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Contents

Abstract

We consider the problem

$$-\Delta u + W(x)u = f(x,u), \quad u \in H_0^1(\Omega),$$

where Ω is an exterior domain in $\mathbb{R}^N$, $N \geq 3$, $W \in \mathcal{C}^0(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} W > 0$, $W(x) \to V_\infty > 0$ as $|x| \to \infty$ and the function f is either the local nonlinearity

$$f(x,u) = |u|^{p-2}u,$$

or the nonlocal one

$$f(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u.$$

In the first case we assume that $2 < p < 2^* := \frac{2N}{N-2}$, while in the second one we assume that $\alpha \in (0,N)$ and $\frac{2N-\alpha}{N} < p < \frac{2N-\alpha}{N-2}$.

Under symmetry assumptions on Ω and W, and appropriate assumptions on the decay of W at infinity, we establish the existence of a positive solution and multiple sign changing solutions to this problem, having small energy (in the symmetric sense). Moreover, we show that there is an effect of the topology of the orbit space of certain symmetric subsets of the domain on the number of low energy sign changing solutions to this problem.

Chapter 1

Introduction

We consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = f(x,u), \\ u \in H_0^1(\Omega), \end{cases}$$

where $N \geq 3$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$, whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty. The potential $V_\infty + V$ is assumed to satisfy

(V_0) $\quad V \in \mathcal{C}^0(\mathbb{R}^N)$, $\;V_\infty \in (0,\infty)$, $\;\inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0$, $\lim_{|x|\to\infty} V(x) = 0$.

The function f can be either the local nonlinearity

$$f(x,u) = |u|^{p-2}u,$$

or the nonlocal one

$$f(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u.$$

In the first case we assume that $2 < p < 2^* := \frac{2N}{N-2}$, while in the second one we assume that $\alpha \in (0,N)$ and $\frac{2N-\alpha}{N} < p < \frac{2N-\alpha}{N-2}$.

In this thesis, we are interested in obtaining positive and sign changing solutions to this problem.

We are going to consider separately the local case and the nonlocal one. In both of them, we analyse two model situations: first, we assume that V tends to its limit at infinity exponentially from below. Then, we consider the case in which V tends exponentially to its limit at infinity taking on values greater than its limit (which includes the autonomous case $V = 0$). The speed of convergence depends on the distance between the elements of the orbits in a certain symmetric subset of the domain. Weaker conditions on the decay of the potential require stronger conditions on the symmetries.

The main results of this thesis, here revised and extended, are contained in two joint works with M. Clapp ([26] and [27]).

1.1 The local problem

We consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is an unbounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty, and $2 < p < 2^* := \frac{2N}{N-2}$. The potential $V_\infty + V$ is assumed to satisfy (V_0).

1.1.1 On the most closely related known results

Equations of this kind arise naturally in various branches of physics and in some problems in biology as well, see for example [10, 33]. The existence of solutions to (1.1) has been extensively studied during the last 25 years. A detailed account is given in Cerami's survey article [15]. In what follows we make reference to the results more closely related to our study.

The main difficulty in dealing with problem (1.1) by means of variational methods is the lack of compactness. This difficulty does not appear when Ω and V are radially symmetric and we look for radial solutions [55, 10, 31]. However if, either Ω or V do not have symmetries, or if they have symmetries with finite orbits, the lack of compactness prevails.

Remarkable progress was made when P.-L. Lions introduced in [41] his concentration compactness method, which allowed to show the existence of a solution of problem (1.1) in $\mathbb{R}^N$ by a minimization argument for $V \leq 0$. This also applies in an exterior domain Ω, like the one we are considering, when $V < 0$ satisfies a suitable decay assumption at infinity. However, when $V \geq 0$ and $\Omega \neq \mathbb{R}^N$ or when $V > 0$ and $\Omega = \mathbb{R}^N$ the question of the existence cannot be treated by minimization. To handle this situation a deeper understanding of the lack of compactness of the variational problem is needed. Benci and Cerami gave in [9] a complete description of the lack of compactness in terms of the solutions to the limit problem

$$\begin{cases} -\Delta u + V_\infty u = |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.2}$$

associated to (1.1). This allowed them to solve the existence problem for $V \equiv 0$ when the diameter of $\mathbb{R}^N \smallsetminus \Omega$ is small enough. Bahri and Lions in [4] eliminated this restriction and, considering some decay assumptions at infinity on V, they showed the existence of a solution for $V \geq 0$. In all of these cases the solution obtained is positive.

A result concerning the existence of multiple solutions with small energy was obtained by Clapp and Weth in [28] when $\Omega = \mathbb{R}^N$ and V approaches to 0 from below at infinity in a suitable way. However, the techniques employed there, provide no information on whether these solutions change sign or not. Cerami, Devillanova and Solimini established the existence of infinitely many solutions in [17] assuming that $\Omega = \mathbb{R}^N$ and V tends to zero from below at infinity at some suitable rate. Recently, Wei and Yan [58] proved the existence of infinitely many positive solutions to this problem when $\Omega = \mathbb{R}^N$ and V is a radial function tending to 0 at infinity, in a polynomial way. Without any symmetry assumptions on the potential, Cerami, Passaseo and Solimini proved in [19] an analogous result for potentials that decay very slowly.

We are interested in obtaining multiplicity of sign changing solutions to this problem. For $\Omega = \mathbb{R}^N$ and $V \equiv 0$ existence of infinitely many sign changing solutions with large symmetries was shown in [8, 43, 49]. When Ω and V have only finite symmetries, existence of a sign changing solution to problem (1.1) was shown by Cerami and Clapp in [16] and by Carvalho, Maia and Miyagaky in [14], under suitable assumptions. We shall refer to these results later in more detail.

Several multiplicity results have been obtained for the singularly perturbed problem $-\varepsilon\Delta u + (V_\infty + V(x))\,u = |u|^{p-2}u$, $u \in H^1(\mathbb{R}^N)$, for small enough $\varepsilon > 0$. It is well-known that, when $\varepsilon \to 0$, there are solutions to this problem which concentrate at critical points of the potential V, see [2, 29]. Hence, it is not surprising that the topology of certain subsets of critical points of V has an effect on the number of solutions to this problem, as has been shown for example in [23]. Even though a similar concentration phenomenon is not present in the problem we are treating here, we will prove in this thesis that, when looking for sign changing solutions, there is a combined effect of the topology and the symmetries of certain subsets of the domain on the number of solutions to problem (1.1).

1.1.2 Main results: Multiplicity of sign changing solutions

In this subsection we state our existence results for the local problem and give some examples of symmetric situations for which they apply.

We study the case where both Ω and V have some symmetries. If Γ is a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$, we denote by

$$\Gamma x := \{gx : g \in \Gamma\}$$

the Γ-orbit of x, by $\#\Gamma x$ its cardinality, and by

$$\ell(\Gamma) := \min\{\#\Gamma x : x \in \mathbb{R}^N \smallsetminus \{0\}\}.$$

We assume that Ω and V are Γ-invariant, this means that $\Gamma x \subset \Omega$ for every $x \in \Omega$ and that V is constant on Γx for each $x \in \mathbb{R}^N$. We consider a continuous group homomorphism $\phi : \Gamma \to \mathbb{Z}/2$ and we look for solutions which satisfy

$$u(gx) = \phi(g)u(x) \qquad \text{for all } g \in \Gamma \text{ and } x \in \Omega. \tag{1.3}$$

A function u with this property will be called *ϕ-equivariant.* We denote by

$$G := \ker \phi.$$

Note that, if u satisfies (1.3), then u is G-invariant. Moreover, $u(\gamma x) = -u(x)$ for every $x \in \Omega$ and $\gamma \in \phi^{-1}(-1)$. Therefore, if ϕ is an epimorphism (i.e. if it is surjective), every nontrivial solution to (1.1) which satisfies (1.3) changes sign. If $\phi \equiv 1$ is the trivial homomorphism, then $\Gamma = G$, and (1.3) simply says that u is G-invariant.

If Z is a Γ-invariant subset of $\mathbb{R}^N$ and ϕ is an epimorphism, the group $\mathbb{Z}/2$ acts on the G-orbit space $Z/G := \{Gx : x \in Z\}$ of Z as follows: we choose $\gamma \in \Gamma$ such that $\phi(\gamma) = -1$ and we define

$$(-1) \cdot Gx := G(\gamma x) \qquad \text{for all } x \in Z.$$

This action is well defined and it does not depend on the choice of γ. We denote by

$$\Sigma := \{x \in \mathbb{R}^N : |x| = 1,\ \#\Gamma x = \ell(\Gamma)\}, \qquad \Sigma_0 := \{x \in \Sigma : Gx = G(\gamma x)\}.$$

If Z is a nonempty Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$, the action of $\mathbb{Z}/2$ on its G-orbit space Z/G is free and the *Krasnoselskii genus* of Z/G, denoted $\mathrm{genus}(Z/G)$, is defined to be the smallest $k \in \mathbb{N}$ such that there exists a continuous map

$$f : Z/G \to \mathbb{S}^{k-1} := \{x \in \mathbb{R}^k : |x| = 1\}$$

which is $\mathbb{Z}/2$-equivariant, i.e. $f((-1)\cdot Gz) = -f(Gz)$ for every $z \in Z$. We define $\mathrm{genus}(\emptyset) := 0$.

For each subgroup K of $O(N)$ and each K-invariant subset Z of $\mathbb{R}^N \smallsetminus \{0\}$ we set

$$\mu(Kz) := \begin{cases} \inf\{|gz - hz| : g, h \in K,\ gz \neq hz\} & \text{if } \#Kz \geq 2, \\ 2\,|z| & \text{if } \#Kz = 1, \end{cases}$$

$$\mu_K(Z) := \inf_{z \in Z} \mu(Kz) \qquad \text{and} \qquad \mu^K(Z) := \sup_{z \in Z} \mu(Kz).$$

In what follows, we will assume that Ω is Γ-invariant, that V is a Γ-invariant function and that (V_0) holds. We will also assume that $\ell(\Gamma) < \infty$, because otherwise, as we are going to see later, problem (1.1) has infinitely many solutions.

We denote by $\hat{c}_\infty$ the energy of the positive solution to the limit problem (1.2). We shall look for solutions with small energy, i.e. which satisfy

$$\frac{p-2}{2p} \int_\Omega |u|^p < \ell(\Gamma)\, \hat{c}_\infty. \tag{1.4}$$

We shall prove the following result.

Theorem 1.1. *If $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, Z is a Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$, and V satisfies the following:*

(V_1) *There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu_\Gamma(Z)\sqrt{V_\infty})$ such that*

$$V(x) \leq -c_0 e^{-\lambda|x|} \qquad \textit{for all } x \in \mathbb{R}^N \textit{ with } |x| \geq r_0,$$

then problem (1.1) has at least $\mathrm{genus}(Z/G)$ *pairs of sign changing solutions $\pm u$, which satisfy (1.3) and (1.4).*

Let us look at some examples.

Example 1. Let Γ be the group spanned by the reflection $\gamma : \mathbb{R}^N \to \mathbb{R}^N$ on a linear subspace W of $\mathbb{R}^N$ of dimension $0 \leq \dim W < N$. If Ω and V are invariant under this reflection, we may take $\phi : \Gamma \to \mathbb{Z}/2$ to be the epimorphism given by $\phi(\gamma) := -1$ and Z to be the unit sphere in the orthogonal complement of W. Then, Theorem 1.1 yields

$$\mathrm{genus}(Z) = N - \dim W$$

pairs of solutions to problem (1.1) provided (V_1) holds for some $\lambda \in (0, 2\sqrt{V_\infty})$.

Under analogous assumptions to those of the previous theorem, Carvalho, Maia and Miyagaki proved in [14] the existence of a solution to (1.1) satisfying (1.3) and (1.4) in the case considered in the above example. Note that in our example $\mu_\Gamma(Z) = 2$, so our assumption (V_1) is less restrictive than the one in [14] where $\lambda \in (0, \sqrt{V_\infty})$ is required.

Another interesting example is the following:

Example 2. If $N = 2n$ we identify $\mathbb{R}^N$ with $\mathbb{C}^n$ and take Γ to be the cyclic group of order $2m$ spanned by

$$\rho(z_1, \dots, z_n) := (e^{\pi i/m} z_1, \dots, e^{\pi i/m} z_n)$$

and $\phi : \Gamma \to \mathbb{Z}/2$ to be the epimorphism given by $\phi(\rho) := -1$. Then $G := \ker \phi$ is the cyclic subgroup of order m spanned by ρ^2. Since the action is free, we have that $\Sigma = \mathbb{S}^{N-1}$ and $\Sigma_0 = \emptyset$, so we may take $Z := \mathbb{S}^{N-1}$. The genus of $\mathbb{S}^{N-1}/G$ can be estimated in many cases. For example, if $m = 2^k$, Lemma C.1 below together with Theorem 1.2 of [5] give

$$\text{genus}(\mathbb{S}^{N-1}/G) \geq \frac{N-1}{2^k} + 1.$$

Since $\mu_\Gamma(\mathbb{S}^{N-1}) = \left| e^{\pi i/m} - 1 \right|$, condition (V_1) becomes more restrictive as m increases. So, if condition (V_1) holds for $m = 2^k$, it will also hold for $m = 2^j$ with $0 \leq j < k$. Now, if u_j is a solution provided by Theorem 1.1 for $m = 2^j$, then u_j satisfies (1.3), i.e.

$$u_j(e^{\pi i l/(2^j)} z) = (-1)^l u_j(z) \qquad \forall\, l = 0, \dots, 2^{j+1} - 1, \quad z \in \Omega \subset \mathbb{C}^n.$$

This implies that $u_k \neq u_j$ if $k > j$. Indeed, if $k > j$ and $u_k(z) = u_j(z) \neq 0$ at some $z \in \Omega$ then, since $u_j(e^{\pi i/(2^j)} z) = -u_j(z)$ and

$$u_k(e^{\pi i/(2^j)} z) = u_k(e^{\pi i(2^{k-j})/(2^k)} z) = (-1)^{2^{k-j}} u_k(z) = u_j(z),$$

we have that $u_k(e^{\pi i/(2^j)} z) \neq u_j(e^{\pi i/(2^j)} z)$. Therefore, Theorem 1.1 provides at least

$$\sum_{j=0}^{k} \frac{N-1}{2^j} + k + 1 = (N-1)\frac{2^{k+1} - 1}{2^k} + k + 1$$

pairs of sign changing solutions in this case.

On the other hand, similar actions in odd dimensions give no solutions. For example, if we take polygonal symmetry in $\mathbb{R}^3$ given by $\rho(z,t) := (e^{\pi i/m} z, t)$, $(z,t) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$—as considered in [56] for a related problem—and $\phi(\rho) := -1$, then

$$\Sigma = \{\pm(0,0,1)\} = \Sigma_0.$$

So Theorem 1.1 gives no information in this case. However, if we consider the group Γ generated by ρ and the reflection $\tau(z,t) := (z, -t)$, and take $\phi(\rho) := 1$ and $\phi(\tau) := -1$, then $\Sigma = \{\pm(0,0,1)\}$ and $\Sigma_0 = \emptyset$ and Theorem 1.1 yields one pair of sign changing solutions.

For potentials with an analogous behavior at infinity, but without requiring any symmetry property neither on the domain nor on the potential, in [28] it was shown that problem (1.1) has at least $\frac{N}{2} + 1$ pairs of solutions. However, the argument used there gives no precise information whether the solutions obtained change sign or not. If ϕ is an epimorphism, property (1.3) asserts that u changes sign and, as we have seen, in some cases Theorem 1.1 yields more than $\frac{N}{2}+1$ pairs of solutions.

We shall prove also the following multiplicity result of sign changing solutions, with a different condition on the potential.

Theorem 1.2. *Let Z be a Γ-invariant subset of Σ. Assume that the following hold:*

(Z_0) *There exists $a_0 > 1$ such that*

$$\operatorname{dist}(\gamma z, Gz) \geq a_0 \mu(Gz) \qquad \text{for all } z \in Z \text{ and } \gamma \in \Gamma \smallsetminus G,$$

(V_2) *There exist $c_0 > 0$ and $\kappa > \mu^{\Gamma}(Z)\sqrt{V_\infty}$ such that*

$$V(x) \leq c_0 e^{-\kappa|x|} \qquad \text{for all } x \in \mathbb{R}^N.$$

Then (1.1) has at least $\operatorname{genus}(Z/G)$ *pairs of sign changing solutions $\pm u$, which satisfy (1.3) and (1.4).*

Let us pointed out that this theorem corresponds to [26, Theorem 1.2]. However, there we assumed Z to be a compact Γ-invariant subset of Σ which satisfies the slightly different condition

$(\hat{Z}_0)$ *dist*$(\gamma z, Gz) > \mu(Gz)$ for all $z \in Z$ and $\gamma \in \Gamma \smallsetminus G$.

We noticed that the compactness assumption for Z can be removed just asking for condition (Z_0) above.

Theorem 1.2 is an extension of the result obtained by Cerami and Clapp in [16], which states the existence of a sign changing solution to the autonomous problem $V \equiv 0$ if (Z_0) holds for some $z \in \Sigma$. Note that (Z_0) implies that $Z \subset \Sigma \smallsetminus \Sigma_0$. Note also that condition (Z_0) cannot be realized if $N = 3$ or if $\ell(G) = 1$. However, we next give an example which illustrates the situation in Theorem 1.2 for higher dimensions.

Example 3. We identify $\mathbb{R}^{4n}$ with $\mathbb{C}^n \times \mathbb{C}^n$ and consider the subgroup Γ of $O(4n)$ spanned by ρ and γ, where $\rho(y,z) := (e^{\pi i/m}y, e^{\pi i/m}z)$ and $\gamma(y,z) := (-\overline{z}, \overline{y})$ for $(y,z) \in \mathbb{C}^n \times \mathbb{C}^n$ and some $m \geq 3$. We define $\phi : \Gamma \to \mathbb{Z}/2$ by $\phi(\rho) = 1$, $\phi(\gamma) = -1$. Then $G := \ker \phi$ is the cyclic subgroup of order $2m$ spanned by ρ. Since $m \geq 3$, property (Z_0) holds for $Z := \mathbb{S}^{4n-1}$. We will prove in Appendix C that

$$\operatorname{genus}(\mathbb{S}^{4n-1}/G) \geq 2n + 1.$$

Consequently, if Ω and V are Γ-invariant and (V_2) holds, Theorem 1.2 yields $2n+1$ pairs of sign changing solutions to problem (1.1). Note that $\mu^G(\mathbb{S}^{4n-1}) = \left|e^{\pi i/m} - 1\right|$, hence (V_2) becomes less restrictive as m increases.

1.2 The nonlocal problem

We consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{1.5}$$

where $N \geq 3$, $\alpha \in (0, N)$, $p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty. We also assume that (V_0) is satisfied.

1.2.1 A brief historical background

A special case of (1.5), relevant in physical applications, is the Choquard equation

$$- \Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u, \quad u \in H^1(\mathbb{R}^3), \tag{1.6}$$

which models an electron trapped in its own hole, and was proposed by Choquard in 1976 as an approximation to Hartree-Fock theory of a one-component plasma [38]. This equation arises in many interesting situations related to the quantum theory of large systems of nonrelativistic bosonic atoms and molecules, see for example [34, 40] and the references therein. It was also proposed by Penrose in 1996 as a model for the self-gravitational collapse of a quantum mechanical wave-function [53]. In this context, problem (1.6) is usually called the nonlinear Schrödinger-Newton equation, see also [46, 47].

In 1976 Lieb [38] proved the existence and uniqueness (modulo translations) of a minimizer to problem (1.6) by using symmetric decreasing rearrangement inequalities. Later, in [42], Lions showed the existence of infinitely many radially symmetric solutions to (1.6). Further results for related problems may be found in [1, 22, 24, 45, 51, 54, 57] and the references therein.

In 2010, Ma and Zhao [44] considered the generalized Choquard equation

$$- \Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1.7}$$

and proved that, for $p \geq 2$, every positive solution of it is radially symmetric and monotone decreasing about some point, under the assumption that a certain set of real numbers, defined in terms of N, α and p, is nonempty. Under the same assumption, Cingolani, Clapp and Secchi [21] recently gave some existence and multiplicity results in the electromagnetic case, and established the regularity and some decay asymptotics at infinity of the ground states of (1.7). Moroz and van Schaftingen [48] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and derived decay asymptotics at infinity for them, as well. These results will play an important role in our study.

1.2.2 Main results: Positive and sign changing solutions

In this subsection we state our existence results for the nonlocal problem. We still use the same notation as in the statement of the main results for the local problem (see subsection 1.1.2). The only difference is that in the special case where $K = G$ and $Z = \Sigma$, we simply write

$$\mu_G := \mu_G(\Sigma) \qquad \text{and} \qquad \mu^G := \mu^G(\Sigma).$$

We just consider the case $\ell(\Gamma) < \infty$, because if all Γ-orbits of Ω are infinite it was already shown in [21, Theorem 1.1] that (1.5) has infinitely many solutions. In this case, $\mu_G > 0$.

We denote by c_∞ the energy of a ground state of the problem

$$\begin{cases} -\Delta u + V_\infty u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.8}$$

We shall look for solutions with small energy, i.e. which satisfy

$$\frac{p-1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} dx\, dy < \ell(\Gamma) c_\infty. \tag{1.9}$$

In what follows, we assume that V satisfies (V_0). We shall prove the following results:

Theorem 1.3. *If $p \geq 2$, Ω is G-invariant and V is a G-invariant function which satisfies*

(V_3) *There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu^G \sqrt{V_\infty})$ such that*

$$V(x) \leq -c_0 e^{-\lambda |x|} \qquad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0,$$

then (1.5) has at least one positive solution u which is G-invariant and satisfies (1.9) with $\Gamma = G$.

Theorem 1.4. *If $p \geq 2$, Ω is Γ-invariant, $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, Z is a Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$ and V is a Γ-invariant function which satisfies*

(V_1) *There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu_\Gamma(Z)\sqrt{V_\infty})$ such that*

$$V(x) \leq -c_0 e^{-\lambda |x|} \qquad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0,$$

then problem (1.5) has at least genus(Z/G) *pairs of sign changing solutions $\pm u$, which satisfy (1.3) and (1.9).*

Theorem 1.5. *If $p \geq 2$, $\ell(G) \geq 3$, Ω is G-invariant and V is a G-invariant function which satisfies*

(V_4) *There exist $c_0 > 0$ and $\kappa > \mu_G \sqrt{V_\infty}$ such that*

$$V(x) \leq c_0 e^{-\kappa |x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

then (1.5) has at least one positive solution u which is G-invariant and satisfies (1.9) with $\Gamma = G$.

Theorem 1.6. *If $p \geq 2$, Ω is Γ-invariant, $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, Z is a Γ-invariant subset of Σ, V is a Γ-invariant function and the following hold:*

(Z_0) *There exists $a_0 > 1$ such that*

$$\operatorname{dist}(\gamma z, Gz) \geq a_0 \mu(Gz) \qquad \text{for all } z \in Z \text{ and } \gamma \in \Gamma \smallsetminus G,$$

(V_2) *There exist $c_0 > 0$ and $\kappa > \mu^\Gamma(Z)\sqrt{V_\infty}$ such that*

$$V(x) \leq c_0 e^{-\kappa |x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

then (1.5) has at least genus(Z/G) *pairs of sign changing solutions $\pm u$, which satisfy (1.3) and (1.9).*

Let us point out that conditions (V_1), (V_2) and (Z_0) are the same ones we required in the local case.

Theorem 1.3 was proved in [21] for $\Omega = \mathbb{R}^N$, under additional assumptions on α and p. As far as we know, Theorem 1.5 is the first existence result for potentials V which are nontrivial and take nonnegative values at infinity. In the local case, Bahri and Lions proved existence for this type of potentials without any symmetries [4]. Unfortunately, some of the facts used in their proof are not available in the nonlocal case.

As we mentioned before, the existence of infinitely many solutions is known in the radial case [42] and in the case where every Γ-orbit in Ω is infinite [21]. In contrast, Theorems 1.4 and 1.6 provide multiple solutions when the data have only finite symmetries.

The examples which illustrate the results for the local case continue being valid in this context. To be precise, if Γ and ϕ are as in the Example 1 and we choose Z in the same way, then Theorem 1.4 yields $\mathrm{genus}(Z) = N - \dim W$ pairs of solutions to problem (1.5) provided (V_1) holds for some $\lambda \in (0, 2\sqrt{V_\infty})$. Furthermore, if Γ and ϕ are as in the Example 2 and condition (V_1) holds for $m = 2^k$, taking $Z := \mathbb{S}^{N-1}$, Theorem 1.4 provides at least

$$\sum_{j=0}^{k} \frac{N-1}{2^j} + k + 1 = (N-1)\frac{2^{k+1}-1}{2^k} + k + 1$$

pairs of sign changing solutions satisfying (1.3) and (1.9).

The group G in Example 2 satisfies $\ell(G) = m$. This shows that there are many groups satisfying the symmetry assumption in Theorem 1.5 when N is even. If N is odd not many groups satisfy $\ell(G) \geq 3$. For example, if $N = 3$, the only subgroups of $O(3)$ which satisfy this condition are the rotation groups of the icosahedron, octahedron and tetrahedron, I, O and T, and the groups $I \times \mathbb{Z}_2^c$, $O \times \mathbb{Z}_2^c$, $T \times \mathbb{Z}_2^c$ and O^- described in [20, Appendix A].

Note that (Z_0) implies that $Z \subset \Sigma \smallsetminus \Sigma_0$. Condition (Z_0) cannot be realized if $N = 3$. In the context of Example 3 we can see that property (Z_0) holds for $Z := \mathbb{S}^{4n-1}$. Therefore, if Ω and V are Γ -invariant and (V_2) holds, Theorem 1.6 yields $2n + 1$ pairs of sign changing solutions to problem (1.5).

1.3 Some open problems

Here we indicate some of the open problems which are motivated by the work of this thesis and some application of the technics developed there to related problems that we plan to study in the near future.

1.3.1 Further questions concerning the nonlocal problem

1. In the nonlocal problem that we considered in [27] the symmetries played an important role to prove the existence of a positive solution for potentials V which are nontrivial and take nonnegative values at infinity. This is, as far as we know, the first existence result in this situation. The problem of existence without symmetries is open, and seems to be nowhere studied in the literature.

In the local case, Bahri and Lions [4] proved existence for this type of potentials without any symmetries. Unfortunately, some of the facts used in their proof are not available in the nonlocal case. Particularly, it is not known whether the ground state of the limit problem (1.8) with the nonlocal nonlinearity

$$f(x,u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u$$

is, in general, the only positive solution (up to translations).

Recently, Ma and Zhao [44] showed that in the classical case $N = 3$, $\alpha = 1$, $p = 2$, the ground state is the only positive solution. We plan to investigate whether, at least in this case, a positive solution to problem (1.5) exists for this type of potentials without any symmetry assumption.

2. Recently, S. Cingolani, M. Clapp and S. Secchi considered the stationary nonlinear magnetic Choquard problem

$$\begin{cases} (-\mathrm{i}\nabla + A(x))^2 u + V(x)u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in L^2(\mathbb{R}^N, \mathbb{C}), \\ \nabla u + \mathrm{i}A(x)u \in L^2(\mathbb{R}^N, \mathbb{C}^N), \end{cases}$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ is a $\mathcal{C}^1$-vector potential, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous scalar potential, $N \geq 3$, $\alpha \in (0, N)$ and $p \in [2, \frac{2N-\alpha}{N-2})$. Under symmetry assumptions on the data and some additional condition on α and p, they proved in [21] that, if V tends to its limit at infinity exponentially from below at an appropriate speed which depends on the symmetries, there exists a complex-valued solution to this problem exhibiting a vortex-type behavior.

We would like to work on an extension of the results obtained in [27] to the magnetic problem above, with the following specific goals: 1) to eliminate the additional condition on α and p, 2) to allow scalar potentials which take on values greater than its limit at infinity, and 3) to obtain multiplicity of vortex-type solutions to this problem.

3. In addition to this, we are interested in obtaining solutions to the nonlocal problem in the symmetric case when $p \in (\frac{2N-\alpha}{N}, 2)$. In this case, solutions should be possible but the arguments used in this thesis do not apply since the energy functional associated to this problem is nowhere twice Fréchet-differentiable. However, one should be able to apply the mountain pass method in order to obtain existence results. Decay asymptotics for the ground state of the limit problem are available, but they are not exponential in this case. They were recently obtained by Moroz and van Schaftingen in [48].

1.3.2 The local and nonlocal problem in domains with unbounded boundary

We are also interested in studying problem (1.5) when Ω is an unbounded smooth domain having unbounded boundary. In 2009, Cerami and Molle [18] considered the problem of finding positive solutions $u \in H_0^1(\Omega)$ of the equation

$$-\Delta u + W(x)u = |u|^{p-2}u,$$

where $p \in (2, 2^*)$, Ω is either $\mathbb{R}^N$ or an unbounded domain which is periodic in the first q coordinates and whose complement is contained in a cylinder

$$\{(x', x'') \in \mathbb{R}^q \times \mathbb{R}^{N-q} : |x''| < R\}.$$

Under appropriate decay assumptions at infinity on the potential W, they showed the existence of one solution when the potential approaches its limit at infinity from below and of $q+1$ solutions when the potential takes on values larger than its limit at infinity. Our purpose is to obtain multiplicity of sign changing solutions of problem (1.5) for this kind of domains, both in the local and the nonlocal case. It is worth mentioning that this problem is particularly interesting because in unbounded smooth domains having unbounded boundary compactness may fail at all energy levels, as shown in [45].

1.3.3 Other related problems

We believe that the methods developed in this thesis may be useful for other problems. For example, recently, Felmer, Quaas and Tan [32] proved the existence of positive ground states of the fractionary laplacian problem

$$\begin{cases} (-\Delta)^\alpha u + u = f(x,u) & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

under suitable assumptions on f. Particularly, they showed that the ground states are radially symmetric and, in contrast with the case $\alpha = 1$, they proved that when $0 < \alpha < 1$ the decay of the ground state at infinity is not exponential, but it is a power-type decay. Using this information, we wish to investigate whether, under suitable assumptions, it is possible to obtain the appropriate asymptotic estimates we need to prove that the nonautonomous problem

$$\begin{cases} (-\Delta)^\alpha u + W(x)u = f(x,u) & \text{in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

has multiple sign changing solutions.

This thesis is organized as follows: In Chapter 2 we set the variational framework for problems (1.1) and (1.5), with an emphasis on the nonlocal case, where some facts are not widely explained in the literature. In Chapter 3 we provide a detailed account of the main tools for proving our existence results. We begin with a careful analysis of the behavior of the Palais–Smale sequences satisfying some symmetry properties, which refines that given in [9]. This allow us to establish a lower bound for the lack of compactness of the variational funcional associated to our problem in the appropriate symmetric subspaces of $H_0^1(\Omega)$. Then we derive some delicate asymptotic estimates which enable us to control the energy of the interaction between the positive and negative ground states of the limit problem

$$\begin{cases} -\Delta u + V_\infty u = f(x,u), \\ u \in H_0^1(\Omega), \end{cases} \tag{1.10}$$

which appear as summands in the test functions we use. The behavior of the ground states in the nonlocal case was recently described in [21, 48]. This yields the existence of positive symmetric solutions. We conclude this chapter with the relation between the Krasnoselskii genus and the critical point theory with symmetries.

Chapter 4 is devoted to the proof of the main results of this thesis, once more, we focus on the nonlocal case and add some remarks which describe the situation in the local one. In the first section, we consider potentials which are strictly negative at infinity and prove Theorems 1.3 and 1.4. In the second section, we consider potentials which take on nonnegative values at infinity and prove Theorems 1.5 and 1.6. In contrast with the semiclassical case considered in [29], our problem exhibits no concentration. Nevertheless, to obtain multiplicity, we are able to apply a new variant of a variational principle which has been successfully used in problems in which concentration occurs [23, 7, 22]. We show there is an effect of the topology of some symmetric subsets of the domain on the number of sign changing solutions. More precisely, the Krasnoselskii genus of the orbit space Z/G provides a lower bound for the number of sign changing solutions with a specific type of symmetries.

Finally, to provide examples of our multiplicity results, in the Appendix, we prove a topological result which relates the Krasnoselskii genus of the orbit space Z/G with the generalized genus of Z, thus allowing the use of well-known estimates for the generalized genus of a representation sphere, like those given in [5, 6], to obtain estimates of the Krasnoselskii genus of its orbit space.

Chapter 2

The variational setting

Throughout this chapter we mainly focus in the nonlocal case, because the variational framework for the local problem is well-known in the literature. More precisely we consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{2.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \setminus \Omega$ is bounded, possibly empty. We continue to assume that the potential $V_\infty + V$ satisfies

(V_0) $\quad V \in \mathcal{C}^0(\mathbb{R}^N)$, $\;V_\infty \in (0, \infty)$, $\;\inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0$, $\lim_{|x|\to\infty} V(x) = 0$.

From now on we shall assume without loss of generality that $V_\infty = 1$.

Notice that even though the support of the function $\left(\frac{1}{|x|^\alpha} * |u|^p\right)$ is not contained inside Ω, the support of $\left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u$ is a subset of Ω for every $u \in H_0^1(\Omega)$.

Observe that if u satisfies

$$-\Delta u + (1 + V(x))\, u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u,$$

multiplying each side of this equation by $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and integrating, we obtain

$$-\int_\Omega (\Delta u)\varphi + \int_\Omega (1 + V(x))\, u\varphi = \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u\varphi \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Applying the Green formula to the first integral in the left-hand side, we conclude that

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega (1 + V(x))\, u\varphi = \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u\varphi \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

A function $u \in H_0^1(\Omega)$ which satisfies the above is called a *weak solution* of (2.1). Throughout this thesis, we shall refer to a weak solution just as a solution.

We consider the functional $J_V : H^1_0(\Omega) \to \mathbb{R}$ given by

$$J_V(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + (1+V(x))u^2\right) - \frac{1}{2p}\int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p.$$

We write

$$\langle u,v\rangle_V := \int_\Omega \nabla u\cdot\nabla v + \int_\Omega (1+V(x))\,uv \tag{2.2}$$

and

$$\|u\|_V := \left(\int_\Omega \left(|\nabla u|^2 + (1+V(x))\,u^2\right)\right)^{1/2}. \tag{2.3}$$

If $V=0$ we write $\langle u,v\rangle$ and $\|u\|$ instead of $\langle u,v\rangle_0$ and $\|u\|_0$.

Proposition 2.1. *If V satisfies (V_0), then $\langle\cdot,\cdot\rangle_V$ is a scalar product in $H^1_0(\Omega)$ and the induced norm $\|u\|_V$ is equivalent to the usual one.*

Proof. Assumption (V_0) guarantees that there exist V_1, $V_2>0$ such that

$$V_1 \le 1+V(x) \le V_2 \qquad \forall x\in\mathbb{R}^N.$$

Using the first inequality one can easily check that $\langle\cdot,\cdot\rangle_V$ is a scalar product in $H^1_0(\Omega)$. On the other hand, the following holds true:

$$\int_\Omega \left(|\nabla u|^2 + (1+V(x))u^2\right) \ge \int_\Omega \left(|\nabla u|^2 + V_1u^2\right) \ge \min\{V_1,1\}\|u\|^2$$

$$\int_\Omega \left(|\nabla u|^2 + (1+V(x))u^2\right) \le \int_\Omega \left(|\nabla u|^2 + V_2u^2\right) \le \max\{V_2,1\}\|u\|^2.$$

Therefore, taking $C_1 := \min\{\sqrt{V_1},1\}$ and $C_2 := \max\{\sqrt{V_2},1\}$ we obtain

$$C_1\|u\| \le \|u\|_V \le C_2\|u\|.$$

□

As usual, we identify $u\in H^1_0(\Omega)$ with its extension to $\mathbb{R}^N$ obtained by setting $u=0$ in $\mathbb{R}^N\smallsetminus\Omega$ and denote by $|u|_q := \left(\int_{\mathbb{R}^N}|u|^q\right)^{1/q}$ the norm in $L^q(\mathbb{R}^N)$.

We define

$$\mathbb{D}(u) := \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}dx\,dy$$

and set $r := \frac{2N}{2N-\alpha}$. Since $p\in(\frac{2N-\alpha}{N},\frac{2N-\alpha}{N-2})$, one has that $pr\in(2,\frac{2N}{N-2})$. Hence, the continuous Sobolev embedding $H^1_0(\Omega)\hookrightarrow L^{pr}(\Omega)$ holds.

The classical Hardy-Littlewood-Sobolev inequality [39, Theorem 4.3] implies

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\varphi(x)\psi(y)}{|x-y|^\alpha}dx\,dy\right| \le \bar{C}|\varphi|_r|\psi|_r, \tag{2.4}$$

for some positive constant $\bar{C}=\bar{C}(\alpha,N)$ and all φ, $\psi\in L^r(\mathbb{R}^N)$. In particular,

$$\mathbb{D}(u) \le \bar{C}|u|^{2p}_{pr} \qquad \text{for all } \ u\in H^1_0(\Omega). \tag{2.5}$$

This shows that $\mathbb{D}$ is well defined.

We can rewrite the functional J_V as

$$J_V(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{2p}\mathbb{D}(u).$$

The proof of the following proposition is given in Appendix B.

Proposition 2.2. *If $p \geq 2$, the functional J_V is of class $\mathcal{C}^2$ and*

$$J_V'(u)v = \langle u, v\rangle_V - \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}uv.$$

Consequently, u is a solution of problem (2.1) if and only if u is a critical point of J_V.

2.1 The variational framework for the symmetric problem

From now on, we shall assume that $p \geq 2$. As in the Introduction, we consider a closed subgroup Γ of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and denote by

$$\Gamma x := \{gx : g \in \Gamma\}$$

the Γ-orbit of x.

Throughout this section we shall assume that Ω and V are Γ-invariant, this means that $\Gamma x \subset \Omega$ for every $x \in \Omega$ and that V is constant on Γx for each $x \in \mathbb{R}^N$. We consider a continuous group homomorphism

$$\phi : \Gamma \to \mathbb{Z}/2$$

and we look for solutions to (2.1) which satisfy

$$u(gx) = \phi(g)u(x) \qquad \text{for all } g \in \Gamma \text{ and } x \in \Omega. \tag{2.6}$$

A function u with this property will be called *ϕ-equivariant.* We denote by

$$G := \ker\phi.$$

Note that, if u satisfies (2.6), then u is G-invariant. Moreover,

$$u(\gamma x) = -u(x) \quad \text{for every } x \in \Omega \text{ and } \gamma \in \phi^{-1}(-1).$$

Therefore, if ϕ is an epimorphism (i.e. if it is surjective), every nontrivial solution to (2.1) which satisfies (2.6) changes sign. If $\phi \equiv 1$ is the trivial homomorphism, then $\Gamma = G$ and (2.6) simply says that u is G-invariant.

The homomorphism ϕ induces an action of Γ on $H_0^1(\Omega)$ as follows: for $\gamma \in \Gamma$ and $u \in H_0^1(\Omega)$ we define $\gamma u \in H_0^1(\Omega)$ by

$$(\gamma u)(x) := \phi(\gamma)u(\gamma^{-1}x). \tag{2.7}$$

The following lemma asserts that J_V is Γ-invariant under this action.

Lemma 2.3. *For all $u, v \in H_0^1(\Omega)$ and $\gamma \in \Gamma$,*

$$\langle \gamma u, \gamma v \rangle_V = \langle u, v \rangle_V , \quad \mathbb{D}(\gamma u) = \mathbb{D}(u), \quad \textit{and} \quad \mathbb{D}'(\gamma u)(\gamma v) = \mathbb{D}'(u)v.$$

Consequently, $J_V(\gamma u) = J_V(u)$ and $J_V'(\gamma u)(\gamma v) = J_V'(u)(v)$.

Proof. Let $\gamma \in \Gamma$ and $u, v \in H_0^1(\Omega)$. Since $\gamma \in O(N)$, $|\det \gamma| = 1$ and $(\gamma x) \cdot (\gamma y) = x \cdot y$ for all $x, y \in \mathbb{R}^N$. We also have that

$$\nabla(\gamma u)(x) = \phi(\gamma)\gamma \nabla u(\gamma^{-1}x).$$

Thus, as $\gamma(\Omega) = \Omega$ and V is Γ-invariant, the change of variable $\tilde{x} = \gamma^{-1}x$ yields

$$\begin{aligned}
\langle \gamma u, \gamma v \rangle_V &= \int_\Omega [\nabla(\gamma u) \cdot \nabla(\gamma v) + (1 + V(x))(\gamma u)(\gamma v)] \\
&= \int_\Omega \left[(\phi(\gamma))^2 \gamma \nabla u(\gamma^{-1}x) \cdot \gamma \nabla v(\gamma^{-1}x) + (1 + V(x))(\phi(\gamma))^2 u(\gamma^{-1}x) v(\gamma^{-1}x)\right] dx \\
&= \int_\Omega \left[\nabla u(\gamma^{-1}x) \cdot \nabla v(\gamma^{-1}x) + (1 + V(x)) u(\gamma^{-1}x) v(\gamma^{-1}x)\right] dx \\
&= \int_\Omega [\nabla u(\tilde{x}) \cdot \nabla v(\tilde{x}) + (1 + V(\gamma \tilde{x})) u(\tilde{x}) v(\tilde{x})] \, |\det \gamma| d\tilde{x} \\
&= \int_\Omega [\nabla u(\tilde{x}) \cdot \nabla v(\tilde{x}) + (1 + V(\tilde{x})) u(\tilde{x}) v(\tilde{x})] \, d\tilde{x} \\
&= \langle u, v \rangle_V .
\end{aligned}$$

Consequently,

$$\|\gamma u\|_V^2 = \|u\|_V^2 \quad \forall u \in H_0^1(\Omega), \, \gamma \in \Gamma.$$

Similarly, the change of variables $\tilde{x} = \gamma^{-1}x$, $\tilde{y} = \gamma^{-1}y$ implies

$$\begin{aligned}
\mathbb{D}(\gamma u) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(\gamma u)(x)|^p |(\gamma u)(y)|^p}{|x - y|^\alpha} dx \, dy \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(\gamma) u(\gamma^{-1}x)|^p |\phi(\gamma) u(\gamma^{-1}y)|^p}{|x - y|^\alpha} dx \, dy \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\phi(\gamma)|^{2p} |u(\tilde{x})|^p |u(\tilde{y})|^p}{|\gamma \tilde{x} - \gamma \tilde{y}|^\alpha} d\tilde{x} \, d\tilde{y} \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(\tilde{x})|^p |u(\tilde{y})|^p}{|\tilde{x} - \tilde{y}|^\alpha} d\tilde{x} \, d\tilde{y} \\
&= \mathbb{D}(u).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{D}'(\gamma u)(\gamma v) &= \int_\Omega \left(\frac{1}{|x|^\alpha} * |\gamma u|^p\right) |\gamma u|^{p-2}(\gamma u)(\gamma v) \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(\gamma u)(y)|^p |(\gamma u)(x)|^{p-2}(\gamma u)(x)(\gamma v)(x)}{|x-y|^\alpha} dx\, dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(\gamma)u(\gamma^{-1}y)|^p |\phi(\gamma)u(\gamma^{-1}x)|^{p-2}\phi(\gamma)u(\gamma^{-1}x)\phi(\gamma)v(\gamma^{-1}x)}{|x-y|^\alpha} dx\, dy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\phi(\gamma)|^{2p}|u(\tilde{y})|^p |u(\tilde{x})|^{p-2}u(\tilde{x})v(\tilde{x})}{|\gamma\tilde{x}-\gamma\tilde{y}|^\alpha} d\tilde{x}\, d\tilde{y} \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(\tilde{y})|^p |u(\tilde{x})|^{p-2}u(\tilde{x})v(\tilde{x})}{|\tilde{x}-\tilde{y}|^\alpha} d\tilde{x}\, d\tilde{y} \\
&= \mathbb{D}'(u)v.
\end{aligned}$$

□

Now, we consider the fixed point space of $H_0^1(\Omega)$ under the action defined in (2.7), namely

$$\begin{aligned}
H_0^1(\Omega)^\phi :&= \{u \in H_0^1(\Omega) : \gamma u = u \ \ \forall \gamma \in \Gamma\} \\
&= \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \ \ \forall \gamma \in \Gamma, \forall x \in \Omega\}.
\end{aligned}$$

Observe that $H_0^1(\Omega)^\phi$ is a closed linear subspace of $H_0^1(\Omega)$, and so, $H_0^1(\Omega)^\phi$ is a Hilbert space.

Next, we have a particular case of the well-known principle of symmetric criticality due to Palais [52, 59], which states that the critical points of the restriction of J_V to the fixed point space $H_0^1(\Omega)^\phi$ are the solutions to problem (2.1) that satisfy (2.6).

Theorem 2.4 (**Principle of symmetric criticality**). *The following hold true:*

(*a*) $\nabla J_V : H_0^1(\Omega) \to H_0^1(\Omega)$ *is* ϕ*-equivariant, i.e.*

$$\nabla J_V(\gamma u) = \gamma \nabla J_V(u) \quad \forall u \in H_0^1(\Omega),\ \gamma \in \Gamma.$$

Consequently, if $u \in H_0^1(\Omega)^\phi$*, then* $\nabla J_V(u) \in H_0^1(\Omega)^\phi$*.*

(*b*) *If* $u \in H_0^1(\Omega)^\phi$ *is a critical point of the restriction* $J_V|_{H_0^1(\Omega)^\phi} : H_0^1(\Omega)^\phi \to \mathbb{R}$*, then* u *is a critical point of* J_V*.*

Proof. Let $\gamma \in \Gamma$ and $u \in H_0^1(\Omega)$. From Lemma 2.3 we have that

$$\begin{aligned}
\langle \nabla J_V(\gamma u), v\rangle_V &= J_V'(\gamma u)v \\
&= J_V'(u)(\gamma^{-1}v) \\
&= \langle \nabla J_V(u), \gamma^{-1}v\rangle_V \\
&= \langle \gamma\nabla J_V(u), v\rangle_V \qquad \forall v \in H_0^1(\Omega).
\end{aligned}$$

Hence, $\nabla J_V(\gamma u) = \gamma\nabla J_V(u)$. In particular, if $u \in H_0^1(\Omega)^\phi$, then $\gamma u = u$ and so

$$\nabla J_V(u) = \gamma\nabla J_V(u) \qquad \forall \gamma \in \Gamma.$$

That is, $\nabla J_V(u) \in H_0^1(\Omega)^\phi$ for all $u \in H_0^1(\Omega)^\phi$. Accordingly,

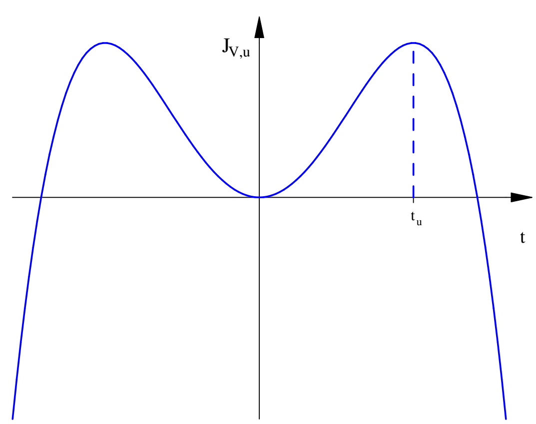
$$\nabla\big(J_V|_{H_0^1(\Omega)^\phi}\big)(u) = \nabla J_V(u) \qquad \forall u \in H_0^1(\Omega)^\phi.$$

This proves (b). $\square$

Next, we analyse the graph of the functional $J_V|_{H_0^1(\Omega)^\phi}$ in order to find some information about the critical points. To do that, we fix a direction $u \in H_0^1(\Omega)^\phi$, $u \neq 0$ and study how the graph of J_V looks like on the line generated by u. More precisely, we consider the function $J_{V,u} : \mathbb{R} \to \mathbb{R}$ given by

$$J_{V,u}(t) := J_V(tu) = \left(\frac{1}{2}\|u\|_V^2\right)t^2 - \left(\frac{1}{2p}\mathbb{D}(u)\right)t^{2p}. \tag{2.8}$$

Notice that this is a polynomial function of t. Since $2p > 2$ and the sign on the leading coefficient is negative, the graph will be down on both ends and, near to zero, the graph will behave roughly like a positive quadratic. Actually, the graph of $J_{V,u}$ has the following shape:



Hence, $J_V|_{H_0^1(\Omega)^\phi}$ is not bounded below and has a local minimum point at 0. Clearly, 0 is a solution of problem (2.1), but we are interested in nontrivial solutions.

The unique critical point of $J_{V,u}$ over $(0,\infty)$ corresponds to a maximum. The set of maximum points of $J_{V,u}$ for all directions $u \in H_0^1(\Omega)^\phi$, $u \neq 0$, is the set

$$\begin{aligned}\mathcal{N}^\phi_{\Omega,V} :&= \left\{u \in H_0^1(\Omega)^\phi : J_V'(u)u = 0\right\}\\ &= \left\{u \in H_0^1(\Omega)^\phi : u \neq 0,\ \|u\|_V^2 = \mathbb{D}(u)\right\},\end{aligned}$$

which is called the *Nehari manifold*. Note that the Nehari manifold contains all of the nontrivial critical points of $J_V|_{H_0^1(\Omega)^\phi}$.

2.2 The Nehari manifold

We shall assume from now on that $p \in \left[2, \frac{2N-\alpha}{N-2}\right)$. We denote by $T_u\mathcal{N}^{\phi}_{\Omega,V}$ the tangent space to the Nehari manifold $\mathcal{N}^{\phi}_{\Omega,V}$ at the point $u \in \mathcal{N}^{\phi}_{\Omega,V}$.

Proposition 2.5. *$\mathcal{N}^{\phi}_{\Omega,V}$ has the following properties:*

(*a*) *There exists $d_0 > 0$ such that $\|u\|_V \geq d_0$ for all $u \in \mathcal{N}^{\phi}_{\Omega,V}$. Consequently, $\mathcal{N}^{\phi}_{\Omega,V}$ is a closed subset of $H^1_0(\Omega)$.*

(*b*) *$\mathcal{N}^{\phi}_{\Omega,V}$ is a submanifold of class $\mathcal{C}^2$ of $H^1_0(\Omega)^{\phi}$.*

(*c*) *$u \notin T_u\mathcal{N}^{\phi}_{\Omega,V}$ for every $u \in \mathcal{N}^{\phi}_{\Omega,V}$.*

(*d*) *For each $u \in H^1_0(\Omega)^{\phi}$, $u \neq 0$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}^{\phi}_{\Omega,V}$. Furthermore, t_u is the only point in $(0,\infty)$ which satisfies*

$$\max_{t\geq 0} J_V(tu) = J_V(t_u u).$$

Proof. (*a*): Inequality (2.5), together with the continuous Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{pr}(\Omega)$ and Proposition 2.1, implies that there exists $C > 0$ such that

$$C \leq \frac{\|u\|^{2p}_V}{\mathbb{D}(u)} \qquad \forall u \in H^1_0(\Omega) \smallsetminus \{0\}.$$

Therefore,

$$C \leq \frac{(\|u\|^2_V)^p}{\mathbb{D}(u)} = \|u\|^{2(p-1)}_V \qquad \forall u \in \mathcal{N}^{\phi}_{\Omega,V}.$$

Hence, taking $d_0 := C^{\frac{1}{2(p-1)}}$, we have that

$$\|u\|_V \geq d_0 \qquad \forall u \in \mathcal{N}^{\phi}_{\Omega,V}.$$

Consequently,

$$\mathcal{N}^{\phi}_{\Omega,V} = \left\{u \in H^1_0(\Omega)^{\phi} : \|u\|_V \geq d_0 \text{ and } \|u\|^2_V - \mathbb{D}(u) = 0\right\},$$

which is clearly a closed subset of $H^1_0(\Omega)^{\phi}$.

(*b*) and (*c*): Consider the function $F : H^1_0(\Omega)^{\phi} \smallsetminus \{0\} \to \mathbb{R}$ given by

$$F(u) := \|u\|^2_V - \mathbb{D}(u).$$

Notice that $\mathcal{N}^{\phi}_{\Omega,V} = F^{-1}(0)$.

As in the proof of Proposition 2.2 (see Appendix B), F is of class $\mathcal{C}^2$ and its derivative is given by

$$F'(u)v = 2\langle u, v\rangle_V - \mathbb{D}'(u)v \qquad \forall u, v \in H^1_0(\Omega)^{\phi}.$$

Moreover, 0 is a regular value of F since

$$F'(u)u = 2\|u\|_V^2 - 2p\mathbb{D}(u) = 2(1-p)\|u\|_V^2 \neq 0 \qquad \forall u \in \mathcal{N}^{\phi}_{\Omega,V}.$$

This shows that $\mathcal{N}^{\phi}_{\Omega,V}$ is a submanifold of class $\mathcal{C}^2$ of $H^1_0(\Omega)^{\phi}$ and that $u \notin \ker F'(u) = T_u\mathcal{N}^{\phi}_{\Omega,V}$.

(d): Let $u \in H^1_0(\Omega)^{\phi}$, $u \neq 0$. Let $J_{V,u} : (0,\infty) \to \mathbb{R}$ be the function given by (2.8). This function has exactly one critical point over $(0,\infty)$, which corresponds to a maximum point. Furthermore, for $t \in (0,\infty)$, the following holds true:

$$J'_{V,u}(t) = J'_V(tu)u = 0 \iff J'_V(tu)tu = 0 \iff tu \in \mathcal{N}^{\phi}_{\Omega,V}.$$

Thus, $J_{V,u}$ has a maximum point at t if and only if $tu \in \mathcal{N}^{\phi}_{\Omega,V}$. This proves ($d$). □

Observe that

$$J_V(u) = \frac{p-1}{2p}\|u\|_V^2 = \frac{p-1}{2p}\mathbb{D}(u) \qquad \forall u \in \mathcal{N}^{\phi}_{\Omega,V}. \tag{2.9}$$

From the above proposition we can conclude the following:

Corollary 2.6. (a) $\inf_{u\in\mathcal{N}^{\phi}_{\Omega,V}} J_V(u) > 0$.

(b) *If $u \in \mathcal{N}^{\phi}_{\Omega,V}$ is a critical point of J_V on $\mathcal{N}^{\phi}_{\Omega,V}$, then u is a nontrivial critical point of $J_V : H^1_0(\Omega) \to \mathbb{R}$ and, consequently, a nontrivial solution of problem (2.1).*

Proof. The statement (a) is an immediate consequence of the identity (2.9) and Proposition 2.5 (a).

(b) If $u \in \mathcal{N}^{\phi}_{\Omega,V}$ is a critical point of J_V on $\mathcal{N}^{\phi}_{\Omega,V}$, then

$$J'_V(u)v = 0 \qquad \forall v \in T_u\mathcal{N}^{\phi}_{\Omega,V}.$$

In addition, from the definition of $\mathcal{N}^{\phi}_{\Omega,V}$ it follows that $J'_V(u)u = 0$. As the orthogonal complement of $T_u\mathcal{N}^{\phi}_{\Omega,V}$ in $H^1_0(\Omega)^{\phi}$ has dimension 1 and $u \notin T_u\mathcal{N}^{\phi}_{\Omega,V}$ (Proposition 2.5 (c)), one has that

$$H^1_0(\Omega)^{\phi} = T_u\mathcal{N}^{\phi}_{\Omega,V} \oplus \{tu : t \in \mathbb{R}\}.$$

Consequently,

$$J'_V(u)v = 0 \qquad \forall v \in H^1_0(\Omega)^{\phi},$$

this means, u is a critical point of $J_V|_{H^1_0(\Omega)^{\phi}} : H^1_0(\Omega)^{\phi} \to \mathbb{R}$. So, by Theorem 2.4 ($b$), we can conclude that u is a critical point of J_V. □

The Nehari manifold $\mathcal{N}^{\phi}_{\Omega,V}$ is radially diffeomorphic to the unit sphere in $H^1_0(\Omega)^{\phi}$. The radial projection $\pi : H^1_0(\Omega)^{\phi} \smallsetminus \{0\} \to \mathcal{N}^{\phi}_{\Omega,V}$ is given by

$$\pi(u) := \left(\frac{\|u\|_V^2}{\mathbb{D}(u)}\right)^{\frac{1}{2(p-1)}} u. \tag{2.10}$$

Accordingly, for every $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$,

$$J_V(\pi(u)) = \frac{p-1}{2p}\left(\frac{\|u\|_V^2}{\mathbb{D}(u)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}. \tag{2.11}$$

Remark 2.7. The solutions of problem (1.1) are the critical points of the functional $\hat{J}_V : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\hat{J}_V(u) := \frac{1}{2}\|u\|_V^2 - \frac{1}{p}|u|_p^p,$$

where $|u|_p := \left(\int_\Omega |u|^p\right)^{1/p}$ is the norm in $L^p(\Omega)$.

From Claim 1 in the proof of Proposition 2.2 and [59, Proposition 1.12] it follows that, if $p \in (2, 2^*)$, the functional $\hat{J}_V$ is of class $\mathcal{C}^2$ and

$$\hat{J}_V'(u)v = \langle u, v\rangle_V - \int_\Omega |u|^{p-2}uv.$$

Consequently, u is a solution of problem (1.1) if and only if u is a critical point of $\hat{J}_V$. On the other hand, a suitable change of variable, like in Lemma 2.3, allows us to conclude that the functional $\hat{J}_V$ is Γ-invariant under the action defined in (2.7). So, by the Principle of Symmetric Criticality (which still works for the functional $\hat{J}_V$), the critical points of the restriction of $\hat{J}_V$ to the fixed point space of this action, namely,

$$H_0^1(\Omega)^\phi = \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \ \ \forall \gamma \in \Gamma,\, x \in \Omega\},$$

are the solutions of problem (1.1) that satisfy (1.3). The nontrivial ones lie on the Nehari manifold

$$\hat{\mathcal{N}}_{\Omega,V}^\phi := \left\{u \in H_0^1(\Omega)^\phi : u \neq 0,\ \|u\|_V^2 = |u|_p^p\right\},$$

which is of class $\mathcal{C}^2$ and is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\phi$. Actually, it is easy to check that $\hat{\mathcal{N}}_{\Omega,V}^\phi$ and $\hat{J}_V$ satisfy properties analogous to those established in Proposition 2.5 and Corollary 2.6.

The radial projection $\hat{\pi} : H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \hat{\mathcal{N}}_{\Omega,V}^\phi$ is given by

$$\hat{\pi}(u) := \left(\frac{\|u\|_V^2}{|u|_p^p}\right)^{\frac{1}{p-2}} u.$$

Observe that, for every $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$,

$$\hat{J}_V(\hat{\pi}(u)) = \frac{p-2}{2p}\left(\frac{\|u\|_V^2}{|u|_p^2}\right)^{\frac{p}{p-2}} = \max_{t\geq 0} \hat{J}_V(tu).$$

We set

$$c_{\Omega,V}^\phi := \inf_{\mathcal{N}_{\Omega,V}^\phi} J_V.$$

If $\phi \equiv 1$ is the trivial homomorphism, then $\Gamma = G := \ker \phi$. In this case we shall write $H_0^1(\Omega)^G$, $\mathcal{N}_{\Omega,V}^G$ and $c_{\Omega,V}^G$ instead of $H_0^1(\Omega)^\phi$, $\mathcal{N}_{\Omega,V}^\phi$ and $c_{\Omega,V}^\phi$. If $G = \{1\}$ is the trivial group, we shall omit it from the notation and write simply $H_0^1(\Omega)$, $\mathcal{N}_{\Omega,V}$ and $c_{\Omega,V}$.

The problem

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{2.12}$$

plays a special role: it is the limit problem for (2.1). In this case we write J_∞, $\mathcal{N}_\infty$ and c_∞ instead of J_0, $\mathcal{N}_{\mathbb{R}^N,0}$ and $c_{\mathbb{R}^N,0}$.

2.3 Non-existence of minimizers for nonnegative potentials

It is known that c_∞ is attained at a positive function $\omega \in H^1(\mathbb{R}^N)$ (see for example [48, Theorem 3]). The following result shows, however, that $c_{\Omega,V}^\phi$ is not necessarily attained.

Proposition 2.8. *If $V \geq 0$, then $c_{\Omega,V} = c_\infty$. If, additionally, $V \not\equiv 0$ when $\Omega = \mathbb{R}^N$, then $c_{\Omega,V}$ is not attained.*

The proof of Proposition 2.8 is based on the following three lemmas and Theorem 2.12 below.

Lemma 2.9. *If $v_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, then after passing to a subsequence, we have that*

$$\lim_{n\to\infty} (\|v_n\|_V^2 - \|v_n\|^2) = 0.$$

Proof. Let $\epsilon > 0$. Since (v_n) is bounded in $H_0^1(\Omega)$ a subsequence satisfies that $v_n \to 0$ strongly in $L^2_{loc}(\Omega)$. Let $C > 0$ be such that $|v_n|_2^2 < C$ for all $n \in \mathbb{N}$. Set

$$A_\epsilon := \left\{ x \in \Omega : |V(x)| \geq \frac{\epsilon}{2C} \right\}.$$

Assumption (V_0) guarantees that A_ϵ is a bounded set and, since $v_n \to 0$ strongly in $L^2_{loc}(\Omega)$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{A_\epsilon} |V(x)||v_n|^2 \leq \sup_{\mathbb{R}^N} |V| \int_{A_\epsilon} |v_n|^2 < \frac{\epsilon}{2} \quad \text{if } n \geq n_0.$$

On the other hand,

$$\int_{\Omega \smallsetminus A_\epsilon} |V(x)||v_n|^2 \leq \frac{\epsilon}{2C} \int_{\Omega \smallsetminus A_\epsilon} |v_n|^2 \leq \frac{\epsilon}{2}.$$

Consequently,

$$\int_\Omega |V(x)||v_n|^2 < \epsilon \quad \text{if } n \geq n_0.$$

From $\|v_n\|_V^2 - \|v_n\|^2 = \int_\Omega V(x)|v_n|^2$ we obtain the conclusion. $\square$

We choose a radially symmetric cut-off function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $0 \le \chi \le 1$, $\chi(x) = 1$ if $|x| \le 1$ and $\chi(x) = 0$ if $|x| \ge 2$. Let $S > 0$ and $u : \mathbb{R}^N \to \mathbb{R}$ given. We define

$$\chi^S(x) := \chi\left(\frac{x}{S}\right) \quad \text{and} \quad u^S := \chi^S u.$$

Observe that $\chi^S(x) = 1$ if $|x| \le S$ and $\chi^S(x) = 0$ if $|x| \ge 2S$.

Lemma 2.10. *If $u \in H^1(\mathbb{R}^N)$, then*

$$\begin{aligned} u^S &\to u && \textit{strongly in } H^1(\mathbb{R}^N),\\ \mathbb{D}\left(u^S\right) &\to \mathbb{D}(u) && \textit{in } \mathbb{R} \end{aligned}$$

as $S \to \infty$.

Proof. Note first that

$$\begin{aligned} \int_{\mathbb{R}^N} \left|u - u^S\right|^2 &= \int_{\mathbb{R}^N} \left|1 - \chi^S\right|^2 |u|^2 \\ &\le \int_{|x| \ge S} |u|^2. \end{aligned} \tag{2.13}$$

Let $C > 0$ be such that $|\nabla\chi(x)| \le C$ for all $x \in \mathbb{R}^N$. Then

$$\begin{aligned} \int_{\mathbb{R}^N} \left|\nabla\left(u - u^S\right)\right|^2 &= \int_{\mathbb{R}^N} \left|\nabla\left(\left(1 - \chi^S\right)u\right)\right|^2 \\ &= \int_{\mathbb{R}^N} \left|\left(1 - \chi^S\right)\nabla u - u\nabla\chi^S\right|^2 \\ &\le 4\left(\int_{\mathbb{R}^N} \left(1 - \chi^S\right)^2 |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2 \left|\nabla\chi^S\right|^2\right) \\ &\le 4\left(\int_{|x| \ge S} |\nabla u|^2 + \frac{C^2}{S^2}\int_{\mathbb{R}^N} |u|^2\right). \end{aligned} \tag{2.14}$$

Consequently, since $u \in H^1(\mathbb{R}^N)$,

$$\|u - u^S\|^2 \le 4\int_{|x| \ge S} |\nabla u|^2 + \frac{4C^2}{S^2}\int_{\mathbb{R}^N} |u|^2 + \int_{|x| \ge S} |u|^2 \to 0 \quad \text{as } S \to \infty.$$

This shows that $u^S \to u$ strongly in $H^1(\mathbb{R}^N)$.

Now, using the Hardy– Littlewood–Sobolev inequality (2.4) one has

$$\begin{aligned}\left|\mathbb{D}(u)-\mathbb{D}\left(u^S\right)\right| &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left||u(x)|^p|u(y)|^p-|u^S(x)|^p|u^S(y)|^p\right|}{|x-y|^\alpha}\,dx\,dy\\ &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left||u(x)|^p|u(y)|^p-|u(x)|^p|u^S(y)|^p\right|}{|x-y|^\alpha}\,dx\,dy\\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left||u(x)|^p|u^S(y)|^p-|u^S(x)|^p|u^S(y)|^p\right|}{|x-y|^\alpha}\,dx\,dy\\ &\leq 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p\left||u(y)|^p-|u^S(y)|^p\right|}{|x-y|^\alpha}\,dx\,dy\\ &\leq 2\bar{C}|u|_{pr}^p\left||u|^p-|u^S|^p\right|_r \qquad (2.15)\\ &\leq C\left(\int_{|x|\geq S}|u|^{pr}\right)^{\frac{1}{r}},\end{aligned}$$

where $r := \frac{2N}{2N-\alpha}$ and $C := 2\bar{C}|u|_{pr}^p$.

Therefore, since $u \in L^{pr}(\mathbb{R}^N)$, we can deduce that

$$\left|\mathbb{D}(u)-\mathbb{D}\left(u^S\right)\right| \to 0 \qquad \text{as } S\to\infty. \qquad \square$$

Lemma 2.11. *Set $K(x) := \frac{1}{|x|^\alpha}$. Every solution $u \in H^1(\mathbb{R}^N)$ to problem (2.12) has the following properties:*

(*a*) *$u \in L^r(\mathbb{R}^N)$ for every $r \in [2,\infty)$.*

(*b*) *$K * |u|^p$ is continuous on $\mathbb{R}^N$ and $\lim_{|x|\to\infty}(K*|u|^p)(x) = 0$.*

(*c*) *u is of class $\mathcal{C}^2$.*

Proof. (*a*) and (*c*) are shown in [48, Proposition 4.1.]. The proof of (*b*) follows the same lines as the one of [21, Lemma A.1.(iii)]. □

Theorem 2.12 (Unique continuation principle)**.** *Let Ω be a connected open subset of $\mathbb{R}^N$, $N \geq 3$ and $W \in \mathcal{C}^0(\Omega)$. If $u \in H^1(\Omega)$ satisfies*

$$-\Delta u + W(x)u = 0,$$

and $u = 0$ on a nonempty open subset of Ω, then $u = 0$ on Ω.

Proof. See for instance [35, 37]. □

We write

$$B_r(\xi) := \{x \in \mathbb{R}^N : |x-\xi| < r\}.$$

Proof of Proposition 2.8. Since $H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$ and $V \geq 0$ one easily concludes that $c_{\Omega,V} \geq c_\infty$. Let $R > 0$ be such that $(\mathbb{R}^N \setminus \Omega) \subset B_R(0)$, and let (x_n) be a sequence in $\mathbb{R}^N$ such that $|x_n| > R$ and $|x_n| \to \infty$. We choose a cut-off function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| \geq 2$. We define $r_n := \frac{1}{2}(|x_n| - R)$ and

$$u_n(x) := \chi\Big(\frac{x - x_n}{r_n}\Big)\omega(x - x_n).$$

Then $u_n \in H_0^1(\Omega)$, $u_n \neq 0$, $u_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and $u_n \to 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$. From Lemma 2.9 we obtain that

$$\lim_{n\to\infty} \|u_n\|_V^2 = \lim_{n\to\infty} \|u_n\|^2$$

and since $u_n(x) = \omega^{r_n}(x - x_n)$, from Lemma 2.10 we deduce that

$$\lim_{n\to\infty} \|u_n\|_V^2 = \lim_{n\to\infty} \|u_n\|^2 = \|\omega\|^2 \qquad \text{and} \qquad \lim_{n\to\infty} \mathbb{D}(u_n) = \mathbb{D}(\omega).$$

Consequently, from (2.11) we obtain that $J_V(\pi(u_n)) \to J_\infty(\omega) = c_\infty$. Therefore $c_{\Omega,V} \leq c_\infty$, and hence $c_{\Omega,V} = c_\infty$.

Now, if there were $u \in \mathcal{N}_{\Omega,V}$ satisfying $J_V(u) = c_{\Omega,V}$, then u would be a nontrivial solution of problem (2.12) with minimum energy and $\|u\|_V^2 = \|u\|^2$. Therefore, u would satisfy

$$-\Delta u + W(x)u = 0,$$

where

$$W(x) = 1 - \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}.$$

From Lemma 2.11 (b) and (c) we would have that W is continuous. We distinguish two cases: (1) If $\Omega = \mathbb{R}^N$ then, by assumption, V is strictly positive on some open set U of $\mathbb{R}^N$. Since

$$0 = \|u\|_V^2 - \|u\|^2 = \int_{\mathbb{R}^N} V(x)u^2 \geq \int_U V(x)u^2 \geq 0,$$

we conclude that $u = 0$ in U. (2) If $\Omega \neq \mathbb{R}^N$ then $u = 0$ in $\mathbb{R}^N \setminus \Omega$. In both cases, we obtain a contradiction to the unique continuation principle (Theorem 2.12). As a result, $c_{\Omega,V}$ is not attained. □

Remark 2.13. In the local case, it is well-known that also the existence of ground states (i.e. minimum points of $\hat{J}_V$ on the Nehari manifold) turns out to heavily depend on the sign of the potential. In fact, it has been proved that no ground state is allowed if, either $V \geq 0$ is strictly positive on a set of positive measure or if $V \geq 0$ and $\mathbb{R}^N \setminus \Omega$ is nonempty, while a positive ground state solution exists if $V \leq 0$ is strictly negative on a set of positive measure.

Chapter 3

Main tools for proving existence

3.1 Representation of Palais-Smale sequences

This section is mainly devoted to the description of the lack of compactness to the nonlocal problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{3.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $p \in \left[2, \frac{2N-\alpha}{N-2}\right)$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty. The potential $V_\infty + V$ is assumed to satisfy

(V_0) $\quad V \in \mathcal{C}^0(\mathbb{R}^N)$, $\;V_\infty \in (0, \infty)$, $\;\inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0$, $\lim_{|x|\to\infty} V(x) = 0$.

From now on we shall assume without loss of generality that $V_\infty = 1$.

As usual, we identify $u \in H_0^1(\Omega)$ with its extension to $\mathbb{R}^N$ obtained by setting $u \equiv 0$ in $\mathbb{R}^N \smallsetminus \Omega$.

Recall that the energy functional $J_V : H_0^1(\Omega) \to \mathbb{R}$ associated to problem (3.1) is given by

$$J_V(u) := \frac{1}{2}\,\|u\|_V^2 - \frac{1}{2p}\mathbb{D}(u),$$

where

$$\mathbb{D}(u) := \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^p = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha}dx\,dy$$

and $\|\cdot\|_V^2$ is the norm induced by the scalar product

$$\langle u, v\rangle_V := \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (1 + V(x))\, uv.$$

If $V = 0$ we write $\langle u, v\rangle$ and $\|u\|$ instead of $\langle u, v\rangle_0$ and $\|u\|_0$.

In the nonsymmetric case, Benci and Cerami [9] described the lack of compactness of the functional $\hat{J}_V$ associated to the local problem (1.1). They showed that the Palais-Smale sequences which do not converge to a solution of problem (1.1) approach a sum of a possibly trivial solution of (1.1) plus nontrivial solutions of the limit problem (1.2) translated by sequences of points in the domain which go to infinity.

We analyze next the Palais-Smale sequences for the functional J_V belonging to

$$H_0^1(\Omega)^\phi := \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x) \ \ \forall \gamma \in \Gamma, \forall x \in \Omega\},$$

where, as in the previous chapter, Γ is a closed subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and $\phi : \Gamma \to \mathbb{Z}/2$ is a continuous group homomorphism.

We shall give a precise description of the relation between the symmetries of the translation points and those of the corresponding solution to the limit problem

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{3.2}$$

This plays an important role in the proof of Corollary 3.7, which will be crucial for our results.

Recall that the $\Gamma-$*orbit* of a point $x \in \mathbb{R}^N$ is the set

$$\Gamma x = \{\eta x : \eta \in \Gamma\}$$

and that the Γ-*isotropy group* of x is the subgroup

$$\Gamma_x := \{\eta \in \Gamma : \eta x = x\}$$

of Γ. We write

$$(\mathbb{R}^N)^H = \{x \in \mathbb{R}^N : \eta x = x \text{ for all } \eta \in H\}$$

for the *fixed point space* of the action of a closed subgroup H of Γ on $\mathbb{R}^N$.

Recall that the subgroups H and K of Γ are called *conjugate* in Γ if and only if there exists $\eta \in \Gamma$ such that $H = \eta K \eta^{-1}$. The *conjugacy class* of H in Γ is the set

$$(H) = \{\eta H \eta^{-1} : \eta \in \Gamma\}.$$

The relation

$$(L) \leq (M) \quad \text{if and only if} \quad \eta L \eta^{-1} \subseteq M \quad \text{for some } \eta \in \Gamma$$

defines a partial order on the set of conjugacy classes of closed subgroups of Γ. The conjugacy class (Γ_x) of an isotropy group Γ_x is called an *isotropy class*.

Next, we collect some known results on spaces with group actions which will be used in the proof of the following lemma.

(F_1) The Γ-orbit Γx of x is Γ-homeomorphic to the homogeneous space Γ/Γ_x. The homeomorphism is given by

$$\eta_x : \Gamma/\Gamma_x \to \Gamma x, \quad \eta\Gamma_x \mapsto \eta x.$$

See for instance [11, I.4.(4.1)] or [30, I.3.(3.19)].

(F_2) Isotropy groups satisfy

$$\Gamma_{\eta x} = \eta \Gamma_x \eta^{-1}.$$

Hence, the conjugate groups to an isotropy group are isotropy groups and so, if Γx is finite, there is only a finite number of groups conjugate to Γ_x. See for instance [11, I.2.(2.1)].

(F_3) Every finite–dimensional vector space has only finitely many isotropy classes (see [30, I.5.(5.11)] and [11, IV.10]).

(F_4) If $K \subset \Gamma_\zeta$ and $(K) = (\Gamma_\zeta)$, then $K = \Gamma_\zeta$. Indeed, let $\eta \in \Gamma$ be such that $\eta\Gamma_\zeta\eta^{-1} = K$. Since $K \subset \Gamma_\zeta$ one has that $(\mathbb{R}^N)^{\Gamma_\zeta} \subset (\mathbb{R}^N)^K$. On the other hand, the map $x \mapsto \eta x$ is an isomorphism between $(\mathbb{R}^N)^{\Gamma_\zeta}$ and $(\mathbb{R}^N)^K$ and hence both spaces have the same dimension. Consequently, $(\mathbb{R}^N)^{\Gamma_\zeta} = (\mathbb{R}^N)^K$ and, since $\eta\zeta \in (\mathbb{R}^N)^K$, we conclude that $\Gamma_\zeta \subset \Gamma_{\eta\zeta} = K$. This proves that $K = \Gamma_\zeta$.

The following lemma and its proof are taken from [21, Lemma 3.2]. We just add some details.

Lemma 3.1. *Given a sequence* (y_n) *in* $\mathbb{R}^N$ *there exist a sequence* (ζ_n) *in* $\mathbb{R}^N$ *and a closed subgroup* K *of* Γ *such that for some subsequence of* (y_n)*, which we still denote in the same way, the following hold:*
(a) $\operatorname{dist}(\Gamma y_n, \zeta_n))$ *is bounded.*
(b) $\Gamma_{\zeta_n} = K$ *for all* $n \in \mathbb{N}$*.*
(c) If $|\Gamma/K| < \infty$ *then* $|\eta\zeta_n - \tilde{\eta}\zeta_n| \to \infty$ *for any* $[\eta], [\tilde{\eta}] \in \Gamma/K$ *with* $[\eta] \neq [\tilde{\eta}]$*.*
(d) If $|\Gamma/K| = \infty$ *then there exists a closed subgroup* K' *of* Γ *such that* $K \subset K'$*,* $|\Gamma/K'| = \infty$ *and* $|\eta\zeta_n - \tilde{\eta}\zeta_n| \to \infty$ *for any* $[\eta], [\tilde{\eta}] \in \Gamma/K'$ *with* $[\eta] \neq [\tilde{\eta}]$*.*

Proof. Set
$$V := \{x \in \mathbb{R}^N : |\Gamma/\Gamma_x| < \infty\}.$$
Note that V is a Γ-invariant linear subspace of $\mathbb{R}^N$. Indeed, let $x, y \in V$ and $a \in \mathbb{R}$. From (F_1) one has that Γx and Γy are finite and so, $\Gamma(x+y)$ and $\Gamma(ax)$ are finite too. Moreover $\Gamma(\eta x) = \Gamma x$ for all $\eta \in \Gamma$. Using again (F_1) the claim follows.

Next, two cases are considered.

Case 1. The sequence $(\operatorname{dist}(y_n, V))$ is bounded.

Let $\Im$ be the set of isotropy classes (Γ_x) such that $x \in V$ and, for some $\eta \in \Gamma$, $\big(\operatorname{dist}\big(y_n, (\mathbb{R}^N)^{\eta\Gamma_x\eta^{-1}}\big)\big)$ contains a bounded subsequence.

Let us see that $\Im \neq \emptyset$. Indeed, if z_n is the orthogonal projection of y_n onto V, from (F_3) there exists an isotropy class (L) such that after passing to a subsequence $(\Gamma_{z_n}) = (L)$ for all $n \in \mathbb{N}$. Moreover, (F_1) implies that the $\Gamma-$orbit of every point in V is finite and then (F_2) yields that the isotropy class of each element in V has only finitely many groups. Therefore, after passing to another subsequence one can assume that $\Gamma_{z_n} = L$ for all $n \in \mathbb{N}$. Note that $(\mathbb{R}^N)^L \subseteq V$. Indeed, if $x \in (\mathbb{R}^N)^L$, then $L \subseteq \Gamma_x$ and so $|\Gamma/\Gamma_x| \leq |\Gamma/L| < \infty$. Therefore,
$$\operatorname{dist}(y_n, (\mathbb{R}^N)^L) = |y_n - z_n| = \operatorname{dist}(y_n, V),$$
and hence $(L) \in \Im$.

Now, choose K and a subsequence of (y_n)—which will be denoted in the same way—such that (K) is a maximal element of $\Im$ (i.e. if $(H) \in \Im$ is such that $(K) \leq (H)$, then $(K) = (H)$) and
$$\operatorname{dist}(y_n, (\mathbb{R}^N)^K) < c < \infty \qquad \forall n \in \mathbb{N}.$$
Let ζ_n be the orthogonal projection of y_n onto $(\mathbb{R}^N)^K$.

(a) is trivially satisfied since

$$\mathrm{dist}(\Gamma y_n, \zeta_n) \le |y_n - \zeta_n| = (\mathrm{dist}(y_n, (\mathbb{R}^N)^K) < c \qquad \forall n \in N.$$

Since $(\mathbb{R}^N)^K \subset V$, by the same argument as above, passing to a subsequence, one can assume that $\Gamma_{\zeta_n} = L$ for all $n \in N$. Since $K \subset L$, $(\mathbb{R}^N)^L \subset (\mathbb{R}^N)^K$. Then

$$\mathrm{dist}(y_n, (\mathbb{R}^N)^L) = |y_n - \zeta_n| < c \qquad \forall n \in N.$$

Therefore $(L) \in \Im$. Since $K \subset \Gamma_{\zeta_n}$ and (K) is maximal, one can conclude that $(K) = (\Gamma_{\zeta_n})$. It follows from (F_4) that $\Gamma_{\zeta_n} = K$. This proves (b).

Since $|\Gamma/K| < \infty$, in order to prove (c) it suffices to show that, if $\eta \notin K$, then $(\eta\zeta_n - \zeta_n)$ does not contain a bounded subsequence. Arguing by contradiction, assume that there exist $\tilde{\eta} \notin K$ and a bounded subsequence of $(\tilde{\eta}\zeta_n - \zeta_n)$. Let L be the subgroup of Γ generated by $K \cup \{\tilde{\eta}\}$, $W := (\mathbb{R}^N)^L$ and $W^\perp$ be the orthogonal complement of W in $(\mathbb{R}^N)^K$. Write

$$\zeta_n = \zeta_n^1 + \zeta_n^2 \quad \text{with} \quad \zeta_n^1 \in W \quad \text{and} \quad \zeta_n^2 \in W^\perp.$$

Then

$$\tilde{\eta}\zeta_n - \zeta_n = \left(\tilde{\eta}\zeta_n^1 - \zeta_n^1\right) + \left(\tilde{\eta}\zeta_n^2 - \zeta_n^2\right) = \tilde{\eta}\zeta_n^2 - \zeta_n^2.$$

Since $\tilde{\eta} \notin K$, assertion (b) implies that $\tilde{\eta}\zeta_n \neq \zeta_n$. Hence $\zeta_n^2 \neq 0$ and, passing to a subsequence, one has

$$\frac{\zeta_n^2}{|\zeta_n^2|} \to \zeta.$$

If (ζ_n^2) is unbounded, a subsequence satisfies

$$\left|\frac{\tilde{\eta}\zeta_n^2}{|\zeta_n^2|} - \frac{\zeta_n^2}{|\zeta_n^2|}\right| = \frac{|\tilde{\eta}\zeta_n - \zeta_n|}{|\zeta_n^2|} \to 0.$$

Therefore $\tilde{\eta}\zeta = \zeta$. Moreover, since $\frac{\zeta_n^2}{|\zeta_n^2|} \in (\mathbb{R}^N)^K$ for all $n \in \mathbb{N}$, then $\zeta \in (\mathbb{R}^N)^K$. Hence $\zeta \in W$, which is a contradiction.

If, on the other hand, (ζ_n^2) is bounded then, passing to a subsequence such that $\Gamma_{\zeta_n^1} = L_1$ for all $n \in \mathbb{N}$, the following holds true:

$$\mathrm{dist}(y_n, (\mathbb{R}^N)^{L_1}) = |y_n - \zeta_n^1| \le |y_n - \zeta_n| + |\zeta_n^2| \le \tilde{c} < \infty,$$

where $\tilde{c}$ is a positive constant. Therefore $(L_1) \in \Im$. Note that $K \subset L \subset L_1$. Since (K) is maximal one has that $(L_1) = (K)$ and by (F_4) one infers that $L_1 = K$, which is again a contradiction.

Case 2. The sequence $(\mathrm{dist}(y_n, V))$ is unbounded.

Passing to a subsequence, one can assume that $\mathrm{dist}(y_n, V) \to \infty$ and by (F_3) one can also assume that there exists an isotropy class (K) such that $(\Gamma_{y_n}) = (K)$ for all $n \in \mathbb{N}$. Choosing $\zeta_n \in \Gamma y_n$ such that $\Gamma_{\zeta_n} = K$, it immediately follows that (a) and (b) hold. Note that $y_n \notin V$. Moreover, $\eta y_n \notin V$ for any $\eta \in \Gamma$ and then $|\Gamma/K| = \infty$.

Let us see that (d) holds. Let $V^\perp$ be the orthogonal complement of V in $\mathbb{R}^N$ and ξ_n be the orthogonal projection of ζ_n onto $V^\perp$. Passing to a subsequence, one has that

$$\frac{\xi_n}{|\xi_n|} \to \xi.$$

Set $K' := \Gamma_\xi$. Thus $K \subset K'$. Moreover, since $\xi \in V^\perp$, $|\Gamma/K'| = \infty$. If $[\eta]$, $[\tilde\eta] \in \Gamma/K'$ are such that $[\eta] \neq [\tilde\eta]$, then $d := |\eta\xi - \tilde\eta\xi| > 0$. Let $n_0 \in \mathbb{N}$ be such that $\left|\frac{\xi_n}{|\xi_n|} - \xi\right| < \frac{d}{4}$ for $n \geq n_0$. Taking into account that η and $\tilde\eta$ are isometries on $\mathbb{R}^N$ one has that

$$\begin{aligned} d &= |\eta\xi - \tilde\eta\xi| \\ &\leq \left|\eta\xi - \frac{\eta\xi_n}{|\xi_n|}\right| + \left|\frac{\eta\xi_n}{|\xi_n|} - \frac{\tilde\eta\xi_n}{|\xi_n|}\right| + \left|\frac{\tilde\eta\xi_n}{|\xi_n|} - \tilde\eta\xi\right| \\ &= \frac{|\eta\xi_n - \tilde\eta\xi_n|}{|\xi_n|} + 2\left|\frac{\xi_n}{|\xi_n|} - \xi\right| \\ &\leq \frac{|\eta\xi_n - \tilde\eta\xi_n|}{|\xi_n|} + \frac{d}{2} \quad \forall n \geq n_0. \end{aligned}$$

Hence,

$$\frac{d}{2}|\xi_n| \leq |\eta\xi_n - \tilde\eta\xi_n| \quad \forall n \geq n_0.$$

Consequently,

$$\frac{d}{2}\operatorname{dist}(\zeta_n, V) = \frac{d}{2}|\xi_n| \leq |\eta\xi_n - \tilde\eta\xi_n| \leq |\eta\zeta_n - \tilde\eta\zeta_n| \quad \forall n \geq n_0.$$

Since $\operatorname{dist}(\zeta_n, V) \to \infty$, assertion (d) holds. □

The following lemma says that $\mathbb{D}$ is invariant under translations.

Lemma 3.2. *For all $u \in H^1(\mathbb{R}^N)$ and $z \in \mathbb{R}^N$,*

$$\mathbb{D}(u(\cdot + z)) = \mathbb{D}(u).$$

Proof. The change of variables $\tilde x = x + z$, $\tilde y = y + z$ yields

$$\begin{aligned} \mathbb{D}(u(\cdot + z)) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x+z)|^p|u(y+z)|^p}{|x-y|^\alpha}dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(\tilde x)|^p|u(\tilde y)|^p}{|(\tilde x - z) - (\tilde y - z)|^\alpha}d\tilde x\,d\tilde y \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(\tilde x)|^p|u(\tilde y)|^p}{|\tilde x - \tilde y|^\alpha}d\tilde x\,d\tilde y \\ &= \mathbb{D}(u). \end{aligned}$$

□

Lemma 3.3. *Let (u_n) be a sequence in $H_0^1(\Omega)$ such that $J_V(u_n) \to c$ and $J_V'(u_n) \to 0$ in $H^{-1}(\Omega)$. Then (u_n) is a bounded sequence in $H_0^1(\Omega)$ and $c \geq 0$.*

Proof. For n sufficiently large one has that

$$\begin{aligned} \frac{p-1}{2p}\|u_n\|_V^2 &= J_V(u_n) - \frac{1}{2p}J_V'(u_n)u_n \qquad (3.3) \\ &\leq |J_V(u_n)| + \frac{1}{2p}\|\nabla J_V(u_n)\|_V\|u_n\|_V \\ &\leq |c| + 1 + \|u_n\|_V. \end{aligned}$$

Consequently, $(\|u_n\|_V)$ is bounded. Therefore, from (3.3) it follows that

$$\frac{p-1}{2p}\|u_n\|_V^2 \to c$$

and so $c \geq 0$. $\square$

As in the previous chapter we denote by J_0 the functional associated to problem (3.1) with $V \equiv 0$, i.e.

$$J_0(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2p}\mathbb{D}(u).$$

The following proposition corresponds to a slight variant of [1, Lemma 3.5] which states a Brezis-Lieb lemma for a large class of nonlocal functions. A proof of it can be found in Appendix A.

Proposition 3.4. *Let (u_n) be a sequence in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. The following hold:*

1. *$\mathbb{D}'(u_n)v \to \mathbb{D}'(u)v$ for all $v \in H_0^1(\Omega)$.*

2. *After passing to a subsequence, we have*

$$\begin{aligned} \mathbb{D}(u_n) - \mathbb{D}(u_n - u) &\to \mathbb{D}(u) \quad \text{in} \quad \mathbb{R}, \\ \mathbb{D}'(u_n) - \mathbb{D}'(u_n - u) &\to \mathbb{D}'(u) \quad \text{in } H^{-1}(\Omega). \end{aligned}$$

The proof of the following lemma follows exactly the same lines as the proof of [59, Lemma 8.2.]. However, we include it here for the sake of completeness.

Lemma 3.5. *If*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in} \quad H_0^1(\Omega), \\ u_n &\to u \quad \text{a.e. on} \quad \Omega, \\ J_V(u_n) &\to c, \\ J_V'(u_n) &\to 0 \quad \text{in} \quad H^{-1}(\Omega), \end{aligned}$$

then $J_V'(u) = 0$ and $v_n := u_n - u$ is such that

$$\begin{aligned} J_0(v_n) &\to c - J_V(u), \\ J_0'(v_n) &\to 0 \quad \text{in} \quad H^{-1}(\Omega). \end{aligned}$$

Proof. From Lemma 2.9, one has that

$$J_V(v_n) - J_0(v_n) = \frac{1}{2}(\|v_n\|_V^2 - \|v_n\|^2) = o(1).$$

Moreover, from Proposition 3.4 it follows that

$$\begin{aligned} J_V(v_n) &= \frac{1}{2}\|u_n - u\|_V^2 - \frac{1}{2p}\mathbb{D}(u_n - u) \\ &= \frac{1}{2}\left(\|u_n\|_V^2 - \|u\|_V^2\right) - \frac{1}{2p}\left(\mathbb{D}(u_n) - \mathbb{D}(u)\right) + o(1) \\ &= J_V(u_n) - J_V(u) + o(1) \\ &= c - J_V(u) + o(1). \end{aligned}$$

Therefore,

$$J_0(v_n) = c - J_V(u) + o(1).$$

Let $v \in H_0^1(\Omega)$. Since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, Proposition 3.4 yields

$$J'_V(u_n)v = \langle u_n, v\rangle_V - \mathbb{D}'(u_n)v \to \langle u, v\rangle_V - \mathbb{D}'(u)v = J'_V(u)v \quad \text{in } \mathbb{R}.$$

On the other hand, by the hypotheses, we have that $J'_V(u_n)v \to 0$ in $\mathbb{R}$. As the limit must be unique, we conclude that $J'_V(u)v = 0$. Hence, $J'_V(u) = 0$.

Set $w \in H_0^1(\Omega)$. In what follows, C denotes a positive constant, possibly different at each occurrence. By assumption (V_0) we have that, for $\varepsilon > 0$ given, there exists $R > 0$ such that $|V(x)| \leq \varepsilon$ for all $x \in \mathbb{R}^N \smallsetminus B_R$ and then

$$\begin{aligned} |J'_V(v_n)w - J'_0(v_n)w| &= \left|\int_\Omega V v_n w\right| \\ &\leq \sup_{B_R}|V| \int_{B_R} |v_n w| + \varepsilon \int_{\mathbb{R}^N \smallsetminus B_R} |v_n w| \\ &\leq C|v_n|_{2,B_R}|w|_2 + \varepsilon|v_n|_2|w|_2 \\ &\leq C(|v_n|_{2,B_R} + \varepsilon|v_n|_2)\|w\|. \end{aligned}$$

Since $v_n \rightharpoonup 0$ in $H_0^1(\Omega)$, (v_n) is a bounded sequence in $L^2(\Omega)$ and so

$$\|J'_V(v_n) - J'_0(v_n)\| \leq C(|v_n|_{2,B_R} + \varepsilon).$$

Moreover, after passing to a subsequence if necessary, we have that $v_n \to 0$ in $L^2_{loc}(\Omega)$ and then

$$\limsup_{n\to\infty} \|J'_V(v_n) - J'_0(v_n)\| \leq C\varepsilon.$$

Letting $\varepsilon \to 0$ we conclude that

$$\lim_{n\to\infty} \|J'_V(v_n) \to J'_0(v_n)\| = 0.$$

Now, Proposition 3.4, together with Riesz representation theorem, gives

$$\nabla\mathbb{D}(u_n) - \nabla\mathbb{D}(u_n - u) \to \nabla\mathbb{D}(u) \quad \text{in} \quad H_0^1(\Omega)$$

and then, since $\nabla J_V(v_n) = v_n - \frac{1}{2p}\nabla\mathbb{D}(v_n)$ with respect to the scalar product defined in (2.2), we obtain

$$\begin{aligned}\nabla J_V(v_n) &= (u_n - u) - \frac{1}{2p}\nabla\mathbb{D}(u_n - u)\\ &= (u_n - u) - \frac{1}{2p}(\nabla\mathbb{D}(u_n) - \nabla\mathbb{D}(u)) + o(1)\\ &= \nabla J_V(u_n) - \nabla J_V(u) + o(1).\end{aligned}$$

Consequently,

$$\begin{aligned}J_0'(v_n) &= J_V'(v_n) + o(1)\\ &= J_V'(u_n) - J_V'(u) + o(1)\\ &= o(1).\end{aligned}$$

□

Recall that the energy functional $J_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated to problem (3.2) is given by

$$J_\infty(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2p}\mathbb{D}(u).$$

We denote by $B_r(y) := \{x \in \mathbb{R}^N : |x - y| < r\}$. If $v \in H^1(\mathbb{R}^N)$ and $\eta \in \Gamma$ we simply write $v\eta$ for the composition $v \circ \eta$.

Proposition 3.6. *Let (u_n) be a sequence in $H_0^1(\Omega)^\phi$ such that $u_n \rightharpoonup 0$ in $H_0^1(\Omega)$, $J_0(u_n) \to c > 0$ and $J_0'(u_n) \to 0$ in $H^{-1}(\Omega)$. Then there exist a sequence (ζ_n) in Ω, a closed subgroup K of finite index in Γ, a nontrivial solution v to problem (3.2) and a sequence (w_n) in $H_0^1(\Omega)^\phi$ such that*

(a) $\Gamma_{\zeta_n} = K$ *for all* $n \in \mathbb{N}$,

(b) $|\zeta_n| \to \infty$ *and* $|\eta\zeta_n - \tilde{\eta}\zeta_n| \to \infty$ *if* $\tilde{\eta}\eta^{-1} \notin K$, $\tilde{\eta}, \eta \in \Gamma$,

(c) $v(\eta x) = \phi(\eta)v(x)$ *for all* $x \in \mathbb{R}^N$, $\eta \in K$,

(d) $\left\| u_n - w_n - \sum_{[\eta]\in\Gamma/K} \phi(\eta)v\eta^{-1}(\cdot - \eta\zeta_n) \right\| \to 0,$

(e) $w_n \rightharpoonup 0$ *in* $H_0^1(\Omega)$, $J_0(w_n) \to c - |\Gamma/K|\, J_\infty(v)$ *and* $J_0'(w_n) \to 0$ *in* $H^{-1}(\Omega)$.

Proof. Lemma 3.3 guarantees that (u_n) is a bounded sequence in $H_0^1(\Omega)$. Thus

$$\frac{p-1}{2p}\mathbb{D}(u_n) = J_0(u_n) - \frac{1}{2}J_0'(u_n)\, u_n \to c > 0.$$

That is, $\mathbb{D}(u_n) \to \frac{2p}{p-1}c > 0$. From (2.5) and Lions' lemma [59, Lemma 1.21] it follows that

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 > 0.$$

We choose $y_n \in \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |u_n|^2 > \delta/2$$

and, for the sequence (y_n), we choose K and (ζ_n) as in Lemma 3.1. We define $v_n(x) := u_n(x+\zeta_n)$. Passing to a subsequence if necessary, we may assume that

$$\begin{aligned} v_n &\rightharpoonup v && \text{weakly in } H^1(\mathbb{R}^N), \\ v_n &\to v && \text{strongly in } L^2_{loc}(\mathbb{R}^N), \\ v_n &\to v && \text{a.e. on } \mathbb{R}^N. \end{aligned}$$

Fixing $C > 0$ such that $\mathrm{dist}(\Gamma y_n, \zeta_n) \leq C$ for every n, we have that $B_1(\eta_n y_n) \subset B_{C+1}(\zeta_n)$ for some $\eta_n \in \Gamma$. Since $|u_n|$ is Γ-invariant we obtain

$$\int_{B_{C+1}(0)} |v_n|^2 = \int_{B_{C+1}(\zeta_n)} |u_n|^2 \geq \int_{B_1(y_n)} |u_n|^2 > \frac{\delta}{2}.$$

This implies that $v \neq 0$. But $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, so $|\zeta_n| \to \infty$.

We claim that v is a solution to problem (3.2). Indeed, Since J_∞ is invariant under translations, we have that

$$\|\nabla J_\infty(u_n(\cdot + \zeta_n))\| = \|\nabla J_0(u_n)\|$$

Therefore, from $\nabla J_0(u_n) \to 0$ in $H^1_0(\Omega)$ we get that $\nabla J_\infty(u_n(\cdot+\zeta_n)) \to 0$ in $H^1(\mathbb{R}^N)$. On the other hand, since $u_n(\cdot + \zeta_n) \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, Proposition 3.4 asserts that

$$\nabla J_\infty(u_n(\cdot + \zeta_n)) \to \nabla J_\infty(v).$$

Since the limit must be unique we conclude that $\nabla J_\infty(v) = 0$.

Assertion (b) of Lemma 3.1 insures that, for every $\eta \in K$,

$$u_n(\eta\, x + \zeta_n) = u_n\left(\eta(x + \zeta_n)\right) = \phi(\eta) u_n(x + \zeta_n).$$

Hence $v(\eta x) = \phi(\eta) v(x)$.

Let $\eta_1, \eta_2, \ldots \eta_t \in \Gamma$ be such that $|\eta_j \zeta_n - \eta_i \zeta_n| \to \infty$ when $i \neq j$. Then

$$\phi(\eta_j) v_n \eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n + \eta_j\zeta_n) \rightharpoonup \phi(\eta_j) v \eta_j^{-1} \tag{3.4}$$

weakly in $H^1(\mathbb{R}^N)$. Therefore,

$$\begin{aligned} &\left\| \phi(\eta_j) v_n \eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n + \eta_j\zeta_n) \right\|^2 = \\ &\left\| \phi(\eta_j) v_n \eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n + \eta_j\zeta_n) - \phi(\eta_j) v \eta_j^{-1} \right\|^2 + \left\| \phi(\eta_j) v \eta_j^{-1} \right\|^2 + o(1). \end{aligned}$$

Now observe that, since u_n is ϕ-equivariant, for all $\eta \in \Gamma$,

$$u_n(y) = \phi(\eta) u_n(\eta^{-1} y) = \phi(\eta) v_n(\eta^{-1} y - \zeta_n) = \phi(\eta) v_n \eta^{-1}(y - \eta\zeta_n). \tag{3.5}$$

Therefore, the change of variable $y = x + \eta_j\zeta_n$ in the above expression yields

$$\Big\|u_n - \sum_{i=j+1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big\|^2 = \Big\|u_n - \sum_{i=j}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big\|^2 + \|v\|^2 + o(1).$$

Iterating this equality, starting from $j = t$, we obtain

$$\begin{aligned}
\|u_n\|^2 &= \Big\|u_n - \sum_{i=(t-1)+1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big\|^2 + \|v\|^2 + o(1) \\
&= \Big\|u_n - \sum_{i=(t-2)+1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big\|^2 + 2\|v\|^2 + o(1) \\
&\ \ \vdots \\
&= \Big\|u_n - \sum_{i=1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big\|^2 + t\|v\|^2 + o(1).
\end{aligned}$$

Consequently,

$$\|u_n\|^2 - \Big\|u_n - \sum_{i=1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big\|^2 = t\|v\|^2 + o(1). \tag{3.6}$$

Since $\|u_n\|^2 \to \frac{2p}{p-1}c$, letting $n \to \infty$ in the above expression, we deduce that

$$\frac{2p}{p-1}c \geq t\|v\|^2.$$

Hence, assertion (d) of Lemma 3.1 implies that $|\Gamma/K| < \infty$, i.e. K has finite index in Γ. Thus assertion (c) of Lemma 3.1 allows us to take $t := |\Gamma/K|$.

On the other hand, since (3.4) holds, Proposition 3.4 asserts that

$$\begin{aligned}
&\mathbb{D}\Big(\phi(\eta_j)v_n\eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n + \eta_j\zeta_n)\Big) = \\
&\mathbb{D}\Big(\phi(\eta_j)v_n\eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n + \eta_j\zeta_n) - \phi(\eta_j)v\eta_j^{-1}\Big) + \mathbb{D}\Big(\phi(\eta_j)v\eta_j^{-1}\Big) + o(1).
\end{aligned}$$

From Lemma 2.3, we have that $\mathbb{D}(\eta_j v) = \mathbb{D}(v)$, i.e. $\mathbb{D}\big(\phi(\eta_j)v\eta_j^{-1}\big) = \mathbb{D}(v)$. Moreover, from Lemma 3.2 with $z = -\eta_j\zeta_n$, we obtain that

$$\mathbb{D}\big(\phi(\eta_j)v\eta_j^{-1}(\cdot - \eta_j\zeta_n)\big) = \mathbb{D}(v).$$

Therefore, taking into account (3.5), it follows that

$$\mathbb{D}\Big(u_n - \sum_{i=j+1}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big) = \mathbb{D}\Big(u_n - \sum_{i=j}^{t} \phi(\eta_i)v\eta_i^{-1}(\cdot - \eta_i\zeta_n)\Big) + \mathbb{D}(v) + o(1).$$

Iterating this equality, starting from $j = t$, we get

$$\mathbb{D}(u_n) - \mathbb{D}\Big(u_n - \sum_{i=1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n)\Big) = t\,\mathbb{D}(v) + o(1). \tag{3.7}$$

Similarly, by Proposition 3.4 we also have that

$$\mathbb{D}'\Bigg(\phi(\eta_j) v_n \eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n + \eta_j \zeta_n)\Bigg) =$$
$$\mathbb{D}'\Bigg(\phi(\eta_j) v_n \eta_j^{-1} - \sum_{i=j+1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n + \eta_j \zeta_n) - \phi(\eta_j) v \eta_j^{-1}\Bigg) + \mathbb{D}'\left(\phi(\eta_j) v \eta_j^{-1}\right) + o(1).$$

Making the change of variable $\tilde{x} = x - \eta_j \zeta_n$ and taking into account (3.5), we obtain

$$\mathbb{D}'\Bigg(u_n - \sum_{i=j+1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n)\Bigg) =$$
$$\mathbb{D}'\Bigg(u_n - \sum_{i=j}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n)\Bigg) + \mathbb{D}'\left(\phi(\eta_j) v \eta_j^{-1}(\cdot - \eta_j \zeta_n)\right) + o(1).$$

Iterating this equality, starting from $j = t$, we get

$$\mathbb{D}'(u_n) - \mathbb{D}'\Bigg(u_n - \sum_{i=1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n)\Bigg) - \sum_{i=1}^{t} \mathbb{D}'\left(\phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n)\right) = o(1) \tag{3.8}$$

in $H^{-1}(\mathbb{R}^N)$. Setting

$$\tilde{w}_n(x) := u_n(x) - \sum_{i=1}^{|\Gamma/K|} \phi(\eta_i) v\big(\eta_i^{-1}(x - \eta_i \zeta_n)\big),$$

we can rewrite expressions (3.6), (3.7) and (3.8) as:

$$\|u_n\|^2 - \|\tilde{w}_n\|^2 \to |\Gamma/K| \|v\|^2 \tag{3.9}$$
$$\mathbb{D}(u_n) - \mathbb{D}(\tilde{w}_n) \to |\Gamma/K|\,\mathbb{D}(v) \tag{3.10}$$
$$\mathbb{D}'(u_n) - \mathbb{D}'(\tilde{w}_n) - \sum_{i=1}^{t} \mathbb{D}'\left(\phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i \zeta_n)\right) \to 0 \quad \text{in } H^{-1}(\Omega). \tag{3.11}$$

From (3.9) and (3.10) it follows that

$$J_\infty(\tilde{w}_n) \to c - |\Gamma/K| J_\infty(v).$$

Let $\rho \in H_0^1(\Omega)$. From the bilinearity of the scalar product in $H^1(\mathbb{R}^N)$ and (3.11) we obtain

$$\begin{aligned} J'_\infty(\tilde{w}_n)\rho &= \langle \tilde{w}_n, \rho\rangle - \mathbb{D}'(\tilde{w}_n)\rho \\ &= \Big\langle u_n - \sum_{i=1}^{t} \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n), \rho \Big\rangle - \mathbb{D}'(\tilde{w}_n)\rho \\ &= \langle u_n, \rho\rangle - \sum_{i=1}^{t} \langle \phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n), \rho\rangle \\ &\quad - \mathbb{D}'(u_n)\rho + \sum_{i=1}^{t} \mathbb{D}'\left(\phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n)\right)\rho + o(1) \\ &= J'_0(u_n)\rho - \sum_{i=1}^{t} J'_\infty\big(\phi(\eta_i) v \eta_i^{-1}(\cdot - \eta_i\zeta_n)\big)\rho. \end{aligned}$$

Since $J'_\infty(v) = 0$, clearly $J'_\infty\big(\phi(\eta) v \eta^{-1}(\cdot - \eta\zeta_n)\big) \to 0$ in $H^{-1}(\mathbb{R}^N)$ for all $\eta \in \Gamma$. Moreover $J'_0(u_n) \to 0$ in $H^{-1}(\Omega)$ and hence

$$J'_\infty(\tilde{w}_n) \to 0 \text{ en } H^{-1}(\Omega).$$

Finally, we choose $R > 0$ such that $\big(\mathbb{R}^N \smallsetminus \Omega\big) \subset B_R(0)$, and a radially symmetric cut off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $0 \le \chi(x) \le 1$, $\chi(x) = 0$ if $|x| \le R$ and $\chi(x) = 1$ if $|x| \ge 2R$.

We define

$$w_n(x) := u_n(x) - \sum_{i=1}^{|\Gamma/K|} \phi(\eta_i)\chi(x) v\big(\eta_i^{-1}(x - \eta_i\zeta_n)\big).$$

Then $w_n \in H_0^1(\Omega)^\phi$. To see that (w_n) satisfies (d) and (e) it suffices to observe that

$$w_n - \tilde{w}_n \to 0 \quad \text{in } H^1(\mathbb{R}^N) \quad \text{as } |\zeta_n| \to \infty. \qquad \square$$

We shall say that J_V *satisfies the ϕ-equivariant Palais-Smale condition $(PS)_c^\phi$ at the level* c if every sequence (v_n) such that

$$v_n \in H_0^1(\Omega)^\phi, \qquad J_V(v_n) \to c, \qquad J'_V(v_n) \to 0 \text{ in } H^{-1}(\Omega), \tag{3.12}$$

has a convergent subsequence in $H_0^1(\Omega)$. If $\phi \equiv 1$, we write $(PS)_c^\Gamma$ instead of $(PS)_c^\phi$. The proposition above gives us a level below which the functional J_V satisfies the Palais-Smale condition.

Corollary 3.7. *J_V satisfies condition $(PS)_c^\phi$ for all $c < \ell(\Gamma)\, c_\infty$.*

Proof. Let (v_n) be a sequence which satisfies (3.12). From Lemma 3.3, we have that (v_n) is bounded in $H_0^1(\Omega)^\phi$ and then a subsequence satisfies that $v_n \rightharpoonup v_0$ weakly in $H_0^1(\Omega)^\phi$, $v_n \to v_0$ strongly in $L^2_{loc}(\Omega)$ and $v_n(x) \to v_0(x)$ a.e. in Ω. Defining $u_n := v_n - v_0$ we have that $u_n \rightharpoonup 0$ in $H_0^1(\Omega)^\phi$. Furthermore, Lemma 3.5 asserts that

$$J_0(u_n) \to d := c - J_V(v_0), \qquad J'_0(u_n) \to 0 \quad \text{in } H^{-1}(\Omega)$$

and v_0 is a solution of (3.1).

If $d \le 0$, Lemma 3.3 guarantees that $u_n \to 0$ strongly in $H_0^1(\Omega)$. If $d > 0$, there exist $\zeta_n \in \Omega$, a closed subgroup K of finite index in Γ, a nontrivial solution v of (3.2) and a sequence (w_n) in $H_0^1(\Omega)^\phi$ with properties (a)-(e) of Proposition 3.6. In particular,

$$J_0(u_n) = J_0(w_n) + |\Gamma/K|\, J_\infty(v) + o(1).$$

Consequently,

$$c \ge d \ge |\Gamma/K| J_\infty(v) \ge \ell(\Gamma) c_\infty.$$

From this contradiction to our hypothesis, we conclude that $u_n \to 0$ strongly in $H_0^1(\Omega)$. ☐

We denote by ∇J_V the gradient of J_V with respect to the scalar product (2.2), and by $\nabla_{\mathcal{N}} J_V(u)$ the orthogonal projection of $\nabla J_V(u)$ onto the tangent space $T_u\mathcal{N}^\phi_{\Omega,V}$ to the Nehari manifold $\mathcal{N}^\phi_{\Omega,V}$ at the point $u \in \mathcal{N}^\phi_{\Omega,V}$. We shall say that J_V *satisfies condition* $(PS)^\phi_c$ *on* $\mathcal{N}^\phi_{\Omega,V}$ if every sequence (u_n) such that

$$u_n \in \mathcal{N}^\phi_{\Omega,V}, \qquad J_V(u_n) \to c, \qquad \nabla_{\mathcal{N}} J_V(u_n) \to 0, \tag{3.13}$$

contains a convergent subsequence in $H_0^1(\Omega)$.

Corollary 3.8. *J_V satisfies condition $(PS)^\phi_c$ on $\mathcal{N}^\phi_{\Omega,V}$ for all $c < \ell(\Gamma)\, c_\infty$.*

Proof. Let (u_n) be a sequence which satisfies (3.13). In view of Corollary 3.7, we just need to prove that $\nabla J_V(u_n) \to 0$.

If $u \in \mathcal{N}^\phi_{\Omega,V}$, Theorem 2.4 (a) asserts that $\nabla J_V(u) \in H_0^1(\Omega)^\phi$. Moreover, the tangent space $T_u\mathcal{N}^\phi_{\Omega,V}$ is the subspace of $H_0^1(\Omega)^\phi$ which is orthogonal to $\nabla F(u)$, where $F(u) := \|u\|_V^2 - \mathbb{D}(u)$. Since

$$\langle \nabla F(u), v\rangle_V = 2\,\langle u, v\rangle_V - 2p \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}uv,$$

clearly (as in the proof of Theorem 2.4 (a)), $\nabla F(u) \in H_0^1(\Omega)^\phi$. We express $\nabla J_V(u_n)$ as

$$\nabla J_V(u_n) = \nabla_{\mathcal{N}} J_V(u_n) + t_n \nabla F(u_n), \qquad t_n \in \mathbb{R}. \tag{3.14}$$

By taking the scalar product of the above equality with u_n and taking into account Proposition 2.5 (a) one gets

$$\begin{aligned}
\langle \nabla_{\mathcal{N}} J_V(u_n), u_n\rangle_V &= \langle \nabla J_V(u_n), u_n\rangle_V - t_n\, \langle \nabla F(u_n), u_n\rangle_V \\
&= \big(\|u_n\|_V^2 - \mathbb{D}(u_n)\big) - t_n\big(2\|u_n\|_V^2 - 2p\mathbb{D}(u_n)\big) \\
&= 2(p-1)t_n\|u_n\|_V^2 \\
&\ge C_1 t_n,
\end{aligned} \tag{3.15}$$

with $C_1 > 0$. Observe that, by (2.9),

$$\frac{p-1}{2p}\|u_n\|_V^2 = J_V(u_n) \to c,$$

then (u_n) is bounded in $H_0^1(\Omega)$ and, since by the hypotheses $\nabla_{\mathcal{N}} J_V(u_n) \to 0$, one has that $\langle \nabla_{\mathcal{N}} J_V(u_n), u_n \rangle_V \to 0$. Thus, (3.15) yields that $t_n \to 0$. Now, set $r := \frac{2N}{2N-\alpha}$. By the Hardy– Littlewood–Sobolev inequality (2.4) and the Sobolev embedding, one has that there exists a constant $C_2 > 0$ such that

$$|\langle \nabla F(u_n), v \rangle_V| \leq 2\|u_n\|_V \|v\|_V + 2p\, C(N, \alpha)\, |u_n|_{pr}^{2p-1} \, |v|_{pr} \leq C_2 \|v\|_V$$

for all $v \in H_0^1(\Omega)$. In particular, if for each $n \in \mathbb{N}$ we take $v = \nabla F(u_n)$ in the above inequality, we obtain $\|\nabla F(u_n)\|_V \leq C_2$, i.e. $(\nabla F(u_n))$ is bounded. Thus, from identity (3.14), $\nabla J_V(u_n) \to 0$ follows. □

Remark 3.9. To prove that the functional $\hat{J}_V$ associated to problem (1.1) satisfies condition $(PS)_c^\phi$ on $\hat{\mathcal{N}}_{\Omega,V}^\phi$ for all $c < \ell(\Gamma)\, \hat{c}_\infty$, we follow an entirely analogous procedure. The only difference is that we need to use the Brezis-Lieb Lemma [13] instead of Proposition 3.4. See [26, Section 3] for further details.

3.2 Asymptotic estimates

In this section we study some asymptotic estimates for the nonlocal problem, which will be mainly used in the proof of Theorems 1.5 and 1.6 stated in the Introduction.

The ground states of problem (3.2) have been recently studied in [21, 48]. The following result holds true.

Theorem 3.10. *Let ω be a ground state of problem (3.2). Then $\omega \in L^1(\mathbb{R}^N) \cap \mathcal{C}^\infty(\mathbb{R}^N)$, ω does not change sign and it is radially symmetric and monotone decreasing in the radial direction with respect to some fixed point. Moreover, ω has the following asymptotic behavior:*

(i) If $p > 2$, then

$$\lim_{|x|\to\infty} |\omega(x)||x|^{\frac{N-1}{2}} e^{|x|} \in (0, \infty).$$

(ii) If $p = 2$, then

$$\lim_{|x|\to\infty} |\omega(x)||x|^{\frac{N-1}{2}} e^{Q(|x|)} \in (0, \infty),$$

where

$$Q(t) := \int_\delta^t \sqrt{1 - \frac{\delta^\alpha}{s^\alpha}} ds \quad \textit{and} \quad \delta^\alpha := (4-\alpha)c_\infty.$$

Proof. See Theorems 3 and 4 in [48]. Note that ω is a solution of (3.2) if and only if $u := \lambda^{-\frac{1}{2(p-1)}} \omega$ is a solution of problem (1.1) in [48], where $\lambda := \frac{\Gamma(\alpha/2)}{\Gamma((N-\alpha)/2)\pi^{N/2} 2^{N-\alpha}}$ and Γ denotes here (and only here) the gamma function (and not the group). □

In what follows, ω will denote a positive ground state of problem (3.2) which is radially symmetric with respect to the origin. We continue to assume that $p \geq 2$.

Lemma 3.11.

$$\lim_{|x|\to\infty} \omega(x)|x|^{\frac{N-1}{2}}e^{a|x|} = \begin{cases} \infty & \text{if } a > 1, \\ 0 & \text{if } a \in (0,1). \end{cases}$$

Proof. Set $b := \frac{N-1}{2}$. We shall prove this result for $p = 2$. The proof for $p > 2$ is an immediate consequence of Theorem 3.10. Observe that, for every $\nu \in (0,1)$ it holds true that

$$\sqrt{1 - \frac{\delta^\alpha}{s^\alpha}} \leq 1 \quad \text{if } s \geq \delta \qquad \text{and} \qquad \sqrt{1 - \frac{\delta^\alpha}{s^\alpha}} \geq \nu \quad \text{if } s \geq \frac{\delta}{(1-\nu^2)^{1/\alpha}} =: s_\nu,$$

and, hence, that

$$Q(t) \leq t \quad \text{if } t \geq \delta \qquad \text{and} \qquad \nu(t - s_\nu) \leq Q(t) \quad \text{if } t \geq s_\nu.$$

Consequently, if $|x| \geq \delta$ then

$$\omega(x)|x|^b e^{a|x|} = \omega(x)|x|^b e^{Q(|x|)} e^{a|x| - Q(|x|)} \geq \omega(x)|x|^b e^{Q(|x|)} e^{(a-1)|x|}.$$

If $a > 1$, the conclusion follows from Theorem 3.10. If $a \in (0,1)$, we fix $\nu \in (a,1)$. Then, for all $|x| \geq s_\nu$,

$$\omega(x)|x|^b e^{a|x|} = \omega(x)|x|^b e^{Q(|x|)} e^{a|x| - Q(|x|)} \leq \omega(x)|x|^b e^{Q(|x|)} e^{(a-\nu)|x| + \nu s_\nu},$$

and using once more Theorem 3.10 the conclusion follows. □

For $\zeta \in \mathbb{R}^N$ we set

$$\omega_\zeta(x) := \omega(x - \zeta).$$

Lemma 3.12. *For each* $a \in (0,1)$,

$$\lim_{|\zeta|\to\infty} \int_{\mathbb{R}^N} \omega^{p-1}\omega_\zeta |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} = 0.$$

Proof. By Lemma 3.11 we have that, for each $\nu \in (0,1)$, there exists a constant $C_\nu > 0$ such that

$$\omega(x) \leq C_\nu e^{-\nu|x|} \qquad \text{for all } x \in \mathbb{R}^N.$$

We fix $\nu_1, \nu_2 \in (a,1)$ with $\nu_1 < \nu_2$. In what follows, C will denote different positive constants depending only on ν_1 and ν_2. We have that

$$\begin{aligned} \int_{\mathbb{R}^N} \omega^{p-1}\omega_\zeta &\leq C \int_{\mathbb{R}^N} e^{-\nu_1(p-1)|x|} e^{-\nu_2|x-\zeta|}\, dx \leq C \int_{\mathbb{R}^N} e^{-\nu_1|x|} e^{-\nu_2|x-\zeta|}\, dx \\ &= C \int_{\mathbb{R}^N} e^{-\nu_1(|x|+|x-\zeta|)} e^{-(\nu_2-\nu_1)|x-\zeta|}\, dx \leq C e^{-\nu_1|\zeta|} \int_{\mathbb{R}^N} e^{-(\nu_2-\nu_1)|x|}\, dx \\ &= C e^{-\nu_1|\zeta|}. \end{aligned}$$

Therefore,

$$0 \leq \int_{\mathbb{R}^N} \omega^{p-1}\omega_\zeta |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} \leq C|\zeta|^{\frac{N-1}{2}} e^{-(\nu_1 - a)|\zeta|},$$

which implies the result. □

For $\zeta \in \mathbb{R}^N$ we define

$$I(\zeta) := \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \omega^p \right) \omega^{p-1} \omega_\zeta. \tag{3.16}$$

Lemma 3.13. *For each $a \in (0,1)$,*

$$\lim_{|\zeta|\to\infty} I(\zeta)\, |\zeta|^{\frac{N-1}{2}}\, e^{a|\zeta|} = 0.$$

Proof. From Lemma 2.11 (b) we infer that $\frac{1}{|x|^\alpha} * \omega^p \in L^\infty(\mathbb{R}^N)$. Hence,

$$0 \le I(\zeta)|\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} \le C \int_{\mathbb{R}^N} \omega^{p-1} \omega_\zeta |\zeta|^{\frac{N-1}{2}} e^{a|\zeta|}.$$

The conclusion follows from Lemma 3.12. □

Lemma 3.14. *For every $a > 1$, there exists a positive constant k_a such that*

$$I(\zeta)|\zeta|^{\frac{N-1}{2}} e^{a|\zeta|} \ge k_a \qquad \textit{for all } \ |\zeta| \ge 1.$$

Proof. Set $b := \frac{N-1}{2}$. Lemma 3.11 asserts the existence of positive constants C_a, R_a such that

$$C_a |x|^{-b} e^{-a|x|} \le \omega(x) \quad \text{if } \ |x| \ge R_a.$$

Let $C_1 > 0$ be such that

$$\omega(x) \ge C_1 e^{-a|x|} \qquad \text{for all } \ |x| \le R_a.$$

Setting $C_2 := \min\{C_a, C_1\}$ we conclude that

$$\omega(x) \ge C_2 (1 + |x|)^{-b} e^{-a|x|} \quad \text{for all } \ x \in \mathbb{R}^N.$$

Hence,

$$\begin{aligned} \omega(x-\zeta)|\zeta|^b e^{a|\zeta|} &\ge C_2(1+|x-\zeta|)^{-b} e^{-a|x-\zeta|} |\zeta|^b e^{a|\zeta|} \\ &\ge C_2 (1+|x-\zeta|)^{-b} |\zeta|^b e^{-a|x|} \qquad \text{for } x, \zeta \in \mathbb{R}^N. \end{aligned}$$

Note that, if $|x| \le 1 \le |\zeta|$, then $1 + |x-\zeta| \le 1 + |x| + |\zeta| \le 3\,|\zeta|$ and so

$$\omega(x-\zeta)|\zeta|^b e^{a|\zeta|} \ge C_3 e^{-a|x|} \quad \text{for } x, \zeta \in \mathbb{R}^N \text{ with } \ |x| \le 1 \le |\zeta|\,,$$

where $C_3 := 3^{-b} C_2$. Consequently,

$$\begin{aligned} I(\zeta)|\zeta|^b e^{a|\zeta|} &= \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \omega^p \right)(x)\, \omega^{p-1}(x) \omega(x-\zeta) |\zeta|^b e^{a|\zeta|}\, dx \\ &\ge C_3 \int_{|x|\le 1} \left(\frac{1}{|x|^\alpha} * \omega^p \right)(x)\, \omega^{p-1}(x) e^{-a|x|} =: k_a \qquad \text{for } \ |\zeta| \ge 1, \end{aligned}$$

as claimed. □

For $\zeta \in \mathbb{R}^N$ we define

$$A(\zeta) := \int_{\mathbb{R}^N} V^+(x)\omega^2(x-\zeta)dx. \tag{3.17}$$

Lemma 3.15. *Let $M \in (0,2)$. If $V(x) \le ce^{-\iota|x|}$ for all $x \in \mathbb{R}^N$ with $c > 0$ and $\iota > M$, then*

$$\lim_{|\zeta|\to\infty} A(\zeta)\,|\zeta|^{\frac{N-1}{2}}\,e^{M|\zeta|} = 0.$$

Proof. Throughout this proof c will denote possibly distinct positive constants that are independent of ζ. Let us fix $\varepsilon \in (0,1)$ such that $\iota(1-\varepsilon) > M$. Then

$$\begin{aligned}\int_{B_{\varepsilon|\zeta|}(\zeta)} V^+(x)\omega^2(x-\zeta)\,|\zeta|^{\frac{N-1}{2}}\,e^{M|\zeta|}dx &\le c\,|\zeta|^{\frac{N-1}{2}}\,e^{-(\iota(1-\varepsilon)-M)|\zeta|}\int_{\mathbb{R}^N}\omega^2(x)dx \\ &= c\,|\zeta|^{\frac{N-1}{2}}\,e^{-(\iota(1-\varepsilon)-M)|\zeta|}.\end{aligned} \tag{3.18}$$

On the other hand, let us fix $a \in (0,1)$ such that $2a > M$. According to Lemma 3.11, for $x \in \mathbb{R}^N \smallsetminus B_{\varepsilon|\zeta|}(\zeta)$ and $|\zeta|$ large enough,

$$\omega^2(x-\zeta) \le c|x-\zeta|^{-(N-1)}\,e^{-2a|x-\zeta|}.$$

Therefore, making the change of variable $y = \frac{x}{|\zeta|}$ and defining $z := \frac{\zeta}{|\zeta|}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon|\zeta|}(\zeta)} V^+(x)\omega^2(x-\zeta)\,|\zeta|^{\frac{N-1}{2}}\,e^{M|\zeta|}dx \\ &\qquad\le c\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon|\zeta|}(\zeta)} \frac{|\zeta|^{\frac{N-1}{2}}\,e^{-(\iota|x|+2a|x-\zeta|-M|\zeta|)}}{|x-\zeta|^{N-1}}dx \\ &\qquad= c\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon}(z)} \frac{|\zeta|^{\frac{N+1}{2}}\,e^{-|\zeta|(\iota|y|+2a|y-z|-M)}}{|y-z|^{N-1}}dy.\end{aligned} \tag{3.19}$$

Set $\iota_0 := \min\{\iota, 2a\}$ and fix $\delta \in (0,1)$ such that $\iota_0\delta > M$. Then

$$\iota\,|y| + 2a\,|y-z| - M \ge \iota_0(|y|+|y-z|-\delta) + (\iota_0\delta - M) \ge \iota_0\delta - M > 0.$$

Taking into account that $\max_{t\in\mathbb{R}} t^{\frac{N+1}{2}}e^{-dt} = \left(\frac{N+1}{2e}\right)^{\frac{N+1}{2}} d^{-\frac{N+1}{2}}$ for $d > 0$, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon}(z)} \frac{|\zeta|^{\frac{N+1}{2}}\,e^{-|\zeta|(\iota|y|+2a|y-z|-M)}}{|y-z|^{N-1}}dy \\ &\quad\le e^{-(\iota_0\delta-M)|\zeta|}\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon}(z)} \frac{|\zeta|^{\frac{N+1}{2}}\,e^{-\iota_0(|y|+|y-z|-\delta)|\zeta|}}{|y-z|^{N-1}}dy \\ &\quad\le e^{-(\iota_0\delta-M)|\zeta|}\int_{\mathbb{R}^N\smallsetminus B_{\varepsilon}(z)} \frac{dy}{\big(\iota_0(|y|+|y-z|-\delta)\big)^{\frac{N+1}{2}}|y-z|^{N-1}} \\ &\quad= ce^{-(\iota_0\delta-M)|\zeta|}.\end{aligned} \tag{3.20}$$

Now the assertion of Lemma 3.15 follows from inequalities (3.18), (3.19) and (3.20). □

Lemma 3.16. *If $f \in \mathcal{C}_c^0\left(\mathbb{R}^N\right)$, $q > 1$ and $a \in (0,1)$, then*

$$\lim_{|\zeta|\to\infty}\left(\int_{\mathbb{R}^N} f(x)\omega^q(x-\zeta)dx\right)|\zeta|^{\frac{N-1}{2}}e^{qa|\zeta|} = 0.$$

Proof. Set $b := \frac{N-1}{2}$. Let $T > 0$ be such that $\operatorname{supp}(f) \subset B_T(0)$. By Lemma 3.11 there exists $C > 0$ such that

$$\omega(x) \le C(T + |x|)^{-b}e^{-a|x|} \qquad \text{for all } x \in \mathbb{R}^N.$$

Therefore, if $|x| \le T$,

$$\begin{aligned}\omega^q(x-\zeta)\,|\zeta|^b\,e^{qa|\zeta|} &\le C^q(T+|x-\zeta|)^{-qb}e^{-qa|x-\zeta|}\,|\zeta|^b\,e^{qa|\zeta|}\\ &\le C^q(|x|+|x-\zeta|)^{-qb}e^{-qa|x-\zeta|}\,|\zeta|^b\,e^{qa|\zeta|} \le C^q\,|\zeta|^{(1-q)b}\,e^{qa|x|}.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N}|f(x)|\,\omega^q(x-\zeta)\,|\zeta|^b\,e^{qa|\zeta|}dx \le C^q\,|\zeta|^{(1-q)b}\int_{|x|\le T}|f(x)|\,e^{qa|x|}dx =: C_1\,|\zeta|^{(1-q)b},$$

from which the assertion of Lemma 3.16 follows. □

Remark 3.17. (Asymptotic estimates for the local problem)

In order to prove Theorem 1.2 we need the corresponding asymptotic estimates for the local problem.

Let $\hat{\omega} \in H^1(\mathbb{R}^N)$ be the unique positive solution to problem (1.2) (namely, the limit problem associated to the local one) which is radially symmetric about the origin. It is well-known (see [10, 36]) that there exist positive constants b_0, b_1 such that

$$\lim_{|x|\to\infty}\left|D^i\hat{\omega}(x)\right|\,|x|^{\frac{N-1}{2}}\exp|x| = b_i \qquad \text{for } i = 0,1. \tag{3.21}$$

Observe that (3.21) implies an analogous result to that given in Lemma 3.11 because, for $a > 0$, one has

$$\lim_{|x|\to\infty}\hat{\omega}(x)|x|^{\frac{N-1}{2}}e^{a|x|} = \left(\lim_{|x|\to\infty}\hat{\omega}(x)|x|^{\frac{N-1}{2}}e^{|x|}\right)\left(\lim_{|x|\to\infty}e^{(a-1)|x|}\right).$$

Therefore, if we set

$$\hat{I}(\zeta) := \int_{\mathbb{R}^N}\hat{\omega}^{p-1}\hat{\omega}_\zeta\,,$$

we can obtain similar statements to those given in Lemmas 3.13, 3.14 and 3.16, due to the proof of these lemmas relies essentially on Lemma 3.11.

However, we did not proceed in this way in [26]. Thanks to (3.21), in the local case, it is not necessary to consider separately the estimates for $a < 1$ and $a > 1$. It suffices to consider $a = 1$. More precisely, in order to describe the asymptotic behavior of $\hat{I}$ we use the following result of Bahri and Li [3, Proposition 1.2].

Lemma 3.18. *Let $f \in \mathcal{C}^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $h \in \mathcal{C}^0(\mathbb{R}^N)$ be radially symmetric functions satisfying*

$$\lim_{|x|\to\infty} f(x)\,|x|^b\,e^{d|x|} = \tau \qquad \text{and} \qquad \int_{\mathbb{R}^N} |h(x)|\,(1+|x|^b)e^{d|x|}dx < \infty$$

for $d \geq 0$, $b \geq 0$ and $\tau \in \mathbb{R}$. Then

$$\lim_{|y|\to\infty} \left(\int_{\mathbb{R}^N} f(x+y)h(x)dx \right) |y|^b\,e^{d|y|} = \tau \int_{\mathbb{R}^N} h(x)e^{-dx_1}dx.$$

As $\hat{\omega}$ is radially symmetric, from (3.21) and Lemma 3.18 we deduce

$$\lim_{|\xi|\to\infty} \hat{I}(\xi)\,|\xi|^{\frac{N-1}{2}}\,e^{|\xi|} = k_1 > 0. \tag{3.22}$$

This asymptotic estimate plays the same role as Lemmas 3.13 and 3.14 together. Furthermore, (3.21) and Lemma 3.18 implies

$$\lim_{|\zeta|\to\infty} \left(\int_{\mathbb{R}^N} f(x)\hat{\omega}^q(x-\zeta)dx \right) |\zeta|^{\frac{N-1}{2}}\,e^{q|\zeta|} = 0, \tag{3.23}$$

provided $f \in \mathcal{C}_c^0\left(\mathbb{R}^N\right)$ is radially symmetric and $q > 1$. This is the statement corresponding to Lemma 3.16.

On the other hand, Lemma 3.15 continues being valid for the local case without any modifications in the statement; the only difference is that in the proof we need to take $a = 1$ in order to apply (3.21).

3.3 The Krasnoselskii genus and multiplicity of critical points

In this section we introduce the notion of Krasnoselskii's genus, which will be a fundamental tool in finding multiplicity of sign changing solutions to both: the local problem and the nonlocal one.

Recall that a $\mathbb{Z}/2$-space is a topological space Y together with a continuous action

$$\mathbb{Z}/2 \times Y \to Y, \quad (-1, y) \mapsto (-1)\cdot y.$$

Let $\mathcal{S}$ be a subset of a $\mathbb{Z}/2$-space Y which is $\mathbb{Z}/2$-equivariant (i.e. $(-1)\cdot y \in \mathcal{S}$ for all $y \in \mathcal{S}$) and such that $(-1)\cdot y \neq y$ for all $y \in \mathcal{S}$.

Definition 1. *If $\mathcal{S} \neq \emptyset$, the Krasnoselskii genus of $\mathcal{S}$, denoted* $\mathrm{genus}(\mathcal{S})$, *is defined to be the smallest $k \in \mathbb{N}$ such that there exists a continuous map $h : \mathcal{S} \to \mathbb{S}^{k-1}$ which is $\mathbb{Z}/2$-equivariant (i.e. $h((-1)\cdot y) = -h(y)$ for all $y \in \mathcal{S}$). If there is not a map with the above property, then* $\mathrm{genus}(\mathcal{S}) := \infty$. *We set* $\mathrm{genus}(\emptyset) := 0$.

The following lemma states an important property of the Krasnoselskii genus which will be useful for our purposes.

Lemma 3.19 (Monotonicity property of the genus)**.** *Assume that Y_i, $i = 1, 2$, are $\mathbb{Z}/2$-spaces. Let $\mathcal{S}_i$ be a subset of Y_i which is $\mathbb{Z}/2$-equivariant and such that $(-1) \cdot y_i \neq y_i$ for all $y_i \in \mathcal{S}_i$. If $\eta \in \mathcal{C}^0(\mathcal{S}_1, \mathcal{S}_2)$ is $\mathbb{Z}/2$-equivariant (i.e. $\eta((-1) \cdot y) = (-1) \cdot \eta(y)$ for all $y \in \mathcal{S}_1$), then*

$$\text{genus}(\mathcal{S}_1) \leq \text{genus}(\mathcal{S}_2).$$

Proof. See for instance [6, Proposition 2.15]. □

Recall that *the Krasnoselskii genus* provides a lower bound for the number of pairs of critical points of an even functional on a Hilbert Manifold. More precisely, one has the following result:

Theorem 3.20. *Let $\mathcal{M}$ be a submanifold of class $\mathcal{C}^2$ of a Hilbert space H which is symmetric (i.e. $u \in \mathcal{M} \Leftrightarrow -u \in \mathcal{M}$) and does not contain the origin. Let $I : H \to \mathbb{R}$ be an even functional of class $\mathcal{C}^2$ which is bounded from below on $\mathcal{M}$ and satisfies the Palais-Smale condition $(PS)_c$ on $\mathcal{M}$ for all $c < d_0$. If $d < d_0$, then I has at least*

$$\text{genus}(\mathcal{M} \cap I^d)$$

pairs of critical points $\pm u$ on $\mathcal{M}$ with critical value $I(u) \leq d$, where $I^d := \{u \in \mathcal{M} : I(u) \leq d\}$.

Proof. See for instance [6, Theorem 2.19 and Proposition 2.10]. □

The above theorem is true for more general symmetries than those we are considering here. See for instance [25].

Remark 3.21. If Γ is a closed subgroup of $O(N)$ such that $\ell(\Gamma) = \infty$, Ω and V are Γ-invariant, $\phi : \Gamma \to \mathbb{Z}_2$ is a continuous group homomorphism and additionally, $\dim H_0^1(\Omega)^\phi = \infty$ when ϕ is an epimorphism, then problem (1.1) has infinitely many solutions satisfying (1.3). Indeed, since $\ell(\Gamma) = \infty$, Remark 3.9 insures that the functional $\hat{J}_V$ associated to problem (1.1) satisfies condition $(PS)_c^\phi$ on $\hat{\mathcal{N}}_{\Omega,V}^\phi$ for all $c \in \mathbb{R}$. On the other hand, by Remark 2.7, one also has that $\hat{J}_V$ is an even $\mathcal{C}^2$-function which is bounded from below on $\hat{\mathcal{N}}_{\Omega,V}^\phi$ and that $\hat{\mathcal{N}}_{\Omega,V}^\phi$ is a $\mathcal{C}^2$-manifold which is symmetric and does not contain the origin. Therefore, by Theorem 3.20, $\hat{J}_V$ has at least $\text{genus}(\hat{\mathcal{N}}_{\Omega,V}^\phi)$ pairs of critical points.

Now, $\hat{\mathcal{N}}_{\Omega,V}^\phi$ is radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\phi$. It is known that if $\ker \phi = \Gamma$, $H_0^1(\Omega)^\phi$ has infinite dimension, but this is not true, generally, if ϕ is an epimorphism. That is why, in this case, we need additionally to assume that $\dim H_0^1(\Omega)^\phi = \infty$. Finally, if $\mathbb{S}_\Omega^\phi$ denotes the unite sphere in $H_0^1(\Omega)^\phi$, from

$$\text{genus}(\hat{\mathcal{N}}_{\Omega,V}^\phi) = \text{genus}(\mathbb{S}_\Omega^\phi) = \infty,$$

one can deduce the existence of infinitely many solutions to (1.1) satisfying (1.3).

Chapter 4

Existence of positive and sign changing solutions

We consider the problem

$$\begin{cases} -\Delta u + (V_\infty + V(x))\, u = \left(\frac{1}{|x|^\alpha} * |u|^p\right) |u|^{p-2}u, \\ u \in H_0^1(\Omega), \end{cases} \tag{4.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $p \in \left[2, \frac{2N-\alpha}{N-2}\right)$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty. The potential $V_\infty + V$ is assumed to satisfy

(V_0) $\quad V \in \mathcal{C}^0(\mathbb{R}^N)$, $\ V_\infty \in (0,\infty)$, $\ \inf_{x\in\mathbb{R}^N}\{V_\infty + V(x)\} > 0$, $\lim_{|x|\to\infty} V(x) = 0$.

As in the previous chapter, we consider a closed subgroup Γ of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and a continuous group homomorphism $\phi : \Gamma \to \mathbb{Z}/2$. We denote by $G := \ker\phi$, by

$$\ell(\Gamma) := \min\{\#\Gamma x : x \in \mathbb{R}^N \smallsetminus \{0\}\},$$

and by

$$\Sigma := \{x \in \mathbb{R}^N : |x| = 1, \#\Gamma x = \ell(\Gamma)\},$$

where $\Gamma x := \{gx : g \in \Gamma\}$ is the Γ-orbit of x and $\#\Gamma x$ is its cardinality.

Recall that a subset Z of $\mathbb{R}^N$ is Γ-invariant if $\Gamma x \subset Z$ for every $x \in Z$, and a function $u : Z \to \mathbb{R}$ is Γ-invariant if it is constant on each Γ-orbit Γx with $x \in Z$. If Z is Γ-invariant and ϕ is an epimorphism, the group $\mathbb{Z}/2$ acts on the G-orbit space $Z/G := \{Gx : x \in Z\}$ of Z as follows: we choose $\gamma \in \Gamma$ such that $\phi(\gamma) = -1$ and we define

$$(-1) \cdot Gx := G(\gamma x) \qquad \text{for all } x \in Z.$$

This action is well defined and it does not depend on the choice of γ. We denote by

$$\Sigma_0 := \{x \in \Sigma : Gx = G(\gamma x)\}.$$

Thus, if Z is a Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$, the action of $\mathbb{Z}/2$ on its G-orbit space Z/G is free.

For each subgroup K of Γ we set

$$\mu(Kz) := \begin{cases} \inf\{|\alpha_1 z - \alpha_2 z| : \alpha_1, \alpha_2 \in K,\ \alpha_1 z \neq \alpha_2 z\} & \text{if } \#Kz \geq 2, \\ 2\,|z| & \text{if } \#Kz = 1, \end{cases}$$

$$\mu_K(Z) := \inf_{z \in Z} \mu(Kz) \qquad \text{and} \qquad \mu^K(Z) := \sup_{z \in Z} \mu(Kz).$$

In the special case where $K = G$ and $Z = \Sigma$, we simply write

$$\mu_G := \mu_G(\Sigma) \qquad \text{and} \qquad \mu^G := \mu^G(\Sigma).$$

The energy functional $J_V : H_0^1(\Omega) \to \mathbb{R}$ associated to problem (4.1) is given by

$$J_V(u) = \frac{1}{2}\,\|u\|_V^2 - \frac{1}{2p}\mathbb{D}(u),$$

where $\|\cdot\|_V$ is the norm defined in (2.3) and

$$\mathbb{D}(u) = \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p.$$

We are interested in obtaining solutions to problem (4.1) which satisfy

$$u(gx) = \phi(g)u(x) \qquad \text{for all } g \in \Gamma \text{ and } x \in \Omega. \tag{4.2}$$

By the principle of symmetric criticality (Theorem 2.4), the solutions to problem (4.1) that satisfy (4.2) are the critical points of the restriction of J_V to the space

$$H_0^1(\Omega)^\phi = \{u \in H_0^1(\Omega) : u(\gamma x) = \phi(\gamma)u(x)\ \ \forall \gamma \in \Gamma,\ \forall x \in \Omega\}.$$

The nontrivial ones lie on the Nehari manifold

$$\mathcal{N}^\phi_{\Omega,V} := \left\{u \in H_0^1(\Omega)^\phi : u \neq 0,\ \|u\|_V^2 = \mathbb{D}(u)\right\},$$

which is of class $\mathcal{C}^2$ and radially diffeomorphic to the unit sphere in $H_0^1(\Omega)^\phi$. The radial projection $\pi : H_0^1(\Omega)^\phi \smallsetminus \{0\} \to \mathcal{N}^\phi_{\Omega,V}$ is given by

$$\pi(u) := \left(\frac{\|u\|_V^2}{\mathbb{D}(u)}\right)^{\frac{1}{2(p-1)}} u.$$

Accordingly, for every $u \in H_0^1(\Omega)^\phi \smallsetminus \{0\}$,

$$J_V(\pi(u)) = \frac{p-1}{2p}\left(\frac{\|u\|_V^2}{\mathbb{D}(u)^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}.$$

We set

$$c^\phi_{\Omega,V} := \inf_{\mathcal{N}^\phi_{\Omega,V}} J_V.$$

If $\phi \equiv 1$ is the trivial homomorphism, then $\Gamma = G := \ker \phi$. In this case we shall write $H_0^1(\Omega)^G$, $\mathcal{N}_{\Omega,V}^G$ and $c_{\Omega,V}^G$ instead of $H_0^1(\Omega)^\phi$, $\mathcal{N}_{\Omega,V}^\phi$ and $c_{\Omega,V}^\phi$. If $G = \{1\}$ is the trivial group, we shall omit it from the notation and write simply $H_0^1(\Omega)$, $\mathcal{N}_{\Omega,V}$ and $c_{\Omega,V}$. For the special problem

$$\begin{cases} -\Delta u + u = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{4.3}$$

we write J_∞, $\mathcal{N}_\infty$ and c_∞ instead of J_0, $\mathcal{N}_{\mathbb{R}^N,0}$ and $c_{\mathbb{R}^N,0}$.

We shall look for solutions with small energy, i.e. which satisfy

$$\frac{p-1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\alpha}dx\,dy < \ell(\Gamma)c_\infty. \tag{4.4}$$

4.1 Proof of Theorems 1.3 and 1.4

In this section we are concerned with potentials which are strictly negative at infinity. More precisely, we are concerned with potentials V which, for some subset Z of Σ, satisfy

(V_*) There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu_\Gamma(Z)\sqrt{V_\infty})$ such that

$$V(x) \leq -c_0 e^{-\lambda|x|} \qquad \text{for all } x \in \mathbb{R}^N \text{ with } |x| \geq r_0.$$

In particular, no behavior is prescribed to V near the origin and so, it can take on negative and nonnegative values there. The aim of this section is to prove the following results (which correspond to Theorems 1.3 and 1.4 stated in the Introduction, respectively).

Theorem 4.1. *If $p \geq 2$, Ω is G-invariant and V is a G -invariant function which satisfies*

(V_3) *There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu^G\sqrt{V_\infty})$ such that*

$$V(x) \leq -c_0 e^{-\lambda|x|} \qquad \textit{for all } x \in \mathbb{R}^N \textit{ with } |x| \geq r_0,$$

then (4.1) has at least one positive solution u which is G -invariant and satisfies (4.4) with $\Gamma = G$.

Theorem 4.2. *If $p \geq 2$, Ω is Γ-invariant, $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, Z is a Γ-invariant subset of $\Sigma \smallsetminus \Sigma_0$ and V is a Γ-invariant function which satisfies*

(V_1) *There exist $r_0 > 0$, $c_0 > 0$ and $\lambda \in (0, \mu_\Gamma(Z)\sqrt{V_\infty})$ such that*

$$V(x) \leq -c_0 e^{-\lambda|x|} \qquad \textit{for all } x \in \mathbb{R}^N \textit{ with } |x| \geq r_0,$$

then problem (4.1) has at least genus(Z/G) *pairs of sign changing solutions $\pm u$, which satisfy (4.2) and (4.4).*

Let Z be a Γ-invariant subset of Σ and let $\lambda \in (0, \mu_\Gamma(Z))$ be such that (V_1) holds (recall that we are assuming that $V_\infty = 1$). We choose $\nu \in (0,1)$ such that $\lambda \in (0, \mu_\Gamma(Z)\nu)$, $\varepsilon \in \left(0, \frac{\mu_\Gamma(Z)\nu-\lambda}{\mu_\Gamma(Z)\nu+\lambda}\right)$ and a radially symmetric cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $0 \leq \chi \leq 1$, $\chi(x) = 1$ if $|x| \leq 1-\varepsilon$ and $\chi(x) = 0$ if $|x| \geq 1$. Let $\omega \in H^1(\mathbb{R}^N)$ be a positive ground state of problem (4.3) which is radially symmetric about the origin. For $S > 0$ we define $\omega^S \in H^1(\mathbb{R}^N)$ by

$$\omega^S(x) := \chi\left(\frac{x}{S}\right)\omega(x).$$

Lemma 3.11 allows to obtain the following asymptotic estimates. The proof we include here was given by S. Cingolani, M. Clapp and S. Secchi [21, Lemma 4.1].

Lemma 4.3. *As $S \to \infty$,*

$$\left|\|\omega\|^2 - \left\|\omega^S\right\|^2\right| = O\big(e^{-2\nu(1-\varepsilon)S}\big), \qquad \left|\mathbb{D}(\omega) - \mathbb{D}(\omega^S)\right| = O\big(e^{-p\nu(1-\varepsilon)S}\big).$$

Proof. Throughout this proof C will denote some positive constants, not necessarily the same one. From (2.13) and (2.14) one has that

$$\|\omega - \omega^S\|^2 \leq C\left(\int_{|x|\geq(1-\varepsilon)S} |\nabla\omega|^2 + \int_{|x|\geq(1-\varepsilon)S} |\omega|^2\right).$$

Therefore, from Lemma 3.11 and [21, (4.1)] it follows that

$$\begin{aligned}
|\|\omega\|^2 - \|\omega^S\|^2| &\leq C\,\|\omega - \omega^S\|^2 \\
&\leq C\int_{|x|\geq(1-\varepsilon)S} |x|^{-(N-1)}e^{-2\nu|x|}dx \\
&\leq C\int_{(1-\varepsilon)S}^{\infty} e^{-2\nu t}dt \\
&= Ce^{-2\nu(1-\varepsilon)S}.
\end{aligned}$$

On the other hand, if $r := \frac{2N}{2N-\alpha}$, from (2.15), one has that

$$\begin{aligned}
\left|\mathbb{D}(\omega) - \mathbb{D}\left(\omega^S\right)\right| &\leq 2\bar{C}|\omega|_{pr}^p \left|\omega^p - (\omega^S)^p\right|_r \\
&\leq C\left(\int_{|x|\geq(1-\varepsilon)S} \omega^{pr}\right)^{\frac{1}{r}}.
\end{aligned}$$

Therefore, from Lemma 3.11 it follows that

$$\begin{aligned}
\left|\mathbb{D}(\omega) - \mathbb{D}\left(\omega^S\right)\right| &\leq C\left(\int_{|x|\geq(1-\varepsilon)S} |x|^{-\frac{pr}{2}(N-1)}e^{-pr\nu|x|}dx\right)^{\frac{1}{r}} \\
&\leq C\left(\int_{(1-\varepsilon)S}^{\infty} e^{-pr\nu t}dt\right)^{\frac{1}{r}} \\
&= Ce^{-p\nu(1-\varepsilon)S},
\end{aligned}$$

as $S \to \infty$. □

We set $\rho := \frac{\mu_\Gamma(Z)\nu+\lambda}{4\nu}$, and for every $z \in Z$ we consider the function $(\omega^{\rho R})_{Rz} \in H^1(\mathbb{R}^N)$ given by

$$(\omega^{\rho R})_{Rz}(x) := \omega^{\rho R}(x - Rz).$$

Note that $\operatorname{supp}((\omega^{\rho R})_{Rz}) \subset \overline{B_{\rho R}(Rz)}$.

The following lemma is a special case of [21, Lemma 4.2] with $A = 0$.

Lemma 4.4. *There exist $d_0 > 0$ and $R_0 > 0$ such that $(\omega^{\rho R})_{Rz} \in H_0^1(\Omega)$ and*

$$J_V(\pi((\omega^{\rho R})_{Rz})) \leq c_\infty - d_0 e^{-\lambda R} \qquad \textit{for all } z \in Z \textit{ and } R \geq R_0.$$

Proof. Assume without loss of generality that the $r_0 > 0$ of condition (V_1) also satisfies $\left(\mathbb{R}^N \smallsetminus B_{r_0}(0)\right) \subset \Omega$. Note that, since $\mu_\Gamma(Z) \leq 2$, $\rho \in (0,1)$. Therefore,

$$R - \rho R \to \infty \quad \text{as} \quad R \to \infty$$

and so, there exists $R_0 > 0$ such that $R - \rho R \geq r_0$ provided $R \geq R_0$, which implies that $(\omega^{\rho R})_{Ry} \in H_0^1(\Omega)$ for all $y \in Z$ and $R \geq R_0$.

From Proposition 2.5 (d) and (2.10) one has that

$$\max_{t \geq 0} J_V(tu) = J_V(t_u u) \quad \text{if and only if} \quad t_u = \left(\frac{\|u\|_V^2}{\mathbb{D}(u)}\right)^{1/(2p-2)}.$$

By Lemma 2.10, $\omega^{\rho R} \to \omega$ in $H^1(\mathbb{R}^N)$ and $\mathbb{D}(\omega^{\rho R}) \to \mathbb{D}(\omega)$ as $R \to \infty$, and so $t_{\omega^{\rho R}} \to t_\omega$ as $R \to \infty$. Therefore, one may choose $t_1 > t_\omega > t_0 > 0$ and $R_0 > 0$ such that the following also holds:

$$\max_{t \geq 0} J_V(t(\omega^{\rho R})_{Ry}) = \max_{t \in [t_0, t_1]} J_V(t(\omega^{\rho R})_{Ry}) \qquad \text{for all } y \in Z \ \text{ and } \ R \geq R_0.$$

By condition (V_1), for every $t \in [t_0, t_1]$, $y \in Z$ and $R \geq R_0$, one has that

$$\begin{aligned}
\int_\Omega V(x) \left|t\omega^{\rho R}(x - Ry)\right|^2 dx &= \int_{|x| \leq \rho R} V(x + Ry) \left|t\omega^{\rho R}(x)\right|^2 dx \\
&\leq -\left(c_0 t^2 \int_{|x| \leq \rho R} e^{-\lambda|x+Ry|} \left|\omega(x)\right|^2 dx\right) \\
&\leq -\left(c_0 t_0^2 \int_{\mathbb{R}^N} e^{-\lambda|x|} |\omega(x)|^2 dx\right) e^{-\lambda R} =: -C_2 e^{-\lambda R}.
\end{aligned}$$

Using the asymptotic estimates from Lemma 4.3 and taking into account that $p \geq 2$, we conclude that choosing $R_0 > 0$ even larger if necessary, there exists $C_1 > 0$ such that

$$\begin{aligned}
J_V(t(\omega^{\rho R})_{Ry}) =& \frac{1}{2} \left\|t(\omega^{\rho R})_{Ry}\right\|^2 + \frac{1}{2} \int_\Omega V(x) \left|t(\omega^{\rho R})_{Ry}\right|^2 dx - \frac{1}{2p} \mathbb{D}\left(t(\omega^{\rho R})_{Ry}\right) \\
\leq& \frac{1}{2}\left(\|t\omega\|^2 + O(e^{-2\nu(1-\varepsilon)\rho R})\right) - \frac{C_2}{2} e^{-\lambda R} \\
&- \frac{1}{2p}\left(\mathbb{D}(t\omega) + O(e^{-p\nu(1-\varepsilon)\rho R})\right) \\
\leq& \frac{1}{2}\|t\omega\|^2 - \frac{1}{2p}\mathbb{D}(t\omega) + O(e^{-2\nu(1-\varepsilon)\rho R}) - \frac{C_2}{2} e^{-\lambda R} \\
\leq& J_\infty(t\omega) - C_1 e^{-\lambda R} \\
\leq& c_\infty - C_1 e^{-\lambda R},
\end{aligned}$$

because $c_\infty = \max_{t\geq 0} J_\infty(t\omega)$ and

$$2\nu\,(1-\varepsilon)\rho > 2\nu\left(1 - \frac{\mu_\Gamma(Z)\nu - \lambda}{\mu_\Gamma(Z)\nu + \lambda}\right)\frac{\mu_\Gamma(Z)\nu + \lambda}{4\nu} = \lambda.$$

□

We fix $R \geq R_0$, and for $z \in Z$ we define

$$\theta(z) := \sum_{gz\in\Gamma z} \phi(g)(\omega^{\rho R})_{Rgz}. \tag{4.5}$$

Proposition 4.5. *If either $\phi \equiv 1$ or $Z \subset \Sigma \smallsetminus \Sigma_0$, then $\theta(z)$ is well defined. $\theta(z)$ is ϕ-equivariant and*

$$J_V(\pi(\theta(z))) \leq \ell(\Gamma)\left(c_\infty - d_0 e^{-\lambda R}\right) \qquad \textit{for all } z \in Z.$$

If moreover $Z \neq \emptyset$, then $c^\phi_{\Omega,V} < \ell(\Gamma)c_\infty$.

Proof. Let $z \in Z$. If $g_1, g_2 \in \Gamma$ are such that $g_1 z = g_2 z$, then $g_2^{-1}g_1 z = z$. Hence, if either $\phi \equiv 1$ or $z \notin \Sigma_0$, it must be true that $\phi(g_2^{-1}g_1) = 1$. Thus $\phi(g_1) = \phi(g_2)$. This shows that $\theta(z)$ is well defined. It is clearly ϕ-equivariant.

On the other hand, since

$$|Rg_1 z - Rg_2 z| \geq R\mu_\Gamma(Z) > 2\rho R \quad \text{when} \quad g_1 z \neq g_2 z,$$

we have that

$$\operatorname{supp}((\omega^{\rho R})_{Rg_1 z}) \cap \operatorname{supp}((\omega^{\rho R})_{Rg_2 z}) = \emptyset.$$

Consequently,

$$\|\theta(z)\|_V^2 = \ell(\Gamma)\|(\omega^{\rho R})_{Rz}\|_V^2 \quad \text{and} \quad \mathbb{D}(\theta(z)) > \ell(\Gamma)\mathbb{D}((\omega^{\rho R})_{Rz}).$$

From (2.11) and Lemma 4.4 we obtain

$$\begin{aligned} J_V(\pi(\theta(z))) &\leq \frac{p-1}{2p}\left(\frac{\ell(\Gamma)\|(\omega^{\rho R})_{Rz}\|_V^2}{[\ell(\Gamma)\mathbb{D}((\omega^{\rho R})_{Rz})]^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} \\ &= \ell(\Gamma)J_V(\pi((\omega^{\rho R})_{Rz})) \leq \ell(\Gamma)\left(c_\infty - d_0 e^{-\lambda R}\right). \end{aligned}$$

Finally, since $\pi(\theta(z)) \in \mathcal{N}^\phi_{\Omega,V}$, we conclude that $c^\phi_{\Omega,V} < \ell(\Gamma)c_\infty$. □

Proof of Theorem 4.1. Let (u_n) be a minimizing sequence for J_V on $\mathcal{N}^G_{\Omega,V}$. By Ekeland's variational principle [59, Theorem 8.5] we may assume that it is a Palais–Smale sequence for J_V.

Let $\phi \equiv 1$, so that $\Gamma = G$. If assumption (V_3) holds for $\lambda \in (0, \mu^G)$, we choose $\zeta \in \Sigma$ such that $\mu(G\zeta) \in (\lambda, \mu^G]$ and define $Z := G\zeta$. Thus $\mu_G(Z) = \mu(G\zeta)$ and assumption (V_1) holds for $\lambda \in (0, \mu_G(Z))$. Hence, we may apply Proposition 4.5 to these data to conclude that $c^G_{\Omega,V} < \ell(G)c_\infty$. Corollary 3.8 then asserts that J_V satisfies condition $(PS)^G_c$ on $\mathcal{N}^G_{\Omega,V}$ for $c := c^G_{\Omega,V}$. Therefore, there exists $u \in \mathcal{N}^G_{\Omega,V}$ such that $u_n \to u$ strongly in $H^1_0(\Omega)$ and, since J_V is of class $\mathcal{C}^1$, u is a minimum of J_V on $\mathcal{N}^G_{\Omega,V}$. Finally, observe that $|u| \in \mathcal{N}^G_{\Omega,V}$ and $J_V(|u|) = J_V(u)$. Hence, by Corollary 2.6, problem (4.1) has a G-invariant positive solution $|u|$ satisfying $J_V(|u|) < \ell(G)c_\infty$. □

Proof of Theorem 4.2. Proposition 2.5 guarantees that $\mathcal{N}^\phi_{\Omega,V}$ is a $\mathcal{C}^2$-manifold which is symmetric and does not contain the origin. Proposition 2.2, together with Corollary 2.6 and Corollary 3.8, asserts that $J_V : \mathcal{N}^\phi_{\Omega,V} \to \mathbb{R}$ is an even $\mathcal{C}^2$-function, which is bounded from below and satisfies $(PS)^\phi_c$ on $\mathcal{N}^\phi_{\Omega,V}$ for all $c < \ell(\Gamma)c_\infty$. Therefore, by Theorem 3.20 and Corollary 2.6, if $d := \ell(\Gamma)\left(c_\infty - d_0 e^{-\lambda R}\right)$, then J_V has at least

$$\text{genus}(\mathcal{N}^\phi_{\Omega,V} \cap J^d_V)$$

pairs of critical points $\pm u$ with $J_V(u) \le d$, where $J^d_V := \{u \in H^1_0(\Omega) : J_V(u) \le d\}$.

The map $\theta : Z \to \mathcal{N}^\phi_{\Omega,V} \cap J^d_V$ defined by (4.5) is continuous. Furthermore, $\theta(gz) = \theta(z)$ for all $g \in G$ and $\theta(\gamma z) = -\theta(z)$ if $\phi(\gamma) = -1$. Consequently, θ induces a continuous map $\widehat{\theta} : Z/G \to \mathcal{N}^\phi_{\Omega,V} \cap J^d_V$, given by $\widehat{\theta}(Gz) := \theta(z)$, which satisfies

$$\widehat{\theta}((-1)\cdot Gz) = -\widehat{\theta}(Gz) \quad \text{for all } z \in Z.$$

By Lemma 3.19, this implies that

$$\text{genus}(Z/G) \le \ \text{genus}(\mathcal{N}^\phi_{\Omega,V} \cap J^d_V)$$

and concludes the proof. □

Remark 4.6. Theorem 1.1 in the Introduction also considers potentials which are strictly negative at infinity. In order to prove Theorem 1.1, we follow the same procedure as in the proof of Theorem 4.2, using (3.21) instead of Lemma 3.11 and taking $\nu = 1$ in Lemmas 4.3, 4.4 and Proposition 4.5. Note that (3.21) implies

$$\left| |\omega|^p_p - |\omega^S|^p_p \right| = O(e^{-p(1-\varepsilon)S}).$$

Notice also that

$$|\theta(z)|^p_p = \ell(\Gamma)|(\omega^{\rho R})_{Rz}|^p_p.$$

4.2 Proof of Theorems 1.5 and 1.6

The purpose of this section is to prove Theorems 1.5 and 1.6 stated in the Introduction, namely,

Theorem 4.7. *If $p \geq 2$, $\ell(G) \geq 3$, Ω is G-invariant and V is a G-invariant function which satisfies*

(V_4) *There exist $c_0 > 0$ and $\kappa > \mu_G\sqrt{V_\infty}$ such that*

$$V(x) \leq c_0 e^{-\kappa|x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

then (4.1) has at least one positive solution u which is G-invariant and satisfies (4.4) with $\Gamma = G$.

Theorem 4.8. *If $p \geq 2$, Ω is Γ-invariant, $\phi : \Gamma \to \mathbb{Z}/2$ is an epimorphism, Z is a Γ-invariant subset of Σ, V is a Γ-invariant function and the following hold:*

(Z_0) *There exists $a_0 > 1$ such that*

$$\operatorname{dist}(\gamma z, Gz) \geq a_0\mu(Gz) \qquad \text{for all } z \in Z \text{ and } \gamma \in \Gamma \smallsetminus G,$$

(V_2) *There exist $c_0 > 0$ and $\kappa > \mu^\Gamma(Z)\sqrt{V_\infty}$ such that*

$$V(x) \leq c_0 e^{-\kappa|x|} \qquad \text{for all } x \in \mathbb{R}^N,$$

then (4.1) has at least $\operatorname{genus}(Z/G)$ *pairs of sign changing solutions $\pm u$, which satisfy (4.2) and (4.4).*

As you can notice, these theorems only consider potentials which take on nonnegative values at infinity.

As in the previous section, let $\phi : \Gamma \to \mathbb{Z}/2$ be a continuous group homomorphism and set $G := \ker\phi$. Let $\omega \in H^1(\mathbb{R}^N)$ be a positive ground state of problem (4.3) which is radially symmetric about the origin, and let Z be a nonempty Γ-invariant subset of Σ. If ϕ is an epimorphism, we also assume that $Z \subset \Sigma \smallsetminus \Sigma_0$. Thus, for $z \in Z$ and $R > 0$, the function

$$\sigma_{Rz} := \sum_{gz\in\Gamma z} \phi(g)\omega_{Rgz}, \qquad \text{where } \ \omega_\zeta(x) := \omega(x-\zeta),$$

is well defined and ϕ-equivariant (see Proposition 4.5). In addition, we assume that

(Z_*) $\mu^\Gamma(Z) < 2$ and there exists $a_0 > 1$ such that

$$\operatorname{dist}(\gamma z, Gz) \geq a_0\mu(Gz) \quad \text{for any } z \in Z \text{ and } \gamma \in \Gamma \smallsetminus G.$$

We choose $R_0 > 0$ such that $\left(\mathbb{R}^N \smallsetminus \Omega\right) \subset B_{R_0}(0)$, and a radially symmetric cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^N)$ such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 0$ if $|x| \leq R_0$ and $\chi(x) = 1$ if $|x| \geq 2R_0$. Observe that $\chi\sigma_R \in H_0^1(\Omega)^\phi$. We shall prove the following result.

Proposition 4.9. *If Z and V satisfy (Z_*) and (V_2) then there exist $C_0, R_0 > 0$ and $\beta > 1$ such that*

$$\frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}} \leq \left(\ell(\Gamma)\,\|\omega\|^2\right)^{\frac{p-1}{p}} - C_0 e^{-\beta R} \quad \text{for any } \ R \geq R_0, \ z \in Z. \tag{4.6}$$

Consequently, $c^\phi_{\Omega,V} < \ell(\Gamma)c_\infty$.

We require some preliminary lemmas. The first two ones yield that $\mu_\Gamma(\Sigma) > 0$. As a result, $\mu_\Gamma(Z) > 0$ also holds.

Lemma 4.10. *Σ is a compact subset of $\mathbb{R}^N$.*

Proof. Let y_n be a sequence in Σ such that $y_n \to y$ in $\mathbb{S}^{N-1}$. Thus

$$\#\Gamma y \geq \ell(\Gamma) = \#\Gamma y_n.$$

Now, let $g_1, \cdots, g_{\ell(\Gamma)} \in \Gamma$ be such that $g_i y \neq g_j y$ if $i \neq j$, and fix $\delta > 0$ such that

$$B_\delta(g_i y) \cap B_\delta(g_j y) = \emptyset \quad \text{if } i \neq j.$$

Since $g_i y_n \in B_\delta(g_i y)$ for sufficiently large n, we conclude that $\#\Gamma y \leq \#\Gamma y_n$. Therefore, $\#\Gamma y = \ell(\Gamma)$ and so $y \in \Sigma$. This proves that Σ is closed. The conclusion follows because, additionally, Σ is bounded. □

Lemma 4.11. *The function $\Sigma \to \mathbb{R}$, $y \mapsto \mu(\Gamma y)$ is continuous.*

Proof. Let $\varepsilon > 0$ be given. Let (y_n) be a sequence in Σ such that $y_n \to y$. Then, there exists $n_0 \in \mathbb{N}$ such that $|y_n - y| < \frac{\varepsilon}{2}$ if $n \geq n_0$. For every $g \in \Gamma$ one has that

$$|y - gy| \leq |y - y_n| + |y_n - gy_n| + |gy_n - gy| = 2|y_n - y| + |y_n - gy_n|,$$

and so

$$\mu(\Gamma y) \leq \varepsilon + \mu(\Gamma y_n) \quad \text{for all } n \geq n_0.$$

Analogously, we obtain

$$\mu(\Gamma y_n) \leq \varepsilon + \mu(\Gamma y) \quad \text{for all } n \geq n_0.$$

Consequently $\mu(\Gamma y_n) \to \mu(\Gamma y)$. □

Lemma 4.12. (i) *If $p \geq 2$ and $a_1, \ldots, a_n \geq 0$, then*

$$\left| \sum_{i=1}^n a_i \right|^p \geq \sum_{i=1}^n a_i^p + (p-1) \sum_{i \neq k} a_i^{p-1} a_k.$$

(ii) *If $p \geq 2$ and $a, b \geq 0$, then*

$$|a - b|^p \geq a^p + b^p - p\left(a^{p-1} b + a b^{p-1}\right).$$

Proof. See Lemma 4 in [16]. □

Lemma 4.13. *If* $p \geq 2$, $A = \sum_{i=1}^{n} a_i$, $\tilde{A} = \sum_{i=1}^{n} \tilde{a}_i$, $B = \sum_{i=1}^{n} b_i$ *and* $\tilde{B} = \sum_{i=1}^{n} \tilde{b}_i$ *with* $a_i, \tilde{a}_i, b_i, \tilde{b}_i \geq 0$, *then*

$$A^p B^p \geq \sum_{i=1}^{n} a_i^p b_i^p + (p-1)\left(\sum_{j \neq m} a_j^p b_j^{p-1} b_m + \sum_{i \neq k} b_i^p a_i^{p-1} a_k\right), \tag{4.7}$$

$$A^2 B^2 \geq \sum_{i=1}^{n} a_i^2 b_i^2 + 2\left(\sum_{j \neq m} a_j^2 b_j b_m + \sum_{i \neq k} b_i^2 a_i a_k\right), \tag{4.8}$$

$$\begin{aligned} \left|A - \tilde{A}\right|^p \left|B - \tilde{B}\right|^p \geq\ & A^p B^p + \tilde{A}^p \tilde{B}^p \\ & - p n^{p-1}\left(B^p + \tilde{B}^p\right)\left[\left(\sum_{i=1}^{n} a_i^{p-1}\right)\tilde{A} + \left(\sum_{i=1}^{n} \tilde{a}_i^{p-1}\right) A\right] \\ & - p n^{p-1}\left(A^p + \tilde{A}^p\right)\left[\left(\sum_{i=1}^{n} b_i^{p-1}\right)\tilde{B} + \left(\sum_{i=1}^{n} \tilde{b}_i^{p-1}\right) B\right]. \end{aligned} \tag{4.9}$$

Proof. Using Lemma 4.12(i) we obtain

$$\begin{aligned} \left|\sum_{i=1}^{n} a_i\right|^p \left|\sum_{j=1}^{n} b_j\right|^p &\geq \left(\sum_{i=1}^{n} a_i^p + (p-1)\sum_{i \neq k} a_i^{p-1} a_k\right)\left(\sum_{j=1}^{n} b_j^p + (p-1)\sum_{j \neq m} b_j^{p-1} b_m\right) \\ &\geq \sum_{i=1}^{n} a_i^p b_i^p + (p-1)\sum_{j \neq m}(a_j^p + a_m^p) b_j^{p-1} b_m + (p-1)\sum_{i \neq k}\left(b_i^p + b_k^p\right) a_i^{p-1} a_k. \end{aligned}$$

Inequalities (4.7) and (4.8) can be immediately deduced from the above expression.

On the other hand, applying Lemma 4.12 (ii) we obtain

$$|A - \tilde{A}|^p |B - \tilde{B}|^p \geq \left[A^p + \tilde{A}^p - p\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right)\right]|B - \tilde{B}|^p$$

Notice that, if $A^p + \tilde{A}^p - p\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right) \geq 0$ then

$$\begin{aligned} &|A - \tilde{A}|^p |B - \tilde{B}|^p \\ &\geq \left[A^p + \tilde{A}^p - p\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right)\right]|B - \tilde{B}|^p \\ &\geq \left[A^p + \tilde{A}^p - p\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right)\right]\left[B^p + \tilde{B}^p - p\left(B^{p-1}\tilde{B} + B\tilde{B}^{p-1}\right)\right] \\ &\geq A^p B^p + \tilde{A}^p \tilde{B}^p - p\left(B^p + \tilde{B}^p\right)\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right) - p\left(A^p + \tilde{A}^p\right)\left(B^{p-1}\tilde{B} + B\tilde{B}^{p-1}\right). \end{aligned}$$

Otherwise, since $|B - \tilde{B}|^p \leq B^p + \tilde{B}^p$,

$$\begin{aligned} |A - \tilde{A}|^p |B - \tilde{B}|^p &\geq \left[A^p + \tilde{A}^p - p\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right)\right]|B - \tilde{B}|^p \\ &\geq \left[A^p + \tilde{A}^p - p\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right)\right]\left[B^p + \tilde{B}^p\right] \\ &\geq A^p B^p + \tilde{A}^p \tilde{B}^p - p\left(B^p + \tilde{B}^p\right)\left(A^{p-1}\tilde{A} + A\tilde{A}^{p-1}\right). \end{aligned}$$

In any case, inequality (4.9) follows. □

Lemma 4.14. *For every* $u \in H^1(\mathbb{R}^N)$ *the following inequalities hold:*

$$\begin{aligned} &\|\chi u\|_V^2 \leq \|u\|_V^2 - \int_{\mathbb{R}^N} (\chi \Delta \chi) u^2, \\ &\mathbb{D}(\chi u) \geq \mathbb{D}(u) - 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(1 - \chi^p(x))|u(x)|^p |u(y)|^p}{|x - y|^\alpha} dx\, dy. \end{aligned}$$

Proof. For every $u \in H^1(\mathbb{R}^N)$ one has that

$$\begin{aligned}
\|\chi u\|_V^2 &= \int_{\mathbb{R}^N} \left(|\chi \nabla u + u \nabla \chi|^2 + (1 + V(x)) \, |\chi u|^2 \right) \\
&= \int_{\mathbb{R}^N} \chi^2 \left(|\nabla u|^2 + (1 + V(x)) |u|^2 \right) + \int_{\mathbb{R}^N} \left(|\nabla \chi|^2 - \frac{1}{2} \Delta(\chi^2) \right) u^2 \\
&\leq \|u\|_V^2 - \int_{\mathbb{R}^N} (\chi \Delta \chi) u^2.
\end{aligned}$$

Writing $ab = 1 - (1-a) - (1-b) + (1-a)(1-b)$ and taking $a := \chi^p(x)$, $b := \chi^p(y)$, we obtain

$$\begin{aligned}
\mathbb{D}(\chi u) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi^p(x)\chi^p(y)|u(x)|^p|u(y)|^p}{|x-y|^\alpha} dx\, dy \\
&= \mathbb{D}(u) - 2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(1 - \chi^p(x))|u(x)|^p|u(y)|^p}{|x-y|^\alpha} dx\, dy \\
&+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(1 - \chi^p(x))(1 - \chi^p(y))|u(x)|^p|u(y)|^p}{|x-y|^\alpha} dx\, dy.
\end{aligned}$$

Notice that the last summand in the right-hand side of the above expression is nonnegative. Then the second inequality follows. □

We shall apply this lemma to the function σ_{Rz} to derive inequality (4.6). To this purpose we also require some asymptotic estimates, which will be provided by the following four lemmas.

Since ω is a solution of problem (4.3), for any $z, z' \in \mathbb{R}^N$, one has that $J'_\infty(\omega_z)\omega_{z'} = 0$, which is equivalent to

$$\int_{\mathbb{R}^N} [\nabla \omega_z \cdot \nabla \omega_{z'} + \omega_z \omega_{z'}] = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \omega_z^p \right) \omega_z^{p-1} \omega_{z'}.$$

A change of variable in the right-hand side of this inequality allows us to express it as

$$\langle \omega_z, \omega_{z'} \rangle = I(z' - z) \qquad \text{for all} \;\; z, z' \in \mathbb{R}^N, \tag{4.10}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $H^1(\mathbb{R}^N)$ and I is the function defined by

$$I(\zeta) := \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \omega^p \right) \omega^{p-1} \omega_\zeta.$$

We denote by $Fz := \{(gz, hz) \in \Gamma z \times \Gamma z : gz \neq hz\}$ and define

$$\begin{aligned}
\varepsilon_{Rz} &:= \sum_{\substack{(gz,hz) \in Fz \\ \phi(g) = \phi(h)}} I(Rgz - Rhz), \\
\widehat{\varepsilon}_{Rz} &:= \sum_{\substack{(gz,hz) \in Fz \\ \phi(g) \neq \phi(h)}} I(Rgz - Rhz) \;\; \text{if } \phi \not\equiv 1, \quad \text{and} \quad \widehat{\varepsilon}_{Rz} := 0 \;\; \text{if } \phi \equiv 1.
\end{aligned}$$

We choose $g_z, h_z \in G$ such that

$$|g_z z - h_z z| = \mu(\Gamma z) := \min\{|gz - hz| : g,h \in \Gamma,\ gz \neq hz\}$$

and set

$$\xi_z := g_z z - h_z z.$$

Lemma 4.15. *If (Z_*) holds, then*

$$\widehat{\varepsilon}_{Rz} = o(\varepsilon_{Rz})$$

uniformly in $z \in Z$.

Proof. For $a_0 > 1$ as in condition (Z_*) we fix $\widehat{a} \in (0,1)$ such that $a := \widehat{a}a_0 > 1$. Thus, $a\,|\xi_z| = a\mu(Gz) \leq \widehat{a}\,|gz - hz|$ for any $z \in Z$, $g,h \in \Gamma$ with $gz \neq hz$ and $\phi(g) \neq \phi(h)$. Lemma 3.14 yields a constant $k_a > 0$ such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \qquad \text{if } \ R \geq \mu_\Gamma(Z)^{-1},$$

where $b := \frac{N-1}{2}$. So, setting $C := k_a^{-1}$ we obtain

$$\begin{aligned} \frac{I(Rgz - Rhz)}{I(R\xi_z)} &\leq \frac{I(Rgz - Rhz)\,|Rgz - Rhz|^b\, e^{\widehat{a}|Rgz - Rhz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C I(Rgz - Rhz)\,|Rgz - Rhz|^b\, e^{\widehat{a}|Rgz-Rhz|} \qquad \text{if } \ R \geq \mu_\Gamma(Z)^{-1}. \end{aligned}$$

Let $\varepsilon > 0$. Lemma 3.13 asserts that there exists $S > 0$ such that $I(\zeta)\,|\zeta|^b\, e^{\widehat{a}|\zeta|} < \varepsilon$ if $|\zeta| > S$. As $\widehat{a}\,|Rgz - Rhz| \geq Ra\mu_G > 0$, taking $R_0 := \max\{\frac{\widehat{a}S}{a\mu_G}, \mu_\Gamma(Z)^{-1}\}$ we conclude that

$$0 \leq \frac{\widehat{\varepsilon}_{Rz}}{\varepsilon_{Rz}} \leq \sum_{\substack{gz \neq hz \in \Gamma z \\ \phi(g) \neq \phi(h)}} \frac{I(Rgz - Rhz)}{I(R\xi_z)} \leq \ell(G)^2 C\varepsilon \qquad \text{if } R \geq R_0,$$

which proves the assertion. ∎

Lemma 4.16. *If (Z_*) holds then, for any $g,h \in \Gamma$ such that $\phi(g) \neq \phi(h)$ and $\gamma \in \Gamma \smallsetminus G$, we have that*

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \Big(|\sum_{\zeta \in Gz} \omega_{R\zeta}|^p + |\sum_{\zeta \in Gz} \omega_{R\gamma\zeta}|^p \Big) \right) \omega_{Rgz}^{p-1} \omega_{Rhz} = o(\varepsilon_{Rz})$$

uniformly in $z \in Z$.

Proof. Since $\frac{1}{|x|^\alpha} * \omega^p \in L^\infty(\mathbb{R}^N)$, we have that $\frac{1}{|x|^\alpha} * \Big(|\sum_{\zeta \in Gz} \omega_{R\zeta}|^p + |\sum_{\zeta \in Gz} \omega_{R\gamma\zeta}|^p \Big)$ is bounded on $\mathbb{R}^N$ uniformly in z. Hence,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\alpha} * \Big(|\sum_{\zeta \in Gz} \omega_{R\zeta}|^p + |\sum_{\zeta \in Gz} \omega_{R\gamma\zeta}|^p \Big) \right) \omega_{Rgz}^{p-1} \omega_{Rhz} \\ &\leq C \int_{\mathbb{R}^N} \omega_{Rgz}^{p-1} \omega_{Rhz} = C \int_{\mathbb{R}^N} \omega^{p-1} \omega_{R(hz-gz)}. \end{aligned}$$

Arguing as in Lemma 4.15, using this time Lemma 3.12, we obtain the conclusion. ∎

Lemma 4.17. *If Z and V satisfy (Z_*) and (V_2), then*

$$\int_{\mathbb{R}^N} V^+\sigma_{Rz}^2 = o(\varepsilon_{Rz})$$

uniformly in $z \in Z$.

Proof. Let $\kappa > \mu^\Gamma(Z)$ be as in assumption (V_2) (recall that $V_\infty = 1$ is assumed). We fix $a > 1$ such that $M := a\mu^\Gamma(Z) < \min\{2, \kappa\}$. Lemma 3.14 implies that there exists a positive constant k_a such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \qquad \text{if } \; R \geq \mu_\Gamma(Z)^{-1},$$

where $b := \frac{N-1}{2}$. Observing that $M|Rz| = MR = aR\mu^\Gamma(Z) \geq a|R\xi_z|$ for all $z \in Z$, we conclude that

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} V^+\sigma_{Rz}^2}{\varepsilon_{Rz}} &\leq C \sum_{gz\in\Gamma z} \frac{A(Rgz)}{I(R\xi_z)} \leq C \sum_{gz\in\Gamma z} \frac{A(Rgz)|Rgz|^b e^{M|Rgz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C \sum_{gz\in\Gamma z} A(Rgz)|Rgz|^b e^{M|Rgz|} \qquad \text{if } \; R \geq \mu_\Gamma(Z)^{-1},\end{aligned}$$

where C denotes different positive constants and A is the map defined in (3.17). Taking Lemma 3.15 into account, we obtain that

$$\lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} V^+\sigma_{Rz}^2}{\varepsilon_{Rz}} = 0$$

uniformly in $z \in Z$, as claimed. $\square$

Lemma 4.18. *If $f \in \mathcal{C}_c^0\left(\mathbb{R}^N\right)$ and $q > \max\{\mu^\Gamma(Z), 1\}$, then*

$$\int_{\mathbb{R}^N} f\sigma_{Rz}^q = o(\varepsilon_{Rz})$$

uniformly in $z \in Z$.

Proof. Let us fix $a > 1$ such that $\widehat{a} := \frac{a\mu^\Gamma(Z)}{q} < 1$. Lemma 3.14 yields that there exists $k_a > 0$ such that

$$I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|} \geq k_a \qquad \text{if } \; R \geq \mu_\Gamma(Z)^{-1},$$

where $b := \frac{N-1}{2}$. Since $q\widehat{a}|Rz| = q\widehat{a}R = aR\mu^\Gamma(Z) \geq a|R\xi_z|$ for all $z \in Z$, we conclude that

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |f|\,\sigma_{Rz}^q}{\varepsilon_{Rz}} &\leq C \sum_{gz\in\Gamma z} \frac{\int_{\mathbb{R}^N} |f|\,\omega_{Rgz}^q}{I(R\xi_z)} \leq C \sum_{gz\in\Gamma z} \frac{\int_{\mathbb{R}^N} |f|\,\omega_{Rgz}^q |Rgz|^b e^{q\widehat{a}|Rgz|}}{I(R\xi_z)|R\xi_z|^b e^{a|R\xi_z|}} \\ &\leq C \sum_{gz\in\Gamma z} \int_{\mathbb{R}^N} |f|\,\omega_{Rgz}^q |Rgz|^b e^{q\widehat{a}|Rgz|} \qquad \text{if } \; R \geq \mu_\Gamma(Z)^{-1},\end{aligned}$$

where C denote distinct positive constants. Hence, from Lemma 3.16 we get

$$\lim_{R\to\infty} \frac{\int_{\mathbb{R}^N} f\sigma_{Rz}^q}{\varepsilon_{Rz}} = 0$$

uniformly in $z \in Z$. $\square$

Finally, we need the following result.

Lemma 4.19. *Let $\psi : (0,\infty) \to \mathbb{R}$ be the function given by*

$$\psi(t) := \frac{a+t+o(t)}{(a+bt+o(t))^{\beta}},$$

where $a > 0$, $\beta \in (0,1)$ and $b\beta > 1$. Then, there exist constants $C_0, t_0 > 0$ such that

$$\psi(t) \leq a^{1-\beta} - C_0 t \quad \textit{for all } t \in (0,t_0).$$

Proof. Taking $\frac{1}{\beta} < q < b$ and $1 < s < r < \beta q$, we have that there exists $t_1 \in (0,1)$ such that

$$\psi(t) \leq \frac{a+st}{(a+qt)^{\beta}} = \frac{a+rt}{(a+qt)^{\beta}} - \frac{(r-s)t}{(a+qt)^{\beta}} \quad \text{for all } t \in (0,t_1).$$

We denote by $f(t) := \frac{a+rt}{(a+qt)^{\beta}}$. Since $f'(0) = (r-\beta q)\, a^{-\beta} < 0$, there exists $t_0 \in (0,t_1)$ such that

$$f(t) \leq f(0) = a^{1-\beta} \quad \text{for all } t \in (0,t_0).$$

Consequently,

$$\psi(t) \leq a^{1-\beta} - \frac{(r-s)}{(a+q)^{\beta}} t \quad \text{for all } t \in (0,t_0),$$

which concludes the proof. □

Proof of Proposition 4.9. Let $\gamma \in \Gamma \smallsetminus G$. If $Gz = \{z_1, \ldots, z_\ell\}$ with $\ell := \ell(G)$, we write

$$\sigma_{Rz} = \sigma^1_{Rz} - \sigma^2_{Rz} \qquad \text{with} \quad \sigma^1_{Rz} := \sum_{i=1}^{\ell} \omega_{Rz_i} \quad \text{and} \quad \sigma^2_{Rz} := \sum_{i=1}^{\ell} \omega_{R\gamma z_i}.$$

Applying Lemma 4.13 to $a_i := \omega_{Rz_i}(x)$, $\hat{a}_i := \omega_{R\gamma z_i}(x)$, $b_i := \omega_{Rz_i}(y)$, $\hat{b}_i := \omega_{R\gamma z_i}(y)$ and using Lemma 4.16 we conclude that

$$\begin{aligned}
\mathbb{D}(\sigma_{Rz}) &\geq \mathbb{D}(\sigma^1_{Rz}) + \mathbb{D}(\sigma^2_{Rz}) + o(\varepsilon_{Rz}) \\
&\geq \begin{cases} \ell(\Gamma)\mathbb{D}(\omega) + 2(p-1)\varepsilon_{Rz} + o(\varepsilon_{Rz}) & \text{if } p > 2, \\ \ell(\Gamma)\mathbb{D}(\omega) + 4\varepsilon_{Rz} + o(\varepsilon_{Rz}) & \text{if } p = 2. \end{cases}
\end{aligned}$$

Note that, since $\frac{1}{|x|^{\alpha}} * \omega^p \in L^{\infty}(\mathbb{R}^N)$, $\frac{1}{|x|^{\alpha}} * |\sigma_{Rz}|^p$ is bounded uniformly in z. So, since $\mu^{\Gamma}(Z) < 2 \leq p$, $\chi\Delta\chi \in \mathcal{C}^0_c(\mathbb{R}^N)$ and $1 - \chi^p \in \mathcal{C}^0_c(\mathbb{R}^N)$, Lemma 4.18 yields that

$$\int_{\mathbb{R}^N} (\chi\Delta\chi)\sigma^2_{Rz} = o(\varepsilon_{Rz}) \qquad \text{and} \qquad \int_{\mathbb{R}^N} (1-\chi^p)\left(\frac{1}{|x|^{\alpha}} * |\sigma_{Rz}|^p\right)\sigma^p_{Rz} = o(\varepsilon_{Rz})$$

uniformly in z. This, together with Lemmas 4.14, 4.15 and 4.17 and expression (4.10), yields

$$\begin{aligned}\|\chi\sigma_{Rz}\|_V^2 &\leq \|\sigma_{Rz}\|^2 + \int_{\mathbb{R}^N} V\sigma_{Rz}^2 - \int_{\mathbb{R}^N}(\chi\Delta\chi)\sigma_{Rz}^2 \\ &\leq \ell(\Gamma)\,\|\omega\|^2 + \varepsilon_{Rz} - \widehat{\varepsilon}_{Rz} + \int_{\mathbb{R}^N} V^+\sigma_{Rz}^2 + o(\varepsilon_{Rz}) \\ &\leq \ell(\Gamma)\,\|\omega\|^2 + \varepsilon_{Rz} + o(\varepsilon_{Rz}), \\ \mathbb{D}(\chi\sigma_{Rz}) &\geq \ell(\Gamma)\mathbb{D}(\omega) + b_p\varepsilon_{Rz} + o(\varepsilon_{Rz}) - 2\int_{\mathbb{R}^N}(1-\chi^p)\left(\frac{1}{|x|^\alpha} * |\sigma_{Rz}|^p\right)\sigma_{Rz}^p \\ &\geq \ell(\Gamma)\mathbb{D}(\omega) + b_p\varepsilon_{Rz} + o(\varepsilon_{Rz}),\end{aligned}$$

where $b_p := 2(p-1)$ if $p > 2$ and $b_p := 4$ if $p = 2$. Consequently, since $\|\omega\|^2 = \mathbb{D}(\omega)$ and $\varepsilon_{Rz} \to 0$ as $R \to \infty$ uniformly in z, Lemma 4.19 insures that there exist $c_1, R_1 > 0$ such that

$$\frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}} \leq \frac{\ell(\Gamma)\,\|\omega\|^2 + \varepsilon_{Rz} + o(\varepsilon_{Rz})}{(\ell(\Gamma)\mathbb{D}(\omega) + b_p\varepsilon_{Rz} + o(\varepsilon_{Rz}))^{\frac{1}{p}}} \leq \left(\ell(\Gamma)\,\|\omega\|^2\right)^{\frac{p-1}{p}} - c_1\varepsilon_{Rz}$$

for $R \geq R_1$ and $z \in Z$. Using Lemma 3.14 we conclude that there exist $C_0, R_0 > 0$ and $\beta > 1$ such that

$$\frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}} \leq \left(\ell(\Gamma)\,\|\omega\|^2\right)^{\frac{p-1}{p}} - C_0e^{-\beta R} \quad \text{for any} \quad R \geq R_0,\ z \in Z,$$

which is inequality (4.6). Finally, since $\pi(\chi\sigma_{Rz}) \in \mathcal{N}_{\Omega,V}^{\phi}$ and

$$J_V(\pi(\chi\sigma_{Rz})) = \frac{p-1}{2p}\left(\frac{\|\chi\sigma_{Rz}\|_V^2}{\mathbb{D}(\chi\sigma_{Rz})^{\frac{1}{p}}}\right)^{\frac{p}{p-1}} < \frac{p-1}{2p}\,\ell(\Gamma)\,\|\omega\|^2 = \ell(\Gamma)c_\infty,$$

one has that $c_{\Omega,V}^{\phi} < \ell(\Gamma)c_\infty$. □

Remark 4.20. The reason why we require condition (Z_0) is because, as we have seen, the energy of the function $\pi(\chi\sigma_{Rz})$ decreases and remains below the level $\ell(\Gamma)c_\infty$ when the concentration points of at least two positive terms of σ_{Rz} are closer than any pair of concentration points of contrary sign terms.

Now we are ready to proof Theorems 4.7 and 4.8.

Proof of Theorem 4.7. Let $\phi \equiv 1$, so that $\Gamma = G$. If assumption (V_4) holds for $\kappa > \mu_G$, we choose $\zeta \in \Sigma$ such that $\mu(G\zeta) \in [\mu_G, \kappa)$ and set $Z := G\zeta$. Thus $\mu^G(Z) = \mu(G\zeta)$ and assumption (V_2) holds for κ. Moreover, since $\ell(G) \geq 3$, $\mu^G(Z) = \mu(G\zeta) < 2$. Therefore (Z_*) holds and we can apply Proposition 4.9 to these data to conclude that $c_{\Omega,V}^G < \ell(G)c_\infty$. Corollary 3.8 then insures that J_V satisfies condition $(PS)_c^G$ on $\mathcal{N}_{\Omega,V}^G$ for $c := c_{\Omega,V}^G$. Consequently, there exists $u \in \mathcal{N}_{\Omega,V}^G$ such that $J_V(u) = c_{\Omega,V}^G$. Since $|u| \in \mathcal{N}_{\Omega,V}^G$ and $J_V(|u|) = J_V(u)$, by Corollary 2.6, $|u|$ is a positive solution of (4.1) which is G-invariant and satisfies $J_V(|u|) < \ell(G)c_\infty$. □

Proof of Theorem 4.8. If ϕ is an epimorphism and (Z_0) holds, then $Z \subset \Sigma \smallsetminus \Sigma_0$ and $2 > \frac{2}{a_0} \geq \mu(Gz) = \mu(\Gamma z)$. Therefore, $\mu^\Gamma(Z) < 2$, and hence (Z_*) holds. We choose $R > R_0$ and set

$$d := \frac{p-1}{2p}\left[\left(\ell(\Gamma)\,\|\omega\|^2\right)^{\frac{p-1}{p}} - C_0\varepsilon^{-\beta R}\right]^{\frac{p}{p-1}}.$$

Proposition 4.9 then asserts that the map $\sigma : Z \to \mathcal{N}^\phi_{\Omega,V} \cap J^d_V$ given by $\sigma(z) := \pi(\chi\sigma_{Rz})$ is well defined. Furthermore, $\sigma(gz) = \sigma(z)$ for all $g \in G$ and $\sigma(\gamma z) = -\sigma(z)$ if $\phi(\gamma) = -1$. Consequently, σ induces a continuous map $\widehat{\sigma} : Z/G \to \mathcal{N}^\phi_{\Omega,V} \cap J^d_V$, given by $\widehat{\sigma}(Gz) := \sigma(z)$, which satisfies $\widehat{\sigma}((-1)\cdot Gz) = -\widehat{\sigma}(Gz)$ for all $z \in Z$. This implies that

$$\operatorname{genus}(Z/G) \leq \operatorname{genus}(\mathcal{N}^\phi_{\Omega,V} \cap J^d_V).$$

Since $\mathcal{N}^\phi_{\Omega,V}$ is a $\mathcal{C}^2$-manifold (see Proposition 2.5) and $J_V : \mathcal{N}^\phi_{\Omega,V} \to \mathbb{R}$ is an even $\mathcal{C}^2$-function which is bounded from below and satisfies condition $(PS)^\phi_c$ on $\mathcal{N}^\phi_{\Omega,V}$ for all $c < \ell(\Gamma)c_\infty$ (see Proposition 2.2, Corollary 2.6 and Corollary 3.8), Theorem 3.20 and Corollary 2.6, allows us to conclude that J_V has at least genus(Z/G) pairs of critical points $\pm u$ with $J_V(u) \leq d$. □

Remark 4.21. (Some comments about the proof of Theorem 1.2)

As we have mentioned in the Introduction, Theorem 1.2 corresponds to [26, Theorem 1.2] and it also considers potentials which take on nonnegative values at infinity. To prove this theorem we may follow the same lines of the proof of Theorem 4.8, taking into account Remark 3.17 and making the obvious modifications derived of considering the term $|u|_p^p$ instead of $\mathbb{D}(u)$ in the energy functional.

However, this is not exactly the proof that we gave in [26]. What we did there was to give a proof in the same style as the proof of Theorem 4.8, but working directly with $a = 1$, using estimates (3.22) and (3.23) and taking into account Remarks 2.7 and 3.9. See [26, Section 5] for further details.

It is also worth mentioning that in [26, Theorem 1.2] we assumed Z to be a compact Γ-invariant subset of Σ which satisfies the slightly different condition

$(\hat{Z}_0)$ $\operatorname{dist}(\gamma z, Gz) > \mu(Gz)$ for all $z \in Z$ and $\gamma \in \Gamma \smallsetminus G$.

However, we realized that the compactness assumption for Z can be removed just asking for condition

(Z_0) There exists $a_0 > 1$ such that $\operatorname{dist}(\gamma z, Gz) \geq a_0\mu(Gz)$ for all $z \in Z$ and $\gamma \in \Gamma \smallsetminus G$.

Indeed, if (Z_0) holds, setting $c := (a_0 - 1)\mu_G(\Sigma) > 0$, we obtain that

$$\operatorname{dist}(\gamma z, Gz) - \mu(Gz) \geq (a_0 - 1)\mu(Gz) \geq c \quad \text{for all } z \in Z \text{ and } \gamma \in \Gamma \smallsetminus G.$$

Moreover, (Z_0) yields that $2 > \frac{2}{a_0} \geq \mu(Gz)$. Therefore,

$$M := \mu^G(Z) = \sup_{z\in Z} \mu(Gz) < 2.$$

Additionally,

$$m := \mu_G(Z) = \inf_{z\in Z} \mu(Gz) \geq \mu_G(\Sigma) > 0.$$

The above are precisely the facts that we need in order to prove Proposition 5.1 in [26], which is fundamental for the proof of Theorem 1.2 in [26].

Appendix A

A Brezis-Lieb lemma for the nonlocal term of the energy functional

The main purpose of this section is to prove Proposition A.1 below. It corresponds to a slight variant of [1, Lemma 3.5] which states a Brezis-Lieb lemma for a large class of nonlocal functions. We follow the same lines of Ackermann's proof, the main differences are that we use Lemma A.6 below instead of [1, Lemma 3.2] and that we are only interested in the special function

$$\mathbb{D} : H_0^1(\Omega) \to \mathbb{R}, \qquad \mathbb{D}(u) = \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p \right) |u|^p,$$

and its derivative. Throughout this section we assume that $N \geq 3$, $\alpha \in (0, N)$, $p \in \left[2, \frac{2N-\alpha}{N-2}\right)$ and Ω is an unbounded smooth domain in $\mathbb{R}^N$ whose complement $\mathbb{R}^N \smallsetminus \Omega$ is bounded, possibly empty.

Proposition A.1. *Let (u_n) be a sequence in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. The following hold:*

1. $\mathbb{D}'(u_n)v \to \mathbb{D}'(u)v$ *for all* $v \in H_0^1(\Omega)$.
2. *After passing to a subsequence, we have*

$$\mathbb{D}(u_n) - \mathbb{D}(u_n - u) \to \mathbb{D}(u) \quad \text{in} \quad \mathbb{R},$$
$$\mathbb{D}'(u_n) - \mathbb{D}'(u_n - u) \to \mathbb{D}'(u) \quad \text{in } H^{-1}(\Omega).$$

In the sequel, for $\Lambda \subseteq \mathbb{R}^N$ and $u \in L^q(\Lambda)$, let $|u|_{q,\Lambda} := \left(\int_\Lambda |u|^q\right)^{\frac{1}{q}}$ and set $|u|_q = |u|_{q,\mathbb{R}^N}$. Also set $L^q := L^q(\mathbb{R}^N)$ and $B_R := \{x \in \mathbb{R}^N : |x| < R\}$.

Lemma A.2. *Let $r, s, q \in [1, \infty)$ with $\frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and (u_n) be a bounded sequence in L^r. If $u_n \to u$ in L^r_{loc} and $v \in L^s$, then $u_n v \to uv$ in L^q.*

Proof. We may assume without loss of generality that $u_n \to 0$ in L^r_{loc}. Let $\varepsilon > 0$. Since $v \in L^s$ and $s < \infty$ there exists $R > 0$ such that

$$|v|_{s,\mathbb{R}^N \smallsetminus B_R} \leq \varepsilon.$$

Therefore, from the Hölder inequality, taking into account that $(|u_n|_r)$ is bounded, one has that

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n v|^q &= \int_{B_R} |u_n v|^q + \int_{\mathbb{R}^N \setminus B_R} |u_n v|^q \\ &\leq |u_n|^q_{r,B_R} |v|^q_s + |u_n|^q_r |v|^q_{s,\mathbb{R}^N \setminus B_R} \\ &\leq C_1 |u_n|^q_{r,B_R} + C_2 \varepsilon^q,\end{aligned}$$

where C_i denotes positive constants. Letting $n \to \infty$ and then $\varepsilon \to 0$ we reach the conclusion of this lemma. $\square$

The following lemma has been shown by Ackermann (see [1, Lemma 3.1]).

Lemma A.3. *Let $r \in [1,\infty)$ and $K \in L^r$. Set $s := \frac{2r}{2r-1}$ and let s' be the conjugate exponent for s. If $t \in [s,\infty)$ and μ is given by $\frac{1}{s'} + \frac{1}{t} = \frac{1}{\mu}$, then the bilinear map $L^s \times L^t \to L^\mu$, sending (u,v) to $(K * u)v$, is well defined and continuous, with*

$$|(K * u)v|_\mu \leq |K * u|_{s'} |v|_t \leq |K|_r |u|_s |v|_t.$$

If $(u_n) \subset L^s$ and $(v_n) \subset L^t$ are bounded and either $u_n \to u$ in L^s and $v_n \to v$ in L^t_{loc} or $u_n \to u$ in L^s_{loc} and $v_n \to v$ in L^t, then $(K * u_n)v_n \to (K * u)v$ in L^μ.

Proof. Throughout this proof C_i will denote positive constants. Let $u \in L^s$ and $v \in L^t$. Since $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{s'}$, the Young convolution theorem [12, Theorem 4.33] asserts that $K * u \in L^{s'}$ and

$$|K * u|_{s'} \leq |K|_r |u|_s.$$

From $t \geq s$ it follows that $\mu \geq 1$. So, the Hölder inequality implies

$$|(K * u)v|_\mu \leq |K * u|_{s'} |v|_t \leq |K|_r |u|_s |v|_t, \tag{A.1}$$

which yields the continuity of the bilinear map $(u,v) \mapsto (K * u)v$.

On the other hand, let (u_n) and (v_n) be given as in the statement of this lemma. In the case that $u_n \to u$ in L^s, it may be assumed without loss of generality that $v_n \to 0$ in L^t_{loc}. From (A.1), taking into account that $(|v_n|_t)$ is bounded, one has that

$$\begin{aligned}|(K * u_n)v_n|_\mu &\leq |(K * (u_n - u))v_n|_\mu + |(K * u)v_n|_\mu \\ &\leq |K|_r |u_n - u|_s |v_n|_t + |(K * u)v_n|_\mu \\ &\leq C_0 |u_n - u|_s + |(K * u)v_n|_\mu.\end{aligned}$$

Now, since $K * u \in L^{s'}$ and (v_n) satisfies the assumptions of Lemma A.2, $(K * u)v_n \to 0$ in L^μ. Therefore,

$$(K * u_n)v_n \to 0 \quad \text{in } L^\mu.$$

In the case that $v_n \to v$ in L^t, again one can assume that $u_n \to 0$ in L^s_{loc}. From the Hölder inequality and the fact that $|K * u_n|_{s'}$ is bounded one has that

$$\begin{aligned}|(K * u_n)v_n|_\mu &\leq |(K * u_n)(v_n - v)|_\mu + |(K * u_n)v|_\mu \\ &\leq |K * u_n|_{s'} |v_n - v|_t + |(K * u_n)v|_\mu \\ &\leq C_3 |v_n - v|_t + |(K * u_n)v|_\mu.\end{aligned}$$

Hence it suffices to show that

$$(K * u_n)v \to 0 \quad \text{in } L^\mu. \tag{A.2}$$

First let us see that

$$K * u_n \to 0 \quad \text{in } L^{s'}_{loc}. \tag{A.3}$$

Fix $R_1 > 0$. Since $K \in L^r$ and $r < \infty$, for any $\varepsilon > 0$, there is $R_2 > 0$ such that

$$|K|_{r,\mathbb{R}^N \setminus B_{R_2}} \le \varepsilon.$$

Put $K_1 := \chi_{B_{R_2}} K$ and $K_2 := K - K_1$ (here $\chi_{B_{R_2}}$ denotes the characteristic function of B_{R_2}). The following holds:

$$\begin{aligned}
|K_1 * u_n|^{s'}_{s',B_{R_1}} &\le \int_{B_{R_1}} \left(\int_{\mathbb{R}^N} |K_1(x-y)u_n(y)| dy \right)^{s'} dx \\
&\le \int_{B_{R_1}} \left(\int_{B_{R_1+R_2}} |K_1(x-y)u_n(y)| dy \right)^{s'} dx \\
&\le |K_1|^{s'}_r |u_n|^{s'}_{s,B_{R_1+R_2}}.
\end{aligned}$$

The last inequality follows from [50, Theorem 3.1.], a generalized form of the Young theorem on convolutions. Since $|K_2|_r \le \varepsilon$ and $(|u_n|_s)$ is bounded, it follows that

$$\begin{aligned}
|K * u_n|_{s',B_{R_1}} &\le |K_1 * u_n|_{s',B_{R_1}} + |K_2 * u_n|_{s',B_{R_1}} \\
&\le |K_1|_r |u_n|_{s,B_{R_1+R_2}} + |K_2|_r |u_n|_s \\
&\le C_4 |u_n|_{s,B_{R_1+R_2}} + C_5 \varepsilon.
\end{aligned}$$

Letting $n \to \infty$ and then $\varepsilon \to 0$ one gets (A.3) because R_1 was arbitrary.

Finally, since $v \in L^t$ and $(K * u_n)$ is a bounded sequence in $L^{s'}$ which satisfies (A.3), Lemma A.2 yields (A.2). □

Lemma A.4. *Let Λ be an open set in $\mathbb{R}^N$. Let $p \ge 2$ and $q \in [p-1, \infty)$. Set $r := \frac{q}{p-1}$. Then the map $f : L^q(\Lambda) \to L^r(\Lambda)$ given by $f(u) := |u|^{p-2}u$ is continuous.*

Proof. Let $u \in L^q(\Lambda)$. We first claim that any sequence (u_n) such that $u_n \to u$ in $L^q(\Lambda)$ has a subsequence (u_{n_k}) such that $f(u_{n_k}) \to f(u)$ in $L^r(\Lambda)$. Indeed, let (u_n) be a sequence in $L^q(\Lambda)$ such that $u_n \to u$ in $L^q(\Lambda)$. Lemma A.1 in [59] asserts that, there exist a subsequence (u_{n_k}) of (u_n) and $g \in L^q(\Lambda)$ such that,

$$u_{n_k} \to u \quad \text{a.e. on } \Lambda \quad \text{ and } \quad |u_{n_k}|,\, |u| \le g \quad \text{a.e. on } \Lambda.$$

This yields that $f(u_{n_k}) - f(u) \to 0$ a.e. on Λ and

$$|f(u_{n_k}) - f(u)|^r = \left| |u_{n_k}|^{p-2}u_{n_k} - |u|^{p-2}u \right|^{\frac{q}{p-1}} \le C(|u_{n_k}|^q + |u|^q) \le C g^q \quad \text{a.e. on } \Lambda,$$

where C denotes different positive constants. Thus, from the Lebesgue dominated convergence theorem we obtain that $f(u_{n_k}) \to f(u)$ in $L^r(\Lambda)$.

The above claim yields that f is continuous at u. Indeed, if there were (u_n) in $L^q(\Lambda)$ such that $u_n \to u$ in $L^q(\Lambda)$ and $f(u_n)$ does not converge to $f(u)$ in $L^r(\Lambda)$ then it would exist $\varepsilon_0 > 0$ and a subsequence (v_n) of (u_n) such that

$$|f(v_n) - f(u)|_r \geq \varepsilon_0 \quad \forall n \in \mathbb{N}. \tag{A.4}$$

Since $v_n \to u$ in $L^q(\Lambda)$, by the first claim, v_n would have a subsequence (v_{n_k}) such that $f(v_{n_k}) \to f(u)$ in $L^r(\Lambda)$. It contradicts (A.4). Therefore, f is continuous at u. □

Lemma A.5. *Let $q > 1$ and $s \in [q, \infty)$. Set $r := \frac{s}{q}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable function with*

$$|f'(u)| \leq C|u|^{q-1} \quad \textit{for all} \quad u \in \mathbb{R}.$$

If (u_n) is a bounded sequence in L^s such that $u_n \to u$ in L^s_{loc}, then

$$f(u_n) - f(u_n - u) \to f(u) \quad \textit{in } L^r.$$

Proof. Since for every $t \in (0,1)$ one has that $|tu_n + (1-t)(u_n - u)| = |u_n - (1-t)u| \leq |u_n| + |u|$, the mean value theorem asserts that, almost everywhere on $\mathbb{R}^N$,

$$|f(u_n) - f(u_n - u)| \leq C[|u_n| + |u|]^{q-1}|u|.$$

For $R > 0$, from the Hölder inequality one gets that

$$\begin{aligned} |f(u_n) - f(u_n - u)|_{r,\mathbb{R}^N \setminus B_R} &\leq C_0 \left|[|u_n| + |u|]^{q-1}|u|\right|_{r,\mathbb{R}^N \setminus B_R} \\ &\leq C_1 \left[|u_n|_s^{q-1} + |u|_s^{q-1}\right] |u|_{s,\mathbb{R}^N \setminus B_R} \\ &\leq C_2 |u|_{s,\mathbb{R}^N \setminus B_R} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

Here C_i denotes positive constants. Since $|f(u)| \leq \frac{C}{q}|u|^q$ for all $u \in \mathbb{R}$, one also has that

$$|f(u)|_{r,\mathbb{R}^N \setminus B_R} \leq \frac{C}{q} |u|^q_{s,\mathbb{R}^N \setminus B_R}.$$

Therefore, since $|u|_{s,\mathbb{R}^N \setminus B_R} \to 0$ as $R \to \infty$, for $\varepsilon > 0$ given, there exists $R > 0$ such that

$$\begin{aligned} |f(u_n) - f(u_n - u) - f(u)|_{r,\mathbb{R}^N \setminus B_R} &\leq |f(u_n) - f(u_n - u)|_{r,\mathbb{R}^N \setminus B_R} + |f(u)|_{r,\mathbb{R}^N \setminus B_R} \\ &\leq \varepsilon/2. \end{aligned} \tag{A.5}$$

On the other hand, since $u_n \to u$ strongly in $L^s(B_R)$, Lemma A.4 insures that

$$f(u_n) - f(u_n - u) \to f(u) \quad \text{strongly in } L^r(B_R),$$

i.e. for n sufficiently large one has that

$$|f(u_n) - f(u_n - u) - f(u)|_{r,B_R} \leq \varepsilon/2. \tag{A.6}$$

Finally, from (A.5) and (A.6) we obtain the conclusion. □

Lemma A.6. *Let $q \geq 2$ and $s \in [q-1, \infty)$ Set $r := \frac{s}{q-1}$. If (u_n) is a bounded sequence in L^s such that $u_n \to u$ in L^s_{loc}, then*

$$|u_n|^{q-2}u_n - |u_n - u|^{q-2}(u_n - u) \to |u|^{q-2}u \quad \text{in } L^r. \tag{A.7}$$

$$|u_n|^{q-1} - |u_n - u|^{q-1} \longrightarrow |u|^{q-1} \quad \text{in } L^r. \tag{A.8}$$

Proof. If $q > 2$, (A.7) and (A.8) follow from Lemma A.5 taking $f(u) := |u|^{q-2}u$ and $f(u) := |u|^{q-1}$, respectively. If $q = 2$, (A.7) clearly holds, while (A.8) is an easy consequence of the Lebesgue dominated convergence theorem, since

$$\begin{aligned} \left||u_n|^{q-1} - |u_n - u|^{q-1} - |u|^{q-1}\right|^r &= \left||u_n| - |u_n - u| - |u|\right|^s \\ &\leq \left(||u_n| - |u_n - u|| + |u|\right)^s \\ &\leq (2|u|)^s. \end{aligned}$$

□

Lemma A.7. *Let $p \geq 2$ and $q \in [p, \infty)$. Let $v \in L^q$. If (u_n) is a bounded sequence in L^q such that $u_n \to u$ in L^q_{loc}, then*

$$|u_n|^{p-2}u_n v \to |u|^{p-2}uv \quad \text{in } L^{\frac{q}{p}}.$$

Proof. Set $r := \frac{q}{p-1}$ and consider the map $f : L^q \to L^r$ given by

$$f(w) := |w|^{p-2}w.$$

Lemma A.4 asserts that f is continuous and hence

$$f(u_n) \to f(u) \quad \text{in } L^r_{loc}.$$

Moreover, $(|f(u_n)|_r)$ is bounded because (u_n) is a bounded sequence in L^q. Therefore, from Lemma A.2

$$f(u_n)v \longrightarrow f(u)v \quad \text{in} \quad L^{\frac{q}{p}}.$$

□

We now have the ingredients to prove Proposition A.1.

Proof of Proposition A.1. First note that

$$\frac{2N-\alpha}{N-2} - \left(\frac{2N}{N-2}\right)\left(\frac{2N-\alpha-2\delta}{2N-2\delta}\right) = \frac{2\alpha\delta}{(N-2)(2N-2\delta)} \to 0 \quad \text{as } \delta \to 0.$$

Therefore, as $p < \frac{2N-\alpha}{N-2}$, we can choose $\delta \in (0, N-\alpha)$ such that

$$p < \left(\frac{2N}{N-2}\right)\left(\frac{2N-\alpha-2\delta}{2N-2\delta}\right). \tag{A.9}$$

Let us set $K(x) := \frac{1}{|x|^\alpha}$, $r_1 := \frac{N-\delta}{\alpha}$ and $r_2 := \frac{N+\delta}{\alpha}$. Write $K := K_1 + K_2$ with $K_1 \in L^{r_1}$ and $K_2 \in L^{r_2}$. For example, you can take $K_1 := \chi_{B_{r_1}} K$ and $K_2 := K - K_1$ (here $\chi_{B_{r_1}}$ denotes the characteristic function of B_{r_1}).

For $i = 1, 2$, consider $\mathbb{D}_i : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathbb{D}_i(u) := \int_{\mathbb{R}^N} (K_i * |u|^p)\, |u|^p.$$

We claim that this map is well defined and continuous. Indeed, setting $s_i := \frac{2r_i}{2r_i - 1}$ one has that $1 < s_2 < s_1$ and then, by (A.9), $s_i p \in (2, 2^*)$. One also has the continuous operator

$$\begin{array}{ccc} L^{s_i p}(\Omega) & \longrightarrow & L^{s_i}(\Omega) \\ u & \longmapsto & |u|^p. \end{array}$$

Continuity of $\mathbb{D}_i$ is then a consequence of continuous Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^{s_i p}(\Omega)$$

and Lemma A.3 with $t = s = s_i$, $r = r_i$ and $\mu = 1$.

Observe that $\mathbb{D}_i' : H_0^1(\Omega) \to H^{-1}(\Omega)$ is given by

$$\mathbb{D}_i'(u)(v) = 2p \int_{\mathbb{R}^N} (K_i * |u|^p)\, |u|^{p-2} uv.$$

Let (u_n) be a sequence in $H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then u_n is a bounded sequence in $H_0^1(\Omega)$ and so in $L^{s_i p}(\Omega)$. Moreover, after passing to a subsequence we have that $u_n \to u$ a.e. on Ω and $u_n \to u$ in $L^{s_i p}_{loc}(\Omega)$.

Let $v \in H_0^1(\Omega)$. Then $v \in L^{s_i p}(\Omega)$ and by Lemma A.7 with $q = s_i p$,

$$|u_n|^{p-2} u_n v \to |u|^{p-2} uv \quad \text{in } L^{s_i}(\Omega).$$

We also have

$$|u_n|^p \to |u|^p \quad \text{in } L^{s_i}_{loc}(\Omega). \tag{A.10}$$

Thus, from Lemma A.3 with $t = s = s_i$, $r = r_i$ and $\mu = 1$, we conclude

$$\mathbb{D}'(u_n)v \to \mathbb{D}'(u)v \quad \text{for all } v \in H_0^1(\Omega).$$

Now note that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{D}_i(u_n) - \mathbb{D}_i(u_n - u) &= \int_{\mathbb{R}^N} (K_i * |u_n|^p)\, |u_n|^p - \int_{\mathbb{R}^N} (K_i * |u_n - u|^p)\, |u_n - u|^p \\ &= \int_{\mathbb{R}^N} (K_i * (|u_n|^p - |u_n - u|^p))\, (|u_n|^p - |u_n - u|^p) \\ &+ 2 \int_{\mathbb{R}^N} (K_i * (|u_n|^p - |u_n - u|^p))\, |u_n - u|^p. \end{aligned}$$

Applying Lemma A.6 with $q = p + 1$ and $s = s_i p$, we obtain that

$$|u_n|^p - |u_n - u|^p \to |u|^p \quad \text{strongly in } L^{s_i}(\Omega). \tag{A.11}$$

Moreover, since $|u_n - u|^p \to 0 \in L^{s_i}_{loc}(\Omega)$, by Lemma A.3 with $t = s = s_i$, $r = r_i$ and $\mu = 1$, we have that

$$\begin{aligned} (K_i * (|u_n|^p - |u_n - u|^p))\,(|u_n|^p - |u_n - u|^p) &\to (K_i * |u|^p)\,|u|^p && \text{in } L^1 \\ (K_i * (|u_n|^p - |u_n - u|^p))\,|u_n - u|^p &\to 0 && \text{in } L^1. \end{aligned}$$

Therefore,

$$\mathbb{D}_i(u_n) - \mathbb{D}_i(u_n - u) \to \mathbb{D}_i(u) \quad \text{in} \quad \mathbb{R}.$$

On the other hand, observe that, for every $n \in \mathbb{N}$,

$$\begin{aligned} (K_i * |u_n|^p)\,|u_n|^{p-2}u_n &- (K_i * |u_n - u|^p)\,|u_n - u|^{p-2}(u_n - u) \\ &= (K_i * |u_n|^p)\,(|u_n|^{p-2}u_n - |u_n - u|^{p-2}(u_n - u)) \\ &\qquad + (K_i * (|u_n|^p - |u_n - u|^p))\,|u_n - u|^{p-2}(u_n - u). \end{aligned}$$

Applying Lemma A.6 with $q = p$ and $s = s_i p$, we obtain that

$$|u_n|^{p-2}u_n - |u_n - u|^{p-2}(u_n - u) \to |u|^{p-2}u \quad \text{in } L^{\frac{s_i p}{p-1}}(\Omega).$$

Moreover, since

$$|u_n - u|^{p-2}(u_n - u) \to 0 \quad \text{in } L^{\frac{s_i p}{p-1}}_{loc}$$

and (A.10), (A.11) hold, Lemma A.3 with $t = \frac{s_i p}{p-1}$, $r = r_i$ and $\mu = (s_i p)'$ (the conjugate exponent for $s_i p$), yields

$$\begin{aligned} (K_i * |u_n|^p)\,(|u_n|^{p-2}u_n - |u_n - u|^{p-2}(u_n - u)) &\to (K_i * |u|^p)|u|^{p-2}u && \text{in } L^{(s_i p)'}(\Omega) \\ (K_i * (|u_n|^p - |u_n - u|^p))\,|u_n - u|^{p-2}(u_n - u) &\to 0 && \text{in } L^{(s_i p)'}(\Omega). \end{aligned}$$

Therefore,

$$(K_i * |u_n|^p)\,|u_n|^{p-2}u_n - (K_i * |u_n - u|^p)\,|u_n - u|^{p-2}(u_n - u) \to (K_i * |u|^p)\,|u|^{p-2}u$$

in $L^{(s_i p)'}(\Omega)$, and from the continuous embedding $L^{(s_i p)'}(\Omega) \hookrightarrow H^{-1}(\Omega)$, we deduce that

$$\mathbb{D}_i'(u_n) - \mathbb{D}_i'(u_n - u) \to \mathbb{D}_i'(u) \quad \text{in } H^{-1}(\Omega).$$

Finally, since $\mathbb{D}(u) = \mathbb{D}_1(u) + \mathbb{D}_2(u)$ and $\mathbb{D}'(u) = \mathbb{D}_1'(u) + \mathbb{D}_2'(u)$, the conclusion of this lemma follows. $\square$

Appendix B

Proof of Proposition 2.2

This appendix is devoted to the proof of Proposition 2.2, namely

Proposition B.1. *If $p \geq 2$, the functional*

$$J_V(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{2p}\mathbb{D}(u)$$

is of class $\mathcal{C}^2$ and

$$J'_V(u)v = \langle u, v\rangle_V - \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}uv.$$

We shall split the proof of Proposition 2.2 in the proof of some claims.

Claim 1. *The functional*

$$\psi : H_0^1(\Omega) \to \mathbb{R}, \qquad \psi(u) = \|u\|^2,$$

is of class $\mathcal{C}^\infty$,

$$\psi'(u)v = 2\langle u, v\rangle, \qquad \psi''(u)(v, w) = 2\langle v, w\rangle,$$

and $\psi^{(k)} = 0$ for all $k \geq 3$.

Proof of claim 1. Let $u, v \in H_0^1(\Omega)$. From

$$\|u + tv\|^2 = \|u\|^2 + 2t\langle u, v\rangle + t^2\|v\|^2,$$

we obtain

$$\lim_{t\to 0} \frac{\|u + tv\|^2 - \|u\|^2}{t} = 2\langle u, v\rangle$$

and, since the function $v \mapsto \langle u, v\rangle$ is linear and continuous, we conclude that ψ is Gâteaux differentiable and $\psi'(u)v = 2\langle u, v\rangle$ for all $v \in H_0^1(\Omega)$. Notice that

$$\psi' : H_0^1(\Omega) \to H^{-1}(\Omega), \qquad \psi'(u) = 2\langle u, \cdot\rangle$$

is a linear map. From

$$|\psi'(u)v| = 2|\langle u, v\rangle| \le 2\|u\|\|v\| = 2\|u\| \qquad \text{if} \quad \|v\| = 1,$$

it follows that

$$\|\psi'(u)\|_{H^{-1}(\Omega)} := \sup_{\|v\|=1} |\psi'(u)v| \le 2\|u\| \qquad \forall u \in H_0^1(\Omega).$$

Therefore, $\psi' : H_0^1(\Omega) \to H^{-1}(\Omega)$ is continuous, and so ψ is of class $\mathcal{C}^1$. As ψ' is linear and continuous, we have that ψ is of class $\mathcal{C}^2$, $\psi''(u) = \psi'$ and that $\psi^{(k)} = 0$ for all $k \ge 3$. This concludes the proof. □

Claim 2. *If $p \ge 2$, the functional*

$$\mathbb{D} : H_0^1(\Omega) \to \mathbb{R}, \qquad \mathbb{D}(u) = \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p \right) |u|^p,$$

is Gâteaux differentiable and

$$\mathbb{D}'(u)v = 2p \int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p \right) |u|^{p-2} uv \quad \forall v \in H_0^1(\Omega).$$

Proof of claim 2. Let $u, v \in H_0^1(\Omega)$. Set $r := \frac{2N}{2N-\alpha}$. Since $pr \in (2, \frac{2N}{N-2})$, one has that $u, v \in L^{pr}(\Omega)$.

For each $x \in \Omega$, consider the function $f : [-1, 1] \to \mathbb{R}$ given by

$$f(t) = \left(\frac{1}{|x|^\alpha} * |u + tv|^p \right) |u(x) + tv(x)|^p.$$

This function is of class $\mathcal{C}^1$ and its derivative is given by

$$\begin{aligned} f'(t) = &\left(\frac{1}{|x|^\alpha} * p|u + tv|^{p-2}(u + tv)v \right) |u(x) + tv(x)|^p \\ &+ \left(\frac{1}{|x|^\alpha} * |u + tv|^p \right) p|u(x) + tv(x)|^{p-2}(u(x) + tv(x))v(x). \end{aligned}$$

By the mean value theorem, for each $0 < |t| < 1$, there exists $s_x \in (0, 1)$ such that

$$\left| \frac{f(t) - f(0)}{t} \right| = |f'(s_x t)|.$$

Therefore, since

$$\begin{aligned} |f'(s_x t)| \le &\, p \left(\frac{1}{|x|^\alpha} * (|u| + |v|)^{p-1}|v| \right) (|u(x)| + |v(x)|)^p \\ &+ p \left(\frac{1}{|x|^\alpha} * (|u| + |v|)^p \right) (|u(x)| + |v(x)|)^{p-1}|v(x)| =: h(x), \end{aligned}$$

we obtain

$$\left| \frac{\left(\frac{1}{|x|^\alpha} * |u+tv|^p\right)|u(x)+tv(x)|^p - \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u(x)|^p}{t} \right| \le h(x).$$

Since $\frac{\alpha}{N} + \frac{2}{r} = 2$ and $|u+tv|^p \in L^r$ for all $t \in [-1,1]$, the Hardy-Littlewood-Sobolev inequality (2.4) guarantees that

$$\frac{\left(\frac{1}{|x|^\alpha} * |u+tv|^p\right)|u+tv|^p - \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^p}{t} \in L^1(\Omega) \quad \text{for all } 0<|t|<1.$$

Moreover, since $(|u|+|v|)^{p-1} \in L^{\frac{pr}{p-1}}$ and $|v| \in L^{pr}(\Omega)$, by the Hölder inequality, $(|u|+|v|)^{p-1}|v| \in L^r(\Omega)$. Therefore, by the Hardy-Littlewood-Sobolev inequality (2.4), $h \in L^1(\Omega)$.

Finally, since, for each $x \in \Omega$, f is differentiable in $t=0$, we have that

$$\begin{aligned}\lim_{t\to 0} &\frac{\left(\frac{1}{|x|^\alpha} * |u+tv|^p\right)|u(x)+tv(x)|^p - \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u(x)|^p}{t} \\ &= p\left(\frac{1}{|x|^\alpha} * |u|^{p-2}uv\right)|u(x)|^p + p\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u(x)|^{p-2}u(x)v(x).\end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}\lim_{t\to 0}\frac{\mathbb{D}(u+tv)-\mathbb{D}(u)}{t} &= p\int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^{p-2}uv\right)|u(x)|^p dx \\ &\quad + p\int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u(x)|^{p-2}u(x)v(x)dx \\ &= 2p\int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}uv.\end{aligned}$$

Now, for each $u \in H_0^1(\Omega)$, the function $T : H_0^1(\Omega) \to \mathbb{R}$, given by

$$Tv := 2p\int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}uv$$

is clearly linear. Furthermore, observe that $|u|^{p-2}uv \in L^r(\Omega)$ and so, by the Hardy-Littlewood-Sobolev inequality (2.4) and the Hölder inequality, we obtain

$$\begin{aligned}|Tv| &\le 2pC||u|^p|_r||u|^{p-2}uv|_r \\ &\le 2pC|u|^p_{pr}\left(|u|^{p-2}_{\frac{pr}{p-2}}|u|_{pr}|v|_{pr}\right) \\ &\le 2pC|u|^{p+1}_{pr}|u|^{p-2}_{\frac{pr}{p-2}}|v|_{pr},\end{aligned}$$

where C is a positive constant. This proves that T is continuous. Consequently, $\mathbb{D}$ is Gâteaux differentiable and

$$\mathbb{D}'(u)v = 2p\int_\Omega \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}uv. \qquad \square$$

Let $u \in L^{pr}(\Omega)$. The Hölder inequality asserts that $|u|^{p-2}u \in L^{\frac{pr}{p-1}}(\Omega)$. Moreover, since $\frac{\alpha}{N} + \frac{1}{r} = 1 + \frac{\alpha}{2N}$, then

$$\left(\frac{1}{|x|^\alpha} * |u|^p\right) \in L^{\frac{2N}{\alpha}}(\Omega),$$

cf.[39, Section 4.3(9)]. Hence, if q is given by $\frac{1}{q} = \frac{\alpha}{2N} + \frac{p-1}{pr}$, the Hölder inequality insures that the map

$$\Phi : L^{pr}(\Omega) \to L^q(\Omega), \qquad \Phi(u) = \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u,$$

is well defined.

Now, if X, Y are Banach spaces, we denote by

$$\mathcal{B}(X,Y) := \{T : X \to Y : T \text{ is linear and continuous}\}.$$

Recall that this is a Banach space under the norm

$$\|T\|_{\mathcal{B}(X,Y)} := \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

Consider the map

$$L : L^q(\Omega) \to \mathcal{B}(L^{pr}(\Omega), \mathbb{R}), \qquad u \mapsto L_u,$$

where L_u is given by

$$L_u(v) := \int_\Omega uv.$$

Since $q = \frac{2Np}{2Np-2N+\alpha} > 1$ and $\frac{1}{q} + \frac{1}{pr} = 1$, we can deduce that L is a linear isometry (see for example [12, Theorem 4.11]).

The map $\mathbb{D}' : H_0^1(\Omega) \to H^{-1}(\Omega)$ is obtained as

$$\mathbb{D}' = 2p(L \circ \Phi \circ \iota),$$

where $\iota : H_0^1(\Omega) \hookrightarrow L^{pr}(\Omega)$ is the continuous Sobolev embedding. Therefore, to see that $\mathbb{D}$ is of class $\mathcal{C}^1$, it suffices to show the following claim.

Claim 3. *If $p \geq 2$, Φ is continuous.*

Proof. Let $u \in L^{pr}(\Omega)$. We first assert that any sequence (u_n) such that $u_n \to u$ in $L^{pr}(\Omega)$ has a subsequence (u_{n_k}) such that $\Phi(u_{n_k}) \to \Phi(u)$ in $L^q(\Omega)$. Indeed, let (u_n) be a sequence in $L^{pr}(\Omega)$ such that $u_n \to u$ in $L^{pr}(\Omega)$. Lemma A.1 in [59] insures the existence of a subsequence (u_{n_k}) of (u_n) and $g \in L^{pr}(\Omega)$ such that,

$$u_{n_k} \to u \quad \text{a.e. on } \Omega \qquad \text{and} \qquad |u_{n_k}|,\, |u| \leq g \quad \text{a.e. on } \Omega.$$

This yields that $\Phi(u_{n_k}) - \Phi(u) \to 0$ a.e. on Ω and

$$\begin{aligned}
|\Phi(u_{n_k}) - \Phi(u)| &= \left|\left(\frac{1}{|x|^\alpha} * |u_{n_k}|^p\right)|u_{n_k}|^{p-2}u_{n_k} - \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}u\right| \\
&\leq \left(\frac{1}{|x|^\alpha} * |u_{n_k}|^p\right)|u_{n_k}|^{p-1} + \left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-1} \\
&\leq 2\left(\frac{1}{|x|^\alpha} * g^p\right)g^{p-1} \quad \text{a.e. on } \Omega.
\end{aligned}$$

Since the map in the right-hand side of this inequality belongs to $L^q(\Omega)$, the Lebesgue dominated convergence theorem in L^q guarantees that $\Phi(u_{n_k}) \to \Phi(u)$ in $L^q(\Omega)$.

The above assertion yields that Φ is continuous at u. Indeed, if there were (u_n) in $L^{pr}(\Omega)$ such that $u_n \to u$ in $L^{pr}(\Omega)$ and $\Phi(u_n)$ does not converge to $\Phi(u)$ in $L^q(\Omega)$ then it would exist $\varepsilon_0 > 0$ and a subsequence (v_n) of (u_n) such that

$$|\Phi(v_n) - \Phi(u)|_q \geq \varepsilon_0 \quad \forall n \in \mathbb{N}. \tag{B.1}$$

Since $v_n \to u$ in $L^{pr}(\Omega)$, by the first assertion, v_n would have a subsequence (v_{n_k}) such that $\Phi(v_{n_k}) \to \Phi(u)$ in $L^q(\Omega)$. It contradicts (B.1). Therefore, Φ is continuous at u. □

Finally, since $\mathbb{D}' = 2p(L \circ \Phi \circ \iota)$ and L, ι are of class $\mathcal{C}^\infty$ (because they are linear and continuous), in order to see that $\mathbb{D}$ is of class $\mathcal{C}^2$, it suffices to show the following claim.

Claim 4. *If $p \geq 2$, Φ is of class $\mathcal{C}^1$ and its derivative $\Phi'(u) : L^{pr}(\Omega) \to L^q(\Omega)$ is given by*

$$\Phi'(u)v = p\left(\frac{1}{|x|^\alpha} * |u|^{p-2}uv\right)|u|^{p-2}u + (p-1)\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}v.$$

Proof. Let $u, v \in L^{pr}(\Omega)$. For each $x \in \Omega$, consider the function $f : [-1,1] \to \mathbb{R}$ given by

$$f(t) = \left(\frac{1}{|x|^\alpha} * |u+tv|^p\right)|u(x)+tv(x)|^{p-2}(u(x)+tv(x)).$$

This function is of class $\mathcal{C}^1$ and its derivative is given by

$$\begin{aligned} f'(t) = &\left(\frac{1}{|x|^\alpha} * p|u+tv|^{p-2}(u+tv)v\right)|u(x)+tv(x)|^{p-2}(u(x)+tv(x)) \\ &+ \left(\frac{1}{|x|^\alpha} * |u+tv|^p\right)(p-1)|u(x)+tv(x)|^{p-2}v(x). \end{aligned}$$

Set

$$Tv := p\left(\frac{1}{|x|^\alpha} * |u|^{p-2}uv\right)|u|^{p-2}u + (p-1)\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}v.$$

From

$$\lim_{t\to 0} \frac{f(t)-f(0)}{t} = f'(0),$$

we can deduce that

$$\frac{\Phi(u+tv)-\Phi(u)}{t} \to Tv \quad \text{a.e in } \Omega \quad \text{as } t \to 0.$$

On the other hand, by the mean value theorem, for each $0 < |t| < 1$, there exists $s_x \in (0,1)$ such that

$$\left|\frac{f(t)-f(0)}{t}\right| = |f'(s_x t)|.$$

Therefore, since

$$\begin{aligned} |f'(s_x t)| \leq\ & p\left(\frac{1}{|x|^\alpha} * (|u|+|v|)^{p-1}|v|\right)(|u(x)|+|v(x)|)^{p-1} \\ &+ (p-1)\left(\frac{1}{|x|^\alpha} * (|u|+|v|)^p\right)(|u(x)|+|v(x)|)^{p-2}|v(x)| := h(x), \end{aligned}$$

we obtain
$$\left|\frac{\Phi(u+tv)-\Phi(u)}{t}\right| \le h \quad \text{a.e in } \Omega.$$
Notice that $h \in L^q(\Omega)$. Thus, by the Lebesgue dominated convergence theorem in L^q, we obtain that
$$\frac{\Phi(u+tv)-\Phi(u)}{t} \to Tv \quad \text{in } L^q(\Omega) \quad \text{as } t \to 0.$$
Accordingly,
$$\lim_{t\to 0}\frac{\Phi(u+tv)-\Phi(u)}{t} = Tv \quad \text{for all } v \in L^{pr}(\Omega).$$
Now, for each $u \in H_0^1(\Omega)$, the function T is clearly linear. Moreover, the Hölder inequality, together with [39, Section 4.3(9)], implies

$$\begin{aligned}
|Tv|_q &\le p\left|\frac{1}{|x|^\alpha} * |u|^{p-2}uv\right|_{\frac{2N}{\alpha}} \left||u|^{p-1}\right|_{\frac{pr}{p-1}} + (p-1)\left|\frac{1}{|x|^\alpha} * |u|^p\right|_{\frac{2N}{\alpha}} \left||u|^{p-2}v\right|_{\frac{pr}{p-1}} \\
&\le pC_1\left||u|^{p-2}uv\right|_r \left||u|^{p-1}\right|_{\frac{pr}{p-1}} + (p-1)C_2\left||u|^p\right|_r \left||u|^{p-2}v\right|_{\frac{pr}{p-1}} \\
&\le pC_1|u|_{pr}^{p-1}|v|_{pr}|u|_{pr}^{p-1} + (p-1)C_2|u|_{pr}^p|u|_{pr}^{p-2}|v|_{pr} \\
&= (pC_1+(p-1)C_2)|u|_{pr}^{2p-2}|v|_{pr}.
\end{aligned}$$

where C_i are positive constants. Thus, T is continuous. This proves that Φ is Gâteaux differentiable and that
$$\Phi'(u)v = p\left(\frac{1}{|x|^\alpha} * |u|^{p-2}uv\right)|u|^{p-2}u + (p-1)\left(\frac{1}{|x|^\alpha} * |u|^p\right)|u|^{p-2}v.$$
Arguing in a similar way as in the proof of Claim 3, we can conclude that
$$\Phi' : L^{pr}(\Omega) \to \mathcal{B}(L^{pr}(\Omega), L^q(\Omega))$$
is continuous. □

Proof of Proposition B.1. The statement of this proposition is an immediate consequence of the above claims. □

Appendix C

The genus of an orbit space

In Theorems 1.1, 1.2, 1.4 and 1.6, the number of solutions is given in terms of the genus of the orbit space Z/G. We shall give estimates for it in terms of the Γ-genus of Z.

Let us recall the notion of Γ-genus, see [6] for further details. Let Γ be a compact Lie group. The *join* of the Γ-spaces $X_1, \ldots, X_m$ is the space

$$X_1 * \cdots * X_m := \left\{ [s_1, x_1, \ldots, s_m, x_m] : s_i \in [0,1],\ \sum_{i=1}^{m} s_i = 1,\ x_i \in X_i \right\}$$

where $[s_1, x_1, \ldots, s_m, x_m] = [t_1, y_1, \ldots, t_m, y_m]$ if, for each $i = 1, ..., m$, either $s_i = t_i$ and $x_i = y_i$ or $s_i = t_i = 0$. This is again a Γ-space with the action

$$g[s_1, x_1, \ldots, s_m, x_m] := [s_1, gx_1, \ldots, s_m, gx_m].$$

The Γ-*genus* of a nonempty Γ-space X is the smallest $m \in \mathbb{N}$ such that there exist closed subgroups $\Gamma_1, ..., \Gamma_m$ of Γ with $\Gamma_i \neq \Gamma$ and a continuous Γ-equivariant map

$$f : X \to \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m,$$

i.e. $f(gx) = gf(x)$ for all $x \in X$, $g \in \Gamma$. We denote it by Γ-genus(X). If no such map exists we set Γ-genus$(X) := \infty$.

If $\Gamma = \mathbb{Z}/2$ then $\mathbb{Z}/2 * \cdots * \mathbb{Z}/2 \cong \mathbb{S}^{m-1}$ with the action given by multiplication, so that the $\mathbb{Z}/2$-genus is just the Krasnoselskii genus.

Let Γ and Λ be compact Lie groups, $\phi : \Gamma \to \Lambda$ be a continuous epimorphism, $K := \ker \phi$ and X a Γ-space. Then Λ acts on the orbit space X/K as follows: for each $x \in X$, $g \in \Lambda$ and some $\gamma \in \Gamma$ such that $\phi(\gamma) = g$ we define

$$g \cdot Kx := K(\gamma x). \tag{B.1}$$

This action is well defined because K is a normal subgroup of Γ. The quotient map $q : X \to X/K$ satisfies that $q(\gamma x) = \phi(\gamma) \cdot q(x)$ for any $\gamma \in \Gamma$, $x \in X$. The following result holds:

Lemma C.1. Γ-*genus*$(X) = \Lambda$-*genus*(X/K).

Proof. Let $\Lambda_1, ..., \Lambda_m$ be closed subgroups of Λ, $\Lambda_i \neq \Lambda$, and let

$$f : X/K \to \Lambda/\Lambda_1 * \cdots * \Lambda/\Lambda_m$$

be a continuous Λ-equivariant map. We define $\Gamma_i := \{\gamma \in \Gamma : \phi(\gamma) \in \Lambda_i\}$. Then ϕ induces homeomorphisms $\phi_i : \Gamma/\Gamma_i \to \Lambda/\Lambda_i$ that satisfy $\phi_i(\gamma\Gamma_i) = \phi(\gamma)\Lambda_i$, which in turn induce an homeomorphism

$$\phi_1 * \cdots * \phi_m : \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m \to \Lambda/\Lambda_1 * \cdots * \Lambda/\Lambda_m,$$

defined in the obvious way. The map $F : X \to \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m$ given by

$$F := (\phi_1 * \cdots * \phi_m)^{-1} \circ f \circ q$$

is continuous and Γ-equivariant. Hence,

$$\Gamma\text{-genus}(X) \leq \Lambda\text{-genus}(X/K).$$

Conversely, let $\Gamma_1, ..., \Gamma_m$ be closed subgroups of Γ, $\Gamma_i \neq \Gamma$, and let

$$F : X \to \Gamma/\Gamma_1 * \cdots * \Gamma/\Gamma_m$$

be a continuous Γ-equivariant map. We define $\Lambda_i := \phi(\Gamma_i)$ and set $\phi_1 * \cdots * \phi_m$ as above. Observe that $(\phi_1 * \cdots * \phi_m) \circ F$ is continuous and constant on $q^{-1}(Kx)$ for each $x \in X$. Hence, it induces a map

$$f : X/K \to \Lambda/\Lambda_1 * \cdots * \Lambda/\Lambda_m$$

which is continuous and Λ-equivariant. Therefore,

$$\Lambda\text{-genus}(X/K) \leq \Gamma\text{-genus}(X).$$

□

Let us look at an example. Let Γ be the subgroup of $O(4n)$ spanned by ρ and γ, where

$$\rho(y, z) := (e^{\pi i/m}y, e^{\pi i/m}z), \qquad \gamma(y, z) := (-\overline{z}, \overline{y}), \qquad \forall (y, z) \in \mathbb{C}^n \times \mathbb{C}^n \equiv \mathbb{R}^{4n},$$

$\overline{z} = (\overline{z}_1, ..., \overline{z}_n)$ and $\overline{z}_i$ is the conjugate of z_i. Note that ρ is of order $2m$, γ is of order 4, $\rho^m = \gamma^2$ and $\gamma\rho = \rho^{-1}\gamma$. Let us consider the homomorphism $\phi : \Gamma \to \mathbb{Z}/2$ given by $\phi(\rho) = 1$ and $\phi(\gamma) = -1$. Then $G := \ker \phi$ is the cyclic subgroup spanned by ρ. The following holds:

Proposition C.2. *(a) genus*$(\mathbb{S}^{4n-1}/G) \geq 2n + 1$.
(b) If $m \geq 3$, then there exists $a_0 > 1$ such that dist$(\gamma x, Gx) \geq a_0\mu(Gx)$ *for all* $x \in \mathbb{S}^{4n-1}$.

Proof. (a) Let us consider the cyclic subgroup of order 4 of Γ spanned by γ and denote it by $\mathbb{Z}/4$. The kernel of the restriction of ϕ to $\mathbb{Z}/4$ is the group $K = \{1, \gamma^2\}$. Lemma C.1, together with Theorem 1.2 of [5], yields

$$\text{genus}(\mathbb{S}^{4n-1}/K) = \mathbb{Z}/4\text{-genus}(\mathbb{S}^{4n-1}) \geq 2n + \frac{1}{2}.$$

As $K \subset G$ the quotient map $\mathbb{S}^{4n-1}/K \to \mathbb{S}^{4n-1}/G$ is well defined and is $\mathbb{Z}/2$-equivariant for the action defined in (B.1). Therefore,

$$\operatorname{genus}(\mathbb{S}^{4n-1}/G) \geq \operatorname{genus}(\mathbb{S}^{4n-1}/K).$$

Combining both inequalities one obtains the assertion.
(b) Note that $\gamma x \cdot \rho^j x = 0$ and, consequently, $\left|\gamma x - \rho^j x\right| = \sqrt{2}$ for any $x \in \mathbb{S}^{4n-1}$, $j = 1, ..., 2m$. On the other hand, $\mu(Gx) = \left|e^{\pi i/m} - 1\right| < \sqrt{2}$ if $m \geq 3$. Hence, taking $a_0 > 1$ such that $a_0 \left|e^{\pi i/m} - 1\right| < \sqrt{2}$, we get the conclusion. □

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