.*

PROOF. Reed and Simon's proof [77] uses orthonormal bases. ■

We remark that both of above propositions have natural generalizations to arbitrary finite tensor products of operators. This can be proven directly or by using the associativity of the tensor product of Hilbert spaces.

We turn now to questions of self-adjointness and spectrum. Let $\{A_k\}_{k=1}^N$ be a family of operators, A_k self-adjoint on the Hilbert space $\mathcal{H}_k$, $k = 1, \ldots N$. We will denote the closure of $I \otimes \ldots \otimes A_k \otimes \ldots I$ on $D = \otimes D_k$ by A_k also. Let $P(x_1, x_N)$ be a polynomial of degree n_k in x_k. The the operator $P(A_1, \ldots, A_N)$ makes sense on $\otimes D(A^{n_k})$, since $D(A^{n_k}) \subset D(A^l)$ for all $l \leq n_k$. In fact, P is essentially self-adjoint in that domain.

THEOREM III.10.6. *Let A_k be a self-adjoint operator on $\mathcal{H}_k$. Let $P(x_1, \ldots, x_N)$ a polynomial with real coefficients of degree n_k in the kth variable and suppose that D_k^l is a domain of essential self-adjointness for $A_k^{n_k}$. Then,*

(*a*) *$P(A_1, \ldots, A_N)$ is essentially self-adjoint on*

$$D^l = \otimes_{k=1}^N D_k^l.$$

(*b*) *The spectrum of $\overline{P(A_1, \ldots, A_N)}$ is the closure of the range of P on the product of the spectra of the A_k. That is,*

$$\sigma(\overline{P(A_1, \ldots, A_N)}) = \overline{P(\sigma(A_1), \ldots, \sigma(A_N))}.$$

PROOF. We will first prove that $P(A_1, \ldots, A_N)$ is essentially self-adjoint on $D = \otimes_{k=1}^N D(A_k^{n_k})$. By the spectral Theorem, there is a measure space $\langle M_k, \mu_k \rangle$ so that A_k is unitarily equivalent to multiplication by a real-valued measurable function f_k on $L^2(M_k, \mu_k)$. By proposition 3 in Section VIII.3 Reed and Simon [77] we may assume that μ_k is finite and that $f_k \in \cap_{1 \leq p < \infty} L^p(M_k, \mu_k)$. Furthermore, by Theorem III.9.6, $\otimes_{k=1}^N L^2(M_k, \mu_k)$ is naturally isomorphic to $L^2(\times_{k=1}^N M_k, \otimes_{k=1}^N \mu_k)$. Under this isomorphism $P(A_1, \ldots, A_N)$ corresponds to multiplication by $P(f_1, \ldots, f_N)$ and D corresponds to the set of finite linear combinations of finite linear combinations of functions $\phi_1(m_1), \phi_2(m_2), \ldots, \phi_N(m_N)$ such that $f_k^{n_k}\phi_k \in L^2(M_k, \mu_k)$.

To prove essential self-adjointness we use Proposition III.7.6. First, since μ_k is finite and $f_k^{n_k} \in L^p(M_k, \mu_k)$ we conclude that $f_k^l \in L^p(M_k, \mu_k)$, for $l \leq p < \infty$. From this it follows immediatly that $P(f_1, \ldots, f_n)$ is in L^p for all such p. In particular $P(f_1, \ldots, f_n)$ is in $L^4(\times_{k=1}^N M_k, \otimes_{k=1}^N \mu_k)$. Since $f_k^{n_k}$ is self-adjoint in D_k, D_k, contains all the characteristic functions of measurable sets in M_k. Thus D contains all finite linear combinations of the characteristic functions of rectangles. It is a fact that the characteristic function of any measurable set in $\times_{k=1}^N M_k$ is equal to such finite linear combination except on a set of arbitrarily small $\otimes_{k=1}^N \mu_k$ measure. Thus, the simple functions on $\times_{k=1}^N M_k$ can be approximated in the L^p sense $(1 \leq p < \infty)$ by elements of D. In particular, D is dense in $L^4(\times_{k=1}^N M_k, \otimes_{k=1}^N \mu_k)$. Essential self-adjointness now follows by Proposition III.7.6.

To show that P is essentially self-adjoint on D^l we need only show (by problem 14 in Reed and Simon's book [77]) that $\overline{P \restriction D^l}$ extends $P \restriction D$. Suppose $\otimes_{k=1}^n \phi_k \in D$. Then $\phi_k \in D(A_k^{n_k})$, so since D_k^l is a domain of essential self-adjointness of $A_k^{n_k}$ there is a sequence $\{\phi_k^l\}_{l=1}^\infty$ so that $\phi_k^l \to \phi_k$, and $A_k^{n_k}\phi_k^l \to A_k^{n_k}\phi_k$. An easy estimate shows that this implies that $A_k^m \phi_k^l \to A_k^m \phi_k$ for all $1 \leq m \leq n_k$. Therefore $\otimes_{k=1}^N \phi_k^l \to \otimes_{k=1}^N \phi_k$ and $P(A_1, \ldots, A_N)\left(\otimes_{k=1}^N \phi_k^l\right) \to P(A_1, \ldots, A_N)\left(\otimes_{k=1}^N \phi_k\right)$. The same argument works for finite linear combinations of vectors of the form $\otimes_{k=1}^N \phi_k$ so $\overline{P \restriction D^l}$ extends $P \restriction D$. This completes the proof of (*a*).

To prove (*b*), suppose that $\lambda \in \overline{P(\sigma(A_1), \ldots, \sigma(A_N))}$. If I is any open interval about λ then $P^{-1}(I)$ contains a product $\times_{k=1}^N I_k$ of open intervals so that $I_k \cap \sigma(A_k) \neq \emptyset$. Since $\sigma(A_k) =$ ess range $f_k^{n_k}$, $\mu_k[(f_k^{n_k})^{-1}(I_k)] \neq 0$ so

$$\mu[P\left((f_1, \ldots, f_N)^{-1}(I)\right] \neq 0.$$

That is, $\lambda \in$ ess range $P(f_1, \ldots, f_N)$, which equals $\sigma(\overline{P(A_1, \ldots, A_N)})$ by the first proposition in Section VIII.3 in Reed and Simon [77]. Conversely if $\lambda \notin \overline{P(\sigma(A_1), \ldots, \sigma(A_N))}$ then $(\lambda - P(f_1, \ldots, f_N))^{-1}$ is bounded a.e. on $\times_{k=1}^N M_k$ so $\lambda \in \rho\left(\overline{P(A_1, \ldots, A_N)}\right)$. ■

If $A_1, \ldots, A_N$ are bounded, $P(\sigma(A_1), \ldots, \sigma(A_N))$ is closed, but in general is not (Problem 43 in Reed and Simon [77]). The following Corollary displays the two most important special case of Theorem III.10.6.

COROLLARY III.10.7. *Let $A_1, \ldots, A_N$ be self-adjoint operators on $\mathcal{H}_1, \ldots, \mathcal{H}_N$ and suppose that, for each k, D_k is a domain of essential self-adjointness for A_k. Then,*

(*a*) *The operators $A_\Pi = A_1 \otimes \ldots \otimes A_N$ and $A_\Sigma = A_1 + \ldots + A_N$ are essentially self-adjoint on $D = \otimes_{k=1}^N D_k$.*

(*b*) $\sigma(A_\Pi) = \overline{\Pi_{k=1}^N \sigma(\overline{A_k})}$ *and* $\sigma(\overline{A_\Sigma}) = \overline{\Sigma_{k=1}^N \sigma(\overline{A_k})}$.

XI Perturbation Theory. Schrödinger Hamiltonians

In this section, we give conditions on the potential v that allows to consider V, the multiplication operator associated to v, as a "small" perturbation of H_0, in such a way that the sum operator $H_0 + V$ be self-adjoint. Clearly a sufficient condition is that $\|v\|_\infty < \infty$, thanks to $V \in \mathcal{B}(\mathcal{H})$. Nevertheless, it is important to deal with unbounded operators, since these potentials appear in quantum mechanics, as an example we have Coulomb potential.

First we give some abstract results, which will be applied to Schrödinger operators. If A and B are self-adjoint and, at least, one of them is bounded, say B, then $A + B$ is self-adjoint with $D(A+B) = D(A)$. If both A and B are unbounded but $D(A+B) = D(A) \cap D(B)$ is dense in $\mathcal{H}$, then $A + B$ is symmmetric but in general it is neither self-adjoint nor essentially self-adjoint.

Next, we introduce the concept of relative boundedness that allows to compare two unbounded operators.

DEFINITION III.11.1. *Let A and B be two linear operators. We say that B is A-bounded if*

(*a*) $D(A) \subset D(B)$,

(*b*) *There exist two numbers β and γ in $[0, \infty)$ such that*

$$\|Bf\| \leq \beta \|Af\| + \gamma \|f\| \qquad \forall f \in D(A). \tag{III.11.68}$$

The infimum of all numbers β such that (III.11.68) *is true is called the A bounde of B.*

REMARK III.11.2. *γ in* (III.11.68) *can be different for different values β. The A bound of B is determined uniquely considering all the possible values of β.*

As an immediate consequence of Definition III.11.1 we have that if $D(A) \subset D(B)$ and A is a bounded operator then A is B-bounded with relative bound 0, because for any $\phi \in D(B)$,

$$\|A\phi\| \leq 0\|B\phi\| + \|A\| \, \|\phi\|. \tag{III.11.69}$$

LEMMA III.11.3. *Assume that $A = A^*$.*

(*i*) *The following three statements are equivalent:*

(*a*) *B es A bounded.*

(*b*) *$B(A - zI)^{-1} \in \mathcal{B}(\mathcal{H})$ for some $z \in \rho(A)$.*

(*c*) *$D(A) \subset D(B)$ and*

$$\|Bf\|^2 \leq \beta_0^2 \|Af\|^2 + \gamma_0^2 \|f\|^2 \qquad \forall f \in D(A), \tag{III.11.70}$$

where β_0, γ_0 are numbers is $[0, \infty)$.

The A bound of B is also equal to the infimum of all numbers β_0 such that (III.11.70) *holds.*

(*ii*) *The following two statements are equivalent::*

(*d*) *B is A bounded with A bound $\nu < 1$.*

(*e*) *There exists a number $z \in \rho(A)$ such that $\|B(A - zI)^{-1}\| < 1$.*

PROPOSITION III.11.4. *If B is A bounded, then $B(A - zI)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $z \in \rho(A)$.*

Moreover, we have a more precise result taken from [64] Proposition 1.3 (a).

PROPOSITION III.11.5. *Assume A to be self-adjoint and $D(A) \subset D(B)$. Then B is A-bounded if and only if $B(A + i)^{-1}$ is bounded. The A-bound of B is equal to*

$$lim_{\gamma \to \infty} \|B(A + i\gamma)^{-1}\|.$$

PROPOSITION III.11.6. *Kato-Rellich Theorem. Let A self-adjoint, B symmetric and A-bounded with A-bound $\nu < 1$. Then $A + B$ is self-adjoint in $D(A)$. Moreover, if A is bounded by below, so $A + B$ is. (A is called bounded by below if $A + \mu \geq 0$, or equivalently if $(-\infty, \mu) \in \rho(A)$, for some $\mu \in \mathbb{R}$).*

PROPOSITION III.11.7. *Under the hypothesis of Proposition III.11.6, B is $(A+B)$ bounded. Moreover, we have the second resolvent equation for all $z \in \rho(A) \cap \rho(A + B)$:*

$$\begin{aligned} (A + B - z)^{-1} &= (A - z)^{-1} - (A - z)^{-1} B (A + B - z)^{-1} \\ &= (A - z)^{-1} - (A + B - z)^{-1} B (A - z)^{-1}. \end{aligned} \tag{III.11.71}$$

Example III.11.8. *Let $A = A^*, B = B^*$ and B belong to $\mathcal{B}(\mathcal{H})$. Then we know that $A+B$ is self-adjoint in $D(A)$. This results, of course, also from Proposition III.11.6. In fact, B is A-bounded with A-bound $\nu = 0$, because we can set $\beta = 0, \gamma = \|B\|$ in* (III.11.68).

Example III.11.9. *Let $A = A^*$ be unbounded, and $B = -\lambda A$ with $\lambda \geq 0$. Then $\|Bf\| \leq \lambda \|Af\|, \forall f \in D(A)$, in consequence B is A-bounded with A-bound λ. If $\lambda < 1, A + B = (1 - \lambda)A$ which is self-adjoint. On the other hand, if $\lambda = 1$, $A + B$ is the restriction of the zero operator with domain $\mathcal{H}$. This shows that the hypothesis $\nu < 1$ cannot be made weaker in Proposition III.11.6 (If $\nu = 1$, one can show, however, that $A + B$ is essentially self-adjoint).*

Now we will apply Proposition III.11.6 to Schrödinger operators in $L^2(\mathbb{R}^n)$. We begin with an auxiliary estimation.

LEMMA III.11.10. *Let $D = L^2(\mathbb{R}^n)$. Let $2 \leq p \leq \infty$ and $\phi, \psi \in L^p(\mathbb{R}^n)$. Denote by $\phi(\bar{P})$ the multiplication operator by $\phi(\bar{k})$ in $\tilde{L}^2(\mathbb{R}^n)$, and define $A_{\phi\psi} = \phi(\bar{P})\psi(\bar{Q}), B_{\phi\psi} = \psi(\bar{Q})\phi(\bar{P})$. Then the closures of $A_{\phi\psi}$ and $B_{\phi\psi}$ lie in $\mathcal{B}(L^2(\mathbb{R}^n))$, and*

$$\|A_{\phi\psi}\| \leq \|\phi\|_p \|\psi\|_p, \tag{III.11.72}$$

$$\|B_{\phi\psi}\| \leq \|\phi\|_p \|\psi\|_p. \tag{III.11.73}$$

We introduce a class of potentials often used. A measurable function $v : \mathbb{R}^n \to \mathbb{R}$ will be called class $v_\oslash$ if can be written as $v = v_1 + v_2$ with $v_1 \in L^\infty(\mathbb{R}^n)$ and $v_2 \in L^p(\mathbb{R}^n)$ for some p that satisfies $p \geq 2$ and $p > n/2$.

Examples:

1. Squared well or barrier:

$$V(\bar{x}) = \begin{cases} V_0 & |\bar{x}| \leq a, \\ 0 & |\bar{x}| > a, \end{cases}$$

   where $V_0 \in \mathbb{R}$, we can take $V_1 = V, V_2 = 0$ or $V_1 = 0, V_2 = V$.

2. Yukawa potential:

$$V(\bar{x}) = \alpha |\bar{x}|^{-1} \exp(-\mu |\bar{x}|),$$

   with $\alpha \in \mathbb{R}$ and $\mu > 0$. If $n = 3$ one can take $V_1 = 0, V_2 = V$ and $p = 2$.

3. Coulomb potential:

$$V(\bar{x}) = \alpha|\bar{x}|^{-1}, \quad \alpha \in \mathbb{R}.$$

   Here, we take:

$$V_1(\bar{x}) = \begin{cases} V(\bar{x}) & |\bar{x}| > 1, \\ 0 & |\bar{x}| \leq 1, \end{cases} \quad V_2(\bar{x}) = V(\bar{x}) - V_1(\bar{x}), p = 2.$$

PROPOSITION III.11.11. *Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $n = 1, 2, ...$ Let $v \in v_\oslash$. Then $D(H_0) \subset D(V)$, V is H_0 bounded with H_0 bound $V = 0$, and $V(H_0 - z)^{-1} \in \mathcal{B}(\mathcal{H}), \forall z \in \rho(H_0)$.*

PROPOSITION III.11.12. *Under the hypothesis of Proposition III.11.11 $H = H_0 + V$ is self-adjoint and bounded by below.*

LEMMA III.11.13. *Let $\mathcal{H} = L^2(\mathbb{R}^n)$. Assume that $v, w \in v_\oslash$, let $H = H_0 + V$ and denote by W the multiplication operator by $w(\bar{x})$. Then, for $z \in \rho(\mathcal{H})$, the operator $W(H - z)^{-1}$ and the closure of $(H - z)^{-1}W$ belong to $\mathcal{B}(\mathcal{H})$.*

Chapter 4

Appendix

In this chapter we are going to present more definitions, computations and proofs with a more depth of detail than in the chapters below. This level of detail has not been found by the author of this thesis in the literature even though the results are well known.

I N-Body Kinematics

PROPOSITION IV.1.1. *Assertions given in* (II.1.2), *and in* (II.1.3) *are true. In particular: Fourier transform maps unitarily* $L^2(\mathbf{X})$ *onto* $L^2(\hat{\mathbf{X}})$.

Proof.

This approach is based in the following references, Adachi [60], Deift et al [65], Enss [67], Rudin [79] and Sigalov and Sigal [81].

We study a system of N distinguishable particles of masses m_j, charges q_j each moving in n-dimensional space. The positions for the N particles can be considered as elements in the Hilbert space $\mathbb{R}^{nN}$ with the scalar product

$$(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})_1 = \sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j \cdot \tilde{\mathbf{y}}_j, \tag{IV.1.1}$$

where $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N)$, $\tilde{\mathbf{y}} = (\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_N)$ and $\cdot$ is the usual scalar product in $\mathbb{R}^n$. This scalar product defines a norm,

$$|||\tilde{\mathbf{x}}||| = \|\tilde{\mathbf{x}}\|_G = (\tilde{\mathbf{x}}, \tilde{\mathbf{x}})_1^{1/2} = \left[\sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j^2 \right]^{1/2}. \tag{IV.1.2}$$

To prove that $\mathbf{X} \cong \mathbb{R}^{n(N-1)}$, let us rewrite (II.1.2):

$$\mathbf{X} = \left\{ \tilde{\mathbf{x}} = \left(-m_1^{-1} \sum_{j=2}^{N} m_j \, \tilde{\mathbf{x}}_j \, , \tilde{\mathbf{x}}_2, \ldots, \tilde{\mathbf{x}}_N \right) \in \mathbb{R}^{nN} \right\}.$$

Then, a Hamel base for $\mathbf{X}$ is the following set, $\mathcal{B}(\mathbf{X}) := \left\{ -(m_j/m_1)\mathbf{e}_{n-i} + \mathbf{e}_{j(n)-i} \,\middle|\, 2 \leq j \leq N, i = 1, \ldots, n \right\}$, where $\mathbf{e}_q$, $q = 1 \ldots, nN$ are the canonical vectors in $\mathbb{R}^{nN}$. The cardinality of $\mathcal{B}(\mathbf{X})$ is $n(N-1)$.

We can define the space $\mathbf{Y}$ as follows:

$$\mathbf{Y} = \left\{ \tilde{\mathbf{y}} = (\tilde{\mathbf{y}}_1, \ldots, \tilde{\mathbf{y}}_N) \in \mathbb{R}^{nN} \,\middle|\, \tilde{\mathbf{y}}_1 = \ldots = \tilde{\mathbf{y}}_N \in \mathbb{R}^n \right\} \cong \mathbb{R}^n. \tag{IV.1.3}$$

For all $\tilde{\mathbf{z}} = (\tilde{\mathbf{z}}_1, \ldots, \tilde{\mathbf{z}}_N) \in \mathbb{R}^{nN}$,

$$\tilde{\mathbf{z}} = \tilde{\mathbf{x}} + \tilde{\mathbf{y}},$$

where

$$\begin{aligned}
\tilde{\mathbf{x}} &= \left(\tilde{\mathbf{z}}_1 - M^{-1} \sum_{j=1}^{N} m_j \, \tilde{\mathbf{z}}_j, \; \ldots, \; \tilde{\mathbf{z}}_N - M^{-1} \sum_{j=1}^{N} m_j \, \tilde{\mathbf{z}}_j \right) \in \mathbf{X}, \\
\tilde{\mathbf{y}} &= \left(M^{-1} \sum_{j=1}^{N} m_j \, \tilde{\mathbf{z}}_j, \; \ldots, \; M^{-1} \sum_{j=1}^{N} m_j \, \tilde{\mathbf{z}}_j \right) \in \mathbf{Y}.
\end{aligned}$$

Hence, $\mathbb{R}^{nN}$ is the direct sum of $\mathbf{X}$ and $\mathbf{Y} : \mathbb{R}^{nN} = \mathbf{X} \oplus \mathbf{Y}$. It is evident that both $\mathbf{X}$ and $\mathbf{Y}$ are closed spaces. In physics terms, $\mathbf{X}$ describes the positions of the N particles with respect to the center of mass of the whole system, i.e. the center of mass is the origin. Besides, $\mathbf{Y}$ represents the set of all possible centers of mass where all particles are considered as one body, that is, all particle's position are equal to the center of mass. With the scalar product $(\cdot, \cdot)_1$, $\mathbf{X}$ and $\mathbf{Y}$ are mutually orthogonal spaces because, for $\tilde{\mathbf{x}} \in \mathbf{X}$ and $\tilde{\mathbf{y}} \in \mathbf{Y}$,

$$(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})_1 = \left(\sum_{j=1}^{N} m_j \, \tilde{\mathbf{x}}_j \right) \cdot \tilde{\mathbf{y}}_1 = 0.$$

Denote $G = \mathbb{R}^{nN}$, $G_j = \mathbb{R}^n_{\tilde{x}_j}$ where the subindex means that the variable we use in this copy of $\mathbb{R}^n$ is $\tilde{x}_j$. With the notation given in subsection (V.1) we have the following isomorphisms (homeomorphisms) between locally compact Abelian (LCA) groups:

$$\begin{aligned}
G &\cong \bigoplus_{j=1}^{N} G_j && \text{(IV.1.4)} \\
G &\cong \mathbf{X} \oplus \mathbf{Y} && \text{(IV.1.5)} \\
\mathbf{X} &\cong G/\mathbf{Y}. && \text{(IV.1.6)}
\end{aligned}$$

At present, (IV.1.4) and (IV.1.5) are clear. To prove (IV.1.6) we establish an isomorphism between $\mathbf{X}$ and $G/\mathbf{Y}$. Let $\zeta : \mathbf{X} \to G/\mathbf{Y}$ such that $\tilde{x} \mapsto \tilde{x} + \mathbf{Y}$. It is not difficult to see that $\zeta(\tilde{x} + \tilde{y}) = \zeta(\tilde{x}) + \zeta(\tilde{y})$. It is one to one: If $\tilde{x}, \tilde{y} \in \mathbf{X}$ and $\tilde{x} + \mathbf{Y} = \tilde{y} + \mathbf{Y}$, then $\tilde{x} - \tilde{y} \in \mathbf{X} \cap \mathbf{Y}$, therefore $\tilde{x} = \tilde{y}$.

Let $d\mu(\tilde{x})$ denote the element of volume with respect to the norm $|||\cdot|||$. We can think of μ as the product measure defined in the Cartesian product $\mathbb{R}^{nN}_{\tilde{\mathbf{x}}} \cong \bigotimes_{j=1}^{N} \mathbb{R}^n_{\tilde{\mathbf{x}}_j}$. In each $\mathbb{R}^n_{\tilde{\mathbf{x}}_j}$ the norm $|||\cdot|||$ induces the metric $ds_j^2 = (\sqrt{m_j} d\tilde{x}_j)^2$, that is the metric tensor is represented by the matrix $m_j I_{n \times n}$, and thus, the measure in the j-th copy of $\mathbb{R}^n_{\tilde{\mathbf{x}}_j}$ is $\mu^j := \sqrt{\det(m_j I_{n\times n})}\lambda$ where λ is the Lebesgue measure in $\mathbb{R}^n$. By the fact that Lebesgue measure is $\sigma-$finite, that Borel $\sigma-$algebra in $\mathbb{R}^{nN}$ is generated by the Cartesian product of N copies of the Borel $\sigma-$algebra in $\mathbb{R}^n$, and the measure product Theorem [61], we conclude that

$$d\mu(\tilde{x}) = (m_1^{n/2} d\tilde{x}_1) \cdots (m_N^{n/2} d\tilde{x}_N) = \left(\prod_{j=1}^{N} m_j^{n/2} \right) d\tilde{x}_1 \cdots d\tilde{x}_N, \qquad \text{(IV.1.7)}$$

where $d\tilde{x}_j$ is the element of volume given by the Lebesgue measure in $\mathbb{R}^n$.

Let Γ, Γ_j be the dual of G, G_j, $1 \le j \le N$, respectively. Let $\hat{\mathbf{X}}$ be the annihilator of $\mathbf{Y}$. Then, by Theorem III.5.19, Theorem III.5.20 and its corollary III.5.21, $\hat{\mathbf{X}}$ is the dual of $\mathbf{X}$, $\Gamma/\hat{\mathbf{X}}$ is the dual of $\mathbf{Y}$, $\Gamma_j \cong \mathbb{R}^n_{\tilde{p}_j}$,

$$\begin{aligned}
\Gamma &\cong \bigoplus_{j=1}^{N} \Gamma_j, \\
\Gamma &\cong \hat{\mathbf{X}} \oplus (\Gamma/\hat{\mathbf{X}}).
\end{aligned}$$

By extending the calculations done to get (III.5.47), the following is valid: For any $\gamma \in \Gamma$ there a unique $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_N) \in \mathbb{R}^{nN}$ such that

$$\gamma(\tilde{x}) = e^{i\tilde{p}\cdot\tilde{x}}$$

where $\cdot$ is the usual scalar product in $\mathbb{R}^{nN}$. These set of $\tilde{p}$-s are going to be considered as the elements of Γ. Having said that, the annihilator $\hat{\mathbf{X}}$ of $\mathbf{Y}$ are all $\tilde{p} \in \mathbb{R}^{nN}$ such that $e^{i\tilde{p}\cdot\tilde{x}} = 1$ for all $\tilde{x} \in \mathbf{Y}$, this implies the following computations:

$$\begin{aligned} \tilde{p}\cdot\tilde{x} &= 0 \\ \sum \tilde{p}_j \cdot \tilde{x}_j &= 0 \\ \left(\sum \tilde{p}_j\right)\cdot \tilde{x}_1 &= 0 \\ \sum \tilde{p}_j &= 0. \end{aligned}$$

That is why the conjugated momenta $\tilde{\mathbf{p}}$, in the center of mass frame, are elements of the dual space $\hat{\mathbf{X}}$, as in (II.1.3),

$$\hat{\mathbf{X}} = \left\{ \tilde{\mathbf{p}} = (\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \dots, \tilde{\mathbf{p}}_N) \in \mathbb{R}^{nN} \,\middle|\, \sum_{j=1}^{N} \tilde{\mathbf{p}}_j = 0 \in \mathbb{R}^n \right\} \cong \mathbb{R}^{n(N-1)}. \tag{IV.1.8}$$

Recalling the Plancherel Theorem III.5.17, the Fourier transform is an isometry from $L^2(G) = L^2(\mathbb{R}^{nN})$ onto $L^2(\Gamma) \cong L^2(\mathbb{R}^{nN})$, with the Haar measures μ in G and ν in Γ, both being equivalent to the Lebesgue Measure λ in $\mathbb{R}^{nN}$ by Theorem III.5.24 (generalized to the n-dimensional case) and Theorem III.5.7. Thus $\mu = \alpha\lambda$ and $\mu = \beta\lambda$, where these $\alpha, \beta > 0$ must hold (III.5.49) and also μ has to take into account (IV.1.7). There is no a unique choice of $\alpha, \beta > 0$, but we prefer that both μ and ν share the same factor of 2π, then

$$\begin{aligned} \mu &= (2\pi)^{-nN/2} \left(\prod_{j=1}^{N} m_j^{n/2} \right) \lambda, \\ \nu &= (2\pi)^{-nN/2} \left(\prod_{j=1}^{N} m_j^{-n/2} \right) \lambda. \end{aligned}$$

We can drop the $(2\pi)^{-nN/2}$ from the measure and consider it as part of the definition on the Fourier transform, we put in these cases, where λ_j is the usual lebesgue measure in the j-th copy of $\mathbb{R}^n$,

$$\mu = \bigotimes_{j=1}^{N} m_j^{n/2} \lambda_j, \tag{IV.1.9}$$

$$\nu = \bigotimes_{j=1}^{N} m_j^{-n/2} \lambda_j. \tag{IV.1.10}$$

Therefore, we modify (III.5.43) and the transition to the momentum representation is, for any $\phi \in L^1(\mathbf{X}) \cap B(\mathbf{X})$, where $B(\mathbf{X})$ is like in Definition III.5.15,

$$\begin{aligned} \hat{\phi}(p) &= (2\pi)^{-nN/2} \int_{\mathbf{X}} \phi(x)\, e^{-ip\cdot x}\, d\mu(x), \quad p \in \hat{\mathbf{X}} \\ \phi(x) &= (2\pi)^{-nN/2} \int_{\hat{\mathbf{X}}} \hat{\phi}(p)\, e^{p\cdot x}\, d\nu(p), \quad x \in \mathbf{X}, \end{aligned}$$

We conclude, by the Plancherel Theorem III.5.17, that Fourier transform maps unitarily $L^2(\mathbf{X})$ onto $L^2(\hat{\mathbf{X}})$.

Using the same reasoning the related measure and metric as in (IV.1.7), the measure ν corresponds to the following metric in Γ.

$$|||\tilde{\mathbf{p}}|||_\Gamma = \left[\sum_{j=1}^{N} m_j^{-1}\tilde{\mathbf{p}}_j^2\right]^{1/2}.$$

■

COMPUTATION IV.1.2. *Proof of* (II.1.5).

PROOF.
We extend the definitions of Q_j and M_j in (II.1.6) and set $Q_N = Q$ and $M_N = M$.

The first part is motivated in Reed y Simon [36], pp. 78. and as a problem 52b. We give here the solution with our notation:

We consider the change of coordinates given by (II.1.4). Additionally, we set

$$\xi_N = -\left(\sum_{k=1}^{N} m_k\right)^{-1}\left(\sum_{k=1}^{M} m_k\tilde{\mathbf{x}}_k\right), \tag{IV.1.11}$$

and we write $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N)$ and $\xi = (\xi_1, \ldots, \xi_N)$.

Let

$$\begin{array}{rcl} g : \mathbb{R}^{nN} & \to & \mathbb{R}^{nN} \\ \tilde{\mathbf{x}} & \mapsto & \xi \end{array} \tag{IV.1.12}$$

be the map that transforms the $\tilde{\mathbf{x}}$ into the ξ coordinates given by (II.1.4) and (IV.1.11). It is not difficult to see that it is a linear function and its kernel is $\{0\}$, thus, g is an isomorphism (bijective map) because the dimensions of their domain and codomain are the same. Being g an isomorphism we have that g^{-1} is also an isomorphism, in particular it is a linear map, this linearity implies g is an homeomorphism with respect the usual topology in $\mathbb{R}^{nN}$. From our further calculations, its Jacobian is equal to 1. This establishes that the map $U : L^2(\mathbb{R}^{nN}) \to L^2(\mathbb{R}^{nN})$, given by $U\psi = \psi \circ g$ is unitary.

Let be f any twice continuously differentiable function in $\mathbb{R}^{nN}_\xi$. We want to find $\nabla_{\tilde{\mathbf{x}}_j} Uf$, for each $j = 1, \ldots N$. To do that we compute $\nabla_{\tilde{\mathbf{x}}} = (\nabla_{\tilde{\mathbf{x}}_1}, \ldots, \nabla_{\tilde{\mathbf{x}}_N})$. By the chain rule:

$$\begin{aligned} \nabla_{\tilde{\mathbf{x}}} Uf &= (\nabla_{\xi_1} f, \ldots, \nabla_{\xi_N} f)_{(1\times nN)} \begin{pmatrix} \nabla_{\tilde{\mathbf{x}}_1}\xi_1 & \ldots & \nabla_{\tilde{\mathbf{x}}_N}\xi_1 \\ \vdots & & \vdots \\ \nabla_{\tilde{\mathbf{x}}_1}\xi_N & \ldots & \nabla_{\tilde{\mathbf{x}}_N}\xi_N \end{pmatrix}_{(nN\times nN)} \\ &= \sum_{j=1}^{N}\left(\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_1}\nabla_{\xi_j} f, \ldots, \frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_N}\nabla_{\xi_j} f\right)_{(1\times nN)} \end{aligned}$$

because

$$\nabla_{\tilde{\mathbf{x}}_k}\xi_j = \begin{pmatrix} \nabla_{\tilde{\mathbf{x}}_k}(\xi_j\cdot\mathbf{e}_1) \\ \vdots \\ \nabla_{\tilde{\mathbf{x}}_k}(\xi_j\cdot\mathbf{e}_n) \end{pmatrix}_{(n\times n)} = \frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k} I_{n\times n},$$

where

$$\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k} = \begin{cases} 0 & \text{if } k > j+1, \\ 1 & \text{if } k = j+1, , \\ -\frac{m_k}{M_j} & \text{if } k \le j, \end{cases}$$

$\mathbf{e}_i, i = 1, \ldots, n$, the canonical vectors in $\mathbb{R}^n$, and I the identity matrix.

Thus, based on [74] page 240, the first part of (II.1.1) can be written as follows.

$$
\begin{aligned}
\sum_{k=1}^{N}(2m_k)^{-1}\tilde{\mathbf{p}}_k^2 &= -\sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_k)^{-1}\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k}\frac{\partial\xi_{j'}}{\partial\tilde{\mathbf{x}}_k}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&= -\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_1)^{-1}\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_1}\frac{\partial\xi_{j'}}{\partial\tilde{\mathbf{x}}_1}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&\quad -\sum_{k=2}^{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_k)^{-1}\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k}\frac{\partial\xi_{j'}}{\partial\tilde{\mathbf{x}}_k}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&= -\sum_{j=1}^{N}\sum_{j'=1}^{N}\frac{1}{2m_1}\frac{m_1}{M_j}\frac{m_1}{M_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&\quad -\sum_{k=2}^{N}\sum_{j=k-1}^{N}\frac{1}{2m_k}\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k}\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k}\nabla_{\xi_j}\cdot\nabla_{\xi_j} \\
&\quad -2\sum_{k=2}^{N}\sum_{j=k-1}^{N}\sum_{j'>j}^{N}\frac{1}{2m_k}\frac{\partial\xi_j}{\partial\tilde{\mathbf{x}}_k}\frac{\partial\xi_{j'}}{\partial\tilde{\mathbf{x}}_k}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&= -\sum_{j=1}^{N}\frac{m_1}{2M_j^2}\triangle_{\xi_j} - \sum_{j=1}^{N}\sum_{j'>j}^{N}\frac{m_1}{M_jM_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} - \sum_{k=2}^{N}\frac{1}{2m_k}\triangle_{\xi_{k-1}} \\
&\quad -\sum_{k=2}^{N}\sum_{j=k}^{N}\frac{m_k}{2M_j^2}\triangle_{\xi_j} + \sum_{k=2}^{N}\sum_{j'>k-1}^{N}\frac{1}{M_{j'}}\nabla_{\xi_{k-1}}\cdot\nabla_{\xi_{j'}} \\
&\quad -\sum_{k=2}^{N}\sum_{j=k}^{N}\sum_{j'>j}^{N}\frac{m_k}{M_jM_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&= -\sum_{k=1}^{N}\sum_{j=k}^{N}\sum_{j'>j}^{N}\frac{m_k}{M_jM_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} + \sum_{k=2}^{N}\sum_{j'>k-1}^{N}\frac{1}{M_{j'}}\nabla_{\xi_{k-1}}\cdot\nabla_{\xi_{j'}} \\
&\quad -\sum_{k=1}^{N}\sum_{j=k}^{N}\frac{m_k}{2M_j^2}\triangle_{\xi_j} - \sum_{k=2}^{N}\frac{1}{2m_k}\triangle_{\xi_{k-1}} \\
&= -\sum_{j=1}^{N-1}\sum_{j'>j}^{N}\sum_{k=1}^{j}\frac{m_k}{M_jM_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} + \sum_{j=1}^{N-1}\sum_{j'>j}^{N}\frac{1}{M_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&\quad -\sum_{j=1}^{N}\sum_{k=1}^{j}\frac{m_k}{2M_j^2}\triangle_{\xi_j} - \sum_{j=1}^{N-1}\frac{1}{2m_{j+1}}\triangle_{\xi_j} \\
&= -\sum_{j=1}^{N-1}\sum_{j'>j}^{N}\frac{1}{M_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} + \sum_{j=1}^{N-1}\sum_{j'>j}^{N}\frac{1}{M_{j'}}\nabla_{\xi_j}\cdot\nabla_{\xi_{j'}} \\
&\quad -\sum_{j=1}^{N}\frac{1}{2M_j}\triangle_{\xi_j} - \sum_{j=1}^{N-1}\frac{1}{2m_{j+1}}\triangle_{\xi_j} \\
&= -\frac{1}{2M}\triangle_{\xi_N} - \frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{1}{M_j}+\frac{1}{m_{j+1}}\right)\triangle_{\xi_j}
\end{aligned}
$$

It is more convenient to replace the variable ξ_N by the center of mass $\mathbf{X}_{CM}$. Because $\mathbf{X}_{CM} = -\xi_N$, we have:

$$(\mathbf{P}_{CM})^2 = -\triangle_{\xi_N}.$$

Hence, by (II.1.6),

$$\sum_{j=1}^{N-1}((2\nu_j)^{-1}\hat{\mathbf{p}}_j^2 = \sum_{k=1}^{N}(2m_k)^{-1}\tilde{\mathbf{p}}_k^2 - (2M)^{-1}(\mathbf{P}_{CM})^2. \tag{IV.1.13}$$

For the second part of (II.1.5):

$$\begin{aligned}
\sum_{j=1}^{N-1} q_j^R\mathbf{E}\cdot\xi_j &= \sum_{j=1}^{N-1}\left(\frac{q_{j+1}M_j - m_{j+1}Q_j}{M_{j+1}}\right)\mathbf{E}\cdot\xi_j \\
&= \sum_{j=1}^{N-1}\left(\frac{q_{j+1}M_j + q_{j+1}m_{j+1} - q_{j+1}m_{j+1} - m_{j+1}Q_j}{M_{j+1}}\right)\mathbf{E}\cdot\xi_j \\
&= \sum_{j=1}^{N-1}\left(\frac{q_{j+1}M_{j+1} - m_{j+1}Q_{j+1}}{M_{j+1}}\right)\mathbf{E}\cdot\xi_j \\
&= \sum_{j=2}^{N}\left(q_j - \frac{m_jQ_j}{M_j}\right)\mathbf{E}\cdot\xi_{j-1} \\
&= \sum_{j=2}^{N}\left(q_j - \frac{m_jQ_j}{M_j}\right)\mathbf{E}\cdot\left[\tilde{\mathbf{x}}_j - \left(\frac{1}{M_{j-1}}\right)\sum_{k=1}^{j-1}m_k\tilde{\mathbf{x}}_k\right] \\
&= \sum_{j=2}^{N}\left(q_j - \frac{m_jQ_j}{M_j}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j - \sum_{j=2}^{N}\sum_{k=1}^{j-1}\left(q_j - \frac{m_jQ_j}{M_j}\right)\left(\frac{1}{M_{j-1}}\right)m_k\mathbf{E}\cdot\tilde{\mathbf{x}}_k \\
&= \sum_{j=2}^{N}\left(q_j - \frac{m_jQ}{M} + \frac{m_jQ}{M} - \frac{m_jQ_j}{M_j}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j \\
&\quad -\sum_{j=2}^{N}\sum_{k=1}^{j-1}\left(q_j - \frac{m_jQ_j}{M_j}\right)\left(\frac{1}{M_{j-1}}\right)m_k\mathbf{E}\cdot\tilde{\mathbf{x}}_k \\
&= \sum_{j=2}^{N}\left(q_j - \frac{m_jQ}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j + \sum_{j=2}^{N}m_j\left(\frac{Q}{M} - \frac{Q_j}{M_j}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j \\
&\quad -\sum_{k=1}^{N-1}\sum_{j=k+1}^{N}\left(q_j - \frac{m_jQ_j}{M_j}\right)\left(\frac{1}{M_{j-1}}\right)m_k\mathbf{E}\cdot\tilde{\mathbf{x}}_k \\
&= \sum_{j=1}^{N}\left(q_j - \frac{m_jQ}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j + \sum_{j=1}^{N}m_j\left(\frac{Q}{M} - \frac{Q_j}{M_j}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j \\
&\quad -\sum_{j=1}^{N-1}m_j\mathbf{E}\cdot\tilde{\mathbf{x}}_j\sum_{k=j+1}^{N}\left(q_k - \frac{m_kQ_k}{M_k}\right)\left(\frac{1}{M_{k-1}}\right) \\
&= \sum_{j=1}^{N}\left(q_j - \frac{m_jQ}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j \\
&\quad +\sum_{j=1}^{N-1}m_j\mathbf{E}\cdot\tilde{\mathbf{x}}_j\left[\frac{Q}{M} - \frac{Q_j}{M_j} - \sum_{k=j+1}^{N}\left(q_k - \frac{m_kQ_k}{M_k}\right)\left(\frac{1}{M_{k-1}}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N}\left(q_j - \frac{m_j Q}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j \\
&\quad + \sum_{j=1}^{N-1} m_j \mathbf{E}\cdot\tilde{\mathbf{x}}_j \left[\frac{Q}{M} - \frac{Q_j}{M_j} - \sum_{k=j+1}^{N}\left(Q_k - Q_{k-1} - \frac{m_k Q_k}{M_k}\right)\frac{1}{M_{k-1}}\right] \\
&= \sum_{j=1}^{N}\left(q_j - \frac{m_j Q}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j \\
&\quad + \sum_{j=1}^{N-1} m_j \mathbf{E}\cdot\tilde{\mathbf{x}}_j \left[\frac{Q}{M} - \frac{Q_j}{M_j} - \sum_{k=j+1}^{N}\left(1 - \frac{m_k}{M_k}\right)\frac{Q_k}{M_{k-1}} + \sum_{k=j+1}^{N}\frac{Q_{k-1}}{M_{k-1}}\right] \\
&= \sum_{j=1}^{N}\left(q_j - \frac{m_j Q}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j + \sum_{j=1}^{N-1} m_j \mathbf{E}\cdot\tilde{\mathbf{x}}_j \left[\frac{Q}{M} - \sum_{k=j+1}^{N}\frac{M_{k-1}}{M_k}\frac{Q_k}{M_{k-1}} + \sum_{k=j+2}^{N}\frac{Q_{k-1}}{M_{k-1}}\right] \\
&= \sum_{j=1}^{N}\left(q_j - \frac{m_j Q}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j + \sum_{j=1}^{N-1} m_j \mathbf{E}\cdot\tilde{\mathbf{x}}_j \left[\frac{Q}{M} - \sum_{k=j+1}^{N}\frac{Q_k}{M_k} + \sum_{k=j+2}^{N}\frac{Q_{k-1}}{M_{k-1}}\right] \\
&= \sum_{j=1}^{N}\left(q_j - \frac{m_j Q}{M}\right)\mathbf{E}\cdot\tilde{\mathbf{x}}_j + \sum_{j=1}^{N-1} m_j \mathbf{E}\cdot\tilde{\mathbf{x}}_j \left[-\sum_{k=j+1}^{N-1}\frac{Q_k}{M_k} + \sum_{k=j+1}^{N-1}\frac{Q_k}{M_k}\right] \\
&= \sum_{j=1}^{N} q_j \mathbf{E}\cdot\tilde{\mathbf{x}}_j - Q\mathbf{E}\cdot\sum_{j=1}^{N}\frac{m_j}{M}\tilde{\mathbf{x}}_j.
\end{aligned}
$$

Therefore

$$
\sum_{j=1}^{N-1} q_j^R \mathbf{E}\cdot\xi_j = \sum_{j=1}^{N}(q_j - m_j Q/M)\mathbf{E}\cdot\tilde{\mathbf{x}}_j. \tag{IV.1.14}
$$

By (IV.1.13) and (IV.1.14) we obtain (II.1.5).

■

PROPOSITION IV.1.3. *The passage from the $\tilde{x}_j$ to x_j variables.*

PROOF.

We turn our attention, now, to the change from the variable with ˜ to the ones without. Let us prove, using (II.1.9) and (II.1.10) that

$$
\mathbf{X} \cong \mathbf{X}_{12} := \left\{(\mathbf{x}, \mathbf{x}_3, \ldots, \mathbf{x}_N) \in \mathbb{R}^{n(N-1)}\right\},
$$

We consider the change of coordinates given by (II.1.9) and (II.1.10).

Let

$$
\begin{aligned}
h : \mathbf{X} &\rightarrow \mathbb{R}^{n(N-1)} \\
\tilde{\mathbf{x}} &\mapsto (\mathbf{x}, \mathbf{x}_3, \ldots, \mathbf{x}_N)
\end{aligned} \tag{IV.1.15}
$$

be the map that transforms the $\tilde{\mathbf{x}}$ in $\mathbf{X}$ into the $\{\mathbf{x}_j\}_{3\le j\le N+1}$ coordinates given by (II.1.9) and (II.1.10). It is not difficult to see that it is a linear function. It is one to one because, if $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbf{X}$ such that, for all $3 \le j \le N$,

$$
\begin{aligned}
\tilde{\mathbf{y}}_2 - \tilde{\mathbf{y}}_1 &= \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \\
\tilde{\mathbf{y}}_j - \frac{m_1\tilde{\mathbf{y}}_1 + m_2\tilde{\mathbf{y}}_2}{m_1 + m_2} &= \tilde{\mathbf{x}}_j - \frac{m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2}{m_1 + m_2}.
\end{aligned}
$$

Then

$$\tilde{\mathbf{y}}_2 - \tilde{\mathbf{x}}_2 = \tilde{\mathbf{y}}_1 - \tilde{\mathbf{x}}_1,$$

and, for all $3 \leq j \leq N$,

$$\begin{aligned}
\tilde{\mathbf{y}}_j - \tilde{\mathbf{x}}_j &= \frac{m_1\tilde{\mathbf{y}}_1 + m_2\tilde{\mathbf{y}}_2}{m_1+m_2} - \frac{m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2}{m_1+m_2} \\
&= \frac{m_1(\tilde{\mathbf{y}}_1 - \tilde{\mathbf{x}}_1) + m_2(\tilde{\mathbf{y}}_2 - \tilde{\mathbf{x}}_2)}{m_1+m_2} \\
&= \tilde{\mathbf{y}}_1 - \tilde{\mathbf{x}}_1.
\end{aligned}$$

The condition $\sum_{j=1}^{N} m_j(\tilde{\mathbf{y}}_j - \tilde{\mathbf{x}}_j) = 0$ implies that $\tilde{\mathbf{y}}_j - \tilde{\mathbf{x}}_j = 0$, for all $1 \leq j \leq N$.

Therefore, h is an isomorphism (bijective map) because the dimensions of their domain and codomain are the same and it is injective. Being h an isomorphism we have that h^{-1} is also an isomorphism, in particular it is a linear map, this linearity implies h is an homeomorphism with respect the usual topology in $\mathbb{R}^{n(N-1)}$.

To compute the Hamiltonian in the x_j variables we have to define another mapping g that can be seen as an extension of h given in (IV.1.15). To take advantage of the similarity between the ξ_j and the $\mathbf{x}_j$ coordinates we introduce a new set of variables:

$$\begin{aligned}
\mathbf{z}_1 &= \mathbf{x}, \\
\mathbf{z}_j &= \mathbf{x}_{j+1},\ 2 \leq j \leq N-1, \\
\mathbf{z}_N &= \xi_N,
\end{aligned}$$

where $\mathbf{x}$, $\{\mathbf{x}_j\}_{3\leq j\leq N}$ are the coordinates given by (II.1.9), (II.1.10) and ξ_N is defined in (IV.1.11). The mapping g is defined as follows:

$$\begin{aligned}
g : \mathbb{R}^{nN} &\rightarrow \mathbb{R}^{nN} \\
\tilde{\mathbf{x}} &\mapsto (\mathbf{z}_1, \ldots, \mathbf{z}_N)
\end{aligned} \tag{IV.1.16}$$

For the same reasons as h, g is an homeomorphism with respect the usual topology from $\mathbb{R}^{nN}$ to itself. By means of the map $U : L^2(\mathbb{R}^{nN}) \rightarrow L^2(\mathbb{R}^{nN})$, given by $U\psi = \psi \circ g$ we make the change of variables. Because g is linear, we can take U as a unitary map by adjusting the measure in the copy of $\mathbb{R}^{nN}$ considered in the codomain of U. It is important that U is unitary in order to preserve probability when considering quantum states, i.e., the change of variables should not change the probability of finding the particle in the Universe.

Let be f any twice continuously differentiable function in $\mathbb{R}^{nN}$. We want to find $\nabla_{\tilde{\mathbf{x}}_j} Uf$, for each $j = 1, \ldots N$. To do that we compute $\nabla_{\tilde{\mathbf{x}}} = (\nabla_{\tilde{\mathbf{x}}_1}, \ldots, \nabla_{\tilde{\mathbf{x}}_N})$. By the chain rule:

$$\begin{aligned}
\nabla_{\tilde{\mathbf{x}}} Uf &= (\nabla_{\mathbf{z}_1} f, \ldots, \nabla_{\mathbf{z}_N} f)_{(1\times nN)} \begin{pmatrix} D_{\tilde{\mathbf{x}}_1}\mathbf{z}_1 & \ldots & D_{\tilde{\mathbf{x}}_N}\mathbf{z}_1 \\ \vdots & & \vdots \\ D_{\tilde{\mathbf{x}}_1}\mathbf{z}_N & \ldots & D_{\tilde{\mathbf{x}}_N}\mathbf{z}_N \end{pmatrix}_{(nN\times nN)} \\
&= \sum_{j=1}^{N} \left(\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1} \nabla_{\mathbf{z}_j} f, \ldots, \frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_N} \nabla_{\mathbf{z}_j} f \right)_{(1\times nN)}
\end{aligned}$$

where

$$D_{\tilde{\mathbf{x}}_k}\mathbf{z}_j = \begin{pmatrix} \nabla_{\tilde{\mathbf{x}}_k}(\mathbf{z}_j \cdot \mathbf{e}_1) \\ \vdots \\ \nabla_{\tilde{\mathbf{x}}_k}(\mathbf{z}_j \cdot \mathbf{e}_n) \end{pmatrix}_{(n\times n)} = \frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k} I_{n\times n},$$

and

$$\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k} = \begin{cases} -1 & \text{if } k=1,\ j=1, \\ -\frac{m_k}{m_1+m_2} & \text{if } 1 \le k \le 2,\ 2 \le j \le N-1, \\ 1 & \text{if } k=j+1,\ 1 \le j \le N-1, \\ 0 & \text{if } k \neq j+1,\ 3 \le k \le N,\ 1 \le j \le N-1, \\ -\frac{m_k}{M} & \text{if } j=N, \end{cases}$$

$\mathbf{e}_i, i=1,\ldots,n$, the canonical vectors in $\mathbb{R}^n$, and I the identity matrix.

Then the free Hamiltonian without considering the electric field in (II.1.1) is

$$\begin{aligned}
\sum_{k=1}^{N} \frac{\tilde{\mathbf{p}}_k^2}{2m_k} &= -\sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_k)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_k}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&= -\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_1)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&\quad -\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_2)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&\quad -\sum_{k=3}^{N}\sum_{j=1}^{N}\sum_{j'=1}^{N}(2m_k)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_k}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&= -\sum_{j=1}^{N}(2m_1)^{-1}\left(\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1}\right)^2\triangle_{\mathbf{z}_j} - 2\sum_{j=1}^{N-1}\sum_{j'>j}^{N}(2m_1)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&\quad -\sum_{j=1}^{N}(2m_2)^{-1}\left(\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\right)^2\triangle_{\mathbf{z}_j} - 2\sum_{j=1}^{N-1}\sum_{j'>j}^{N}(2m_2)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&\quad -\sum_{k=3}^{N}\sum_{j=1}^{N}(2m_k)^{-1}\left(\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\right)^2\triangle_{\mathbf{z}_j} - 2\sum_{k=3}^{N}\sum_{j=1}^{N-1}\sum_{j'>j}^{N}(2m_k)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_k}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
&= -(2m_1)^{-1}\left(\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_1}\right)^2\triangle_{\mathbf{z}_1} - \sum_{j=2}^{N-1}(2m_1)^{-1}\left(\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1}\right)^2\triangle_{\mathbf{z}_j} - (2m_1)^{-1}\left(\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_1}\right)^2\triangle_{\mathbf{z}_N} \\
&\quad -2\sum_{j'=2}^{N-1}(2m_1)^{-1}\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_1}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_1}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}} - 2(2m_1)^{-1}\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_1}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_1}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N} \\
&\quad -2\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}(2m_1)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - 2\sum_{j=2}^{N-1}(2m_1)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_1}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N} \\
&\quad -(2m_2)^{-1}\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_2}\triangle_{\mathbf{z}_1} - \sum_{j=2}^{N-1}(2m_2)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\triangle_{\mathbf{z}_j} - (2m_2)^{-1}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_2}\triangle_{\mathbf{z}_N} \\
&\quad -2\sum_{j'=2}^{N-1}(2m_2)^{-1}\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_2}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}} - 2(2m_2)^{-1}\frac{\partial \mathbf{z}_1}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_2}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N} \\
&\quad -2\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}(2m_2)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - 2\sum_{j=2}^{N-1}(2m_2)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_2}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{k=3}^{N}\sum_{j=1}^{N-1}(2m_k)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\triangle_{\mathbf{z}_j} - \sum_{k=3}^{N}(2m_k)^{-1}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_k}\triangle_{\mathbf{z}_N} \\
& -2\sum_{k=3}^{N}\sum_{j=1}^{N-2}\sum_{j'>j}^{N-1}(2m_k)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_{j'}}{\partial \tilde{\mathbf{x}}_k}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - 2\sum_{k=3}^{N}\sum_{j=1}^{N-1}(2m_k)^{-1}\frac{\partial \mathbf{z}_j}{\partial \tilde{\mathbf{x}}_k}\frac{\partial \mathbf{z}_N}{\partial \tilde{\mathbf{x}}_k}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N} \\
= \; & -(2m_1)^{-1}\triangle_{\mathbf{z}_1} - \sum_{j=2}^{N-1}\frac{m_1}{2(m_1+m_2)^2}\triangle_{\mathbf{z}_j} - \frac{m_1}{2M^2}\triangle_{\mathbf{z}_N} \\
& -\sum_{j'=2}^{N-1}\frac{1}{m_1+m_2}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}} - \frac{1}{M}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N} \\
& -\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}\frac{m_1}{(m_1+m_2)^2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - \sum_{j=2}^{N-1}\frac{1}{m_1+m_2}\frac{m_1}{M}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N} \\
& -(2m_2)^{-1}\triangle_{\mathbf{z}_1} - \sum_{j=2}^{N-1}\frac{m_2}{2(m_1+m_2)^2}\triangle_{\mathbf{z}_j} - \frac{m_2}{2M^2}\triangle_{\mathbf{z}_N} \\
& +\sum_{j'=2}^{N-1}\frac{1}{m_1+m_2}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}} + \frac{1}{M}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N} \\
& -\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}\frac{m_2}{(m_1+m_2)^2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - \sum_{j=2}^{N-1}\frac{m_2}{(m_1+m_2)M}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N} \\
& -\sum_{k=3}^{N}\frac{1}{2m_k}\triangle_{\mathbf{z}_{k-1}} - \sum_{k=3}^{N}\frac{m_k}{2M^2}\triangle_{\mathbf{z}_N} \\
& -2\sum_{k=3}^{N}\sum_{j=1}^{N-2}\sum_{j'>j}^{N-1}(2m_k)^{-1}(0)\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} \\
& -2\sum_{k=3}^{N}(2m_k)^{-1}(1)\frac{-m_k}{M}\nabla_{\mathbf{z}_{k-1}}\cdot\nabla_{\mathbf{z}_N} \\
= \; & -(2m_1)^{-1}\triangle_{\mathbf{z}_1} - \sum_{j=2}^{N-1}\frac{m_1}{2(m_1+m_2)^2}\triangle_{\mathbf{z}_j} - \frac{m_1}{2M^2}\triangle_{\mathbf{z}_N} \\
& -\sum_{j'=2}^{N-1}\frac{1}{m_1+m_2}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}} - \frac{1}{M}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N} \\
& -\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}\frac{m_1}{(m_1+m_2)^2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - \sum_{j=2}^{N-1}\frac{1}{m_1+m_2}\frac{m_1}{M}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N} \\
& -(2m_2)^{-1}\triangle_{\mathbf{z}_1} - \sum_{j=2}^{N-1}\frac{m_2}{2(m_1+m_2)^2}\triangle_{\mathbf{z}_j} - \frac{m_2}{2M^2}\triangle_{\mathbf{z}_N} \\
& +\sum_{j'=2}^{N-1}\frac{1}{m_1+m_2}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}} + \frac{1}{M}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N} \\
& -\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}\frac{m_2}{(m_1+m_2)^2}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}} - \sum_{j=2}^{N-1}\frac{m_2}{(m_1+m_2)M}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N}
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{j=2}^{N-1}\frac{1}{2m_{j+1}}\triangle_{\mathbf{z}_j}-\sum_{k=3}^{N}\frac{m_k}{2M^2}\triangle_{\mathbf{z}_N}+\sum_{j=2}^{N-1}\frac{1}{M}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N} \qquad \text{(IV.1.17)}\\
= & -\frac{1}{2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\triangle_{\mathbf{z}_1}-\sum_{j=2}^{N-1}\frac{1}{2}\left[\frac{1}{m_{j+1}}+\frac{m_1}{(m_1+m_2)^2}+\frac{m_2}{(m_1+m_2)^2}\right]\triangle_{\mathbf{z}_j}\\
& -\frac{1}{2}\left[\frac{m_1}{M^2}+\frac{m_2}{M^2}+\frac{\sum_3^N m_k}{M^2}\right]\triangle_{\mathbf{z}_N}-\left[\frac{1}{M}-\frac{1}{M}\right]\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_N}\\
& -\left[\frac{m_1}{(m_1+m_2)^2}+\frac{m_2}{(m_1+m_2)^2}\right]\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}}\\
& -\left[\frac{m_1}{(m_1+m_2)M}+\frac{m_2}{(m_1+m_2)M}-\frac{1}{M}\right]\sum_{j=2}^{N-1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_N}\\
& -\left[\frac{1}{m_1+m_2}-\frac{1}{m_1+m_2}\right]\sum_{j'=2}^{N-1}\nabla_{\mathbf{z}_1}\cdot\nabla_{\mathbf{z}_{j'}}\\
= & -\frac{1}{2\mu_{12}}\triangle_{\mathbf{z}_1}-\sum_{j=2}^{N-1}\frac{1}{2\mu_{j+1}}\triangle_{\mathbf{z}_j}-\frac{1}{2M}\triangle_{\mathbf{z}_N}-\frac{1}{(m_1+m_2)}\sum_{j=2}^{N-2}\sum_{j'>j}^{N-1}\nabla_{\mathbf{z}_j}\cdot\nabla_{\mathbf{z}_{j'}}\\
= & -\frac{1}{2\mu_{12}}\triangle_{\mathbf{x}}-\sum_{j=3}^{N}\frac{1}{2\mu_j}\triangle_{\mathbf{x}_j}-\frac{1}{2M}\triangle_{\xi_N}-\frac{1}{(m_1+m_2)}\sum_{j=3}^{N-1}\sum_{j'>j}^{N}\nabla_{\mathbf{x}_j}\cdot\nabla_{\mathbf{x}_{j'}} \qquad \text{(IV.1.18)}
\end{aligned}
$$

It is important to remark that in (IV.1.18), there are neither $\nabla_{\mathbf{x}}\cdot\nabla_{\xi_N}$ nor $\nabla_{\mathbf{x}_j}\cdot\nabla_{\xi_N}$ terms, as Reed and Simon's book [36] requires in the section named "Quantum scattering II: N-Body case".

To change variables electric in the field part of in (II.1.1), we perform the following computations:

$$
\begin{aligned}
\sum_{j=1}^{N} q_j\tilde{\mathbf{x}}_j-Q\mathbf{X}_{CM} &= \sum_{j=1}^{N} q_j\tilde{\mathbf{x}}_j-\sum_{k=1}^{N} q_k\frac{1}{M}\sum_{j=1}^{N} m_j\tilde{\mathbf{x}}_j=\sum_{j=1}^{N}\left(q_j-\frac{Qm_j}{M}\right)\tilde{\mathbf{x}}_j\\
&= \sum_{j=1}^{N}\left(q_j-\frac{Qm_j}{M}\right)\left(\tilde{\mathbf{x}}_j-\frac{m_1\tilde{\mathbf{x}}_1+m_2\tilde{\mathbf{x}}_2}{m_1+m_2}\right)\\
&\quad+\left(\sum_{j=1}^{N} q_j-\frac{Q}{M}\sum_{j=1}^{N} m_j\right)\frac{m_1\tilde{\mathbf{x}}_1+m_2\tilde{\mathbf{x}}_2}{m_1+m_2}\\
&= \left(q_1-\frac{Qm_1}{M}\right)\frac{-m_2}{m_1+m_2}\left(\tilde{\mathbf{x}}_2-\tilde{\mathbf{x}}_1\right)+\left(q_2-\frac{Qm_2}{M}\right)\frac{m_1}{m_1+m_2}\left(\tilde{\mathbf{x}}_2-\tilde{\mathbf{x}}_1\right)\\
&\quad+\sum_{j=3}^{N}\left(q_j-\frac{Qm_j}{M}\right)\mathbf{x}_j\\
&= \left(-q_1m_2+\frac{Qm_1m_2}{M}+q_2m_1-\frac{Qm_1m_2}{M}\right)\frac{\mathbf{x}}{m_1+m_2}+\sum_{j=3}^{N}\left(q_j-\frac{Qm_j}{M}\right)\mathbf{x}_j\\
&= \frac{q_2m_1-q_1m_2}{m_1+m_2}\mathbf{x}+\sum_{j=3}^{N}\left(q_j-\frac{Qm_j}{M}\right)\mathbf{x}_j \qquad \text{(IV.1.19)}
\end{aligned}
$$

■

PROPOSITION IV.1.4. (II.1.9)-(II.1.22) *are true.*

Proof.

We recall that $\mathbf{p} = -i\nabla_x$. Besides, $\mathbf{p}_j = -i\nabla_{x_j}$ must be true because p and x are conjugate variables and so they are, p_j and x_j. By (IV.1.18) and a direct computation we have that:

$$\begin{aligned}\mathbf{p} &= 2\mu\, i[H_0,\, \mathbf{x}], && \text{(IV.1.20)}\\ \mathbf{p}_j &= 2\mu_j\, i[H_0,\, \mathbf{x}_j],\ j = 1,\ldots,N. && \text{(IV.1.21)}\end{aligned}$$

By (IV.1.28) and (IV.1.29) below, we can obtain (IV.1.21) for $j = 1, 2$. In general, we can construct momentum operators by equations similar to (IV.1.20) and (IV.1.21). For more details, please check Berg [62].

Similarly, because $\tilde{\mathbf{p}}_j = -i\nabla_{\tilde{x}_j}$, we have that

$$\tilde{\mathbf{p}}_j = 2m_j\, i[H_0,\, \tilde{\mathbf{x}}_j],\ j = 1,\ldots,N. \tag{IV.1.22}$$

Therefore, we have (II.1.9) by substituting $\mathbf{x} := \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1$ and (IV.1.22) into (IV.1.20). Equation (II.1.10) is true by definition, if we substitute (II.1.10) and (IV.1.22) into (IV.1.21) we obtain (II.1.11).

Considering that $\mathbf{p}_j/\mu_j$ is the velocity operator of the jth particle with respect to the center of mass of the pair $(1, 2)$, $\tilde{\mathbf{p}}_j/m_j$ is the velocity operator of the jth particle and $(\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2)/(m_1 + m_2)$ is the velocity operator of the pair $(1, 2)$, we obtain (II.1.11). Moreover, $\mathbf{x}_j, \mathbf{p}_j$, $j = 3,\ldots,N$, $\mathbf{x}$ and $\mathbf{p}$ satisfy commutation relations. To see that, and without loss of generality, let us assume that $n = 1$,

$$\begin{aligned}[\mathbf{x}_j,\, \mathbf{p}_j] &= [\tilde{\mathbf{x}}_j - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1+m_2),\, \mu_j(\tilde{\mathbf{p}}_j/m_j - (\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2)/(m_1+m_2))]\\ &= \frac{\mu_j}{m_j}[\tilde{\mathbf{x}}_j,\, \tilde{\mathbf{p}}_j] - \frac{\mu_j}{m_1+m_2}\left([\tilde{\mathbf{x}}_j,\, \tilde{\mathbf{p}}_1] + [\tilde{\mathbf{x}}_j,\, \tilde{\mathbf{p}}_2]\right) - \frac{\mu_j}{m_j(m_1+m_2)}\left(m_1[\tilde{\mathbf{x}}_1,\, \tilde{\mathbf{p}}_j] + m_2[\tilde{\mathbf{x}}_2,\, \tilde{\mathbf{p}}_j]\right)\\ &\quad + \frac{\mu_j m_1}{(m_1+m_2)^2}\left([\tilde{\mathbf{x}}_1,\, \tilde{\mathbf{p}}_1] + [\tilde{\mathbf{x}}_1,\, \tilde{\mathbf{p}}_2]\right) + \frac{\mu_j m_2}{(m_1+m_2)^2}\left([\tilde{\mathbf{x}}_2,\, \tilde{\mathbf{p}}_1] + [\tilde{\mathbf{x}}_2,\, \tilde{\mathbf{p}}_2]\right)\\ &= \frac{m_1+m_2}{m_1+m_2+m_j} i\delta_{jj} - \frac{m_j}{m_1+m_2+m_j} i\left(\delta_{1j} + \delta_{2j}\right) - \frac{1}{m_1+m_2+m_j} i\left(m_1\delta_{1j} + m_2\delta_{2j}\right)\\ &\quad + \frac{m_j m_1}{(m_1+m_2+m_j)(m_1+m_2)}(i) + \frac{m_j m_2}{(m_1+m_2+m_j)(m_1+m_2)}(i)\\ &= i\left(1 - \frac{(m_1+m_j)\delta_{1j} + (m_2+m_j)\delta_{2j}}{m_1+m_2+m_j}\right)\\ &= i\begin{cases}1, & \text{if } j = 3,\ldots,N,\\ \frac{m_2}{m_1+m_2+m_j}, & \text{if } j = 1,\\ \frac{m_1}{m_1+m_2+m_j}, & \text{if } j = 2.\end{cases}\end{aligned}$$

$$\begin{aligned}[\mathbf{x},\, \mathbf{p}] &= [\tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1,\, \mu_{12}(\tilde{\mathbf{p}}_2/m_2 - \tilde{\mathbf{p}}_1/m_1)]\\ &= \frac{\mu_{12}}{m_2}[\tilde{\mathbf{x}}_2,\, \tilde{\mathbf{p}}_2] - \frac{\mu_{12}}{m_1}[\tilde{\mathbf{x}}_2,\, \tilde{\mathbf{p}}_1] - \frac{\mu_{12}}{m_2}[\tilde{\mathbf{x}}_1,\, \tilde{\mathbf{p}}_2] + \frac{\mu_{12}}{m_1}[\tilde{\mathbf{x}}_1,\, \tilde{\mathbf{p}}_1] = \frac{\mu_{12}}{m_2} + \frac{\mu_{12}}{m_1} = 1.\end{aligned}$$

If $3 \le j, k \le N$, $j \ne k$ then $\mathbf{x}_j$ and $\mathbf{p}_k$ are independent variables, so are $\mathbf{x}$ and $\mathbf{p}_j$, and $\mathbf{x}_j$ and $\mathbf{p}$. In summary $\{\mathbf{x}, \mathbf{x}_3, \ldots \mathbf{x}_N\}$ and $\{\mathbf{p}, \mathbf{p}_3, \ldots \mathbf{p}_N\}$ satisfy canonical commutation relations. Please also see Enss [16] equations 3.2 and 3.3, and equations 393-395 in Dollard [66].

Now, let us see why $\{\mathbf{p}, \mathbf{p}_3, \ldots \mathbf{p}_N\}$ is linearly independent is a set of $N - 1$ independent n-dimensional variables relative to the center of mass frame. We identify the n-dimensional variables $\tilde{\mathbf{p}}_j$, $j = 1,\ldots,N$ as variables in a $(N - 1)n$ dimensional space as follows

$$\tilde{\mathbf{p}}_1 \sim \sum_{l=1}^{n} \tilde{p}_{1,l}\mathbf{e}_l, \quad \tilde{\mathbf{p}}_j \sim \sum_{l=1}^{n} \tilde{p}_{j,l}\, \mathbf{e}_{(j-2)n+l}, \quad j = 2,\ldots,N, \tag{IV.1.23}$$

where $\tilde{\mathbf{p}}_k = (\tilde{p}_{k,1}, \ldots, \tilde{p}_{k,n}) \in \mathbb{R}^n$, $k = 1, \ldots, N$, and $\mathbf{e}_{(j-2)n+l}$, $l = 1, \ldots, n$, $j = 2, \ldots, N$, are the canonical vectors in $\mathbb{R}^{(N-1)n}$. On the other hand, we have

$$\begin{aligned}
\mathbf{p} &= \mu_{12}\left(m_2^{-1}\tilde{\mathbf{p}}_2 + m_1^{-1}\sum_{j=2}^{N}\tilde{\mathbf{p}}_j\right) = \mu_{12}\left(\left(m_1^{-1} + m_2^{-1}\right)\tilde{\mathbf{p}}_2 + m_1^{-1}\sum_{j=3}^{N}\tilde{\mathbf{p}}_j\right) \\
&= \tilde{\mathbf{p}}_2 + \frac{m_2}{m_1+m_2}\sum_{j=3}^{N}\tilde{\mathbf{p}}_j. \qquad \text{(IV.1.24)}
\end{aligned}$$

From (II.1.11), (IV.1.23) and (IV.1.24) we get the wanted linearly independence. To prove that $\{\mathbf{x}, \mathbf{x}_3, \ldots \mathbf{x}_N\}$ is linearly independent we use (II.1.11).

For the proof of (II.1.14), we note that $\mathbf{v}_1 = -\mathbf{v}\mu_{12}/m_1$ and $\mathbf{v}_2 = \mathbf{v}\mu_{12}/m_2$ mean that in the center of mass frame of particles 1 and 2, these particles have momenta equal in magnitude and are in exactly opposite directions.

We commence with $\mathbf{v}_{1,j}$, for $j = 3, 4, \ldots, N$:

$$\begin{aligned}
\mathbf{v}_{1,j} &= \mathbf{v}_j - \mathbf{v}_1 = \mathbf{d}_j v^2 - (-\hat{\mathbf{v}} v \mu_{12}/m_1) = v^2\left(\mathbf{d}_j + \hat{\mathbf{v}}\mu_{12}/(m_1 v)\right) \\
&= v^2\left(\mathbf{d}_j + \hat{\mathbf{v}}\mu_{12}/(m_1 v)\right).
\end{aligned}$$

Similarly, for $j = 3, 4, \ldots, N$:

$$\mathbf{v}_{2,j} = v^2\left(\mathbf{d}_j - \hat{\mathbf{v}}\mu_{12}/(m_2 v)\right).$$

If $(\mathbf{d}_j + \hat{\mathbf{v}}\mu_{12}/(m_1 v)) = 0$, then $|\mathbf{d}_j| = |\hat{\mathbf{v}}\mu_{12}/(m_1 v)|$, i.e. $v = \mu_{12}/(m_1 d_j)$. Thereby, if we ask $v > \mu_{12}/(m_1 d_j)$ we have $\mathbf{v}_{1,j} \neq 0$. Likewise, if $v > \mu_{12}/(m_2 d_j)$ then $\mathbf{v}_{2,j} \neq 0$.

As a consequence we have that $v_{jk} = O(v^2)$, for $1 \leq j < k$ and $3 \leq k \leq N$. ■

PROPOSITION IV.1.5. *Inequality* (II.1.18) *is true.*

Proof. First, we use (II.1.10)

$$\begin{aligned}
|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j| &= \left|\tilde{\mathbf{x}}_k - \frac{1}{m_1+m_2}(m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2) - \left[\tilde{\mathbf{x}}_j - \frac{1}{m_1+m_2}(m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)\right]\right| \\
&= |\mathbf{x}_k - \mathbf{x}_j| \leq |\mathbf{x}_k| + |\mathbf{x}_j|, \quad j, k = 1, \cdots, N,
\end{aligned}$$

and

$$\begin{aligned}
|\mathbf{x}_1| &= \left|\tilde{\mathbf{x}}_1 - \frac{1}{m_1+m_2}(m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)\right| = \frac{1}{m_1+m_2}\left|m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_1 - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)\right| \\
&= \frac{m_2}{m_1+m_2}|\mathbf{x}| \leq |\mathbf{x}|, \\
|\mathbf{x}_2| &= \left|\tilde{\mathbf{x}}_2 - \frac{1}{m_1+m_2}(m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)\right| = \frac{1}{m_1+m_2}\left|m_1\tilde{\mathbf{x}}_2 + m_2\tilde{\mathbf{x}}_2 - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)\right| \\
&= \frac{m_1}{m_1+m_2}|\mathbf{x}| \leq |\mathbf{x}|.
\end{aligned}$$

At this moment, we estimate the left hand side of (II.1.18). Because $\phi_0 \in \mathcal{S}(\mathbb{R}^{n(N-1)})$, we have for any $1 \leq j, < k \leq N$:

$$\begin{aligned}
\left\|(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^2\Phi_{\mathbf{v}}\right\| &\leq \begin{cases}
\left\|\left(1 + (|\mathbf{x}_k| + |\mathbf{x}_j|)^2\right)^2 \phi_{\mathbf{v}}(\mathbf{x}, \mathbf{x}_3, \ldots, \mathbf{x}_N)\right\|, & \text{if } j, k = 3, \ldots, N, \\
\left\|\left(1 + (|\mathbf{x}_k| + |\mathbf{x}|)^2\right)^2 \phi_{\mathbf{v}}(\mathbf{x}, \mathbf{x}_3, \ldots, \mathbf{x}_N)\right\|, & \text{if } j = 1, 2 \text{ and } k = 3, \ldots, N, \\
\left\|\left(1 + |\mathbf{x}|^2\right)^2 \phi_{\mathbf{v}}(\mathbf{x}, \mathbf{x}_3, \ldots, \mathbf{x}_N)\right\|, & \text{if } j = 1 \text{ and } k = 2.
\end{cases} \\
&\leq C_{jk} \\
&\leq C.
\end{aligned}$$

We can think of $\mathbf{p}_{jk}$, the momentum as an operator of the kth particle with respect to the jth one, as being a multiplication operator in the momentum space and as a derivative, with respect the relative coordinate of the kth particle with respect to the jth one, in the configuration space. This justifies (II.1.15). The relative velocities do not depend on the frame of reference chosen to compute them, that is why (II.1.16) should be true. To prove (II.1.16), by an easy permutation of the change or variables, where now $x := \tilde{x}_k - \tilde{x}_j$, (IV.1.18) can be written as:

$$\sum_{k'=1}^{N} \frac{\tilde{\mathbf{p}}_{k'}^2}{2m_{k'}} = -\frac{1}{2\mu_{jk}} \triangle_{\mathbf{x}} - \sum_{1\le j'\le N}^{j'\neq j, j'\neq k} \frac{1}{2\mu_{j'}} \triangle_{\mathbf{x}_{j'}} - \frac{1}{2M} \triangle_{\xi_N} - C_{jk} \sum_{j''\neq j'}^{\{j',j''\}\cap\{j,k\}=\emptyset} \nabla_{\mathbf{x}_{j'}} \cdot \nabla_{\mathbf{x}_{j''}}. \qquad \text{(IV.1.25)}$$

From (IV.1.25), we get (IV.1.26) in a similar fashion as (IV.1.20) was gotten:

$$\mathbf{p}_{jk} = 2\mu_{jk}\, i[H_0,\, \tilde{x}_k - \tilde{x}_j], \qquad \text{(IV.1.26)}$$

from which it follows the first equality in (II.1.16). The second equality can be derived from (II.1.11).

■

We have to notice that, for $j = 3, \dots, N$, $\mathbf{x}_j$ and $\mathbf{p}_j$ are the relative position and the relative momenta operators with respect the center of mass of the pair $(1, 2)$. Then, we have that $\mathbf{x}_j = i\nabla_{\mathbf{p}_j}$. To prove (II.1.21) we use (II.1.10) to get $\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j = \mathbf{x}_k - \mathbf{x}_j$.

Let us take k=3, ..., N.

$$\begin{aligned}
\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_1 &= \tilde{\mathbf{x}}_k - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1+m_2) + (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1+m_2) - \tilde{\mathbf{x}}_1 \\
&= \mathbf{x}_k + \frac{m_2}{m_1+m_2}(\tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1) = \mathbf{x}_k + \frac{\mu_{12}}{m_1}\mathbf{x} = i\frac{\partial}{\partial \mathbf{p}_k} + \frac{\mu_{12}}{m_1} i \frac{\partial}{\partial \mathbf{p}},
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_2 &= \tilde{\mathbf{x}}_k - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1+m_2) + (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1+m_2) - \tilde{\mathbf{x}}_2 \\
&= \mathbf{x}_k - \frac{m_1}{m_1+m_2}(\tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1) = \mathbf{x}_k - \frac{\mu_{12}}{m_2}\mathbf{x} = i\frac{\partial}{\partial \mathbf{p}_k} - \frac{\mu_{12}}{m_2} i \frac{\partial}{\partial \mathbf{p}},
\end{aligned}$$

Hence we have proved (II.1.22).

COMPUTATION IV.1.6. *Determination of η_{jk}, for $1 \le j < k \le N$ used to define the support of the functions f_{jk} in* (II.1.19).

PROOF. Let us rewrite (II.1.19). By the use (II.1.16), there are functions $f_{jk} \in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$ such that, for all $1 \le j < k \le N$,

$$\begin{aligned}
&\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3, \dots, \mathbf{p}_N - \mu_N\mathbf{v}_N) \\
= \;& f_{jk}\left(\mu_{jk}[(\mathbf{p}_k/\mu_k - \mathbf{v}_k) - (\mathbf{p}_j/\mu_j - \mathbf{v}_j)]\right) \hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3, \dots, \mathbf{p}_N - \mu_N\mathbf{v}_N). \qquad \text{(IV.1.27)}
\end{aligned}$$

We want to find a $R > 0$ such that if $\mathbf{q} \in \mathbb{R}^n \setminus B_{\mu_{12}R}$, then $f_{12}(\mathbf{q}) = 0$. Let us analyze the following cases:

(*a*) $(j, k) = (1, 2)$.

By (IV.1.27):

$$\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3 = f_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\,\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3.$$

From here, we see that we can choose supp f_{12} such that supp $\hat{\phi}_{12} \subset$ supp $f_{12} \subset B_{\mu_{12}\eta}$. Thus $R = \eta$.

(*b*) (j, k), $j < k$; $3 \le k \le N$.

Here, it is important to realize that $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta operators in the center of mass frame of particles 1 and 2, respectively. Thus, by physics arguments, $\mathbf{p} = -\frac{m_1}{\mu_1}\mathbf{p}_1 = \frac{m_2}{\mu_2}\mathbf{p}_2$. Let us verify these equalities, by (II.1.9) and (II.1.11):

$$\begin{aligned}
\mathbf{p} + \frac{m_1}{\mu_1}\mathbf{p}_1 &= \frac{m_1 m_2}{m_1+m_2}(\tilde{\mathbf{p}}_2/m_2 - \tilde{\mathbf{p}}_1/m_1) + \frac{m_1}{\mu_1}\mu_1(\tilde{\mathbf{p}}_1/m_1 - (\tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2)/(m_1+m_2)) \\
&= \frac{m_1}{m_1+m_2}\tilde{\mathbf{p}}_2 - \frac{m_2}{m_1+m_2}\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_1 - \frac{m_1}{m_1+m_2}\tilde{\mathbf{p}}_1 - \frac{m_1}{m_1+m_2}\tilde{\mathbf{p}}_2 \\
\mathbf{p} &= -\frac{m_1}{\mu_1}\mathbf{p}_1, \qquad \text{(IV.1.28)} \\
\mathbf{p} - \frac{m_2}{\mu_2}\mathbf{p}_2 &= \frac{m_1 m_2}{m_1+m_2}(\tilde{\mathbf{p}}_2/m_2 - \tilde{\mathbf{p}}_1/m_1) - \frac{m_2}{\mu_2}\mu_2(\tilde{\mathbf{p}}_2/m_2 - (\tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2)/(m_1+m_2)) \\
&= \frac{m_1}{m_1+m_2}\tilde{\mathbf{p}}_2 - \frac{m_2}{m_1+m_2}\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2 + \frac{m_2}{m_1+m_2}\tilde{\mathbf{p}}_1 + \frac{m_2}{m_1+m_2}\tilde{\mathbf{p}}_2 \\
\mathbf{p} &= \frac{m_2}{\mu_2}\mathbf{p}_2. \qquad \text{(IV.1.29)}
\end{aligned}$$

In the particular case where $j=2$, $k=3,\ldots,N$, (IV.1.27) and (IV.1.29) imply that

$$\begin{aligned}
&\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3,\ldots,\mathbf{p}_N - \mu_N\mathbf{v}_N) \\
= \; & f_{2k}\left(\mu_{2k}[(\mathbf{p}_k/\mu_k - \mathbf{v}_k) - (\mathbf{p}/m_2 - \mathbf{v}\mu_{12}/m_2)]\right)\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3,\ldots,\mathbf{p}_N - \mu_N\mathbf{v}_N).
\end{aligned}$$

We want $f_{2k} = 1$ for $\mathbf{p}_k, \mathbf{p}$ such that $\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})$ and $\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3,\ldots,\mathbf{p}_N - \mu_N\mathbf{v}_N)$ are supported. i.e.

$$\begin{aligned}
|\mathbf{p} - \mu_{12}\mathbf{v}| &< \mu_{12}\eta \\
\mu_{2k}\,|\mathbf{p}/m_2 - \mu_{12}\mathbf{v}/m_2| &< (\mu_{2k}\mu_{12}/m_2)\eta, \\
|\mathbf{p}_k - \mu_k\mathbf{v}_k| &< \mu_k \\
\mu_{2k}\,|\mathbf{p}_k/\mu_k - \mathbf{v}_k| &< \mu_{2k},
\end{aligned}$$

Then

$$\mu_{2k}\,|(\mathbf{p}_k/\mu_k - \mathbf{v}_k) - (\mathbf{p}/m_2 - \mathbf{v}\mu_{12}/m_2)| < \mu_{2k}(1 + \eta\mu_{12}/m_2).$$

Similarly when $j=1$, $k=3,\ldots,N$, $\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})$ and $\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3,\ldots,\mathbf{p}_N - \mu_N\mathbf{v}_N)$ are supported if

$$\mu_{1k}\,|(\mathbf{p}_k/\mu_k - \mathbf{v}_k) + (\mathbf{p}/m_1 - \mathbf{v}\mu_{12}/m_1)| < \mu_{2k}(1 + \eta\mu_{12}/m_1).$$

In the case $3 \le j < k \le N$ $\hat{\phi}_{12}(\mathbf{p} - \mu_{12}\mathbf{v})$ and $\hat{\phi}_3(\mathbf{p}_3 - \mu_3\mathbf{v}_3,\ldots,\mathbf{p}_N - \mu_N\mathbf{v}_N)$ are supported if

$$\mu_{jk}\left|(\mathbf{p}_k/\mu_k - \mathbf{v}_k) + (\mathbf{p}_j/\mu_j - \mathbf{v}_j)\right| < 2\mu_{jk}.$$

By setting $\eta_{1j} = 2(1 + \eta\mu_{12}/m_1)$, $\eta_{2j} = 2(1 + \eta\mu_{12}/m_2)$ and, for $3 \le j < k \le N$, $\eta_{jk} = 4$, we define, for $1 \le j < k$, $3 \le k \le N$, $f_{jk} \in C_0^\infty(B_{\mu_{jk}\eta_{jk}})$ with $f_{jk}(\mathbf{q}) = 1$ if $|\mathbf{q}| < \mu_{jk}\eta_{jk}/2$.

■

II Dynamics

Next, we turn our attention to the subject of self-adjointness of the Hamiltonian in (II.1.41) and why we impose some restrictions to the potentials involved wherein.

LEMMA IV.2.1. *Let A and B be operators in a Hilbert space. If b is any non-negative constant, then A is (B/b)-bounded with relative bound ab if and only if A is B-bounded with relative bound a.*

PROOF. Let $\phi \in D(B)$. Then, there exist α and β non-negative such that

$$\|A\phi\| \leq \alpha\|B\phi\| + \beta\|\phi\| = \alpha b\|(B/b)\phi\| + \beta\|\phi\|.$$

The result follows. ■

LEMMA IV.2.2. *Assume $x \to V(x)$ is a real-valued function defined in $\mathbb{R}^n$ with $x = (x_1, \dots, x_n)$. If $(1+|x_1|)V$ is relatively bounded with respect to the Laplacian with relative bound a, then x_1V is relatively bounded with respect to the Laplacian with relative bound not greater that a and V is relatively bounded with respect to the Laplacian with relative bound not greater that a.*

PROOF. By assumption, there are non-negative real numbers α and β such that

$$\|(1+|x_1|)V\phi\|^2 \leq \alpha^2 \|\triangle\phi\|^2 + \beta^2 \|\phi\|^2 \quad \forall\, \phi \in D(\triangle), \tag{IV.2.30}$$

with a being the infimum of all α such that (IV.2.30) is true. Take $\phi \in C_0^\infty(\mathbb{R}^n)$ then

$$\begin{aligned}
\|x_1V\phi\|^2 &= \int_{\mathbb{R}^n} dx\, |x_1|^2 |V(x)\phi(x)|^2 \\
&\leq \int_{\mathbb{R}^n} dx\, (1+|x_1|)^2 |V(x)\phi(x)|^2 = \|(1+|x_1|)V\phi\|^2 \\
&\leq \alpha^2 \|\triangle\phi\|^2 + \beta^2 \|\phi\|^2,
\end{aligned} \tag{IV.2.31}$$

and

$$\begin{aligned}
\|V\phi\|^2 &= \int_{\mathbb{R}^n} dx\, |x_1|^2 |V(x)\phi(x)|^2 \\
&\leq \int_{\mathbb{R}^n} dx\, (1+|x_1|)^2 |V(x)\phi(x)|^2 = \|(1+|x_1|)V\phi\|^2 \\
&\leq \alpha^2 \|\triangle\phi\|^2 + \beta^2 \|\phi\|^2.
\end{aligned} \tag{IV.2.32}$$

From (IV.2.31) and (IV.2.32) we obtain the result. ■

Now we put Lemma 2 in section 8 "Electric Fieds" in Simon [82] which we can use it to justify the hipothesis imposed on the potential by Adachi and Maehara [2]. We have included the mass m and the magnitud of the electric field E to adjust the lemma for our purposes.

LEMMA IV.2.3. *Let us define $H_0 = -\triangle/(2m) - Ex_1$ where m and E are non-negative real numbers. Suppose that $V = V_1 + V_2$ with V_2 bounded, V_1 $\triangle$-bounded with relative bound a and V_1x_1 $\triangle$-bounded. Then V is H_0-bounded with relative bound $2ma$.*

PROOF. Recall that the Laplacian $\triangle$ is represented in the momentum space as $\mathbf{p}^2 = \sum \mathbf{p}_j^2$. We first observe that

$$\begin{aligned}
x_1\left[(-\triangle/(2m)+ib)^{-1}(H_0+ib)^{-1}\right] &= i\frac{\partial}{\partial p_1}\left[(-\triangle/(2m)+ib)^{-1}(H_0+ib)^{-1}\right] \\
&= i\left[\frac{\partial}{\partial p_1}(-\triangle/(2m)+ib)^{-1}\right](H_0+ib)^{-1} \\
&\quad +(-\triangle/(2m)+ib)^{-1}\left[i\frac{\partial}{\partial p_1}(H_0+ib)^{-1}\right] \\
&= i\left[-1(\mathbf{p}^2/(2m)+ib)^{-2}(\mathbf{p}_1/m)\right](H_0+ib)^{-1} \\
&\quad +(-\triangle/(2m)+ib)^{-1}\left[x_1(H_0+ib)^{-1}\right].
\end{aligned}$$

Then

$$(-\triangle/(2m)+ib)^{-1}x_1(H_0+ib)^{-1} = x_1(-\triangle/(2m)+ib)^{-1}(H_0+ib)^{-1} + i(\mathbf{p}_1/m)(-\triangle/(2m)+ib)^{-2}(H_0+ib)^{-1}. \quad \text{(IV.2.33)}$$

Now we decompose $V_1(H_0+ib)^{-1}$ by (IV.2.33) and estimate it by using Proposition III.11.5 as follows

$$\begin{aligned} V_1(H_0+ib)^{-1} &= V_1(-\triangle/(2m)+ib)^{-1}(-\triangle/(2m)+ib)(H_0+ib)^{-1} \\ &= V_1(-\triangle/(2m)+ib)^{-1}(-\triangle/(2m)-Ex_1+ib)(H_0+ib)^{-1} \\ &\quad +V_1(-\triangle/(2m)+ib)^{-1}Ex_1(H_0+ib)^{-1} \\ &= V_1(-\triangle/(2m)+ib)^{-1}+E\,V_1(-\triangle/(2m)+ib)^{-1}x_1(H_0+ib)^{-1} \\ &= V_1(-\triangle/(2m)+ib)^{-1}+E\,V_1x_1(-\triangle/(2m)+ib)^{-1}(H_0+ib)^{-1} \\ &\quad +iE\,V_1(-\triangle/(2m)+ib)^{-1}(\mathbf{p}_1/m)(-\triangle/(2m)+ib)^{-1}(H_0+ib)^{-1} \end{aligned}$$

Calculating the norm:

$$\begin{aligned} \|V_1(H_0+ib)^{-1}\| &\le \|V_1(-\triangle/(2m)+ib)^{-1}\|+E\|V_1x_1(-\triangle/(2m)+ib)^{-1}\|\,\|(H_0+ib)^{-1}\| \\ &\quad E\|V_1(-\triangle/(2m)+ib)^{-1}\|\,\|(\mathbf{p}_1/m)(-\triangle/(2m)+ib)^{-1}\|\,\|(H_0+ib)^{-1}\| \end{aligned}$$

$$\begin{aligned} \lim_{b\to\infty}\|V_1(H_0+ib)^{-1}\| &\le \lim_{b\to\infty}\|V_1(-\triangle/(2m)+ib)^{-1}\|+E\Big[\lim_{b\to\infty}\|V_1x_1(-\triangle/(2m)+ib)^{-1}\| \\ &\quad +\lim_{b\to\infty}\|V_1(-\triangle/(2m)+ib)^{-1}\|\lim_{b\to\infty}\|(\mathbf{p}_1/m)(-\mathbf{p}^2/(2m)+ib)^{-1}\|\Big] \\ &\quad \times\lim_{b\to\infty}\|(H_0+ib)^{-1}\| \\ &= 2ma+E\Big[\lim_{b\to\infty}\|V_1x_1(-\triangle/(2m)+ib)^{-1}\| \\ &\quad +\lim_{b\to\infty}\|V_1(-\triangle/(2m)+ib)^{-1}\|(0)\Big](0) \\ &= 2ma. \end{aligned}$$

Finally, by (III.11.69)

$$\begin{aligned} \lim_{b\to\infty}\|V(H_0+ib)^{-1}\| &\le 2ma+\lim_{b\to\infty}\|V_2(H_0+ib)^{-1}\| \\ &\le 2ma. \end{aligned}$$

■

In Lemma IV.2.3 we take $V=V_1+V_2$ because there are bounded operators V_2 such that V_2x_1 are not $\triangle$-bounded. Example: the identity operator. Let us take $y=(y_1,0,\ldots,0)$, $\phi\in C_0^\infty$ with supp $\phi\in B_{|y_1|/2}(0)$. Recall that $\tau_y\phi(x)=\phi(x-y)$.

$$\begin{aligned} \|x_1(\tau_y\phi)\|^2 &= \int dx\,|x_1\phi(x-y)|^2 \\ &= \int_{B_{|y_1|/2}(0)} dz\,|(z_1+y_1)\,|\phi(z)|^2 \\ &= \int_{B_{|y_1|/2}(0)} dz\,|\,|y_1|-|z_1|\,|^2\,|\phi(z)|^2 \\ &\ge \int_{B_{|y_1|/2}(0)} dz\,y_1^2/4\,|\phi(z)|^2 \\ &= (y_1^2/4)\|\phi\|^2\to\infty \text{ as } |y_1|\to\infty. \end{aligned}$$

On the other hand, by Proposition III.4.6,

$$\begin{aligned}\|\triangle(\tau_y\phi)\| &= \|\mathcal{F}^{-1}\mathbf{p}^2\mathcal{F}(\tau_y\phi)\| = \|\mathbf{p}^2(\tau_y\phi)\hat{}\,\| = \|\mathbf{p}^2 e^{-i\mathbf{p}\cdot y}\hat{\phi}\| \\ &= \left(\int dp\,|p^2 e^{-iy\cdot p}\hat{\phi}(p)|^2\right)^{1/2} = \left(\int dp\,|p^2\hat{\phi}(p)|^2\right)^{1/2} = \|\mathbf{p}^2\hat{\phi}\| \\ &= \|\triangle\phi\|.\end{aligned}$$

Hence, $\|\triangle(\tau_y\phi)\|$ is constant whereas $\|x_1(\tau_y\phi)\|$ goes to ∞ as $|y_1|$ goes to ∞. That is the reason why Ix_1 is not $\triangle$-bounded.

At this moment we are prepared to justify the hypotheses over the boundedness of the potential given in Definitions II.1.1 and II.1.2 that allow the perturbed Hamiltonian to be essentially self-adjoint and thus we can apply Stone's Theorem in order to define the propagator e^{-itH} with H given in (II.1.37). We will assume that the relative charge q_1^R, given by (II.1.6), is not zero when Jacobi coordinates are based on every pair $1 \leq j < k \leq N$. The case where q_1^R is zero is similar and simpler. We define

$$H_{12} = (2\nu_1)^{-1}\hat{\mathbf{p}}_1^2 + q_1^R\mathbf{E}\cdot\xi_1, \tag{IV.2.34}$$

thus we compactly reexpress (II.1.7) as

$$H_0 = H_{12}\otimes I + I\otimes\hat{H}_0.$$

First we take $\phi_l \in C_0^\infty(\mathbb{R}^n)$, $\psi_l \in C_0^\infty(\mathbb{R}^{(n-1)N})$. The beauty of having based Jacobi coordinates is that this procedure applies to all pairs. Let us consider the potential V_{jk}^{vs} given in (II.2.1). Therefore,

$$V_{jk}^{vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) = V_{jk}^{vs}(\xi_1)\otimes I.$$

By Definition II.1.2 and Lemmas IV.2.1, IV.2.2 and IV.2.3 there exist non-negative constants α_l and β_l such that

$$\begin{aligned}\|(V_{jk}^{vs}\otimes I)(\phi_l\otimes\psi_l)\| &= \|V_{jk}\phi_l\|\,\|\psi_l\| \\ &\leq \alpha_l\|\psi_l\|\,\|H_{12}\phi_l\| + \beta_l\|\psi_l\|\,\|\phi_l\| \\ &= \alpha_l\|(H_{12}\otimes I)(\phi_l\otimes\psi_l)\| + \beta_l\|\phi_l\otimes\psi_l\| \\ &\leq \alpha_l\|(H_{12}\otimes I + I\otimes\hat{H}_0)(\phi_l\otimes\psi_l)\| + \beta_l\|\phi_l\otimes\psi_l\| + \alpha_l\|\phi_l\otimes\hat{H}_0\psi_l\| \\ &\leq \alpha_l\|H_0(\phi_l\otimes\psi_l)\| + \left(\beta_l + \alpha_l\|\hat{H}_0\psi_l\|/\|\psi_l\|\right)\|\phi_l\otimes\psi\|,\end{aligned}$$

where the infimum of all such α_l is zero by assumption in Definition II.1.2.

Taking linear combinations of the $\phi_l\otimes\psi_l$ there exist α and β such that for any η in a dense set of $L^2(\mathbf{X})$

$$\|V_{jk}^{vs}\eta\| \leq \alpha_{jk}\|H_0\eta\| + \beta_{jk}\|\eta\|.$$

We can conclude that the H_0-bound of V_{jk}^{vs} is zero. This applies also to V_{jk}^s and V_{jk}^l as in (II.2.1) because they are bounded. Thus, summing up and using the fact that for all $1 \leq j < k \leq N$, V_{jk}^0 and V_{jk}^E are relatively bounded with respect to H_0 with relative bound zero, we obtain that the operator V, given in (II.1.41) is H_0-bounded with relative bound zero and, hence, H, also given in (II.1.41), is essentially self-adjoint by the celebrated Kato-Rellich Theorem III.11.6. Therefore we can invoke Theorem III.8.1 to get a propagator.

PROPOSITION IV.2.4. *The following estimations are true for* $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

$$|y|_\infty \leq |y|_p \leq n^{1/p}|y|_\infty, \tag{IV.2.35}$$

with $p \geq 1$ *and* $|y|_p^p = \sum_{i=1}^n y_i^p$.

Proof. First inequality follows from $|y|_p \geq |y_i|$, $\forall\ 1 \leq i \leq n$. For the second:

$$\begin{aligned} |y|_\infty &\geq |y_i|, \quad \forall\ 1 \leq i \leq n \\ n|y|_\infty^p &\geq \sum_{i=1}^{n} |y_i|^p \\ n^{1/p}|y|_\infty &\geq |y|_p. \end{aligned}$$

■

The following Lemma is a slightly modified version of Lemma 3.3 in Hörmander [25]. It is used to add more regularity to long range potentials.

LEMMA IV.2.5. *Let $V \in C^k(\mathbb{R}^n)$ and assume that $|D^\alpha V(x)| \leq C(1+|x|)^{-m(|\alpha|)}$, $|\alpha| \leq k$, where $m(0), \ldots, m(k)$ are positive numbers, and suppose that $0 < \delta < \max_{0<j\leq k} \frac{m(j)-1}{j}$, $\delta < 1$. Then, one can split $V = V_1 + V_2$, so that for some $\epsilon > 0$, $|V_1(x)| \leq C(1+|x|)^{-1-\epsilon}$, thus V_1 is of short-range, and for all α, $|D^\alpha V_2(x)| \leq C_\alpha (1+|x|)^{-m'(|\alpha|)}$, where $m'(j) = \max_{i\leq j, i\leq k} \left(m(i) + \delta(j-i)\right)$.*

COMPUTATION IV.2.6. *Decay of the second, third and fourth derivatives of the potential $V^{0,l}$ given in* (II.1.25)*.*

Proof. Let us assume (II.1.24) and consider $1 \leq i \leq n$. Then, by the fundamental Theorem of calculus and the fact that $V^{0,l}(y) \to 0$ as $y \to \infty$:

1. Case $y_i \leq 0$

$$\begin{aligned} \left|V^{0,l}(y_1,\ldots,y_n)\right| &= \left|\int_{-\infty}^{y_i} \partial_i V^{0,l}(y_1,\ldots,y_{i-1},\xi_i,y_{i+1},\ldots,y_n)\, d\xi_i\right| \\ &\leq \int_{-\infty}^{y_i} \left|\partial_i V^{0,l}(y_1,\ldots,y_{i-1},\xi_i,y_{i+1},\ldots,y_n)\right| d\xi_i \\ &\leq C\int_{-\infty}^{y_i} \left(1+|(y_1,\ldots,y_{i-1},\xi_i,y_{i+1},\ldots,y_n)|\right)^{-\gamma_1} d\xi_i \\ &\leq C\int_{-\infty}^{y_i} \left(1+|\xi_i|\right)^{-\gamma_1} d\xi_i \\ &= C\int_{-\infty}^{y_i} \left(1-\xi_i\right)^{-\gamma_1} d\xi_i = \frac{-C}{-\gamma_1+1}\left(1-\xi_i\right)^{-\gamma_1+1}\Big|_{-\infty}^{y_i} \\ &= \frac{C}{\gamma_1-1}\left(1+|y_i|\right)^{-(\gamma_1-1)} \end{aligned}$$

2. Case $y_i > 0$

$$\begin{aligned} \left|V^{0,l}(y_1,\ldots,y_n)\right| &= \left|\int_{y_i}^{\infty} \partial_i V^{0,l}(y_1,\ldots,y_{i-1},\xi_i,y_{i+1},\ldots,y_n)\, d\xi_i\right| \\ &\leq \int_{y_i}^{\infty} \left|\partial_i V^{0,l}(y_1,\ldots,y_{i-1},\xi_i,y_{i+1},\ldots,y_n)\right| d\xi_i \\ &\leq C\int_{y_i}^{\infty} \left(1+|(y_1,\ldots,y_{i-1},\xi_i,y_{i+1},\ldots,y_n)|\right)^{-\gamma_1} d\xi_i \\ &\leq C\int_{y_i}^{\infty} \left(1+|\xi_i|\right)^{-\gamma_1} d\xi_i \\ &\leq C\int_{y_i}^{\infty} \left(1+\xi_i\right)^{-\gamma_1} d\xi_i = \frac{C}{-\gamma_1+1}\left(1+\xi_i\right)^{-\gamma_1+1}\Big|_{y_i}^{\infty} \\ &= \frac{C}{\gamma_1-1}\left(1+|y_i|\right)^{-(\gamma_1-1)} \end{aligned}$$

Thus, by (IV.2.35)

$$\begin{aligned}\left|V^{0,\,l}(y_1,\ldots,y_n)\right| &\le C\,(1+|y_i|)^{-(\gamma_1-1)}\,,\ \forall\ 1\le i\le n\\ \left|V^{0,\,l}(y_1,\ldots,y_n)\right| &\le C\min_{1\le i\le n}(1+|y_i|)^{-(\gamma_1-1)} = C\left(1+\max_{1\le i\le n}|y_i|\right)^{-(\gamma_1-1)}\\ &\le C\,(1+|y|_\infty)^{-(\gamma_1-1)}\le C\,(1+|y|)^{-(\gamma_1-1)}\,.\end{aligned}$$

That is why we can assume that

$$|\partial^\beta V^{0,l}|\le C\,(1+|x|)^{-(\gamma_1-1+|\beta|)}\,,\quad |\beta|\le 1,$$

with $3/2<\gamma_1\le 2$.

Let us define:

$$\begin{aligned}m(0) &= \gamma_1-1,\\ m(1) &= \gamma_1.\end{aligned}$$

Let us apply Lemma 3.3 from Hörmander

$$\begin{aligned}\max_{0<j\le 1}\frac{m(j)-1}{j} &= \frac{m(1)-1}{1}\\ &= \gamma_1-1.\end{aligned}$$

Let δ be such that $0<\delta<\gamma_1-1$ and $\delta<1$. This is equivalent to choose an $0<\epsilon_0<1/2$ with $\delta=\epsilon_0+1/2$, then $0<\epsilon<\gamma_1-3/2\le 1/2$.

Now we compute the new exponents

$$\begin{aligned}m'(0) &= \max_{i\le 0,i\le 1}(m(i)+\delta(0-i))\\ m'(0) &= m(0)\\ &= \gamma_1-1.\\ m'(1) &= \max_{i\le 1,i\le 1}(m(i)+\delta(1-i))\\ m(0)+\delta &= \gamma_1-1+\epsilon_0+1/2=\gamma_1-(1/2-\epsilon_0)\\ m(1)+0 &= \gamma_1\\ m'(1) &= \gamma_1.\end{aligned}$$

So far, we have recovered the same exponents we already had, now we will obtain the new ones. Because $2\epsilon_0<\epsilon_0+1/2$:

$$\begin{aligned}m'(2) &= \max_{i\le 2,i\le 1}(m(i)+\delta(2-i))\\ m(0)+2\delta &= \gamma_1-1+2(\epsilon_0+1/2)=\gamma_1+2\epsilon_0\\ m(1)+\delta &= \gamma_1+(\epsilon_0+1/2)\\ m'(2) &= \gamma_1+(\epsilon_0+1/2)\\ &> \epsilon_0+3/2+\epsilon_0+1/2=1+2(\epsilon_0+1/2).\end{aligned}$$

Because $3\epsilon_0+1/2<2\epsilon_0+1$:

$$\begin{aligned}m'(3) &= \max_{i\le 3,i\le 1}(m(i)+\delta(3-i))\\ m(0)+3\delta &= \gamma_1-1+3(\epsilon_0+1/2)=\gamma_1+3\epsilon_0+1/2\\ m(1)+2\delta &= \gamma_1+2(\epsilon_0+1/2)\\ m'(3) &= \gamma_1+2(\epsilon_0+1/2)\\ &> \epsilon_0+3/2+2(\epsilon_0+1/2)=1+3(\epsilon_0+1/2).\end{aligned}$$

Because $4\epsilon_0 + 1 < 3\epsilon_0 + 3/2$:

$$\begin{aligned} m'(4) &= \max_{i\le 4, i\le 1} (m(i) + \delta(4-i)) \\ m(0) + 4\delta &= \gamma_1 - 1 + 4(\epsilon_0 + 1/2) = \gamma_1 + 4\epsilon_0 + 1 \\ m(1) + 3\delta &= \gamma_1 + 3(\epsilon_0 + 1/2) \\ m'(4) &= \gamma_1 + 3(\epsilon_0 + 1/2) \\ &> \epsilon_0 + 3/2 + 3(\epsilon_0 + 1/2) = 1 + 4(\epsilon_0 + 1/2). \end{aligned}$$

Therefore, we can additionally assume without loss of generality that

$$|\partial^{\alpha_0} V^{0,l}| \le C\,(1+|x|)^{-1-|\alpha_0|(\epsilon_0+1/2)}, \quad 2 \le |\alpha_0| \le 4, \quad V^{0,l} \in C^4(\mathbb{R}^n).$$

■

COMPUTATION IV.2.7. *Decay of the third and fourth derivatives of the potential* $V^{E,l}$ *given in* (II.1.30).

PROOF.
We know that

$$|\partial^\beta V^{E,l}| \le C < x >^{-\gamma_D - \mu|\beta|}, \quad |\beta| \le 2$$

with $0 < \gamma_D \le 1/2$ and $1 - \gamma_D < \mu \le 1$.
Let us define:

$$\begin{aligned} m(0) &= \gamma_D, \\ m(1) &= \gamma_D + \mu, \\ m(2) &= \gamma_D + 2\mu. \end{aligned}$$

Let us apply Lemma 3.3 from Hörmander

$$\begin{aligned} \max_{0<j\le 2} \frac{m(j)-1}{j} &= \max\left\{\frac{m(1)-1}{1}, \frac{m(2)-1}{2}\right\} \\ &= \max\left\{\gamma_D + \mu - 1, \frac{\gamma_D + 2\mu - 1}{2}\right\} \\ &= \frac{\gamma_D + 2\mu - 1}{2}. \end{aligned}$$

Because

$$\begin{aligned} \frac{\gamma_D + 2\mu - 1}{2} - (\gamma_D + \mu - 1) &= (\gamma_D + 2\mu - 1 - 2\gamma_D - 2\mu + 2)/2 \\ &= (1 - \gamma_D)/2 \\ &\ge 1/4 \\ &> 0. \end{aligned}$$

We have to choose δ such that $0 < \delta < \frac{\gamma_D + 2\mu - 1}{2}$ and $\delta < 1$. Let δ be equal to $\mu/2$. Clearly $0 < \delta < 1$. To prove that $\delta < \frac{\gamma_D + 2\mu - 1}{2}$ we do the following estimation:

$$\begin{aligned} \gamma_D + \mu &> 1 \\ \gamma_D + 2\mu - 1 &> 1 + \mu - 1 \\ \frac{\gamma_D + 2\mu - 1}{2} &> \mu/2 = \delta. \end{aligned}$$

Now we compute the new exponents

$$\begin{aligned} m'(0) &= \max_{i\leq 0, i\leq 2}(m(i)+\delta(0-i)) \\ m'(0) &= m(0) \\ &= \gamma_D. \end{aligned}$$

$$\begin{aligned} m'(1) &= \max_{i\leq 1, i\leq 2}(m(i)+\delta(1-i)) \\ m(0)+\delta &= \gamma_D+\mu/2 \\ m(1)+0 &= \gamma_D+\mu \\ m'(1) &= \gamma_D+\mu. \end{aligned}$$

$$\begin{aligned} m'(2) &= \max_{i\leq 2, i\leq 2}(m(i)+\delta(2-i)) \\ m(0)+2\delta &= \gamma_D+\mu \\ m(1)+\delta &= \gamma_D+3\mu/2 \\ m(2)+0 &= \gamma_D+2\mu \\ m'(2) &= \gamma_D+2\mu. \end{aligned}$$

So far, we have recovered the same exponents we already had, now we will obtain the new ones.

$$\begin{aligned} m'(3) &= \max_{i\leq 3, i\leq 2}(m(i)+\delta(3-i)) \\ m(0)+3\delta &= \gamma_D+3\mu/2 \\ m(1)+2\delta &= \gamma_D+2\mu \\ m(2)+\delta &= \gamma_D+5\mu/2 \\ m'(3) &= \gamma_D+\frac{2+3}{2}\mu. \end{aligned}$$

$$\begin{aligned} m'(4) &= \max_{i\leq 4, i\leq 2}(m(i)+\delta(4-i)) \\ m(0)+4\delta &= \gamma_D+2\mu \\ m(1)+3\delta &= \gamma_D+5\mu/2 \\ m(2)+2\delta &= \gamma_D+3\mu \\ m'(4) &= \gamma_D+\frac{2+4}{2}\mu. \end{aligned}$$

Therefore, we can additionally assume without loss of generality that

$$|\partial^\beta V^{E,l}| \leq C < x >^{-\gamma_D-\mu\frac{2+|\beta|}{2}}, \quad 3\leq|\beta|\leq 4, \quad V^{E,l}\in C^4(\mathbb{R}^n).$$

■

Proposition IV.2.8 below has been taken from Cycon et Al [64]. Some computations has been added to render the proof even more clear. This proposition proves (II.1.31).

PROPOSITION IV.2.8. *Let K_0 be the closure of $\mathbf{p}^2/(2m)+qEx_1$ on $\mathcal{S}(\mathbb{R}^n)$. Then K_0 is self-adjoint, and the time evolution is*

$$e^{-it(\mathbf{p}^2/(2m)-qE\mathbf{x}_1)} = e^{iqE\mathbf{x}_1 t}e^{-it^3q^2E^2/(6m)}e^{-i\mathbf{p}_1 qEt^2/(2m)}e^{-it\mathbf{p}^2/(2m)},$$

for $t\in\mathbb{R}$.

PROOF. Consider the decomposition $L^2(\mathbb{R}^n) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{n-1})$, according to the coordinate decomposition $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_\perp)$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_\perp)$, in configuration space as well as in momentum space. Let $\phi(\mathbf{p})$ belong to $\mathcal{S}(\mathbb{R}^n)$. Since x_1 acts as $i\partial/\partial p_1$,

$$\begin{aligned}
e^{-i\mathbf{p}_1^3/(6mqE)}\,(qEx_1)\,e^{i\mathbf{p}_1^3/(6mqE)}\,\phi(\mathbf{p}) &= e^{-i\mathbf{p}_1^3/(6mqE)}\Big(iqE\,\partial/\partial p_1\left(e^{i\mathbf{p}_1^3/(6mqE)}\right)\\
&\quad +iqEe^{i\mathbf{p}_1^3/(6mqE)}\,\partial/\partial p_1\Big)\,\phi(\mathbf{p})\\
&= e^{-i\mathbf{p}_1^3/(6mqE)}\Big(iqE(i\mathbf{p}_1^2/(2mqE))e^{i\mathbf{p}_1^3/(6mqE)}\\
&\quad +e^{i\mathbf{p}_1^3/(6mqE)}\,qEx_1\Big)\,\phi(\mathbf{p})\\
&= \big(iqE(i\mathbf{p}_1^2/(2mqE)) + qEx_1\big)\,\phi(\mathbf{p})\\
e^{-i\mathbf{p}_1^3/(6mqE)}\left(\mathbf{p}_\perp^2/(2m) - qEx_1\right)e^{i\mathbf{p}_1^3/(6mqE)}\,\phi(\mathbf{p}) &= \left(\mathbf{p}_\perp^2/(2m) + \mathbf{p}_1^2/(2m) - qEx_1\right)\,\phi(\mathbf{p})\\
&= K_0\,\phi(\mathbf{p}). \qquad \text{(IV.2.36)}
\end{aligned}$$

So, we have that K_0 is unitarily equivalent to the operator $\left(\mathbf{p}_\perp^2/(2m) + \mathbf{p}_1^2/(2m) + qEx_1\right)$ which, by Fourier transform in the $x_\perp$ variable, in its turn becomes a multiplication operator in the $(x_1, p_\perp)$ variables. We conclude that K_0 is self-adjoint.

Let $\psi(\mathbf{x})$ belong to $\mathcal{S}(\mathbb{R}^n)$ and $t \in \mathbb{R}$. Since $\mathbf{p}_1$ acts as $-i\partial/\partial x_1$,

$$\begin{aligned}
e^{-iqEx_1t}\,\mathbf{p}_1\,e^{iqEx_1t}\,\psi(x) &= e^{-iqEx_1t}\left(-i\partial/\partial x_1 e^{iqEx_1t} + e^{iqEx_1t}\mathbf{p}_1\right)\psi(x)\\
&= (qEt + \mathbf{p}_1)\,\psi(x),
\end{aligned}$$

then, in $\mathcal{S}(\mathbb{R}^n)$

$$e^{-iqE\mathbf{x}_1t}\,\mathbf{p}_1\,e^{iqE\mathbf{x}_1t} = \mathbf{p}_1 + qEt. \qquad \text{(IV.2.37)}$$

By (IV.2.36) and (IV.2.37) and functional calculus, we have in $\mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}
e^{-itK_0} &= e^{-i\mathbf{p}_1^3/(6mqE)}\,e^{-it\left(\mathbf{p}_\perp^2/(2m) - qEx_1\right)}\,e^{i\mathbf{p}_1^3/(6mqE)}\\
&= e^{-i\mathbf{p}_1^3/(6mqE)}\,e^{itqE\mathbf{x}_1}\,e^{i\mathbf{p}_1^3/(6mqE)}\,e^{-it\mathbf{p}_\perp^2/(2m)}\\
&= e^{itqE\mathbf{x}_1}\,e^{-itqE\mathbf{x}_1}\,e^{-i\mathbf{p}_1^3/(6mqE)}\,e^{itqE\mathbf{x}_1}\,e^{i\mathbf{p}_1^3/(6mqE)}\,e^{-it\mathbf{p}_\perp^2/(2m)}\\
&= e^{itqE\mathbf{x}_1}\,e^{-i(\mathbf{p}_1+qEt)^3/(6mqE)}\,e^{i\mathbf{p}_1^3/(6mqE)}\,e^{-it\mathbf{p}_\perp^2/(2m)}\\
&= e^{itqE\mathbf{x}_1}\,e^{-i(\mathbf{p}_1^3+3\mathbf{p}_1^2qEt+3q^2E^2t^2\mathbf{p}_1+(qEt)^3)/(6mqE)}\,e^{i\mathbf{p}_1^3/(6mqE)}\,e^{-it\mathbf{p}_\perp^2/(2m)}\\
&= e^{itqE\mathbf{x}_1}\,e^{-iqEt/(2m)\mathbf{p}_1}\,e^{-it^3q^2E^2/(6m)}\,e^{-it\mathbf{p}_1^2/(2m)}\,e^{-it\mathbf{p}_\perp^2/(2m)}\\
&= e^{iqE\mathbf{x}_1t}e^{-it^3q^2E^2/(6m)}e^{-i\mathbf{p}_1qEt^2/(2m)}e^{-it\mathbf{p}^2/(2m)}.
\end{aligned}$$

Finally we take closures on both sides of the equation. ■

PROPOSITION IV.2.9. *Relations that are obtained under translation in configuration or momentum space. Proof of Equations II.1.32, II.1.33, II.1.34 and II.1.35.*

PROOF.

Let us denote by $\mathcal{F}$ and by $\hat{}$ the Fourier transform in $\mathbb{R}^n$. Let us take $\phi \in L^1(\mathbb{R}^n)$, and $g \in L^\infty(\mathbb{R}^n)$. We note that $g\,(\tau_{\mathbf{v}t}\phi) \in L^1(\mathbb{R}^n)$. Then, by proposition III.4.6,

$$\begin{aligned}
g(x + \mathbf{v}t)\phi(x) &= \tau_{-\mathbf{v}t}\left(g(x)\left(\tau_{\mathbf{v}t}\phi(x)\right)\right)\\
&= \mathcal{F}^{-1}\left[\tau_{-\mathbf{v}t}\left(g(x)\left(\tau_{\mathbf{v}t}\phi(x)\right)\right)\right]^{\hat{}}\\
&= \mathcal{F}^{-1}e^{i\mathbf{p}\cdot\mathbf{v}t}\mathcal{F}g(x)\mathcal{F}^{-1}\mathcal{F}\left(\tau_{\mathbf{v}t}\phi(x)\right)\\
&= \mathcal{F}^{-1}e^{i\mathbf{p}\cdot\mathbf{v}t}g(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{v}t}\mathcal{F}\phi(x)\\
g(x + \mathbf{v}t) &= \mathcal{F}^{-1}e^{i\mathbf{p}\cdot\mathbf{v}t}g(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{v}t}\mathcal{F}\\
\mathcal{F}g(x + \mathbf{v}t)\mathcal{F}^{-1} &= e^{i\mathbf{p}\cdot\mathbf{v}t}g(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{v}t}\\
g(\mathbf{x} + \mathbf{v}t) &= e^{i\mathbf{p}\cdot\mathbf{v}t}g(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{v}t}
\end{aligned}$$

and

$$\begin{aligned}
g(\mathbf{p}+m\mathbf{v})\phi &= \mathcal{F}^{-1}g(p+m\mathbf{v})\mathcal{F}\phi \\
&= \mathcal{F}^{-1}\tau_{-m\mathbf{v}}\left(g(p)\hat{\phi}(p-m\mathbf{v})\right) \\
&= \mathcal{F}^{-1}\tau_{-m\mathbf{v}}\left(\mathcal{F}^{-1}g(p)\hat{\phi}(p-m\mathbf{v})\right)^{\wedge} \\
&= \mathcal{F}^{-1}\left(e^{-im\mathbf{v}\cdot\mathbf{x}}\mathcal{F}^{-1}g(p)\hat{\phi}(p-m\mathbf{v})\right)^{\wedge} \\
&= e^{-im\mathbf{v}\cdot\mathbf{x}}\mathcal{F}^{-1}g(p)\mathcal{F}\mathcal{F}^{-1}\tau_{m\mathbf{v}}\hat{\phi}(p) \\
&= e^{-im\mathbf{v}\cdot\mathbf{x}}g(\mathbf{p})\mathcal{F}^{-1}\left(e^{im\mathbf{v}\cdot\mathbf{x}}\phi\right)^{\wedge} \\
&= e^{-im\mathbf{v}\cdot\mathbf{x}}g(\mathbf{p})e^{im\mathbf{v}\cdot\mathbf{x}}\phi.
\end{aligned}$$

Then, in $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ which is a dense set in $L^2(\mathbb{R}^n)$, one has that $g(\mathbf{x}+\mathbf{v}t)=e^{i\mathbf{p}\cdot\mathbf{v}t}g(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{v}t}$ and $g(\mathbf{p}+m\mathbf{v})=e^{-im\mathbf{v}\cdot\mathbf{x}}g(\mathbf{p})e^{im\mathbf{v}\cdot\mathbf{x}}$ in particular

$$\mathbf{x}+\mathbf{v}t = e^{i\mathbf{p}\cdot\mathbf{v}t}\,\mathbf{x}\,e^{-i\mathbf{p}\cdot\mathbf{v}t}, \qquad \text{(IV.2.38)}$$

$$\mathbf{p}+m\mathbf{v} = e^{-im\mathbf{v}\cdot\mathbf{x}}\,\mathbf{p}\,e^{im\mathbf{v}\cdot\mathbf{x}}. \qquad \text{(IV.2.39)}$$

Now, Let $\hat{\phi}\in C_0^\infty(\mathbb{R}^n)$. We compute as follows:

$$\begin{aligned}
e^{it\mathbf{p}^2/(2m)}\,\mathbf{x}\,e^{-it\mathbf{p}^2/(2m)}\hat{\phi}(\mathbf{p}) &= e^{it\mathbf{p}^2/(2m)}\left(e^{-it\mathbf{p}^2/(2m)}i\nabla_{\mathbf{p}}\,\hat{\phi}(\mathbf{p})+ie^{-it\mathbf{p}^2/(2m)}\left(-it\mathbf{p}/m\right)\hat{\phi}(\mathbf{p})\right) \\
&= \left(\mathbf{x}+t\,\mathbf{p}/m\right)\hat{\phi}(\mathbf{p}).
\end{aligned}$$

Then, in $C_0^\infty(\mathbb{R}^n)$:

$$\begin{aligned}
e^{-it\mathbf{p}^2/(2m)}\left(\mathbf{x}+t\,\mathbf{p}/m\right)e^{it\mathbf{p}^2/(2m)} &= \mathbf{x} \\
e^{it\mathbf{p}^2/(2m)}\left(\mathbf{x}\right)e^{-it\mathbf{p}^2/(2m)} &= \mathbf{x}+t\,\mathbf{p}/m. \qquad \text{(IV.2.40)}
\end{aligned}$$

This means that the operator $\mathbf{x}+t\,\mathbf{p}/m$ is diagonalizable.

By functional calculus, i.e, by (III.7.64), (IV.2.38), (IV.2.39) and (IV.2.40) we obtain (II.1.32), (II.1.33) and (II.1.35):

$$\begin{aligned}
f\left(\mathbf{x}+\mathbf{v}t\right) &= e^{i\mathbf{p}\cdot\mathbf{v}t}\,f(\mathbf{x})\,e^{-i\mathbf{p}\cdot\mathbf{v}t}, \quad \text{(II.1.32)} \\
f\left(\mathbf{p}+m\mathbf{v}\right) &= e^{-im\mathbf{v}\cdot\mathbf{x}}\,f(\mathbf{p})\,e^{im\mathbf{v}\cdot\mathbf{x}}, \quad \text{(II.1.33)} \\
f\left(\mathbf{x}+t\,\mathbf{p}/m\right) &= e^{it\mathbf{p}^2/(2m)}f\left(\mathbf{x}\right)e^{-it\mathbf{p}^2/(2m)}. \quad \text{(II.1.35)}
\end{aligned}$$

We remark that in (III.7.64), f can be any complex measurable function, not necessarily bounded nor continuous.

Finally, by (II.1.33) we get (II.1.34),

$$\begin{aligned}
e^{-im\mathbf{v}\cdot\mathbf{x}}e^{-it\mathbf{p}^2/(2m)}e^{im\mathbf{v}\cdot\mathbf{x}} &= e^{-it(\mathbf{p}+m\mathbf{v})^2/(2m)} = e^{-it(\mathbf{p}^2+2m\mathbf{v}\cdot\mathbf{p}+m^2\mathbf{v})/(2m)} \\
&= e^{-it\mathbf{p}^2/(2m)}\,e^{-it\mathbf{v}\cdot\mathbf{p}}\,e^{-itmv^2/2} \\
&= e^{-i\mathbf{p}\cdot\mathbf{v}t}e^{-it\mathbf{p}^2/(2m)}e^{-imv^2t/2}. \qquad \text{(IV.2.41)}
\end{aligned}$$

■

III Baker-Campbell-Hausdorff Formula

The Baker-Campbell-Hausdorff formula is used in Valencia and Weder [84] (II.2.34). Therein, we mention that our source is Enss [14]. In this section, we will clarify what Baker-Campbell-Hausdorff version we are using and the reason why we need it.

We recall that $\mathbf{x}$ and $t\,\mathbf{p}/m$ do not commute in Quantum Mechanics. For this reason, in order to compute $e^{i(\mathbf{x}+t\,\mathbf{p}/m)\cdot q}$ we use (II.1.35) with the function f equals the exponential function: $e^{i\tau q\cdot \bullet}$. Then, we apply (IV.2.41).

$$\begin{aligned} e^{i\tau q\cdot(\mathbf{x}+t\,\mathbf{p}/m)} &= e^{it\mathbf{p}^2/(2m)}e^{i\tau q\cdot\mathbf{x}}e^{-it\mathbf{p}^2/(2m)} \\ &= e^{it\mathbf{p}^2/(2m)}e^{-i\tau(-q)\cdot\mathbf{x}}e^{-i(t\tau/m)\mathbf{p}^2/(2\tau)}e^{i\tau(-q)\cdot\mathbf{x}}e^{i\tau q\cdot\mathbf{x}} \\ &= e^{it\mathbf{p}^2/(2m)}e^{-i\mathbf{p}\cdot(-q)(t\tau/m)}e^{-i(t\tau/m)\mathbf{p}^2/(2\tau)}e^{-i\tau q^2(t\tau/m)/2}e^{i\tau q\cdot\mathbf{x}} \\ &= e^{i\tau q\cdot(\mathbf{p}/m)t}e^{i\tau q\cdot\mathbf{x}}e^{-i\tau^2q^2t/(2m)}. \end{aligned} \tag{IV.3.42}$$

Our (IV.3.42) is equivalent to equation (13) in Enss [14], taking his velocity operator as $\mathbf{p}/m$.

LEMMA IV.3.1. *Let V be a real function with domain $\mathbb{R}^n$. If $V(\mathbf{x})$ and its first and second order derivatives are continuous and decay towards infinity, $m > 0$ and t is real, then*

$$V(\mathbf{x}+t\,\mathbf{p}/m) - V(t\,\mathbf{p}/m) = \int_0^1 ds\left[(\nabla V)(s\mathbf{x}+t\,\mathbf{p}/m)\cdot\mathbf{x} + \frac{it}{2m}(\Delta V)(s\mathbf{x}+t\,\mathbf{p}/m)\right]. \tag{IV.3.43}$$

Proof. Let us denote $\widehat{\ }$ as Fourier Transform. Because V is measurable and bounded, and both $\mathbf{x}+t\,\mathbf{p}/m$ and $t\,\mathbf{p}/m$ are self-adjoint operators in $S(\mathbb{R}^n)$, we can apply functional calculus:

$$V(\mathbf{x}+t\,\mathbf{p}/m) = \frac{1}{(2\pi)^{n/2}}\int d^nq\,\widehat{V}(q)e^{i(\mathbf{x}+t\,\mathbf{p}/m)\cdot q} \tag{IV.3.44}$$

$$V(t\,\mathbf{p}/m) = \frac{1}{(2\pi)^{n/2}}\int d^nq\,\widehat{V}(q)e^{i(t\,\mathbf{p}/m)\cdot q}. \tag{IV.3.45}$$

By (IV.3.42), for any θ real

$$e^{i(\theta\mathbf{x}+t\,\mathbf{p}/m)\cdot q} = e^{it\,\mathbf{p}/m\cdot q}\,e^{i\theta\mathbf{x}\cdot q}\,e^{-i\frac{\theta t}{2m}q^2}. \tag{IV.3.46}$$

Therefore, by (IV.3.45) and (IV.3.46),

$$V(\mathbf{x}+t\,\mathbf{p}/m) - V(t\,\mathbf{p}/m) = \frac{1}{(2\pi)^{n/2}}\int d^nq\,\widehat{V}(q)e^{it\,\mathbf{p}/m\cdot q}\left[e^{i\mathbf{x}\cdot q}\,e^{-i\frac{t}{2m}q^2} - 1\right]. \tag{IV.3.47}$$

By Duhamel formula:

$$e^{i\mathbf{x}\cdot q}\,e^{-i\frac{t}{2m}q^2} - 1 = \int_0^1 d\theta\,\frac{d}{d\theta}\left[e^{\theta(i\mathbf{x}\cdot q - i\frac{t}{2m}q^2)}\right] = \int_0^1 d\theta\,i(\mathbf{x}\cdot q - \frac{t}{2m}q^2)e^{\theta(i\mathbf{x}\cdot q - i\frac{t}{2m}q^2)}. \tag{IV.3.48}$$

Substituting (IV.3.46), (IV.3.48) into (IV.3.47):

$$V(\mathbf{x}+t\,\mathbf{p}/m) - V(t\,\mathbf{p}/m) = \frac{1}{(2\pi)^{n/2}}\int\int_0^1 \widehat{V}(q)\,e^{it\,\mathbf{p}/m\cdot q}\,e^{i\theta\mathbf{x}\cdot q}\,e^{-i\frac{\theta t}{2m}q^2}\,i(\mathbf{x}\cdot q - \frac{t}{2m}q^2)d\theta\,d^nq,$$

by (IV.3.46), Fubini and inverting Fourier transform:

$$\begin{aligned} V(\mathbf{x}+t\,\mathbf{p}/m) - V(t\,\mathbf{p}/m) &= \frac{1}{(2\pi)^{n/2}}\int\int_0^1 \widehat{V}(q)\,e^{it\,\mathbf{p}/m\cdot q + i\theta\mathbf{x}\cdot q}i(\mathbf{x}\cdot q - \frac{t}{2m}q^2)d\theta\,d^nq, \\ &= \int_0^1 \frac{1}{(2\pi)^{n/2}}\int e^{it\,\mathbf{p}/m\cdot q + i\theta\mathbf{x}\cdot q}\,\widehat{V}(q)\,(i\mathbf{x}\cdot q - \frac{it}{2m}q^2)d^nq\,d\theta, \\ &= \int_0^1 \frac{1}{(2\pi)^{n/2}}\int e^{i(t\,\mathbf{p}/m+\theta\mathbf{x})\cdot q}\,(\mathbf{x}\cdot(iq) + \frac{it}{2m}(-q^2))\widehat{V}(q)d^nq\,d\theta, \\ &= \int_0^1 \left[(\mathbf{x}\cdot\nabla_{\mathbf{x}} + \frac{it}{2m}(\triangle_{\mathbf{x}}))V\right](t\,\mathbf{p}/m+\theta\mathbf{x})\,d\theta, \\ &= \int_0^1 \left[(\nabla V)(t\,\mathbf{p}/m+\theta\mathbf{x})\cdot\mathbf{x} + \frac{it}{2m}(\triangle V)(t\,\mathbf{p}/m+\theta\mathbf{x})\right]d\theta. \end{aligned}$$

The Fourier step is justified by the following relations (given in Enss [16] equation 2.5), with $\mathbf{x}$ and q being the conjugate variables:

$$-i\nabla_{\mathbf{x}} \longleftrightarrow q \iff \nabla_{\mathbf{x}} \longleftrightarrow iq,$$

and

$$-i\nabla_{\mathbf{x}}(-i\nabla_{\mathbf{x}}) \longleftrightarrow q(q) \iff -\triangle_{\mathbf{x}} \longleftrightarrow q^2.$$

■

IV High-velocity estimates

COMPUTATION IV.4.1. *Proof of* (II.2.14).

PROOF.

Let us show (II.2.13). Let r_1 and r_2 be two vectors in $\mathbb{R}^n$. Please note that

$$-2|r_1||r_2| \geq -r_1^2 - r_2^2. \tag{IV.4.49}$$

By (IV.4.49):

$$\begin{aligned}
|r_1 + r_2| &= \sqrt{r_1^2 + r_2^2 + 2r_1 \cdot r_2} \geq \sqrt{r_1^2 + r_2^2 - 2r_1 \cdot r_2} \geq \sqrt{r_1^2 + r_2^2 - 2|r_1||r_2||\hat{r}_1 \cdot \hat{r}_2|} \\
|r_1 + r_2| &\geq \sqrt{r_1^2 + r_2^2}\,\sqrt{1 - |\hat{r}_1 \cdot \hat{r}_2|}.
\end{aligned} \tag{IV.4.50}$$

Taking, $r_1 = \mathbf{v}t$, $r_2 = \mathbf{e}_1 q_{12} E t^2/(2\mu_{12})$ and noting, by (II.2.12), that $1 - |\hat{r}_1 \cdot \hat{r}_2| \geq 1 - \delta > 0$, we obtain (II.2.13).

For all $r \in \mathbb{R}^n$,

$$\sum_{i=2}^{n} |\hat{r} \cdot \mathbf{e}_i|^2 = 1 - |\hat{r} \cdot \mathbf{e}_1|^2. \tag{IV.4.51}$$

For $r = (r \cdot \mathbf{e}_1, r \cdot \mathbf{e}_2, \ldots, r \cdot \mathbf{e}_n) \in \mathbb{R}^n$, we can define a vector $r_\perp = (r \cdot \mathbf{e}_2, \ldots, r \cdot \mathbf{e}_n) \in \mathbb{R}^{n-1}$. By (IV.4.51),

$$|r_\perp|^2 = |r|^2 - |r \cdot \mathbf{e}_1|^2. \tag{IV.4.52}$$

We have the following equivalency: There exists $0 \leq \Delta_1 < 1$ such that $|r \cdot \mathbf{e}_1| \leq \Delta_1 |r|$, if and only if, there exists $0 < \Delta_2 \leq 1$ such that $|r_\perp| \geq \Delta_2 |r|$. We note that $\Delta_2^2 = 1 - \Delta_1^2$.

Now, we make $r_1 = \mathbf{v} + \mathbf{p}/\mu_{12}$, $r_2 = \mathbf{e}_1 q_{12} E t/(2\mu_{12})$, with $|\mathbf{p}| \leq \mu_{12}\eta$, and $\eta/v < \sqrt{1-\delta}/4$. Using (IV.4.52):

$$\begin{aligned}
|r_{1\perp}| &= |(\mathbf{v} + \mathbf{p}/\mu_{12})_\perp| \geq |\mathbf{v}_\perp| - |\left(\mathbf{p}/\mu_{12}\right)_\perp| \geq \sqrt{1-\delta^2}\, v - |\left(\mathbf{p}/\mu_{12}\right)_\perp| \geq \left(\sqrt{1-\delta^2} - \sqrt{1-\delta}/4\right) v \\
&\geq \left(\sqrt{1+\delta} - 1/4\right) \sqrt{1-\delta}\, v \geq 3\sqrt{1-\delta}\, v/4 = \frac{3\sqrt{1-\delta}\, v/4}{|\mathbf{v} + \mathbf{p}/\mu_{12}|} |r_1| \geq \frac{3\sqrt{1-\delta}\, v/4}{|\mathbf{v}| + |\mathbf{p}/\mu_{12}|} |r_1| \\
&> \frac{3\sqrt{1-\delta}}{4 + \sqrt{1-\delta}} |r_1| = \frac{3\sqrt{1-\delta}\left(4 - \sqrt{1-\delta}\right)}{16 - (1-\delta)} |r_1| = \frac{12\sqrt{1-\delta} - 3 + 3\delta}{15 + \delta} |r_1|.
\end{aligned} \tag{IV.4.53}$$

From (IV.4.53), we define $\Delta_2 = \frac{12\sqrt{1-\delta}-3+3\delta}{15+\delta}$. Clearly $0 < \Delta_2$. To see that $\Delta_2 \leq 1$, it is enough to note that: $\delta < 1 < 3$, then $12\sqrt{1-\delta} < 12 < 18 - 2\delta$, then $12\sqrt{1-\delta} - 3 + 3\delta < 15 + \delta$.

Let us observe the following estimate:

$$|r_1| = |\mathbf{v} + \mathbf{p}/\mu_{12}| \geq |\mathbf{v}| - |\mathbf{p}/\mu_{12}| > \left(1 - \sqrt{1-\delta}/4\right) v \geq (1 - 1/4)\, v = 3v/4. \tag{IV.4.54}$$

Hence, by our equivalency above:

$$|\hat{r}_1 \cdot \mathbf{e}_1| \leq \Delta_1 = \sqrt{1-\Delta_2^2} < 1. \tag{IV.4.55}$$

We can now estimate what we want. By (IV.4.50), (IV.4.54) and (IV.4.55);

$$\begin{aligned} |r_1 + r_2|\,|t| &\geq |r_1|\sqrt{1-\Delta_1}\,|t| > \frac{3\sqrt{1-\Delta_1}}{4}\,|\mathbf{v}t|, && \text{(IV.4.56)} \\ |r_1 + r_2|\,|t| &\geq |r_2|\sqrt{1-\Delta_1}\,|t| = \frac{\sqrt{1-\Delta_1}q_{12}E}{2\mu_{12}}\,t^2. && \text{(IV.4.57)} \end{aligned}$$

This proves (II.2.14) with $c_1 = \frac{3\sqrt{1-\Delta_1}}{4}$, and $c_2 = \frac{\sqrt{1-\Delta_1}q_{12}E}{2\mu_{12}}$.

■

LEMMA IV.4.2. *More details for the proof of Lemma II.2.2.*

Proof. In the proof of Lemma II.2.2, the norm of the operator

$$G_{jk} := (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a\ \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-a/2} \tag{IV.4.58}$$

is estimated. Let us see why this estimation is correct.
We rewrite:

$$G_{jk} := (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a\,\tilde{U}_D(t)\,J_{jk}, \quad \text{where} \quad J_{jk} := \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-a/2}.$$

If a is an odd number, we notice that G_{jk} is a mapping from a dense subset of $L^2(\mathbb{R}^n)$ to $\left(L^2(\mathbb{R}^n)\right)^n$. More explicitly, if a is an odd number and ϕ is a function in a dense set $L^2(\mathbb{R}^n)$, we note that:

$$G_{jk}\,\phi = \Big((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_1\,(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^{a-1}\,\tilde{U}_D(t)\,J_{jk}\,\phi,\ \ldots,\ (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_n\,(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^{a-1}\,\tilde{U}_D(t)\,J_{jk}\,\phi\Big).$$

If a is an odd number, the norm of $G_{jk}\,\phi$ is

$$\begin{aligned} \|G_{jk}\,\phi\|^2 &= \int_{\mathbb{R}^n}\sum_{l=1}^n \left|(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l\,(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^{a-1}\,\tilde{U}_D(t)\,J_{jk}\,\phi\right|^2 \\ &= \sum_{l=1}^n \int_{\mathbb{R}^n} \left|G_{jk}\cdot\mathbf{e}_l\,\phi\right|^2 = \sum_{l=1}^n \left\|G_{jk}\cdot\mathbf{e}_l\,\phi\right\|^2. \end{aligned} \tag{IV.4.59}$$

Otherwise, if a is an even number:

$$\|G_{jk}\,\phi\|^2 = \int_{\mathbb{R}^n}\left|(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a\,\tilde{U}_D(t)\,J_{jk}\,\phi\right|^2. \tag{IV.4.60}$$

The norm $\|G_{jk}\|$ is computed in the customary way:

$$\|G_{jk}\| := \sup_{\|\phi\|=1} \|G_{jk}\,\phi\|.$$

When a is even, $(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a$ is a polynomial of degree a, and when a is odd is a vector in $\mathbb{R}^n$ whose entries are polynomials of degree a. These polynomials are of the form:

$$(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^a = \begin{cases} \left((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_1,\ \ldots,\ (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_n\right), & \text{if } a=1, \\ \sum_{r=1}^n\sum_{r'=1}^n \left((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_r\right)^2\left((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{r'}\right)^2, & \text{if } a=4. \end{cases}$$

Therefore, a being 1 or 4, $\|G_{jk}\|$ can be estimated by linear combinations of the norm of

$$(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^{\alpha_b}\, \tilde{U}_D(t)\, J_{jk}, \tag{IV.4.61}$$

where the multi-index $\alpha_b = (\alpha_{b1}, \ldots, \alpha_{bn})$ and $(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^{\alpha_b} = ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_1)^{\alpha_{b1}} \cdots ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_n)^{\alpha_{bn}}$, with α_{br} an even number for all $1 \le r \le n$, if $a = 4$.

We reproduce equations (II.1.21) and (II.1.22)

$$\mathbf{x} = \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1 = i\frac{\partial}{\partial \mathbf{p}}, \quad \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j = i\frac{\partial}{\partial \mathbf{p}_k} - i\frac{\partial}{\partial \mathbf{p}_j}, \quad j,k = 3,\ldots,N, \tag{II.1.21}$$

$$\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_1 = i\frac{\partial}{\partial \mathbf{p}_k} + \frac{\mu_{12}}{m_1} i\frac{\partial}{\partial \mathbf{p}}, \quad \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_2 = i\frac{\partial}{\partial \mathbf{p}_k} - \frac{\mu_{12}}{m_2} i\frac{\partial}{\partial \mathbf{p}}, \quad k = 3,\ldots,N. \tag{II.1.22}$$

Similarly, we recall that

$$\mathbf{p} = -\frac{m_1}{\mu_1}\mathbf{p}_1, \quad \text{(IV.1.28)}$$
$$\mathbf{p} = \frac{m_2}{\mu_2}\mathbf{p}_2. \quad \text{(IV.1.29)}$$

We start with the case $a = 1$, by (II.1.21) and (II.1.22), for $1 \le l \le n$, $((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)$ are first order derivatives. Then, by (IV.4.61) and the Leibnitz rule,

$$\begin{aligned}
\|G_{jk}\| &\le C\sum_{l=1}^{n}\left[\left\|\Big(((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,\tilde{U}_D(t)\Big)\, J_{jk}\right\| + \left\|\tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\, J_{jk}\right\|\right] \\
&\le C\sum_{l=1}^{n}\left[\left\|((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})\right\|\right. \\
&\quad \left. + \left\|((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})\right\| + \left\|\frac{\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j}{(1+|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{1/2}}\right\|\right].
\end{aligned}$$

Because $f_{j'k'}$ are compactly supported infinitely differentiable functions for all j', k' we have that:

$$\|G_{jk}\| \le C\sum_{l=1}^{n}\left[1 + \left\|((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,\tilde{U}_D(t)\right\|\right]. \tag{IV.4.62}$$

We remember that $V : \mathbb{R}^n \to \mathbb{R}$, $(y_1, \ldots, y_n) \mapsto V(y_1, \ldots, y_n)$. We are composing it after the mapping g, from $\left\{(p, p_3, \ldots, p_N)\,\middle|\, p = -\frac{m_1}{\mu_1}p_1 = \frac{m_2}{\mu_2}p_2\right\} \cong \mathbb{R}^{n(N-1)} \subset \mathbb{R}^{nN}$ onto $\mathbb{R}^n$, $(p, p_3, \ldots, p_N) \mapsto s(p_k/\mu_k - p_j/\mu_j) + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk})$, with s any real. The derivative $D(V\circ g)$ is the following:

$$\begin{aligned}
D(V\circ g) &= (\nabla V(g(p, p_3, \ldots, p_N)))_{1\times n}\left(\left(\frac{\partial g}{\partial p_{,1}}\right)^T, \left(\frac{\partial g}{\partial p_{,2}}\right)^T, \ldots, \left(\frac{\partial g}{\partial p_{N,n}}\right)^T\right)_{n\times n(N-1)} \\
&= \left(\sum_{r=1}^{n}\frac{\partial V}{\partial y_r}(g(p, p_3, \ldots, p_N))\frac{\partial g\cdot \mathbf{e}_r}{\partial p_{,1}}, \ldots, \sum_{r=1}^{n}\frac{\partial V}{\partial y_r}(g(p, p_3, \ldots, p_N))\frac{\partial g\cdot \mathbf{e}_r}{\partial p_{N,n}}\right)_{1\times n(N-1)} \\
&= \left(\frac{\partial V}{\partial y_1}(g(p, p_3, \ldots, p_N))\frac{\partial g\cdot \mathbf{e}_1}{\partial p_{,1}}, \ldots, \frac{\partial V}{\partial y_n}(g(p, p_3, \ldots, p_N))\frac{\partial g\cdot \mathbf{e}_n}{\partial p_{,n}}, \ldots,\right. \\
&\quad \left.\frac{\partial V}{\partial y_1}(g(p, p_3, \ldots, p_N))\frac{\partial g\cdot \mathbf{e}_1}{\partial p_{N,1}}, \ldots, \frac{\partial V}{\partial y_n}(g(p, p_3, \ldots, p_N))\frac{\partial g\cdot \mathbf{e}_n}{\partial p_{N,n}}\right)_{1\times n(N-1)}.
\end{aligned}$$

Having computed the derivative of the potential, we have that, for $3 \le l' \le n$, $g_{j'k'} \in C_0^\infty(B_{\mu_{j'k'}\eta_{j'k'}})$ and $g_{j'k'} = 1$ in the support of $f_{j'k'}$.

$$\begin{aligned}
&\left\| \left(\left(\frac{\partial}{\partial \mathbf{p}_{l'}} \cdot \mathbf{e}_l \right) \tilde{U}_D(t) \right) \right. \\
&\left. \times \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'}) \right\| = \left\| \tilde{U}_D(t) \sum_{j'<k'} \int_0^t ds \, \frac{\partial}{\partial p_{l',l}} \left(V^l_{j'k'} \left(s(p_{k'}/\mu_{k'} - p_{j'}/\mu_{j'}) \right. \right. \right. \\
&\qquad \left. \left. \left. + \mathbf{e}_1 q_{j'k'} E s^2/(2\mu_{j'k'}) \right) \right) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'}) \right\| \\
&\le \sum_{j'<k'} \left\| g_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'}) \int_0^t ds \, \nabla V^l_{j'k'} \left(s\mathbf{p}_{j'k'}/\mu_{jk} \right. \right. \\
&\qquad \left. \left. + \mathbf{e}_1 q_{j'k'} E s^2/(2\mu_{j'k'}) \right) \cdot \frac{\partial g}{\partial p_{l',l}} \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'}) \right\| \\
&\le C \sum_{j'<k'} \left\| g_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'}) \int_0^t ds \, |s| \frac{\partial V^l_{j'k'}}{\partial y_l} (s\mathbf{p}_{j'k'}/\mu_{j'k'} \right. \\
&\qquad \left. + \mathbf{e}_1 q_{j'k'} E s^2/(2\mu_{j'k'})) \begin{cases} 1/\mu_{k'}, & \text{if } l' = k', \\ 1/\mu_{j'}, & \text{if } l' = j', \\ 0, & \text{if } l' \ne j' \text{ and } l' \ne k', \end{cases} \right\| \\
&\le C \sum_{j'<k',\, l' \in \{j',k'\}} \left\| \int_0^t |s| \frac{\partial V^l_{j'k'}}{\partial y_l} (s(\mathbf{p}_{j'k'}/\mu_{j'k'} + \mathbf{v}_{j'k'}) \right. \\
&\qquad \left. + \mathbf{e}_1 q_{j'k'} E s^2/(2\mu_{j'k'}))\, g_{j'k'}(\mathbf{p}_{j'k'}) \, ds \right\|, \qquad \text{(IV.4.63)}
\end{aligned}$$

Likewise, by (IV.1.28) and (IV.1.29),

$$\begin{aligned}
\left\| \left(\frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{e}_l \right) \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'}) \right\| &\le C \sum_{j'<k',\, j' \in \{1,2\}} \left\| \int_0^t |s| \frac{\partial V^l_{j'k'}}{\partial y_l} (s(\mathbf{p}_{j'k'}/\mu_{j'k'} + \mathbf{v}_{j'k'}) \right. \\
&\qquad \left. + \mathbf{e}_1 q_{j'k'} E s^2/(2\mu_{j'k'}))\, g_{j'k'}(\mathbf{p}_{j'k'}) \, ds \right\|. \qquad \text{(IV.4.64)}
\end{aligned}$$

In both (IV.4.64) and (IV.4.63), we could interchange the derivative with the integral by using the same estimations given in (II.2.20) that allow to dominate the integrand and its derivatives with respect to the parameters $\mathbf{p}, \mathbf{p}_3, \ldots, \mathbf{p}_N$ by an integrable function depending only on the integration variable s.

Hence, by (IV.4.62), (IV.4.63) and (IV.4.64), for $1 \le j < k \le N$, with I_{β_b} is as in (II.2.18),

$$\left\| (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\, \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-1/2} \right\| \le C \left(1 + \sum_{|\beta_b|=1} I_{\beta_b} \right) \qquad \text{(IV.4.65)}$$

Let us write $I^{j'k'}_{\beta_b}$ instead of I_{β_b} is as in (II.2.18). Then, the terms that are summed in the right hand side of (IV.4.65) are those pairs (j', k') such that $1 \le j' < k' \le N$, and either $\{j', k'\} \cap \{j, k\} \ne \emptyset$ or $j = 1$ and $j' = 2$ or $j = 2$ and $j' = 1$.

Let us define a new set of pairs,

$$N_{jk} := \left\{(j',k')\,\middle|\,1 \leq j' < k' \leq N \text{ and either } j'=j \text{ or } j'=k \text{ or } k'=j \text{ or } k'=k \text{ or } j'+j=3\right\}. \quad \text{(IV.4.66)}$$

Then, (II.2.18) can be reformulated as:

$$\left\|(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\,\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j|^2)^{-1/2}\right\| \leq C\left(1 + \sum_{(j',k')\in N_{jk}}^{|\beta_b|=1} I_{\beta_b}^{j'k'}\right). \quad \text{(IV.4.67)}$$

The presence of $g_{j'k'}(\mathbf{p}_{j'k'})$ allows us to only consider $|\mathbf{p}_{j'k'}| \leq \mu_{j'k'}\eta_{j'k'}$, necessary to apply the estimations given by (II.2.13) and (II.2.14). We use the definition of (II.2.18) and (II.2.19) and the decay of the potentials given in (II.1.24), (II.1.25), (II.1.29), (II.1.30). Remember that we take $\tilde{\sigma} = 0$, if $q_{j'k'} \neq 0$ and $\tilde{\sigma} = 1$, if $q_{j'k'} = 0$. In inequality (II.2.20), we make the change of variable $\tau = v^{\tilde{\sigma}/(2-\tilde{\sigma})}s$. Then

$$\begin{aligned}
I_{\beta_b} &\leq C\int_0^{|t|} s^{|\beta_b|} i_{\beta_b, v_{j'k'}}(s)\,ds \\
&= C v_{j'k'}^{-(|\beta_b|+1)\tilde{\sigma}/(2-\tilde{\sigma})}\int_0^{|t|} (v_{j'k'}^{\tilde{\sigma}/(2-\tilde{\sigma})}s)^{|\beta_b|} i_{\beta_b,1}(v_{j'k'}^{\tilde{\sigma}/(2-\tilde{\sigma})}s)\,d(v_{j'k'}^{\tilde{\sigma}/(2-\tilde{\sigma})}s) \\
&= C v_{j'k'}^{-(|\beta_b|+1)\tilde{\sigma}/(2-\tilde{\sigma})}\int_0^{v_{j'k'}^{\tilde{\sigma}/(2-\tilde{\sigma})}|t|} \tau^{|\beta_b|} i_{\beta_b,1}(\tau)\,d\tau. \quad \text{(II.2.20)}.
\end{aligned}$$

Let us compute I_{β_b}, focusing in $|\beta_b| = 1$.
If $q_{j'k'} \neq 0$, then we have, $2(-\gamma_D - \mu) + 2 < 0$, that

$$\begin{aligned}
I_{\beta_b} &\leq C\int_0^{|t|} \tau(1+\tau^2)^{-\gamma_D-\mu}\,d\tau \leq C\int_0^{|t|} \tau(1+\tau)^{2(-\gamma_D-\mu)}\,d\tau \leq C\int_0^{|t|} (1+\tau)^{2(-\gamma_D-\mu)+1}\,d\tau \\
&\leq C(1+\tau)^{2(-\gamma_D-\mu)+2}\Big|_0^{|t|} \leq C(1+|t|)^{2(-\gamma_D-\mu)+2} < C, \qquad \text{(IV.4.68)}
\end{aligned}$$

else, if $q_{j'k'} = 0$ then we have that

$$I_{\beta_b} \leq C v_{j'k'}^{-2}\int_0^{v_{j'k'}|t|} \tau(1+\tau)^{-\gamma_1}\,d\tau \leq v_{j'k'}^{-2}\int_0^{v_{j'k'}|t|} (1+\tau)^{-\gamma_1+1}\,d\tau$$

$$\leq C v_{j'k'}^{-2}\begin{cases}(1+\tau)^{-\gamma_1+2}\Big|_0^{v_{j'k'}|t|}, & \text{if } \gamma < 2,\\ \ln(1+\tau)\Big|_0^{v_{j'k'}|t|}, & \text{if } \gamma = 2,\end{cases} \leq C v_{j'k'}^{-2}\begin{cases}(1+|v_{j'k'}t|)^{2-\gamma_1}, & \text{if } \gamma < 2,\\ \ln(1+|v_{j'k'}t|), & \text{if } \gamma = 2\end{cases} \quad \text{(IV.4.69)}$$

We want to pass from $v_{j'k'}$ to v_{jk} in (IV.4.69). We have three cases:

(*a*) Either $(j',k') = (j,k) = (1,2)$, or $(j',k') \neq (1,2)$ and $(j,k) \neq (1,2)$. In the first instance, $v_{j'k'} = v_{jk} = v$, and in the second, $v_{j'k'} = Cv^2$, $v_{jk} = Cv^2$. In both cases, $v_{j'k'} = Cv_{jk}$. Then

$$v_{j'k'}^{-2}|v_{j'k'}t|^{2-\gamma_1} = C v_{jk}^{-2}|v_{jk}t|^{2-\gamma_1}.$$

(*b*) $(j',k') \neq (1,2)$ and $(j,k) = (1,2)$. Here, $v_{j'k'} = Cv^2$, $v_{jk} = v$, therefore, $v_{j'k'} = Cv_{jk}^2$ and

$$v_{j'k'}^{-2}|v_{j'k'}t|^{2-\gamma_1} = C v_{jk}^{-4}|v_{jk}^2 t|^{2-\gamma_1} = C v_{jk}^{-4+2-\gamma_1}|v_{jk}t|^{2-\gamma_1} = C v_{jk}^{-(2+\gamma_1)}|v_{jk}t|^{2-\gamma_1}.$$

(c) $(j',k')=(1,2)$ and $(j,k)\neq(1,2)$. In this case, $v_{j'k'}=v$, $v_{jk}=Cv^2$, therefore, $v_{j'k'}=Cv_{jk}^{1/2}$ and

$$v_{j'k'}^{-2}|v_{j'k'}t|^{2-\gamma_1}=Cv_{jk}^{-1}|v_{jk}^{1/2}t|^{2-\gamma_1}=Cv_{jk}^{-1-1+\gamma_1/2}|v_{jk}t|^{2-\gamma_1}=Cv_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{2-\gamma_1}.$$

By L'Hôpital rule:

$$\lim_{\tau\to\infty}\frac{v_{jk}^{-2}\ln(1+\tau)}{v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}\tau)}=\lim_{\tau\to\infty}v_{jk}^{-1}\frac{\frac{1}{1+\tau}}{\frac{v_{jk}^{-1/2}}{1+v_{jk}^{-1/2}\tau}}=\lim_{\tau\to\infty}v_{jk}^{-1/2}\frac{1+v_{jk}^{-1/2}\tau}{1+\tau}\lim_{\tau\to\infty}v_{jk}^{-1/2}\frac{\frac{1}{\tau}+v_{jk}^{-1/2}}{\frac{1}{\tau}+1}=v_{jk}^{-1},$$

$$\lim_{\tau\to\infty}\frac{v_{jk}^{-4}\ln(1+v_{jk}\tau)}{v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}\tau)}=\lim_{\tau\to\infty}v_{jk}^{-3}\frac{\frac{v_{jk}}{1+v_{jk}\tau}}{\frac{v_{jk}^{-1/2}}{1+v_{jk}^{-1/2}\tau}}=\lim_{\tau\to\infty}v_{jk}^{-3/2}\frac{1+v_{jk}^{-1/2}\tau}{1+v_{jk}\tau}=v_{jk}^{-3},$$

and

$$\begin{aligned}\lim_{\tau\to\infty}\frac{v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}\tau)}{1+v_{jk}^{-(2-\gamma_1/2)}\tau^{\epsilon}} &= \lim_{\tau\to\infty}v_{jk}^{1-\gamma_1/2}\frac{\frac{v_{jk}^{-1/2}}{1+v_{jk}^{-1/2}\tau}}{\epsilon\tau^{\epsilon-1}}\\ &= \frac{v_{jk}^{(1-\gamma_1)/2}}{\epsilon}\lim_{\tau\to\infty}\frac{1}{\tau^{\epsilon-1}+v_{jk}^{-1/2}\tau^{\epsilon}}=0. \qquad\text{(IV.4.70)}\end{aligned}$$

Then, by (IV.4.69)

$$\begin{aligned} I_{\beta_b} &\leq C\begin{cases}1+v_{jk}^{-2}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2 \text{ and either } (j',k')=(j,k)=(1,2),\\ & \text{or } (j',k')\neq(1,2) \text{ and } (j,k)\neq(1,2),\\ 1+v_{jk}^{-(2+\gamma_1)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2, (j',k')\neq(1,2) \text{ and } (j,k)=(1,2),\\ 1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2, (j',k')=(1,2) \text{ and } (j,k)\neq(1,2)\\ v_{jk}^{-2}\ln(1+|v_{jk}t|), & \text{if } \gamma_1=2 \text{ and either } (j',k')=(j,k)=(1,2),\\ & \text{or } (j',k')\neq(1,2) \text{ and } (j,k)\neq(1,2),\\ v_{jk}^{-4}\ln(1+v_{jk}|v_{jk}t|), & \text{if } \gamma_1=2, (j',k')\neq(1,2) \text{ and } (j,k)=(1,2),\\ v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}|v_{jk}t|), & \text{if } \gamma_1=2, (j',k')=(1,2) \text{ and } (j,k)\neq(1,2),\end{cases}\\ &\leq C\begin{cases}1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2,\\ v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}|v_{jk}t|), & \text{if } \gamma_1=2.\end{cases} \qquad\text{(IV.4.71)}\end{aligned}$$

By (IV.4.68) and (IV.4.71)

$$I_{\beta_b}\leq C\begin{cases}1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \gamma_1<2,\ q_{j'k'}\neq 0,\\ v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}|v_{jk}t|), & \text{if } \gamma_1=2,\ q_{j'k'}\neq 0,\\ 1, & \text{if } q_{j'k'}=0.\end{cases} \qquad\text{(IV.4.72)}$$

Finally (IV.4.67), (IV.4.70) and (IV.4.72) imply that

$$\begin{aligned} A_{jk} &:= \left\|(\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\,\tilde{U}_D(t)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+|\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j|^2)^{-1/2}\right\|\\ &\leq C\begin{cases}1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{2-\gamma_1}, & \text{if } \zeta_{jk}^a,\\ 1+v_{jk}^{-1}\ln(1+v_{jk}^{-1/2}|v_{jk}t|), & \text{if } \zeta_{jk}^b,\\ 1, & \text{if } \zeta_{jk}^c,\end{cases}\\ &\leq C\left(1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{\theta_{jk}}\right).\end{aligned}$$

This completes the proof of (II.2.16).

At this moment, we are particularly interested in $(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^4$:

$$\begin{aligned}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)^4 &= \left(\sum_{l=1}^{n} ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^2\right)^2 \\ &= \sum_{l=1}^{n} ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^4 + 2\sum_{1\le l<l''\le n} ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^2 ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})^2 \\ &= \sum_{l=1}^{n} \left(\tilde{\mathbf{x}}_{k,l}^4 - 4\tilde{\mathbf{x}}_{k,l}^3\tilde{\mathbf{x}}_{j,l} + 6\tilde{\mathbf{x}}_{k,l}^2\tilde{\mathbf{x}}_{j,l}^2 - 4\tilde{\mathbf{x}}_{k,l}\tilde{\mathbf{x}}_{j,l}^3 + \tilde{\mathbf{x}}_{j,l}^4\right) \\ &\quad +2\sum_{1\le l<l'\le n}\left(\tilde{\mathbf{x}}_{k,l}^2\tilde{\mathbf{x}}_{k,l'}^2 - 2\tilde{\mathbf{x}}_{k,l}^2\tilde{\mathbf{x}}_{k,l'}\tilde{\mathbf{x}}_{j,l'} + \tilde{\mathbf{x}}_{k,l}^2\tilde{\mathbf{x}}_{j,l'}^2 - 2\tilde{\mathbf{x}}_{k,l}\tilde{\mathbf{x}}_{j,l}\tilde{\mathbf{x}}_{k,l'}^2 + 4\tilde{\mathbf{x}}_{k,l}\tilde{\mathbf{x}}_{j,l}\tilde{\mathbf{x}}_{k,l'}\tilde{\mathbf{x}}_{j,l'}\right. \\ &\quad \left. -2\tilde{\mathbf{x}}_{k,l}\tilde{\mathbf{x}}_{j,l}\tilde{\mathbf{x}}_{j,l'}^2 + \tilde{\mathbf{x}}_{j,l}^2\tilde{\mathbf{x}}_{k,l'}^2 - 2\tilde{\mathbf{x}}_{j,l}^2\tilde{\mathbf{x}}_{k,l'}\tilde{\mathbf{x}}_{j,l'} + \tilde{\mathbf{x}}_{j,l}^2\tilde{\mathbf{x}}_{j,l'}^2\right).\end{aligned}$$

We continue with the case $a = 4$. By the same argument used to obtain (IV.4.62), we have that, for non-negative integers a_1, a_2 such that $a_1 + a_2 \le 4$ and all $1 \le l, l'' \le n$,

$$\|((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^{a_1} ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})^{a_2} J_{jk}\| \le C. \tag{IV.4.73}$$

By (II.1.21), (II.1.22), (IV.4.61), (IV.4.73) and the Leibnitz rule, for $1 \le l, l' \le n$,

$$\|G_{jk}\| \le C\left(1 + \sum_{1\le a_1+a_2\le 4}^{1\le l,l\le n} \|(\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^{a_1} ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})^{a_2} \tilde{U}_D(t)\,)\, J_{jk}\|\right). \tag{IV.4.74}$$

It is convenient to write our Dollard-type modifier as

$$\tilde{U}_D(t) = e^{M(t)}, \quad M(t) := -i\sum_{j<k}\int_0^t ds\, V_{jk}^l(s\mathbf{p}_{jk}/\mu_{jk} + \mathbf{e}_1 q_{jk}Es^2/(2\mu_{jk})). \tag{IV.4.75}$$

Let us compute the derivatives of our Dollard-type modifier:

$$((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,\tilde{U}_D(t) = \tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t). \tag{IV.4.76}$$

$$\begin{aligned}((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,\tilde{U}_D(t) &= \tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,M(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t) \\ &\quad +\tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t).\end{aligned} \tag{IV.4.77}$$

$$\begin{aligned}((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^2\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,\tilde{U}_D(t) &= \tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,M(t))^2\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t) \\ &\quad +\tilde{U}_D(t)\left(((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^2\,M(t)\right)((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t) \\ &\quad +\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,M(t)) \\ &\qquad \times ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t) \\ &\quad +\tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,M(t) \\ &\qquad \times ((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t) \\ &\quad +\tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_l)^2\,((\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)\cdot \mathbf{e}_{l''})\,M(t).\end{aligned} \tag{IV.4.78}$$

$$\begin{aligned}((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,\tilde{U}_D(t) &= \tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))^2\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+\tilde{U}_D(t)\left(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,M(t)\right)((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+2\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))\\ &\quad\times((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+\tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t). \end{aligned} \tag{IV.4.79}$$

$$\begin{aligned}((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)^2\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,\tilde{U}_D(t) &= \tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))^2\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t))^2\\ &+2\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))\\ &\quad\times(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))\\ &\quad\times((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))^2\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)^2\,M(t)\\ &+\tilde{U}_D(t)\left(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,M(t)\right)(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t))^2\\ &+\tilde{U}_D(t)\left(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,M(t)\right)\\ &\quad\times((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+\tilde{U}_D(t)\left(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,M(t)\right)((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)^2\,M(t)\\ &+2\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t))\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))\\ &\quad\times((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+2\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t))^2\\ &+2\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,M(t))\\ &\quad\times((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)^2\,M(t)\\ &+\tilde{U}_D(t)\,(((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t))\\ &\quad\times((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)\,M(t)\\ &+\tilde{U}_D(t)\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^2\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)^2\,M(t). \end{aligned} \tag{IV.4.80}$$

We note that

$$\begin{aligned}&\Bigg\|((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_{l''})^{a_1}\,((\tilde{\mathbf{x}}_k-\tilde{\mathbf{x}}_j)\cdot\mathbf{e}_l)^{a_2}\\ &\times\; M(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})\Bigg\| \;\le\; C\sum_{(j',k')\in N_{jk}}\left\|\int_0^t |s|^{a_1+a_2}\frac{\partial^{a_1+a_2}V^l_{j'k'}}{\partial y_{l''}^{a_1}\partial y_l^{a_2}}(s(\mathbf{p}_{j'k'}/\mu_{j'k'}+\mathbf{v}_{j'k'})\right.\\ &\qquad\left.+\mathbf{e}_1 q_{j'k'}Es^2/(2\mu_{j'k'}))\,g_{j'k'}(\mathbf{p}_{j'k'})\;ds\right\|\\ &\qquad = C\sum_{(j',k')\in N_{jk}}^{|\beta_b|=a_1+a_2}\left\|\int_0^t |s|^{|\beta_b|}\left(D^{\beta_b}V^l_{j'k'}\right)(s(\mathbf{p}_{j'k'}/\mu_{j'k'}+\mathbf{v}_{j'k'})\right.\\ &\qquad\left.+\mathbf{e}_1 q_{j'k'}Es^2/(2\mu_{j'k'}))\,g_{j'k'}(\mathbf{p}_{j'k'})\;ds\right\|\\ &\qquad \le C\sum_{(j',k')\in N_{jk}}^{|\beta_b|=a_1+a_2} I^{j',k'}_{\beta_b}. \end{aligned} \tag{IV.4.81}$$

We attain (IV.4.82) in a similar way as in (IV.4.67) by (IV.4.66), (IV.4.74), (IV.4.75), (IV.4.76), (IV.4.77), (IV.4.78), (IV.4.79), (IV.4.80) and (IV.4.81):

$$\|G_{jk}\| \leq C\left(1+\sum_{b'=1}^{4}\prod_{\sum|\beta_b|=b'}\sum_{(j',k')\in N_{jk}} I_{\beta_b}^{j',k'}\right) \tag{IV.4.82}$$

Next, we dedicate ourselves to compute $I_{\beta_b}^{j',k'}$, for $2 \leq |\beta_b| \leq 4$,

If $q_{j'k'} = 0$, then we have $2(-\gamma_D - 2\mu) + 3 < 0$. This is possible because $\gamma_D + \mu > 1$ and $\mu > 1/2$. Then, for $|\beta_b| = 2$,

$$\begin{aligned} I_{\beta_b} &\leq C\int_0^{|t|}\tau^2(1+\tau^2)^{-\gamma_D-2\mu}\,d\tau \leq C\int_0^{|t|}\tau^2(1+\tau)^{2(-\gamma_D-2\mu)}\,d\tau \\ &\leq C\int_0^{|t|}(1+\tau)^{2(-\gamma_D-2\mu)+2}\,d\tau \leq C(1+\tau)^{2(-\gamma_D-2\mu)+3}\Big|_0^{|t|} \\ &\leq C(1+|t|)^{2(-\gamma_D-2\mu)+3} \leq C, \end{aligned} \tag{IV.4.83}$$

for $3 \leq |\beta_b| = 4$,

$$\begin{aligned} I_{\beta_b} &\leq C\int_0^{v|t|}\tau^{|\beta_b|}(1+\tau^2)^{-\gamma_D-\mu(2+|\beta_b|)/2}\,d\tau \leq C\int_0^{v|t|}\tau^{|\beta_b|}(1+\tau)^{2[-\gamma_D-\mu(2+|\beta_b|)/2]}\,d\tau \\ &\leq C\int_0^{v|t|}(1+\tau)^{2[-\gamma_D-\mu(2+|\beta_b|)/2]+|\beta_b|}\,d\tau \leq C(1+\tau)^{2[-\gamma_D-\mu(2+|\beta_b|)/2]+|\beta_b|+1}\Big|_0^{v|t|} \\ &\leq C(1+|vt|)^{|\beta_b|+1-2\gamma_D-(2+|\beta_b|)\mu}. \end{aligned} \tag{IV.4.84}$$

We note that $|\beta_b| + 1 - 2\gamma_D - (2 + |\beta_b|)\mu$ can either be positive or non-positive. For $|\beta_b| = 4$,

$$\min\{5 - 2\gamma_D - 6\mu\} = 5 - 2(1/2) - 6 = -2, \text{ and } \sup\{5 - 2\gamma_D - 6\mu\} = 5 - 2(1) - 4(1/2) = 1.$$

Therefore, for $3 \leq |\beta_b| = 4$, and by (IV.4.84):

$$I_{\beta_b} \leq C\left(1+|vt|^{\max\{|\beta_b|+1-2\gamma_D-(2+|\beta_b|)\mu,\,0\}}\right).$$

Else, i.e. If $q_{j'k'} = 0$, then we have that, for $2 \leq |\beta_b| = 4$,

$$\begin{aligned} I_{\beta_b} &\leq C\int_0^{v|t|}\tau^{|\beta_b|}(1+\tau)^{-1-|\beta_b|(\epsilon_0+1/2)}\,d\tau \leq C\int_0^{v|t|}(1+\tau)^{-1-|\beta_b|(\epsilon_0+1/2-1)}\,d\tau \\ &\leq C(1+\tau)^{-|\beta_b|(\epsilon_0-1/2)}\Big|_0^{v|t|} \leq C(1+|vt|)^{|\beta_b|(-\epsilon_0+1/2)}. \end{aligned}$$

Finally, we want to estimate the right hand side of (IV.4.82). For that matter, we form the following combinations:

(*a*) All the charges $q_{j'k'}$, with $(j',k') \in N_{jk}$, are equal to zero,

$$\sum_{b'=1}^{4}\prod_{\sum|\beta_b|=b'}\sum_{(j',k')\in N_{jk}} I_{\beta_b}^{j',k'} \leq C\sum_{b'=1}^{4}(1+|vt|)^{|b'|(-\epsilon_0+1/2)} \leq C(1+|vt|)^{-4\epsilon_0+2} \leq C(1+|vt|)^{2-\tilde{\epsilon}},$$

with $0 < \tilde{\epsilon} \leq 4\epsilon_0$.

(*b*) All the charges $q_{j'k'}$, with $(j',k') \in N_{jk}$, are different from zero. Observe that $4 - 2\gamma_D - 5\mu \leq 5 - 2\gamma_D - 6\mu$,

$$\begin{aligned} \sum_{b'=1}^{4} \prod_{\sum|\beta_b|=b'} \sum_{(j',k')\in N_{jk}} I_{\beta_b}^{j',k'} &\leq C \sum_{b'=3}^{4} \prod_{\sum|\beta_b|=b'} \sum_{(j',k')\in N_{jk}} I_{\beta_b}^{j',k'} \\ &\leq C \sum_{|\beta_b|=3}^{4} \left(1 + |vt|^{|\beta_b|+1-2\gamma_D-(2+|\beta_b|)\mu}\right) \\ &\leq C \left((1+|vt|)^{4-2\gamma_D-5\mu} + (1+|vt|)^{5-2\gamma_D-6\mu}\right) \leq C\,(1+|vt|)^{5-2\gamma_D-6\mu} \\ &\leq C\,(1+|vt|)^{2-\tilde{\epsilon}}, \end{aligned}$$

provided $2 > 2 - \tilde{\epsilon} \geq 5 - 2\gamma_D - 6\mu$, that is $0 < \tilde{\epsilon} \leq 2\gamma_D + 6\mu - 3$. Note that $\inf\{2\gamma_D + 6\mu - 3\} = 2 + 4(1/2) - 3 = 1$.

(*c*) There are two pairs (j',k'), $(j'',k'') \in N_{jk}$ such that $q_{j'k'} = 0$ and $q_{j'',k''} \neq 0$:

$$\begin{aligned} &\sum_{b'=1}^{4} \prod_{\sum|\beta_b|=b'} \sum_{(j',k')\in N_{jk}} I_{\beta_b}^{j',k'} \\ &\leq C \sum_{b'=1}^{4} \prod_{\sum|\beta_b|=b'} \sum_{(j',k')\in N_{jk}} \begin{cases} (1+|vt|)^{|\beta_b|(-\epsilon_0+1/2)}, & \text{if } q_{j'k'} = 0 \text{ and } 1 \leq |\beta_b| \leq 4, \\ 1, & \text{if } q_{j'k'} \neq 0 \text{ and } 1 \leq |\beta_b| \leq 2, \\ (1+|vt|)^{|\beta_b|+1-2\gamma_D-(|\beta_b|+2)\mu}, & \text{if } q_{j'k'} \neq 0 \text{ and } 3 \leq |\beta_b| \leq 4. \end{cases} \\ &\leq C \sum_{b'=1}^{4} \prod_{\sum|\beta_b|=b'} \left(1 + \sum_{|\beta_b|=1}^{4} (1+|vt|)^{|\beta_b|(-\epsilon_0+1/2)} + \sum_{|\beta_b|=3}^{4} (1+|vt|)^{|\beta_b|+1-2\gamma_D-(2+|\beta_b|)\mu}\right) \end{aligned}$$

$$\begin{aligned} &\leq C\Big(1 + (1+|vt|)^{-\epsilon_0+1/2} + (1+|vt|)^{2(-\epsilon_0+1/2)} + (1+|vt|)^{3(-\epsilon_0+1/2)} + (1+|vt|)^{4-2\gamma_D-5\mu} \\ &\quad + (1+|vt|)^{4(-\epsilon_0+1/2)} + (1+|vt|)^{-\epsilon_0+1/2}\,(1+|vt|)^{4-2\gamma_D-5\mu} + (1+|vt|)^{5-2\gamma_D-6\mu}\Big) \\ &\leq C\left(1 + (1+|vt|)^{4(-\epsilon_0+1/2)} + (1+|vt|)^{-\epsilon_0+1/2}\,(1+|vt|)^{4-2\gamma_D-5\mu} + (1+|vt|)^{5-2\gamma_D-6\mu}\right) \\ &\leq C\,(1+|vt|)^{2-\tilde{\epsilon}}, \end{aligned}$$

with $0 < \tilde{\epsilon} < \min\{4\epsilon_0,\, 2\gamma_D + 5\mu + \epsilon_0 - 5/2,\, 2\gamma_D + 6\mu - 3\}$, because,

(1) $2 > 2 - \tilde{\epsilon} \geq 9/2 - 2\gamma_D - 5\mu - \epsilon_0$, that is $0 < \tilde{\epsilon} \leq 2\gamma_D + 5\mu + \epsilon_0 - 5/2$. Note that $\inf\{2\gamma_D + 5\mu + \epsilon_0 - 5/2\} = 2 + 3(1/2) - 5/2 = 1$.

(2) $2 > 2 - \tilde{\epsilon} \geq 4 - 2\gamma_D - 5\mu$, that is $0 < \tilde{\epsilon} \leq 2\gamma_D + 5\mu - 2$. Recall that $4 - 2\gamma_D - 5\mu \leq 5 - 2\gamma_D - 6\mu$.

(3) Note that if $0 < \epsilon_0 < 1/4$ and $1 \geq \mu > 3/4$, then $2\gamma_D + 5\mu + \epsilon_0 - 5/2 < 2\gamma_D + 6\mu - 3$, conversely, if $1/4 < \epsilon_0 < 1/2$ and $1/2 < \mu < 3/4$, then $2\gamma_D + 5\mu + \epsilon_0 - 5/2 > 2\gamma_D + 6\mu - 3$.

(4) Note that if $0 < \epsilon_0 < 1/4$, then $4\epsilon_0 < 1 < 2\gamma_D + 6\mu - 3$. On the other hand, if $\epsilon_0 = \gamma_D = 7/15$ and $\mu = 9/15$, which are possible values, then $4\epsilon_0 = 28/15 > 14/15 + 3/5 = 14/15 + 6(9/15) - 3 = 2\gamma_D + 6\mu - 3$.

In general we take

$$0 < \tilde{\epsilon} < \min\{4\epsilon_0,\, 2\gamma_D + 5\mu + \epsilon_0 - 5/2,\, 2\gamma_D + 6\mu - 3\} < 2,$$

where the last inequality is true because $0 < \epsilon_0 < 1/2$.

If $q_{j'k'} = 0$ for all (j', k'), then we can take $\gamma_D = 1/2$ and $\mu = 1$; in this case $\min\{4\epsilon_0, 2\gamma_D + 5\mu + \epsilon_0 - 5/2, 2\gamma_D + 6\mu - 3\} = 4\epsilon_0$. On the contrary, If $q_{j'k'} \neq 0$ for all (j', k') then we can take ϵ_0 so big such that $\min\{4\epsilon_0, 2\gamma_D + 5\mu + \epsilon_0 - 5/2, 2\gamma_D + 6\mu - 3\} = \min\{2, 2\gamma_D + 6\mu - 3\}$. It is important that $\tilde{\epsilon}$ be less than 2 to assure the last step in the proof of Lemma II.2.2, i.e.,

$$C(1 + |vt|)^{2-\tilde{\epsilon}} \leq C(1 + |v_{jk}t|)^{2-\tilde{\epsilon}}.$$

At this moment we have the estimations that finish the proof of Lemma II.2.2.

■

LEMMA IV.4.3. *Completion of the proof of Lemma II.2.5.*

Proof. Actually, we do not have that $\operatorname{supp}\left(V^l_{jk,v_{jk}t} - V^l_{jk}\right) \subset B_{cv_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}}$, but $\operatorname{supp}\left(V^l_{jk,v_{jk}t} - V^l_{jk}\right) \subset \overline{B_{cv_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}}}$, where the line over means topological closure. Let us take $x \in \{y \big| (V^l_{jk,v_{jk}t} - V^l_{jk})(y) \neq 0\}$, then $\chi(x/(v_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}})) \neq 1$, and so, $|x| < c\, v_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}$ and

$$\{y \big| (V^l_{jk,v_{jk}t} - V^l_{jk})(y) \neq 0\} \subset B_{cv_{jk}^{\tilde{\sigma}_{jk}}|t|^{2-\tilde{\sigma}_{jk}}}. \tag{IV.4.85}$$

The following estimations are straightforward:

$$\|V^l_{jk,v_{jk}t} - V^l_{jk}\| \leq \|V^l_{jk}\|\, \|(1-\chi)\| \leq \|V^l_{jk}\|.$$

It is not difficult to realize that the sentence "If $q_{jk} \neq 0$ with $\mathbf{p}$ in the support of g, we note, by (II.2.33), that $V^l_{jk,v_{jk}t}(t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})) = V^l_{jk}(t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))$" is true.

In the other hand, the sentence "If $q_{jk} = 0$, $\mathbf{p}$ belongs to the support of $g(\cdot - \mu_{jk}\mathbf{v}_{jk})$, and $v_0 > 2\eta_{jk}$ then $V^l_{jk,v_{jk}t}(t\,\mathbf{p}/\mu_{jk}) = V^l_{jk}(t\,\mathbf{p}/\mu_{jk})$" requires some easy computations. Let us prove the sentence: $v_{jk} > v_0 > 2\eta_{jk}$ implies that $-\eta_{jk} > -v_{jk}/2$. The fact that $\mathbf{p}$ belongs to the support of $g(\cdot - \mu_{jk}\mathbf{v}_{jk})$ means that $|\mathbf{p} - \mu_{jk}v_{jk}| < \mu_{jk}\eta_{jk}$, then:

$$\begin{aligned} |\mathbf{p}| &= |\mu_{jk}v_{jk} - (\mathbf{p} - \mu_{jk}v_{jk})| \geq \mu_{jk}v_{jk} - |\mathbf{p} - \mu_{jk}v_{jk}| \\ &> \mu_{jk}v_{jk} - \mu_{jk}\eta_{jk} > \mu_{jk}v_{jk} - \mu_{jk}v_{jk}/2 > \mu_{jk}v_{jk}/2. \end{aligned}$$

This implies $V^l_{jk,v_{jk}t}(t\,\mathbf{p}/\mu_{jk}) = V^l_{jk}(t\,\mathbf{p}/\mu_{jk})$.

If $q_{jk} = 0$, $\mathbf{p}$ belongs to the support of $g(\cdot - \mu_{jk}\mathbf{v}_{jk})$, and $v_0 > 2\eta_{jk}$ then $V^l_{jk,v_{jk}t}(t\,\mathbf{p}/\mu_{jk}) = V^l_{jk}(t\,\mathbf{p}/\mu_{jk})$. By equations (II.1.31)-(II.1.35):

$$\begin{aligned} & e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}}\, e^{itH_0} \left(V^l_{jk,v_{jk}t}(\mathbf{x}) - V^l_{jk}(t\,\mathbf{p}/\mu_{jk} - \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) e^{-itH_0}\, g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) \\ = {} & e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}}\, e^{it\mathbf{p}^2/(2\mu_{jk})}\, e^{i\mathbf{p}\cdot\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})}\, e^{-iq_{jk}Ex_1t} \left(V^l_{jk,v_{jk}t}(\mathbf{x}) - V^l_{jk}(t\,\mathbf{p}/\mu_{jk} - \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) \\ & \times e^{iq_{jk}Ex_1t}\, e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})}\, e^{-it\mathbf{p}^2/(2\mu_{jk})}\, g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) \\ = {} & e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}}\, e^{it\mathbf{p}^2/(2\mu_{jk})}\, e^{i\mathbf{p}\cdot\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})} \left(V^l_{jk,v_{jk}t}(\mathbf{x}) - V^l_{jk}(t\,\mathbf{p}/\mu_{jk} + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) \\ & \times e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})}\, e^{-it\mathbf{p}^2/(2\mu_{jk})}\, g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) \\ = {} & e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}}\, e^{it\mathbf{p}^2/(2\mu_{jk})} \left(V^l_{jk,v_{jk}t}(\mathbf{x} + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})) - V^l_{jk}(t\,\mathbf{p}/\mu_{jk} + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) e^{-it\mathbf{p}^2/(2\mu_{jk})} \\ & \times g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) \\ = {} & e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}} \left(V^l_{jk,v_{jk}t}(\mathbf{x} + t\,\mathbf{p}/\mu_{jk} + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})) - V^l_{jk}(t\,\mathbf{p}/\mu_{jk} + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) \\ & \times g(\mathbf{p} - \mu_{jk}\mathbf{v}_{jk}) \\ = {} & \left(V^l_{jk,v_{jk}t}(\mathbf{x} + t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})) - V^l_{jk,v_{jk}t}(t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) \\ & \times g(\mathbf{p})\, e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}}. \end{aligned} \tag{IV.4.86}$$

The Baker-Campbell-Hausdorff formula [14] gives us the following equality (equation (IV.4.87) is completely justified by (IV.3.43))

$$\begin{aligned}
&\left(V^l_{jk,v_{jk}t}(\mathbf{x}+t\,\mathbf{p}/\mu_{jk}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))-V^l_{jk,v_{jk}t}(t\,\mathbf{p}/\mu_{jk}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right)\\
=\;&\int_0^1 ds\left[\left(\nabla V^l_{jk,v_{jk}t}\right)(s\mathbf{x}+t\,\mathbf{p}/\mu_{jk}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\cdot\mathbf{x}\right.\\
&\left.+\frac{it}{(2\mu_{jk})}\left(\Delta V^l_{jk,v_{jk}t}\right)(s\mathbf{x}+t\,\mathbf{p}/\mu_{jk}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right].
\end{aligned} \tag{IV.4.87}$$

From the Leibniz rule we have that,

$$\begin{aligned}
|\nabla V^l_{jk,v_{jk}t}(\mathbf{x})| &\le |\nabla V^l_{jk}(\mathbf{x})|+O(v_{jk}^{-\tilde\sigma_{jk}}|t|^{-(2-\tilde\sigma_{jk})})|V^l_{jk}(\mathbf{x})|,\\
|\Delta V^l_{jk,v_{jk}t}(\mathbf{x})| &\le |\Delta V^l_{jk}|+O(v_{jk}^{-\tilde\sigma_{jk}}|t|^{-(2-\tilde\sigma_{jk})})|\nabla V^l_{jk}(\mathbf{x})|+O(v_{jk}^{-2\tilde\sigma_{jk}}|t|^{-2(2-\tilde\sigma_{jk})})|V^l_{jk}(\mathbf{x})|.
\end{aligned}$$

Focusing in the case $q_{jk}=0$, we compute, having in consideration that in the support of $V^l_{jk,v_{jk}t}$ we must have $|\mathbf{x}|\ge(c/2)|v_{jk}t|$:

$$\begin{aligned}
|\nabla V^l_{jk,v_{jk}t}(\mathbf{x})| &\le C\left((1+|v_{jk}t|)^{-\gamma_1}+|v_{jk}t|^{-1}(1+|v_{jk}t|)^{-\gamma_1+1}\right)\;\le\;C(|v_{jk}t|)^{-\gamma_1},\\
|\Delta V^l_{jk,v_{jk}t}(\mathbf{x})| &\le C\left((1+|v_{jk}t|)^{-2-2\epsilon_0}+|v_{jk}t|^{-1}(1+|v_{jk}t|)^{-\gamma_1}+|v_{jk}t|^{-2}(1+|v_{jk}t|)^{-\gamma_1+1}\right)\\
&\le C(|v_{jk}t|)^{-2-2\epsilon_0},
\end{aligned}$$

where $3/2<\gamma_1\le 2$, $0<\epsilon_0<\gamma_1-3/2$ are as in Definition II.1.1 and (II.1.25), respectively.

As in Enss and Weder [20], (II.2.17), (IV.4.86), (IV.4.87), (IV.4.88), (IV.4.88) and $\epsilon_0<\gamma_1-3/2$ imply, for $q_{jk}=0$:

$$\begin{aligned}
I_1 \;\le\;& \int_0^1 ds\left\|\left[\left(\nabla V^l_{jk,v_{jk}t}\right)(s\mathbf{x}+t\,\mathbf{p}/\mu_{jk}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\cdot\mathbf{x}\right.\right.\\
&\left.+\frac{it}{(2\mu_{jk})}\left(\Delta V^l_{jk,v_{jk}t}\right)(s\mathbf{x}+t\,\mathbf{p}/\mu_{jk}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right]g(\mathbf{p})\,e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}}\\
&\left.\times\tilde U_D(t)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2}\right\|\\
\le\;& C\left[|v_{jk}t|^{-\gamma_1}\left\|\mathbf{x}\,\tilde U_D(t)\prod_{j'<k'}f_{j'k'}(\mathbf{p}_{j'k'}-\mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-1/2}\right\|+|v_{jk}t|^{-1-2\epsilon_0}\right]\\
\le\;& C\left[|v_{jk}t|^{-\gamma_1}\begin{cases}(1+|v_{jk}t|)^{2-\gamma_1}, & \text{if } 3/2<\gamma_1<2,\\ 1+\ln(1+|v_{jk}t|), & \text{if } \gamma_1=2,\end{cases}+|v_{jk}t|^{-1-2\epsilon_0}\right]\;\le\;C|v_{jk}t|^{-1-2\epsilon_0},
\end{aligned}$$

The justification of Equation (IV.4.88) is straightforward. To show (IV.4.88) we note, as in Definition II.1.1 that $1+2\epsilon_0<3/2+\epsilon_0<\gamma_1$, then $2+2\epsilon_0<\gamma_1+1$, thus let us estimate as follows:

$$\begin{aligned}
|\Delta V^l_{jk,v_{jk}t}(\mathbf{x})| &\le C\left((1+|v_{jk}t|)^{-2-2\epsilon_0}+|v_{jk}t|^{-1}(1+|v_{jk}t|)^{-\gamma_1}+|v_{jk}t|^{-2}(1+|v_{jk}t|)^{-\gamma_1+1}\right)\\
&\le C\left(|v_{jk}t|^{-2-2\epsilon_0}+|v_{jk}t|^{-\gamma_1-1}+|v_{jk}t|^{-\gamma_1-1}\right)\le C|v_{jk}t|^{-2-2\epsilon_0}.
\end{aligned}$$

In both (IV.4.88) and (IV.4.88) we note that it is important to have $|v_{jk}t|\ge 1$.

When $q_{jk}=0$, the estimation of I_1, given by (II.2.30) is easily followed, but we want to do some computations to make it even more clear. We have to cases:

(a) $3/2 < \gamma_1 < 2$. We know that $\epsilon_0 < \gamma_1 - 3/2$. Then $3 < 2\gamma_1 - 2\epsilon_0$ which is equivalent to $-\gamma_1 < 2 - 2\gamma_1 < -1 - 2\epsilon_0$.

(b) $\gamma_1 = 2$. Here, it is enough to show that: $\lim_{\tau\to\infty} \frac{\tau^{-2}\ln(1+\tau)}{\tau^{-1-2\epsilon_0}} < C$. By L'Hôpital rule and the fact that $\epsilon_0 < \gamma_1 - 3/2 = 2 - 3/2 = 1/2$:

$$\lim_{\tau\to\infty} \frac{\tau^{-2}\ln(1+\tau)}{\tau^{-1-2\epsilon_0}} = \lim_{\tau\to\infty} \frac{\ln(1+\tau)}{\tau^{1-2\epsilon_0}} = \lim_{\tau\to\infty} \frac{\frac{1}{(1+\tau)}}{(1-2\epsilon_0)\tau^{-2\epsilon_0}} < C \lim_{\tau\to\infty} \tau^{2\epsilon_0-1} = 0.$$

Because of the presence of the natural logarithm and the estimation in case (b) above, we need a large $M > 1$ such that that $|v_{jk}t| > M$ in order to conclude that

$$I_1 \le C|v_{jk}t|^{-1-2\epsilon_0}.$$

Additionally, directly from (II.2.30) we have that,

$$I_1 \le 2C \sup_{y\in\mathbb{R}^n} |V(y)| \le C. \tag{IV.4.88}$$

That is, I_1 is uniformly bounded

Then, we compute the integral of I_1,

$$\begin{aligned}\int_{-\infty}^{\infty} I_1\, dt &\le \int_{|v_{jk}t|\le M} I_1\, dt + \int_{|v_{jk}t|>M} I_1\, dt \le C v_{jk}^{-1}\left(\int_{|\tau|\le M} d\tau + \int_{|\tau|>M} |\tau|^{-1-2\epsilon_0}\, d\tau\right)\\ &\le C v_{jk}^{-1}\left(2M + 2\frac{1}{-2\epsilon_0}\tau^{-2\epsilon_0}\Big|_M^{\infty}\right) = C v_{jk}^{-1}\left(2M + \frac{1}{\epsilon_0}M^{-2\epsilon_0}\right) = O(v_{jk}^{-1}).\end{aligned}$$

Now, we will consider $q_{jk} \neq 0$. Thus we complete the details of the estimation of I_1 :

$$\begin{aligned}I_1 \quad \le \quad & \int_0^1 ds \Bigg\| \Bigg[\left(\nabla V^l_{jk,v_{jk}t}\right)(s\mathbf{x} + t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\cdot \mathbf{x} \\ & + \frac{it}{(2\mu_{jk})}\left(\Delta V^l_{jk,v_{jk}t}\right)(s\mathbf{x} + t\,\mathbf{p}/\mu_{jk} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\Bigg] g(\mathbf{p})\, e^{-i\mu_{jk}\mathbf{v}_{jk}\cdot\mathbf{x}} \\ & \times \tilde{U}_D(t) \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-2} \Bigg\|\end{aligned}$$

$$\begin{aligned}\le\ C & \Bigg[\left((1+v_{jk}^{\tilde\sigma_{jk}}|t|^{2-\tilde\sigma_{jk}})^{-(\gamma_D+\mu)} + v_{jk}^{-\tilde\sigma_{jk}}|t|^{-(2-\tilde\sigma_{jk})}(1+v_{jk}^{\tilde\sigma_{jk}}|t|^{2-\tilde\sigma_{jk}})^{-\gamma_D}\right) \\ \times & \left\| \mathbf{x}\,\tilde{U}_D(t)\prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1+\mathbf{x}^2)^{-1/2}\right\| + |t|\left((1+v_{jk}^{\tilde\sigma_{jk}}|t|^{2-\tilde\sigma_{jk}})^{-(\gamma_D+2\mu)}\right. \\ & \left. + v_{jk}^{-\tilde\sigma_{jk}}|t|^{-(2-\tilde\sigma_{jk})}(1+v_{jk}^{\tilde\sigma_{jk}}|t|^{2-\tilde\sigma_{jk}})^{-(\gamma_D+\mu)} + v_{jk}^{-2\tilde\sigma_{jk}}|t|^{-2(2-\tilde\sigma_{jk})}(1+v_{jk}^{\tilde\sigma_{jk}}|t|^{2-\tilde\sigma_{jk}})^{-\gamma_D}\right)\Bigg]\end{aligned}$$

We apply (II.2.16). Thus we arrive to the estimation given in the text,

$$I_1 \le I_{11} + I_{12}.$$

While computing the integral of I_{11}, we require $\tilde{\sigma}_{jk} < 2-(1+\theta_{jk})/(\gamma_D+\mu)$. This inequality is equivalent to the following inequalities:

$$\begin{aligned} 2-\tilde{\sigma}_{jk} &> (1+\theta_{jk})/(\gamma_D+\mu) \\ (2-\tilde{\sigma}_{jk})(\gamma_D+\mu) &> 1+\theta_{jk} \\ 0 &> -(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+\theta_{jk}+1 \\ &\geq -(2-\tilde{\sigma}_{jk})(\gamma_D+1)+\theta_{jk}+1. \end{aligned}$$

Furthermore, because $0<\tilde{\sigma}_{jk}<1$,

$$\begin{aligned} 0 &> -(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+1, \\ 0 &> -(2-\tilde{\sigma}_{jk})(\gamma_D+1)+1. \end{aligned}$$

Then,

$$\begin{aligned} \lim_{t\to\infty} t^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+\theta_{jk}+1} = \lim_{t\to\infty} t^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)+\theta_{jk}+1} &= 0 \\ \lim_{t\to\infty} t^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+1} = \lim_{t\to\infty} t^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)+1} &= 0. \end{aligned}$$

Let us show that $\int_{-\infty}^{\infty} dt\, I_{11} = O(v_{jk}^{-b})$ and determine the value of b :

$$\begin{aligned} I_{11} \leq\ & C\left(\left(v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu)}|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)} + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+1)}|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)}\right)\right. \\ & \left.\times\left(1+v_{jk}^{-(2-\gamma_1/2)}|v_{jk}t|^{\theta_{jk}}\right)F(|t|>v_{jk}^{-b}) + \|V_{jk}^l\|\,F(|t|\leq v_{jk}^{-b})\right) \\ \leq\ & C\left(v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu)}|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)} + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+1)}|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)}\right. \\ & \left.+v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu)-(2-\gamma_1/2)+\theta_{jk}}|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+\theta_{jk}} + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+1)-(2-\gamma_1/2)+\theta_{jk}}|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)+\theta_{jk}}\right) \\ & \times F(|t|>v_{jk}^{-b}) + F(|t|\leq v_{jk}^{-b}). \end{aligned}$$

Integrating,

$$\begin{aligned} \int dt\, I_{11} \leq\ & C\left(v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu)}\left(v_{jk}^{-b}\right)^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+1} + v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+1)}\left(v_{jk}^{-b}\right)^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)+1}\right. \\ & +v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu)-(2-\gamma_1/2)+\theta_{jk}}\left(v_{jk}^{-b}\right)^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu)+\theta_{jk}+1} \\ & \left.+v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+1)-(2-\gamma_1/2)+\theta_{jk}}\left(v_{jk}^{-b}\right)^{-(2-\tilde{\sigma}_{jk})(\gamma_D+1)+\theta_{jk}+1} + v_{jk}^{-b}\right) \end{aligned}$$

$$\begin{aligned} \int dt\, I_{11} \leq\ & Cv_{jk}^{-b}\left(1+v_{jk}^{[-\tilde{\sigma}_{jk}+b(2-\tilde{\sigma}_{jk})](\gamma_D+\mu)} + v_{jk}^{[-\tilde{\sigma}_{jk}+b(2-\tilde{\sigma}_{jk})](\gamma_D+1)}\right. \\ & \left.+v_{jk}^{[-\tilde{\sigma}_{jk}+b(2-\tilde{\sigma}_{jk})](\gamma_D+\mu)-(2-\gamma_1/2)+\theta_{jk}-b\theta_{jk}} + v_{jk}^{[-\tilde{\sigma}_{jk}+b(2-\tilde{\sigma}_{jk})](\gamma_D+1)-(2-\gamma_1/2)+\theta_{jk}-b\theta_{jk}}\right). \end{aligned}$$

Provided $v_{jk}>1$, let us find b such as $\int dt\, I_{11} \leq Cv_{jk}^{-b}$:

$$\left[-\tilde{\sigma}_{jk}+b(2-\tilde{\sigma}_{jk})\right](\gamma_D+\mu) \leq 0 \Leftrightarrow b \leq \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}},$$

$$\left[-\tilde{\sigma}_{jk}+b(2-\tilde{\sigma}_{jk})\right](\gamma_D+1) \leq 0 \Leftrightarrow b \leq \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}},$$

$$\begin{aligned}
[-\tilde{\sigma}_{jk} + b(2-\tilde{\sigma}_{jk})]\,(\gamma_D+\mu) - (2-\gamma_1/2) + \theta_{jk} - b\theta_{jk} &\leq 0 \\
b\,[(2-\tilde{\sigma}_{jk})(\gamma_D+\mu) - \theta_{jk}] &\leq \tilde{\sigma}_{jk}(\gamma_D+\mu) + (2-\gamma_1/2) - \theta_{jk} \\
b &\leq \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+\mu)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+\mu)},
\end{aligned}$$

and, similarly,

$$\begin{aligned}
[-\tilde{\sigma}_{jk} + b(2-\tilde{\sigma}_{jk})]\,(\gamma_D+1) - (2-\gamma_1/2) + \theta_{jk} - b\theta_{jk} &\leq 0 \\
b &\leq \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+1)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+1)}.
\end{aligned}$$

By $\mu \leq 1$ and $2-\gamma_1/2-\theta_{jk} > 0$, we have that

$$\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+1) \leq \tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+\mu),$$

likewise,

$$0 < 2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+\mu) \leq 2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+1).$$

Then,

$$\frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+1)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+1)} \leq \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+\mu)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+\mu)}.$$

On the other hand, it is straightforward that,

$$\frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}} \leq \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+1)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+1)}.$$

We conclude that in order to estimate $\int dt\, I_{11} \leq Cv_{jk}^{-b}$, we should take

$$b = \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}} = \min\left\{\frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}}, \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+1)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+1)}, \frac{\tilde{\sigma}_{jk} + (2-\gamma_1/2-\theta_{jk})/(\gamma_D+\mu)}{2-\tilde{\sigma}_{jk}-\theta_{jk}/(\gamma_D+\mu)}\right\}.$$

We know that $0 < \tilde{\sigma}_{jk} < 1$, then $2-\tilde{\sigma}_{jk} > 1$ and

$$2\tilde{\sigma}_{jk} < 2 \iff \tilde{\sigma}_{jk} < 2-\tilde{\sigma}_{jk} \iff 0 < \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}} < 1,$$

then $b = \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}}$, satisfies $0 < b < 1$.

We have to verify that we can choose $\tilde{\sigma}_{jk}$ such that $0 < \tilde{\sigma}_{jk} < 2 - \max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\}$. We have three cases:

(a) $\max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\} = \frac{1+\theta_{jk}}{\gamma_D+\mu}$. Then, we should have in this case that $0 < 2-\frac{1+\theta_{jk}}{\gamma_D+\mu}$. This true because $2(\gamma_D+\mu) > 2$ and $3/2 > 1+\theta_{jk}$.

(b) $\max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\} = \frac{2}{\gamma_D+2\mu}$. Then, $0 < \tilde{\sigma}_{jk} < 2 - \frac{2}{\gamma_D+2\mu}$. This is possible because $2(\gamma_D+2\mu) > 3 > 2$.

(c) $\max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\} = 1$. Then, $0 < \tilde{\sigma}_{jk} < 1$, which we have by hipothesis.

The cases above show that our hypothesis on $\tilde{\sigma}_{jk}$ are perfectly plausible.

It is also a good moment to see that $\max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\}$ can be either $\frac{1+\theta_{jk}}{\gamma_D+\mu}$ or $\frac{2}{\gamma_D+2\mu}$ or 1. If we consider the minimum values for γ_D and μ, then $\frac{2}{\gamma_D+2\mu} \approx \frac{4}{3}$ and $\frac{1}{\gamma_D+\mu} \approx 1$. Thus, when $\theta_{jk} = 0$, we have that $\frac{1+\theta_{jk}}{\gamma_D+\mu} < 1 < \frac{2}{\gamma_D+2\mu}$ and when $\theta_{jk} = \frac{2}{5}$, (a possible value) we have that $\frac{1+\theta_{jk}}{\gamma_D+\mu} > \frac{2}{\gamma_D+2\mu} > 1$. Else, if we consider the maximum values for $\gamma_D = 1/2$ and $\mu = 1$, and $\theta_{jk} = 0$, we have that $\frac{1+\theta_{jk}}{\gamma_D+\mu} = \frac{2}{3} < \frac{2}{\gamma_D+2\mu} = \frac{4}{5} < 1$.

Because we are using Adachi and Maehara's computations [2] of the last three terms of the integral of I_3 in the proof of their Lemma 3.4, we want to verify these computations.

- First integral. Let $a > 0$ be a constant to be determined.

  By the hypothesis we have that $\tilde{\sigma}_{jk} < 2 - 2/(\gamma_D + 2\mu) \Leftrightarrow -(2 - \tilde{\sigma}_{jk})(\gamma_D + 2\mu) + 2 < 0$, we obtain:

$$\begin{aligned}\int_{|t| \geq v_{jk}^{-a}} dt\, v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+1} &= v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)} \frac{|t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+2}}{-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+2}\Big|_{v_{jk}^{-a}}^{\infty} \\ &= \frac{v^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)-a(-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+2)}}{(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)-2}.\end{aligned}$$

  Then, with $a = \frac{\tilde{\sigma}_{jk}(\gamma_D+2\mu)}{(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)-1}$,

$$\begin{aligned}\int_{-\infty}^{\infty} dt\, v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+1} &= O\left(v^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)-a(-(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)+2)}\right) + O\left(v^{-a}\right) \\ &= O\left(v^{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)/[(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)-1]}\right).\end{aligned}$$

- Second integral.

  Because of $1 \geq \mu$, we have that $2-2/(\gamma_D+\mu+1) \geq 2-2/(\gamma_D+2\mu)$, therefore $\tilde{\sigma}_{jk} < 2-2/(\gamma_D+\mu+1)$, then by similar computations made for the first integral, in the previous item, we have that,

$$\int_{a}^{\infty} dt\, v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+\mu+1)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)+1} = O\left(v^{-\tilde{\sigma}_{jk}(\gamma_D+\mu+1)/[(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)-1]}\right).$$

- Third integral.

  We again use $1 \geq \mu$, to deduce that $2 - 2/(\gamma_D + 2) \geq 2 - 2/(\gamma_D + \mu + 1)$,

$$\int_{a}^{\infty} dt\, v_{jk}^{-\tilde{\sigma}_{jk}(\gamma_D+2)} |t|^{-(2-\tilde{\sigma}_{jk})(\gamma_D+2)+1} = O\left(v^{-\tilde{\sigma}_{jk}(\gamma_D+2)/[(2-\tilde{\sigma}_{jk})(\gamma_D+2)-1]}\right).$$

Then, we observe that

$$\begin{aligned}&\max\{-\tilde{\sigma}_{jk}(\gamma_D+2\mu)/[(2-\tilde{\sigma}_{jk})(\gamma_D+2\mu)-1], \\ &-\tilde{\sigma}_{jk}(\gamma_D+\mu+1)/[(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)-1], \\ &\qquad -\tilde{\sigma}_{jk}(\gamma_D+2)/[(2-\tilde{\sigma}_{jk})(\gamma_D+2)-1]\} = -\tilde{\sigma}_{jk}/[(2-\tilde{\sigma}_{jk})-1/(\gamma_D+2)].\end{aligned}$$

To prove this fact, let us prove only one inequality because the other is similar. Because $1 \geq \mu$, $\Leftrightarrow$ $\gamma_D + 2 \geq \gamma_D + \mu + 1$, we have that,

$$\begin{aligned}(2-\tilde{\sigma}_{jk})(\gamma_D+2)-1 &\geq (2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)-1 > 0 \\ 1/[(2-\tilde{\sigma}_{jk})(\gamma_D+2)-1] &\leq 1/[(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)-1] \\ (\gamma_D+2)/[(2-\tilde{\sigma}_{jk})(\gamma_D+2)-1] &\leq (\gamma_D+\mu+1)/[(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)-1] \\ -\tilde{\sigma}_{jk}(\gamma_D+2)/[(2-\tilde{\sigma}_{jk})(\gamma_D+2)-1] &\geq -\tilde{\sigma}_{jk}(\gamma_D+\mu+1)/[(2-\tilde{\sigma}_{jk})(\gamma_D+\mu+1)-1].\end{aligned}$$

Finally, $\int_{-\infty}^{+\infty} dt\, I_{12} = O\left(v_{jk}^{-\tilde{\sigma}_{jk}/[(2-\tilde{\sigma}_{jk})-1/(\gamma_D+2)]}\right)$.

To get the result (II.2.35), we have to simply note that $-\frac{\tilde{\sigma}_{jk}}{(2-\tilde{\sigma}_{jk})-1/(\gamma_D+2)} \leq -\frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}}$.

For $q_{jk} \neq 0$, if $|\mathbf{x}| \leq (5/8)\, v_{jk}^{\sigma_{jk}}|t|$ and $v_{jk}^{\sigma_{jk}-1} \leq (2/5)\sqrt{1-\delta_{jk}}$, we obtain (II.2.36) as in (II.2.33), taking $\mathbf{x}$ instead of $t\,\mathbf{q}/\mu_{jk}$ and $5v_{jk}^{\tilde{\sigma}_{jk}}/8$ instead of η_{jk}, then $\frac{\eta_{jk}}{v_{jk}} \leq (5v_{jk}^{\tilde{\sigma}_{jk}}/8)v_{jk}^{-1} \leq (5/8)(2/5)\sqrt{1-\delta_{jk}} = \sqrt{1-\delta_{jk}}/4$. By (II.2.36) we have that $\left(V_{jk}^{l} - V_{jk,v_{jk}t}^{l}\right)(\mathbf{x} + \mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})) = 0$ if $q_{jk} \neq 0$, $|\mathbf{x}| \leq (5/8)\, v_{jk}^{\sigma_{jk}}|t|$ and $v_{jk}^{\sigma_{jk}-1} \leq (2/5)\sqrt{1-\delta_{jk}}$.

If $|\mathbf{x}| \leq v_{jk}|t|/2$, then $|\mathbf{x}+\mathbf{v}_{jk}t| \geq |\mathbf{v}_{jk}t| - |\mathbf{x}| \geq v_{jk}|t|/2$. If, additionally, we assume that $q_{jk} = 0$ we have that $\left(V^l_{jk} - V^l_{jk,v_{jk}t}\right)(\mathbf{x}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})) = 0$.

We define $\mathcal{M}' = \{\mathbf{x} \in \mathbb{R}^n \,\big|\, |\mathbf{x}-\mathbf{v}_{jk}t| \geq |v^{\sigma_{jk}}_{jk}t|/2\}$ and $\mathcal{M} = \{\mathbf{x} \in \mathbb{R}^n \,\big|\, |\mathbf{x}| < |v^{\sigma_{jk}}_{jk}t|/8\}$. We will prove that $r \geq |v^{\sigma_{jk}}_{jk}t|/4$ in Proposition II.2.1, provided $v^{\sigma_{jk}}_{jk} > 4\eta_{jk}$, Let us take $x \in \mathcal{M}'$ and $y \in \mathcal{M}+\mathbf{v}_{jk}t$, then $|\mathbf{x}-\mathbf{y}| = \big|(\mathbf{x}-\mathbf{v}_{jk}t)-(\mathbf{y}-\mathbf{v}_{jk}t)\big| \geq |v^{\sigma_{jk}}_{jk}t|/2 - |v^{\sigma_{jk}}_{jk}t|/8 = 3|v^{\sigma_{jk}}_{jk}t|/8$. Thus, $r \geq 3|v^{\sigma_{jk}}_{jk}t|/8 - \eta_{jk}|t| \geq 3|v^{\sigma_{jk}}_{jk}t|/8 - |v^{\sigma_{jk}}_{jk}t|/4 = |v^{\sigma_{jk}}_{jk}t|/8$.

Therefore, we estimate I_2 as follows:

$$
\begin{aligned}
I_2 &\leq C\left\|\left(V^l_{jk} - V^l_{jk,v_{jk}t}\right)(\mathbf{x})\, e^{-i\mathbf{p}\cdot\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk})}\, e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p}-\mu_{jk}\mathbf{v}_{jk})F(|\mathbf{x}| < v^{\sigma_{jk}}|t|/8)\right\| \\
&= C\left\|\left(V^l_{jk} - V^l_{jk,v_{jk}t}\right)(\mathbf{x}+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\, e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p}-\mu_{jk}\mathbf{v}_{jk})F(|\mathbf{x}| < v^{\sigma_{jk}}|t|/8)\right\| \\
&= C\left\|\left(V^l_{jk} - V^l_{jk,v_{jk}t}\right)(\mathbf{x}+\mathbf{v}_{jk}t+\mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\, e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p})F(|\mathbf{x}| < v^{\sigma_{jk}}_{jk}|t|/8)\right\| \\
&\leq C\left\|V^l_{jk}\right\| \left\| F\left(|\mathbf{x}| \geq \begin{cases} 5v^{\sigma_{jk}}_{jk}|t|/8, & \text{if } q_{jk} \neq 0, \\ v_{jk}|t|/2, & \text{if } q_{jk} = 0, \end{cases}\right) e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p})F(|\mathbf{x}| < v^{\sigma_{jk}}_{jk}|t|/8)\right\| \\
&= C\left\| F\left(|\mathbf{x}-\mathbf{v}_{jk}t| \geq \begin{cases} 5v^{\sigma_{jk}}_{jk}|t|/8, & \text{if } q_{jk} \neq 0, \\ v_{jk}|t|/2, & \text{if } q_{jk} = 0, \end{cases}\right) e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p}-\mu_{jk}\mathbf{v}_{jk})F(|\mathbf{x}| < v^{\sigma_{jk}}_{jk}|t|/8)\right\| \\
&\leq C\|F(\mathbf{x}\in\mathcal{M}')\, e^{-it\mathbf{p}^2/(2\mu_{jk})} g(\mathbf{p}-\mu_{jk}\mathbf{v}_{jk})F(\mathbf{x}\in\mathcal{M})\| \\
&\leq C(1+r+|t|)^{-2} \leq C(1+v^{\sigma_{jk}}_{jk}|t|/8+|t|)^{-2} \leq C(1+v^{\sigma_{jk}}_{jk}|t|)^{-2}.
\end{aligned}
$$

We think that we have to mention that $\left(V^l_{jk} - V^l_{jk,v_{jk}t}\right)$ is a bounded function to complete the justification of (II.2.38).

■

COMPUTATION IV.4.4. *Constraints to assure that $\sigma_{jk} > 1/2$ in Theorem II.2.8.*

Proof. Without loss of generality we assume throughout this computation that $q_{jk} \neq 0$. With this new requirement σ_{jk} and $\tilde{\sigma}_{jk}$ must hold: $1/2 < \sigma_{jk} = \frac{\tilde{\sigma}_{jk}}{2-\tilde{\sigma}_{jk}}$ and $0 < \tilde{\sigma}_{jk} < 2 - \max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\}$.

Then, $2-\tilde{\sigma}_{jk} < 2\tilde{\sigma}_{jk} \Leftrightarrow \tilde{\sigma}_{jk} > \frac{2}{3}$, and in consequence, (IV.4.89) below has to be true,

$$\max\{\frac{1+\theta_{jk}}{\gamma_D+\mu}, \frac{2}{\gamma_D+2\mu}, 1\} < \frac{4}{3}. \tag{IV.4.89}$$

Please observe, in view of $\gamma_D + 2\mu > 1+\mu > 3/2$, that

$$\max\{\frac{2}{\gamma_D+2\mu}, 1\} < \frac{4}{3}. \tag{IV.4.90}$$

We have two cases:

(*a*) $\frac{1+\theta_{jk}}{\gamma_D+\mu} < \max\{\frac{2}{\gamma_D+2\mu}, 1\}$. In this case, we have that (IV.4.90) implies (IV.4.89). Then, the condition on θ_{jk} is

$$\theta_{jk} < (\gamma_D+\mu)\max\{\frac{2}{\gamma_D+2\mu}, 1\} - 1. \tag{IV.4.91}$$

(*b*) $\frac{1+\theta_{jk}}{\gamma_D+\mu} \geq \max\{\frac{2}{\gamma_D+2\mu}, 1\}$. In order to satisfy (IV.4.89), θ_{jk} must hold: $\frac{1+\theta_{jk}}{\gamma_D+\mu} < \frac{4}{3} \Leftrightarrow \theta_{jk} < \frac{4}{3}(\gamma_D+\mu)-1$. Then, the condition on θ_{jk} is

$$(\gamma_D+\mu)\max\{\frac{2}{\gamma_D+2\mu}, 1\} - 1 \leq \theta_{jk} < \frac{4}{3}(\gamma_D+\mu) - 1, \tag{IV.4.92}$$

which is never empty by (IV.4.90).

Finally, by the two cases above (IV.4.91) and (IV.4.92), (IV.4.89) is equivalent to $\theta_{jk} < \frac{4}{3}(\gamma_D + \mu) - 1$. ■

PROPOSITION IV.4.5. *The three logical propositions P_1, P_2 and P_3, defined below, are equivalent for $N \geq 1$.*

P_1 : *There exist two pairs (j_1, k_1) and (j_2, k_2) with $1 \leq j_1 < k_1 \leq N$ and $1 \leq j_2 < k_2 \leq N$, such that $q_{j_1k_1} \neq 0$ and $q_{j_2k_2} = 0$.*

P_2 : *There exist two pairs (j, k) and (j', k') with $1 \leq j < k \leq N$, $1 \leq j' < k' \leq N$ such that $q_{jk} \neq 0$, $q_{j'k'} = 0$ and, at least one of the following is true, $j' = j$ or $k' = k$.*

P_3 : *There exist two pairs (j, k) and (j', k') with $1 \leq j < k \leq N$, $1 \leq j' < k' \leq N$ such that $q_{jk} \neq 0$, $q_{j'k'} = 0$ and, at least one of the following is true, $j' = j$ or $j' = k$ or $k' = j$ or $k' = k$ or $j' + j = 3$.*

PROOF.

$P_1 \Rightarrow) P_2$ N=1,2; true by vacuity. $N \geq 3$; without loss of generality, assume that $(j_1, k_1) = (1, 2)$, i.e, $q_{12} \neq 0$. We know that there exists a pair (j_2, k_2), such that $j_2 < k_2$, $3 \leq k_2$ and $q_{j_2k_2} = 0$.

(a) $q_{1k_2} = 0$. Let us set $j' := j := 1, \quad k' := k_2, \quad k := 2$.

(b) $q_{1k_2} \neq 0$. Let us set $j' := j_2, \quad j := 1, \quad k' := k := k_2$.

Then, in both cases, $q_{jk} \neq 0,\ q_{j'k'} = 0,\ j < k,\ j' < k'$, and either $j' = j$ or $k' = k$.

$P_2 \Rightarrow) P_3$ Trivial.

$P_3 \Rightarrow) P_1$ Trivial.

■

LEMMA IV.4.6. *More details to render clearer the proof of Theorem II.2.10.*

PROOF. In Valencia's Thesis [83] Cálculo 8.5, it is shown the simple proof of $[S^D, \mathbf{p}_l] = [S^D, \mathbf{p}_l - \mu_{12}v_l] = [S^D - I_{G,v}, \mathbf{p}_l - \mu_{12}v_l]$ and $(\mathbf{p}_l - \mu_{12}v_l)\Phi_\mathbf{v} = (\mathbf{p}_l\Phi_0)_\mathbf{v}$ where $\mathbf{p}_l$ and v_l are the l-th components of the relative momentum and the velocity $\mathbf{v}$ of the chosen pair $(1, 2)$, respectively. We justify (II.2.46) by interchanging the integral with the scalar product because of the existence of the wave operators in the strong topology:

$$\begin{aligned}
\frac{1}{I_{G,v}}\left(i[S^D, \mathbf{p}_l]\Phi_\mathbf{v}, \Psi_\mathbf{v}\right) &= \left(\left[(I_{G,v})^{-1}\, i\left(S^D - I_{G,v}\right), \mathbf{p}_l - \mu_{12}v_l\right]\Phi_\mathbf{v}, \Psi_\mathbf{v}\right) \\
&= \left(\left[\left(\int_{-\infty}^{+\infty} dt\ \left(U^{D,G,v}(t)\right)^* V_t e^{-iHt}\Omega_-^{D,G,v}\right), \mathbf{p}_l - \mu_{12}v_l\right]\Phi_\mathbf{v}, \Psi_\mathbf{v}\right)
\end{aligned}$$

$$\begin{aligned}
&= \left(\int_{-\infty}^{+\infty} dt\ \left(U^{D,G,v}(t)\right)^* V_t e^{-iHt}\Omega_-^{D,G,v}\left(\mathbf{p}_l\Phi_0\right)_\mathbf{v}, \Psi_\mathbf{v}\right) \\
&\quad - \left(\int_{-\infty}^{+\infty} dt\ \left(U^{D,G,v}(t)\right)^* V_t e^{-iHt}\Omega_-^{D,G,v}\Phi_\mathbf{v}, \left(\mathbf{p}_l\Psi_0\right)_\mathbf{v}\right) \\
&= \int_{-\infty}^{+\infty} dt\left(V_t U^{D,G,v}(t)\left(\mathbf{p}_l\Phi_0\right)_\mathbf{v}, U^{D,G,v}(t)\Psi_\mathbf{v}\right) - \int_{-\infty}^{+\infty} dt\left(V_t U^{D,G,v}(t)\Phi_\mathbf{v}, U^{D,G,v}(t)\left(\mathbf{p}_l\Psi_0\right)_\mathbf{v}\right) \\
&\quad + \int_{-\infty}^{+\infty} dt\left(V_t\left(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t)\right)\left(\mathbf{p}_l\Phi_0\right)_\mathbf{v}, U^{D,G,v}(t)\Psi_\mathbf{v}\right) \\
&\quad - \int_{-\infty}^{+\infty} dt\left(V_t\left(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t)\right)\Phi_\mathbf{v}, U^{D,G,v}(t)\left(\mathbf{p}_l\Psi_0\right)_\mathbf{v}\right).
\end{aligned}$$

Let us compute the following expression:

$$\begin{aligned} e^{-itH_0}\mathbf{p}_l e^{itH_0} &= \left(e^{-it\left[((2\mu_{12}))^{-1}\mathbf{p}^2+q_{12}\mathbf{E}\cdot\mathbf{x}\right]}\otimes e^{-it\hat{H}_0}\right)\left(\mathbf{p}_l\right)\left(e^{it\left[((2\mu_{12}))^{-1}\mathbf{p}^2+q_{12}\mathbf{E}\cdot\mathbf{x}\right]}\otimes e^{it\hat{H}_0}\right) \\ &= \left(e^{-it\left[((2\mu_{12}))^{-1}\mathbf{p}^2+q_{12}\mathbf{E}\cdot\mathbf{x}\right]}\mathbf{p}_l e^{it\left[((2\mu_{12}))^{-1}\mathbf{p}^2+q_{12}\mathbf{E}\cdot\mathbf{x}\right]}\right) \\ &= \left(e^{iq_{12}Ex_1t}e^{-it^3q_{12}^2E^2/(6\mu_{12})}e^{-ip_1q_{12}Et^2/(2\mu_{12})}e^{-it\mathbf{p}^2/(2\mu_{12})}\,\mathbf{p}_l\,e^{it\mathbf{p}^2/(2\mu_{12})}\right. \\ &\quad \left.\cdot e^{ip_1q_{12}Et^2/(2\mu_{12})}e^{it^3q_{12}^2E^2/(6\mu_{12})}e^{-iq_{12}Ex_1t}\right) \\ &= \left(e^{iq_{12}Ex_1t}\,\mathbf{p}_l\,e^{-iq_{12}Ex_1t}\right) = \left(\mathbf{p}_l-\delta_{l,1}q_{12}Et\right). \qquad \text{(IV.4.93)} \end{aligned}$$

Let V_{12}^{sl} be either the operator V_{12}^{s} or V_{12}^{l}. Remembering that $\mathbf{p}_l=-i\partial/\partial x_l$ and using (IV.4.93):

$$\begin{aligned} & i\left(\left(\frac{\partial}{\partial x_l}V_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)(\mathbf{p}_l\Psi_0)_{\mathbf{v}}\right) \\ = & -\left(\left(\mathbf{p}_lV_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(e^{itH_0}V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},\right. \\ & \left.\tilde{U}_D(t)(\mathbf{p}_l-\mu_{12}v_l)\Psi_{\mathbf{v}}\right) \\ = & -\left(\left(\mathbf{p}_lV_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(e^{-itH_0}\mathbf{p}_le^{itH_0}V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},\right. \\ & \left.e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(e^{itH_0}V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(-\mu_{12}v_l)\Phi_{\mathbf{v}},\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ = & -\left(\left(\mathbf{p}_lV_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(\left(\mathbf{p}_l-\delta_{l,1}q_{12}Et\right)V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},\right. \\ & \left.e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(e^{itH_0}V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(-\mu_{12}v_l)\Phi_{\mathbf{v}},\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ = & -\left(\left(\mathbf{p}_lV_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right)+\left(\left(\mathbf{p}_lV_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ & +\left(V_{12}^{sl}(\mathbf{x})\left(\mathbf{p}_l-\delta_{l,1}q_{12}Et\right)e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ & +\left(e^{itH_0}V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(-\mu_{12}v_l)\Phi_{\mathbf{v}},\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ = & \left(V_{12}^{sl}(\mathbf{x})e^{-itH_0}\mathbf{p}_le^{itH_0}e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ & +\left(V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(-\mu_{12}v_l)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right). \end{aligned}$$

This implies that:

$$\begin{aligned} i\left(\left(\frac{\partial}{\partial x_l}V_{12}^{sl}\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) &= \left(V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(\mathbf{p}_l\Phi_0)_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\ &\quad -\left(V_{12}^{sl}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}},e^{-itH_0}\tilde{U}_D(t)(\mathbf{p}_l\Psi_0)_{\mathbf{v}}\right). \qquad \text{(IV.4.94)} \end{aligned}$$

By equations (II.1.31)-(II.1.35), we have the following

$$\begin{aligned} e^{itH_0}\,V_{12}^{vsl}(\mathbf{x})\,e^{-itH_0} &= e^{it\mathbf{p}^2/(2\mu_{12})}e^{i\mathbf{p}\cdot(q_{12}Et^2/(2\mu_{12}))\mathbf{e}_1}\,V_{12}^{vsl}(\mathbf{x})\,e^{-i\mathbf{p}\cdot(q_{12}Et^2/(2\mu_{12}))\mathbf{e}_1}e^{-it\mathbf{p}^2/(2\mu_{12})} \\ &= e^{it\mathbf{p}^2/(2\mu_{12})}\,V_{12}^{vsl}(\mathbf{x}+\mathbf{e}_1q_{12}Et^2/(2\mu_{12}))\,e^{-it\mathbf{p}^2/(2\mu_{12})} \\ &= V_{12}^{vsl}(\mathbf{x}+t\,\mathbf{p}/\mu_{12}+\mathbf{e}_1q_{12}Et^2/(2\mu_{12})), \qquad \text{(IV.4.95)} \end{aligned}$$

where V_{12}^{vsl} represents any of the following operators V_{12}^{vs}, V_{12}^{s}, or V_{12}^{l} defined in (II.2.1).

Using equations (II.1.31)-(II.1.35) and substituting (II.2.45) , (IV.4.94), (IV.4.95) and (II.2.50) in (II.2.48), we get:

$$
\begin{aligned}
l_v(vt) &= \left(V_{12}^{vs}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(\mathbf{p}_l\Phi_0)_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) - \left(V_{12}^{vs}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)(\mathbf{p}_l\Psi_0)_{\mathbf{v}}\right) \\
&\quad + i\left(\left(\frac{\partial}{\partial x_l}V_{12}^s\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&\quad + i\left(\left(\frac{\partial}{\partial x_l}V_{12}^l\right)(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&= \left(e^{itH_0}V_{12}^{vs}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)(\mathbf{p}_l\Phi_0)_{\mathbf{v}}, \tilde{U}_D(t)\Psi_{\mathbf{v}}\right) - \left(e^{itH_0}V_{12}^{vs}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, \tilde{U}_D(t)(\mathbf{p}_l\Psi_0)_{\mathbf{v}}\right) \\
&\quad + i\left(e^{itH_0}\frac{\partial V_{12}^s}{\partial x_l}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, \tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&\quad + i\left(e^{itH_0}\frac{\partial V_{12}^l}{\partial x_l}(\mathbf{x})e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, \tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&= \left(e^{-i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\, V_{12}^{vs}(\mathbf{x} + t\,\mathbf{p}/\mu_{12} + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\tilde{U}_D(\mathbf{v},t)\mathbf{p}_l\Phi_0, \tilde{U}_D(\mathbf{v},t)\Psi_0\right) \\
&\quad - \left(e^{-i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\, V_{12}^{vs}(\mathbf{x} + t\,\mathbf{p}/\mu_{12} + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\tilde{U}_D(\mathbf{v},t)\Phi_0, \tilde{U}_D(\mathbf{v},t)\mathbf{p}_l\Psi_0\right) \\
&\quad + i\left(e^{-i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\, \frac{\partial V_{12}^s}{\partial x_l}(\mathbf{x} + t\,\mathbf{p}/\mu_{12} + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\tilde{U}_D(\mathbf{v},t)\Phi_0, \tilde{U}_D(\mathbf{v},t)\Psi_0\right) \\
&\quad + i\left(e^{-i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\, \frac{\partial V_{12}^l}{\partial x_l}(\mathbf{x} + t\,\mathbf{p}/\mu_{12} + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\tilde{U}_D(\mathbf{v},t)\Phi_0, \tilde{U}_D(\mathbf{v},t)\Psi_0\right) \\
&= \left(V_{12}^{vs}(\mathbf{x} + \mathbf{v}t + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\mathbf{p}_l\Phi_0, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\Psi_0\right) \\
&\quad - \left(V_{12}^{vs}(\mathbf{x} + \mathbf{v}t + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\Phi_0, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\mathbf{p}_l\Psi_0\right) \\
&\quad + i\left(\frac{\partial V_{12}^s}{\partial x_l}(\mathbf{x} + \mathbf{v}t + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\Phi_0, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\Psi_0\right) \\
&\quad + i\left(\frac{\partial V_{12}^l}{\partial x_l}(\mathbf{x} + \mathbf{v}t + \mathbf{e}_1 q_{12}Et^2/(2\mu_{12}))\, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\Phi_0, e^{-it\mathbf{p}^2/(2\mu_{12})}\tilde{U}_D(\mathbf{v},t)\Psi_0\right). \quad \text{(IV.4.96)}
\end{aligned}
$$

it follows that (II.2.51) is true.

Moreover,

$$
\begin{aligned}
\frac{R(v)}{v} &= \sum_{j<k, 3\le k\le N} \int dt\, \left(V_{jk}^{vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Phi_0\right)_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&\quad + \sum_{j<k, 3\le k\le N}^{E} \int dt\, \left(\left(V_{jk}^{s}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^{s}(\mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Phi_0\right)_{\mathbf{v}}, \right. \\
&\qquad \left. e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&\quad + \sum_{j<k, 3\le k\le N} \int dt\, \left(\left(V_{jk}^{l}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^{l}(t\,\mathbf{p}_{jk}/\mu_{jk} - \mathbf{e}_1 q_{jk}Et^2/(2\mu_{jk}))\right) e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Phi_0\right)_{\mathbf{v}}, \right. \\
&\qquad \left. e^{-itH_0}\tilde{U}_D(t)\Psi_{\mathbf{v}}\right) \\
&\quad - \sum_{j<k, 3\le k\le N} \int dt\, \left(V_{jk}^{vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j)e^{-itH_0}\tilde{U}_D(t)\Phi_{\mathbf{v}}, e^{-itH_0}\tilde{U}_D(t)\left(\mathbf{p}_l\Psi_0\right)_{\mathbf{v}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j<k, 3\le k\le N}^{E} \int dt\, \Big(\big(V_{jk}^s(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^s(\mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\big)\, e^{-itH_0} \tilde{U}_D(t)\Phi_{\mathbf{v}}, \\
&\quad e^{-itH_0}\tilde{U}_D(t)\,(\mathbf{p}_l\Psi_0)_{\mathbf{v}} \Big) \\
&- \sum_{j<k, 3\le k\le N} \int dt\, \Big(\big(V_{jk}^l(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^l(t\,\mathbf{p}_{jk}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\big)\, e^{-itH_0} \tilde{U}_D(t)\Phi_{\mathbf{v}}, \\
&\quad e^{-itH_0}\tilde{U}_D(t)\,(\mathbf{p}_l\Psi_0)_{\mathbf{v}} \Big) \\
&+ \sum_{j<k} \int dt\, \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t) \Big) (\mathbf{p}_l\Phi_0)_{\mathbf{v}}, V_{jk}^{vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) U^{D,G,v}(t)\Psi_{\mathbf{v}} \Big) \\
&- \sum_{j<k} \int dt\, \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t) \Big) \Phi_{\mathbf{v}}, V_{jk}^{vs}(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) U^{D,G,v}(t)\,(\mathbf{p}_l\Psi_0)_{\mathbf{v}} \Big) \\
&+ \sum_{j<k}^{E} \int dt\, \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t) \Big) (\mathbf{p}_l\Phi_0)_{\mathbf{v}}, \\
&\quad \big(V_{jk}^s(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^s(\mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\big)\, U^{D,G,v}(t)\Psi_{\mathbf{v}} \Big) \\
&- \sum_{j<k}^{E} \int dt\, \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t) \Big) \Phi_{\mathbf{v}}, \\
&\quad \big(V_{jk}^s(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^s(\mathbf{v}_{jk}t + \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\big)\, U^{D,G,v}(t)\,(\mathbf{p}_l\Psi_0)_{\mathbf{v}} \Big) \\
&+ \sum_{j<k} \int dt\, \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t) \Big) (\mathbf{p}_l\Phi_0)_{\mathbf{v}}, \\
&\quad \big(V_{jk}^l(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^l(t\,\mathbf{p}_{jk}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\big)\, U^{D,G,v}(t)\Psi_{\mathbf{v}} \Big) \\
&- \sum_{j<k} \int dt\, \Big(\Big(e^{-iHt}\Omega_-^{D,G,v} - U^{D,G,v}(t) \Big) \Phi_{\mathbf{v}}, \\
&\quad \big(V_{jk}^l(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_j) - V_{jk}^l(t\,\mathbf{p}_{jk}/\mu_{jk} - \mathbf{e}_1 q_{jk} E t^2/(2\mu_{jk}))\big)\, U^{D,G,v}(t)\,(\mathbf{p}_l\Psi_0)_{\mathbf{v}} \Big).
\end{aligned}
$$

Thus, by Lemmata II.2.3, II.2.4 and II.2.7, if $V_{jk}^l = 0$ for all $1 \le j < k \le N$:

$$
\begin{aligned}
\frac{R(v)}{v} \quad &= \quad O(v^{-2}) + \begin{cases} O(v^{-2\alpha}), & \text{if } \alpha < 1, \sum\limits_{j<k,3\le k\le N} |q_{jk}| > 0, \\ O(v^{-2(1-\epsilon_1)}), & \text{if } \alpha = 1, \sum\limits_{j<k,3\le k\le N} |q_{jk}| > 0, \end{cases} \\
&+ \begin{cases} O(v^{-\alpha}), & \text{if } \alpha < 1, \sum\limits_{j<k,3\le k\le N} |q_{jk}| > 0, \\ O(v^{-(1-\epsilon_1)}), & \text{if } \alpha = 1, \sum\limits_{j<k,3\le k\le N} |q_{jk}| > 0, \\ O(v^{-1}), & \text{if } \sum\limits_{j<k,3\le k\le N} |q_{jk}| = 0, \end{cases} \times \begin{cases} O(v^{-\alpha}), & \text{if } \alpha < 1, \sum\limits_{j<k} |q_{jk}| > 0, \\ O(v^{-(1-\epsilon_1)}), & \text{if } \alpha = 1, \sum\limits_{j<k} |q_{jk}| > 0, \\ O(v^{-1}), & \text{if } \sum\limits_{j<k} |q_{jk}| = 0. \end{cases}
\end{aligned}
$$

Then,

$$
R(v) \quad = \quad \begin{cases} O(v^{1-2\alpha}), & \text{if } \alpha < 1, \sum\limits_{j<k} |q_{jk}| > 0, \\ O(v^{-1+2\epsilon_1}), & \text{if } \alpha = 1, \sum\limits_{j<k} |q_{jk}| > 0, \\ O(v^{-1}), & \text{if } \sum\limits_{j<k} |q_{jk}| = 0. \end{cases}
$$

Finally, the convergence rate of $R(v)$ as $v \to \infty$ is

$$\begin{aligned}\limsup_{v\to\infty} v^{\rho}|R(v)| &\leq C\limsup_{v\to\infty}\begin{cases} v^{1+\rho-2\alpha}, & \text{if } \alpha<1,\ \sum_{j<k}|q_{jk}|>0,\\ v^{\rho-1+2\epsilon_1}, & \text{if } \alpha=1,\ \sum_{j<k}|q_{jk}|>0,\\ v^{-1+\rho}, & \text{if } \sum_{j<k}|q_{jk}|=0.\end{cases}\\ &\leq \begin{cases} 0, & \text{if } 0\leq\rho<2\alpha-1,\\ C, & \text{if } \rho=2\alpha-1<1,\ \sum_{j<k}|q_{jk}|>0,\\ C, & \text{if } \rho=\alpha=1,\ \sum_{j<k}|q_{jk}|=0,\\ \infty, & \text{if } \rho=\alpha=1,\ \sum_{j<k}|q_{jk}|>0.\end{cases}\end{aligned}$$

Then,

$$R(v) = \begin{cases} o(v^{-\rho}), & \text{if } 0\leq\rho<2\alpha-1,\\ O(v^{-\rho}), & \text{if } \rho=2\alpha-1<1,\ \sum_{j<k}|q_{jk}|>0,\\ O(v^{-1}), & \text{if } \rho=1,\ \sum_{j<k}|q_{jk}|=0,\end{cases} \qquad \text{(II.2.56)}$$

Similarly, by Lemmata II.2.3, II.2.4, II.2.5 and II.2.7, if $V_{jk}^{l}\neq 0$ for some $1\leq j<k\leq N$:

$$\begin{aligned}\frac{R(v)}{v} &= O(v^{-2})+O\left(v^{-2\min\{\sigma_{jk}\,|\,j<k,3\leq k\leq N\}}\right)+\begin{cases} O(v^{-2\alpha}), & \text{if } \alpha<1,\ \sum_{j<k,3\leq k\leq N}|q_{jk}|>0,\\ O(v^{-2(1-\epsilon_1)}), & \text{if } \alpha=1,\ \sum_{j<k,3\leq k\leq N}|q_{jk}|>0,\end{cases}\\ &+O\left(v^{-\min\{\alpha,\,\sigma_{jk}\,|\,1\leq j<k\leq N\}}\right)\\ &\times\left(O(v^{-1})+O\left(v^{-\min\{\sigma_{jk}\,|\,1\leq j<k\leq N\}}\right)+\begin{cases} O(v^{-\alpha}), & \text{if } \alpha<1,\ \sum_{j<k}|q_{jk}|>0,\\ O(v^{-(1-\epsilon_1)}), & \text{if } \alpha=1,\ \sum_{j<k}|q_{jk}|>0,\\ O(v^{-1}), & \text{if } \sum_{j<k}|q_{jk}|=0.\end{cases}\right).\end{aligned}$$

Then,

$$R(v) = \begin{cases} O\left(v^{1-2\min\{\alpha,\,\sigma_{jk}\,|\,1\leq j<k\leq N\}}\right), & \text{if } \sum_{j<k}|q_{jk}|>0,\\ O(v^{-1}), & \text{if } \sum_{j<k}|q_{jk}|=0,\end{cases}$$

$$R(v) = \begin{cases} o(v^{-\rho}), & \text{if } 0\leq\rho<2\min\{\alpha,\,\sigma_{jk}\,|\,1\leq j<k\leq N\}-1,\\ O(v^{-\rho}), & \text{if } \rho=2\min\{\alpha,\,\sigma_{jk}\,|\,1\leq j<k\leq N\}-1<1,\\ O(v^{-1}), & \text{if } \rho=1,\ \sum_{j<k}|q_{jk}|=0.\end{cases} \qquad \text{(II.2.57)}$$

Let us analyze the rate of convergence of $h_{\mathbf{v}}^{(1)}$. On one hand, with $t = \tau/v$:

$$\begin{aligned}
& \left\|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\tilde{U}_D(\mathbf{v},\tau/v) - I)\Psi_0\right\|^2 \\
= & \int dp dp_3\cdots dp_N \left|(e^{-ip_1 q_{12} Et^2/(2\mu_{12})} e^{-ip^2 t/(2\mu_{12})}\, \tilde{U}_D(\mathbf{v},\tau/v) - 1)\hat{\psi}_{12}(p)\hat{\psi}_3(p_3,\ldots,p_N)\right|^2 \\
= & \int dp dp_3\cdots dp_N \left|\int_0^t ds \frac{d}{ds}\left(e^{-ip_1 q_{12} Es^2/(2\mu_{12})} e^{-ip^2 s/(2\mu_{12})}\, \tilde{U}_D(\mathbf{v},s)\right)\right|^2 \left|\hat{\psi}_{12}(p)\hat{\psi}_3(p_3,\ldots,p_N)\right|^2 \\
\leq & \int dp dp_3\cdots dp_N \left[\int_0^t ds \left|\frac{d}{ds}\left(e^{-ip_1 q_{12} Es^2/(2\mu_{12}) - ip^2 s/(2\mu_{12})}\, \tilde{U}_D(\mathbf{v},s)\right)\right|\right]^2 \left|\hat{\psi}_{12}(p)\hat{\psi}_3(p_3,\ldots,p_N)\right|^2 \\
\leq & \int \left[\int_0^t ds \left|p_1 q_{12} Es/\mu_{12} + p^2/(2\mu_{12}) + \sum_{j<k} V_{jk}^l(s(p_{jk}/\mu_{jk} + \mathbf{v}_{jk}) + \mathbf{e}_1 q_{jk} Es^2/(2\mu_{jk}))\right|\right]^2 \\
& \times \left|\hat{\psi}_{12}(p)\hat{\psi}_3(p_3,\ldots,p_N)\right|^2 dp dp_3\cdots dp_N \\
\leq & \int dp\, \left[|p_1 q_{12} E| t^2/(2\mu_{12}) + (p^2/(2\mu_{12}) + C)t\right]^2 \left|\hat{\psi}_{12}(p)\right|^2 \\
\leq & \left[C\left|\tau/v\right|\left(1 + \left|\tau/v\right|\right)\right]^2,
\end{aligned}$$

On the other hand:

$$\left\|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\, \tilde{U}_D(\mathbf{v},\tau/v) - I)\Psi_0\right\| \quad \leq \quad 2\left\|\Psi_{12}\right\|.$$

Now we study $|h_{\mathbf{v}}^{(1)}(\tau)|$'s decay as $v \to \infty$ applying Lemma II.2.3, and (II.2.67), (II.2.68) with $a = \rho$:

$$\begin{aligned}
|h_{\mathbf{v}}^{(1)}(\tau)| \quad \leq & \quad \left\|V_{12}^{vs}(\mathbf{x} + \hat{\mathbf{v}}\tau) e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\, \tilde{U}_D(\mathbf{v},\tau/v)\mathbf{p}_l\Phi_0\right\| \\
& \times \|(e^{-i\mathbf{p}_1 q_{12} E\tau^2/(2\mu_{12}v^2)} e^{-i\mathbf{p}^2\tau/(2\mu_{12}v)}\, \tilde{U}_D(\mathbf{v},\tau/v) - I)\Psi_0\| \\
\leq & \quad C\left|\tau/v\right|^\rho \left\|V_{12}^{vs}\left(\mathbf{x} + (\mathbf{p}/(\mu_{12}v) + \hat{\mathbf{v}})\tau + \mathbf{e}_1 q_{12} E\tau^2/(2v^2\mu_{12})\right) e^{-i\mu_{12}\mathbf{v}\cdot\mathbf{x}}\, \tilde{U}_D(\tau/v)\left(\mathbf{p}_l\Phi_0\right)_v\right\| \\
\leq & \quad C\left|\tau/v\right|^\rho \left\|V_{12}^{vs}\left(\mathbf{x} + \tau\,\mathbf{p}/(\mu_{12}v) + \mathbf{e}_1 q_{12} E\tau^2/(2v^2\mu_{12})\right)\tilde{U}_D(\tau/v)\right. \\
& \left.\times \prod_{j'<k'} f_{j'k'}(\mathbf{p}_{j'k'} - \mu_{j'k'}\mathbf{v}_{j'k'})(1 + |\mathbf{x}|^2)^{-2}\right\|.
\end{aligned}$$

Then

$$v^\rho |h_{\mathbf{v}}^{(1)}(\tau)| \leq C|\tau|^\rho h_{12}(|\tau|) \in L^1(-\infty,\infty).\ \text{(II.2.69)} \qquad \text{(IV.4.97)}$$

Hence, for $\rho = 1$

$$v\int |h_{\mathbf{v}}^{(1)}(\tau)| d\tau \leq C.$$

For $0 \leq \rho < 1$, by Lebesgue dominated convergence theorem

$$\lim_{v\to\infty} v^\rho \int h_{\mathbf{v}}^{(1)}(\tau) d\tau = \int \lim_{v\to\infty} v^\rho h_{\mathbf{v}}^{(1)}(\tau) d\tau = 0,$$

where we used that $\lim_{v\to\infty} v^\rho h_{\mathbf{v}}^{(1)}(\tau) = 0$, since by (II.2.67) and (II.2.68) with $a = 1$ we have $v^\rho |h_{\mathbf{v}}^{(1)}(\tau)| \leq C|\tau| v^{\rho-1}$.

As a result

$$\int_{-\infty}^{+\infty} d\tau h_{\mathbf{v}}^{(1)}(\tau) = \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1. \end{cases} \quad \text{(II.2.70)}$$

At this moment, we turn our attention to the rate of convergence of $h_{\mathbf{v}}^{(2)}$. When $|\mathbf{x} + \tau\hat{\mathbf{v}}| \le |\tau|/2$, we have $|\mathbf{x}| \ge |\tau| - |\mathbf{x} + \tau\hat{\mathbf{v}}| \ge |\tau|/2$. With the last inequality we can estimate the second factor in the scalar product of (II.2.65). Let g be a $C_0^\infty(\mathbb{R}^n)$ such that $g(\mathbf{p})\hat{\psi}_{12} = \hat{\psi}_{12}$. By (II.2.67) and (II.2.68):

$$\begin{aligned} v^\rho \int_{-\infty}^{\infty} d\tau\, |h_{\mathbf{v}}^{(2)}(\tau)| &\le C \int_{-\infty}^{\infty} d\tau\, |\tau|^\rho \|V_{12}^{vs}(\mathbf{x} + \hat{\mathbf{v}}\tau)\Psi_0\| \\ &\le C \int_{-\infty}^{+\infty} d\tau |\tau|^\rho \left(\|V_{12}^{vs}(\mathbf{x} + \hat{\mathbf{v}}\tau) g(\mathbf{p}) F(|\mathbf{x} + \hat{\mathbf{v}}\tau| \ge |\tau|/2)\|\right. \\ &\quad \left. + \|V_{12}^{vs}(\mathbf{x} + \hat{\mathbf{v}}\tau) g(\mathbf{p})\|\, \|F(|\mathbf{x}| \ge |\tau|/2)\Psi_{12}\|\right). \quad \text{(II.2.71)} \end{aligned}$$

Due to the short-range condition (II.2.21), the first integral in (II.2.71) is finite; the fast decay in configuration space of Ψ_{12} makes the second integral in (II.2.71) be bounded:

$$\begin{aligned} \int_{-\infty}^{\infty} d\tau\, |\tau|^\rho\, \|F(|\mathbf{x}| \ge |\tau|/2)\Psi_{12}\| &= \int_{-\infty}^{\infty} d\tau\, |\tau|^\rho (1 + |\tau|)^{-3} \left\|(1 + |\tau|)^3 F(|\mathbf{x}| \ge \frac{|\tau|}{2})\Psi_{12}\right\| \\ &\le \int_{-\infty}^{\infty} d\tau\, |\tau|^\rho (1 + |\tau|)^{-3} \|(1 + |2\mathbf{x}|)^3 \Psi_{12}\| \\ &\le C \int_{-\infty}^{\infty} \frac{d\tau}{(1 + |\tau|)^2} < \infty. \end{aligned}$$

Hence, for $\rho = 1$

$$v \int |h_{\mathbf{v}}^{(2)}(\tau)| d\tau \le C,$$

and for $0 \le \rho < 1$, by Lebesgue dominated convergence theorem

$$\lim_{v\to\infty} v^\rho \int h_{\mathbf{v}}^{(2)}(\tau) d\tau = \int \lim_{v\to\infty} v^\rho h_{\mathbf{v}}^{(2)}(\tau) d\tau = 0,$$

where we used that $\lim_{v\to\infty} v^\rho h_{\mathbf{v}}^{(2)}(\tau) = 0$, since by (II.2.67) and (II.2.68) with $a = 1$ we have $v^\rho |h_{\mathbf{v}}^{(2)}(\tau)| \le C|\tau| v^{\rho-1}$.

As a result

$$\int_{-\infty}^{+\infty} d\tau h_{\mathbf{v}}^{(2)}(\tau) = \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1. \end{cases} \quad \text{(II.2.72)}$$

Moreover, when there is at least one pair with non-zero relative charge, we have to estimate the following error, see (II.2.39) and (II.2.46). In this case, $\rho < 1$, and $-(2\gamma - 1) \le -(2\alpha - 1) \le -\rho$, where γ is as in Definition II.1.2. By equation (II.2.13):

$$\begin{aligned} |I_{G,v} - 1| &\le \sum_{j<k}^{E} \int_{-\infty}^{\infty} ds\, |V_{jk}^s(\mathbf{v}_{jk}s + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk}))| \\ &\le C \sum_{j<k}^{E} \int_{-\infty}^{\infty} ds\, \left(1 + |\mathbf{v}_{jk}s + \mathbf{e}_1 q_{jk} E s^2/(2\mu_{jk})|\right)^{-\gamma} \le C \int_0^\infty ds\, \left(1 + |\mathbf{v}s| + s^2\right)^{-\gamma} \\ &= C \left[\int_0^v ds\, (1 + |\mathbf{v}s|)^{-\gamma} + \int_v^\infty ds\, \left(1 + s^2\right)^{-\gamma}\right] \le C \begin{cases} v^{-(2\gamma-1)}, & \text{if } 1/2 < \gamma < 1, \\ \frac{\ln v}{v}, & \text{if } \gamma = 1, \end{cases} \end{aligned}$$

$$= \begin{cases} o(v^{-\rho}), & \text{if } 1/2 < \gamma < 1 \text{ and } \rho < 2\gamma - 1 \text{ or } \gamma = 1, \\ O(v^{-\rho}), & \text{if } 1/2 < \gamma < 1 \text{ and } \rho = 2\gamma - 1, \end{cases}$$

$$= \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 2\min\{\alpha, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1. \end{cases} \quad \text{(II.2.74)}$$

One of the main results of Valencia and Weder [84] (and this thesis) is the error term in (II.2.44). Let us see in detail why this is true. We define $e(v)$, the error term, for large v, as:

$$e(v) = v\left(i[S^D, \mathbf{p}_l]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right) - \lim_{v\to\infty} v\left(i[S^D, \mathbf{p}_l]\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}\right). \quad \text{(IV.4.98)}$$

First, until we say the opposite, we will assume that we do not know whether or not $V^L = 0$.

By assumption of Theorem II.2.8,

$$0 \leq \rho \leq 2\min\{\alpha, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1. \quad \text{(IV.4.99)}$$

Moreover, from Lemma II.2.5, when $q_{jk} \neq 0$, one has $0 < \sigma_{jk} < 1$. Then, (IV.4.99) implies that if we do not know that whether $V^L = 0$ and we know that $\sum_{j<k} |q_{jk}| > 0$, we must have that

$$0 \leq \rho \leq 2\min\{\alpha, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1. \quad \text{(IV.4.100)}$$

Conversely, if we have $\rho = 1$ then $\sum_{j<k} |q_{jk}| = 0$ must be true, because in this case, $\alpha = 1$, and $\sigma_{jk} = 1$, for all $1 \leq j < k \leq N$, see the hypothesis of Theorem II.2.8. We will usually write $\rho = 1$ and $\sum_{j<k} |q_{jk}| = 0$ together even though $\rho = 1$ is enough when we do not know that whether $V^L = 0$.

By (II.2.46), (II.2.51), (II.2.57), (II.2.66), (II.2.73) and (II.2.74), for all $\rho_l < \begin{cases} \gamma_D + \mu - 1, & \text{if } q_{12} \neq 0, \\ \gamma_1 - 1, & \text{if } q_{12} = 0, \end{cases}$ and $\rho_s < \alpha$.

$$\begin{aligned} |e(v)| &= \left| I_{G,v}\left(I(v) + R(v)\right) - \lim_{v\to\infty} I_{G,v}\left(I(v) + R(v)\right) \right| \\ &\leq \left|\left(I_{G,v} - 1\right)\left(I(v) + R(v)\right)\right| \\ &\quad + \left|\left(I(v) + R(v)\right) - \lim_{v\to\infty} I_{G,v}\left(I(v) + R(v)\right)\right| \\ &\leq C\left|\left(I_{G,v} - 1\right)\right| + \sum_{l=1}^{4} \left| J_l(v) - \lim_{v\to\infty} J_l(v) \right| + |R(v)| \end{aligned}$$

$$= C \begin{cases} v^{-(2\gamma-1)}, & \text{if } 1/2 < \gamma < 1, \text{ and } \sum_{j<k} |q_{jk}| > 0, \\ \frac{\ln v}{v}, & \text{if } \gamma = 1, \text{ and } \sum_{j<k} |q_{jk}| > 0, \\ 0, & \text{if } \sum_{j<k} |q_{jk}| = 0, \end{cases}$$

$$+ \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1. \end{cases} + \begin{cases} o(v^{-\rho_s}), & \text{if } q_{12} \neq 0, \\ 0, & \text{if } q_{12} = 0, \end{cases} + \begin{cases} o(v^{-\rho_l}), & \text{if } V_{12}^l \neq 0, \\ 0, & \text{if } V_{12}^l = 0, \end{cases}$$

$$+ \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 2\min\{\alpha, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum_{j<k} |q_{jk}| = 0. \end{cases}$$

From (II.2.57), we see the need of having $\sigma_{jk} > 1/2$ for all $1 \leq j < k \leq N$ in order to have $R(v) \to 0$ as $v \to \infty$.

Furthermore, if $\gamma < 1$,

$$\begin{aligned}\lim_{v\to\infty} \frac{v^{-(2\gamma-1)}}{v^{-\rho}} &= \lim_{v\to\infty} v^{\rho-(2\gamma-1)} \\ &= \begin{cases} 0, & \text{if } 0 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 2\gamma - 1, \\ & \text{or } 0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 = 2\gamma - 1, \\ C, & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 = 2\gamma - 1, \end{cases}\end{aligned}$$

we know that $1/2 < \alpha \leq \gamma < 1$, then, by (IV.4.99), $\rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 \leq 2\alpha - 1 \leq 2\gamma - 1 < \gamma < 1$. If $\gamma = 1$ by L'Hôpital's rule,

$$\lim_{v\to\infty} \frac{\frac{\ln v}{v}}{v^{-\rho}} = \lim_{v\to\infty} \frac{\ln v}{v^{1-\rho}} = \lim_{v\to\infty} \frac{v^{-1}}{(1-\rho)v^{-\rho}} = \lim_{v\to\infty} \frac{v^{\rho-1}}{(1-\rho)} = 0.$$

Then, in view that $\sum_{j<k} |q_{jk}| > 0$ (which implies $\rho < 1$),

$$\begin{aligned}|(I_{G,v} - 1)| &= \begin{cases} o(v^{-\rho}), & \text{if } 1/2 < \gamma < 1, \text{ and } \sum_{j<k} |q_{jk}| > 0, \text{ and} \\ & 0 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1,\, \rho < 2\gamma - 1, \\ O(v^{-\rho}), & \text{if } 1/2 < \gamma < 1, \text{ and } \sum_{j<k} |q_{jk}| > 0, \text{ and} \\ & \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 = 2\gamma - 1, \\ o(v^{-\rho}), & \text{if } \gamma = 1, \text{ and } \sum_{j<k} |q_{jk}| > 0, \text{ and} \\ & 0 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ 0, & \text{if } \sum_{j<k} |q_{jk}| = 0 \text{ and} \\ & 0 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \end{cases} \\ &= \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1,\, \sum_{j<k} |q_{jk}| = 0. \end{cases} \qquad \text{(IV.4.101)}\end{aligned}$$

We give more details to the argument given in Valencia and Weder [84]: When $q_{12} \neq 0$, we do not have an extra error term of the form $o(v^{-\rho_s})$ because in (II.2.56) and (II.2.57) $\rho < \alpha$. Let us prove this last statement: First case, assume that $\rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$ then $\rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 \leq 2\alpha - 1 \leq \alpha$, which implies that $\rho < \alpha$. Second case $\rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1$. Hence we have two possibilities: $\rho = 2\alpha - 1 < 1$ this implies that $2\alpha - 1 < \alpha < 1$, thus $\rho < \alpha$; the second possibility is $\rho < 2\alpha - 1$ but $2\alpha - 1 \leq \alpha$ gives us $\rho < \alpha$. If $\rho < \alpha$, one can always choose ρ_s such that $\rho < \rho_s < \alpha \leq 1$. Therefore,

$$\begin{cases} o(v^{-\rho_s}), & \text{if } q_{12} \neq 0, \\ 0, & \text{if } q_{12} = 0, \end{cases} = \begin{cases} o(v^{-\rho}), & \text{if } 0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1,\, \sum_{j<k} |q_{jk}| = 0. \end{cases} \qquad \text{(IV.4.102)}$$

If $\rho < 1$, and we do not know whether or not any charge q_{jk} is different from zero, then we only know that $0 \leq \rho \leq 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$ must hold. If $0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$

then automatically $\rho < 1$, else, if $\rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1$, then, by ρ being less than 1, $\rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1$ is true. Now, it is no difficult to see that,

$$\begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1, \end{cases} = \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho \le 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum\limits_{j<k} |q_{jk}| = 0 \end{cases}$$
$$= \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum\limits_{j<k} |q_{jk}| = 0. \end{cases}$$

Let us recall the definition of γ_2 given in the the hypothesis of Theorem II.2.8:

$$\gamma_2 = \begin{cases} \gamma_1, & \text{if } q_{12} = 0 \text{ and } V^l_{12} \ne 0, \\ \gamma_D + \mu, & \text{if } q_{12} \ne 0, \text{ and } V^l_{12} \ne 0, \\ 2, & \text{if } V^l_{12} = 0. \end{cases}$$

Let us define two functions,

$$e_1(v) := \begin{cases} o(v^{-\rho_l}), & \text{if } V^l_{12} \ne 0, \text{ for all } \rho_l < \gamma_2 - 1 \\ 0, & \text{if } V^l_{12} = 0, \end{cases}$$
$$e_2(v) := \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum\limits_{j<k} |q_{jk}| = 0. \end{cases}$$

So far we have proven that

$$|e(v)| \le |e_1(v)| + |e_2(v)|.$$

Let us estimate this two errors. We have the following cases:

(a) Case $V^l_{12} = 0\,(\gamma_2 = 2)$. This case applies when we only know that the pair potential, that we are reconstructing, has a long-range part equals zero, but we do not know whether $V^L = 0$. Then

$$|e(v)| = |e_2(v)| = \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < \min\{\gamma_2, 2\alpha, 2\sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < \gamma_2 - 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} = 0. \end{cases}$$

In this case $\gamma_2 - 1 \le \rho \le 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1$ and $V^l_{12} \ne 0$ are not true. Then we can say that

$$|e(v)| = \begin{cases} o(v^{-\rho_l}), & \text{if } \gamma_2 - 1 \le \rho \le 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1, \\ & \quad \text{for any } \rho_l,\ 0 \le \rho_l < \gamma_2 - 1, \\ o(v^{-\rho}), & \text{if } 0 \le \rho < \min\{\gamma_2, 2\alpha, 2\sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < \gamma_2 - 1, \\ o(v^{-\rho_l}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} \ne 0, \\ & \quad \text{for any } \rho_l,\ 0 \le \rho_l < \gamma_1 - 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} = 0. \end{cases}$$

(*b*) Case $V^l_{12} \neq 0$.

It is obvious that either $\gamma_2 - 1 \leq \rho$ or $\rho < \gamma_2 - 1$. If ρ satisfies $0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$ then ρ satisfies one of the following subcases:

(*i*) $\gamma_2 - 1 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$.

(*ii*) $0 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$ and $\rho < \gamma_2 - 1$. We can bind these two inequalities together by writting $0 \leq \rho < \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$.

On the other hand, if ρ satisfies $\rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1$ then ρ satisfies one of the following subcases:

(*iii*) $\gamma_2 - 1 \leq \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1$.

(*iv*) $\rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < \gamma_2 - 1$.

Then, we can estimate e_2 as follows:

$$|e_2(v)| \quad = \quad \begin{cases} o(v^{-\rho}), & \text{if } \gamma_2 - 1 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ o(v^{-\rho}), & \text{if } 0 \leq \rho < \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ O(v^{-\rho}), & \text{if } \gamma_2 - 1 \leq \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < \gamma_2 - 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} \neq 0, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} = 0. \end{cases}$$

It is not difficult to see that $o(v^{-\rho_l}) + o(v^{-\rho}) = o(v^{-\min\{\rho,\rho_l\}})$. We need to prove that $o(v^{-\rho_l}) + O(v^{-\rho}) = \begin{cases} o(v^{-\rho_l}), & \text{if } \rho_l < \rho, \\ O(v^{-\rho}), & \text{if } \rho_l \geq \rho. \end{cases}$ Let $f(v) = o(v^{-\rho_l})$ and $g(v) = O(v^{-\rho})$:

Assume that $\rho_l < \rho$.

$$\lim_{v\to\infty} \frac{f(v) + g(v)}{v^{-\rho_l}} = \lim_{v\to\infty} \frac{f(v)}{v^{-\rho_l}} + \lim_{v\to\infty} \frac{g(v)}{v^{-\rho_l}} = 0 + \lim_{v\to\infty} \frac{g(v)}{v^{-\rho}} \lim_{v\to\infty} v^{\rho_l - \rho} = C(0) = 0.$$

Conversely, suppose that $\rho_l \geq \rho$. Then,

$$\lim_{v\to\infty} \frac{f(v) + g(v)}{v^{-\rho}} = \lim_{v\to\infty} \frac{f(v)}{v^{-\rho_l}} \lim_{v\to\infty} v^{-\rho_l + \rho} + \lim_{v\to\infty} \frac{g(v)}{v^{-\rho}} = 0 + C = C.$$

In order to count $e_1(v)$, we need $V^l_{12} \neq 0$, $\rho_l < \gamma_2 - 1$ and $\rho_l < \rho$. Besides, $e_1(v)$ does not count neither when we have $\rho < \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$ because we can choose ρ_l such that $\rho < \rho_l \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1$, nor when $\rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < \gamma_2 - 1$ because we can choose ρ_l such that $\rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < \rho_l < \gamma_2 - 1$, nor when $\rho = 1$ and $V^l_{12} = 0$. Hence $e(v)$ can be estimated as follows:

$$|e(v)| \quad = \quad \begin{cases} o(v^{-\rho_l}), & \text{if } \rho_l < \gamma_2 - 1 \leq \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ o(v^{-\rho}), & \text{if } 0 \leq \rho < \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1, \\ o(v^{-\rho_l}), & \text{if } \rho_l < \gamma_2 - 1 \leq \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \leq j < k \leq N\} - 1 < \gamma_2 - 1, \\ o(v^{-\rho_l}), & \text{if } \rho_l < \gamma_2 - 1 = \gamma_1 - 1 \leq \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} \neq 0, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V^l_{12} = 0, \end{cases}$$

$$= \begin{cases} o(v^{-\rho_l}), & \text{if } \gamma_2 - 1 \le \rho \le 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1, \\ & \quad \text{for any } \rho_l,\, 0 \le \rho_l < \gamma_2 - 1, \\ o(v^{-\rho}), & \text{if } 0 \le \rho < \min\{\gamma_2, 2\alpha,\, 2\sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < \gamma_2 - 1, \\ o(v^{-\rho_l}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V_{12}^l \ne 0, \\ & \quad \text{for any } \rho_l,\, 0 \le \rho_l < \gamma_1 - 1, \\ O(v^{-1}), & \text{if } \rho = 1, \sum |q_{jk}| = 0 \text{ and } V_{12}^l = 0. \end{cases} \tag{IV.4.103}$$

Now, we consider the special case where we suppose that $V^L = 0$. If $\alpha < 1$ then $\sum\limits_{j<k} |q_{jk}| > 0$ because when $\sum\limits_{j<k} |q_{jk}| = 0$ necessarily $\alpha = 1$. But, if $\alpha = 1$ either $\sum\limits_{j<k} |q_{jk}| = 0$ or $\sum\limits_{j<k} |q_{jk}| > 0$. By (II.2.56) we cannot have $\rho = \alpha = 1$ and $\sum\limits_{j<k} |q_{jk}| > 0$. That is why in (II.2.44) we always put $\rho = 1$ and $\sum\limits_{j<k} |q_{jk}| = 0$ together in order to avoid $\rho = 1$ and $\sum\limits_{j<k} |q_{jk}| > 0$. We note that we can have $\rho < 2\alpha - 1 \le \alpha \le 1$ whether $\sum\limits_{j<k} |q_{jk}| = 0$ or $\sum\limits_{j<k} |q_{jk}| > 0$.

Following the same reasoning used to get (IV.4.101) and recalling that in this situation $\sigma_{jk} = 1$:

$$\begin{aligned} |(I_{G,v} - 1)| &= \begin{cases} o(v^{-\rho}), & \text{if } 1/2 < \gamma < 1, \text{ and } \sum\limits_{j<k} |q_{jk}| > 0, \text{ and} \\ & 0 \le \rho \le 2\alpha - 1,\, \rho < 2\gamma - 1, \\ O(v^{-\rho}), & \text{if } 1/2 < \gamma < 1, \text{ and } \sum\limits_{j<k} |q_{jk}| > 0, \text{ and} \\ & \rho = 2\alpha - 1 = 2\gamma - 1, \\ o(v^{-\rho}), & \text{if } \gamma = 1, \text{ and } \sum\limits_{j<k} |q_{jk}| > 0, \text{ and} \\ & 0 \le \rho \le 2\alpha - 1, \\ 0, & \text{if } \sum\limits_{j<k} |q_{jk}| = 0 \text{ and} \\ & 0 \le \rho \le 2\alpha - 1, \end{cases} \\ &= \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1, \\ O(v^{-\rho}), & \text{if } \rho = 2\min\{\alpha,\, \sigma_{jk} \,|\, 1 \le j < k \le N\} - 1 < 1, \\ O(v^{-1}), & \text{if } \rho = 1,\, \sum\limits_{j<k} |q_{jk}| = 0. \end{cases} \end{aligned} \tag{IV.4.104}$$

Also, (IV.4.102) is valid. Then, if $V^L = 0$, (IV.4.98) is estimated by (II.2.46), (II.2.51), (II.2.56), (II.2.66), (II.2.73), (II.2.74), (IV.4.102) and (IV.4.104) for all $\rho_s < \alpha$.

$$\begin{aligned} |e(v)| &= \left| I_{G,v}\left(I(v) + R(v)\right) - \lim_{v\to\infty} I_{G,v}\left(I(v) + R(v)\right)\right| \\ &\le |(I_{G,v} - 1)\left(I(v) + R(v)\right)| + \left|\left(I(v) + R(v)\right) - \lim_{v\to\infty} I_{G,v}\left(I(v) + R(v)\right)\right| \\ &= C \begin{cases} v^{-(2\gamma-1)}, & \text{if } 1/2 < \gamma < 1, \text{ and } \sum\limits_{j<k} |q_{jk}| > 0, \\ \frac{\ln v}{v}, & \text{if } \gamma = 1, \text{ and } \sum\limits_{j<k} |q_{jk}| > 0, \\ 0, & \text{if } \sum\limits_{j<k} |q_{jk}| = 0, \end{cases} + \begin{cases} o(v^{-\rho}), & \text{if } 0 \le \rho < 1, \\ O(v^{-1}), & \text{if } \rho = 1. \end{cases} \end{aligned}$$

$$
+\begin{cases} o(v^{-\rho_s}), & \text{if } q_{12}\neq 0,\\ 0, & \text{if } q_{12}=0, \end{cases}
+\begin{cases} o(v^{-\rho}), & \text{if } 0\le\rho<2\alpha-1,\\ O(v^{-\rho}), & \text{if } \rho=2\alpha-1<1,\ \sum\limits_{j<k}|q_{jk}|>0\\ O(v^{-1}), & \text{if } \rho=1,\ \sum\limits_{j<k}|q_{jk}|=0. \end{cases}
$$

$$
=\begin{cases} o(v^{-\rho}), & \text{if } 0\le\rho<2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1,\\ O(v^{-\rho}), & \text{if } \rho=2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<1,\\ O(v^{-1}), & \text{if } \rho=1,\ \sum\limits_{j<k}|q_{jk}|=0. \end{cases} \tag{IV.4.105}
$$

Because $V^L=0$, meaning that there are not long-range forces, we have that $V_{12}^l=0$, obviously $V_{12}^l\neq 0$ is not true, and $\gamma_2=2$ is true. This makes impossible to have $\gamma_2-1\le\rho\le 2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<1$. Besides $0\le\rho<2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1$ and $0\le\rho<\min\{\gamma_2,2\alpha,\,2\sigma_{jk}\,|\,1\le j<k\le N\}-1$ are equivalent, so are $\rho=2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<1$ and $\rho=2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<\gamma_2-1$:

$$
|e(v)| = \begin{cases} o(v^{-\rho_l}), & \text{if } \gamma_2-1\le\rho\le 2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<1,\\ & \quad\text{for any } \rho_l,\ 0\le\rho_l<\gamma_2-1,\\ o(v^{-\rho}), & \text{if } 0\le\rho<\min\{\gamma_2,2\alpha,\,2\sigma_{jk}\,|\,1\le j<k\le N\}-1,\\ O(v^{-\rho}), & \text{if } \rho=2\min\{\alpha,\,\sigma_{jk}\,|\,1\le j<k\le N\}-1<\gamma_2-1,\\ o(v^{-\rho_l}), & \text{if } \rho=1,\sum|q_{jk}|=0 \text{ and } V_{12}^l\neq 0,\\ & \quad\text{for any } \rho_l,\ 0\le\rho_l<\gamma_1-1,\\ O(v^{-1}), & \text{if } \rho=1,\sum|q_{jk}|=0 \text{ and } V_{12}^l=0. \end{cases} \tag{IV.4.106}
$$

By (IV.4.103) and (IV.4.106) we obtain the error term given in (II.2.44).

■

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