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## FACULTAD DE CIENCIAS

# TOPOLOGY OF SINGULARITIES OF REAL ANALYTIC FUNCTIONS

## TESIS

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*To a very beautiful soul: José Luis.*

“Never trust the storyteller. Only trust the story.”

*Neil Gaiman* —Fables and Reflections, The Sandman—

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# INTRODUCTION

The purpose of this work is to study open-book fibrations on the sphere $\mathbb{S}^5$ associated to real analytic mappings with an isolated singularity, and show that in many cases these open-books are new in singularity theory, in the sense that they cannot come from holomorphic singularities.

An open-book decomposition of a manifold $M$ consists of a submanifold $L$ of codimension 2 embedded in $M$ with trivial normal bundle, together with a locally trivial fibration of its complement, $\phi : M \setminus L \to \mathbb{S}^1$, which restricted to a neighbourhood $L \times \mathbb{D}^2$ of $L$ is of the form $\phi(x,t) = t/\|t\|$ with $(x,t) \in L \times (\mathbb{D}^2 \setminus \{0\})$. The manifold $L$ is called the binding of the open book and each fibre is called a page. The concept of open-book was introduced by H. E. Winkelnkemper in [68] and it has proved to be a very interesting concept in geometry and topology (see for instance Winkelnkemper's survey article [54]).

In the case of a holomorphic function with an isolated singularity, its Milnor fibration gives an open-book decomposition of every small sphere around the critical point. The binding is the link of the singularity and the pages are the Milnor fibres.

If we let $f\colon (\mathbb{R}^n,0) \to (\mathbb{R}^k,0)$ be a real analytic germ with an isolated critical point (or singularity) at the origin, and $\mathbb{S}^{n-1}_\varepsilon$ is a sufficiently small sphere centred at the origin, then the intersection $L_f = f^{-1}(0) \cap \mathbb{S}^{n-1}_\varepsilon$ is an $(n-k-1)$-submanifold of the sphere, called the link of the singularity. The link determines the topology of the corresponding singular variety $V = f^{-1}(0)$ in the sense that given $\varepsilon > 0$ small enough, there is a homeomorphism between the pair $\left(\mathbb{B}_\varepsilon, V \cap \mathbb{B}_\varepsilon\right)$ and the pair $\left(\operatorname{cone}(\mathbb{S}_\varepsilon), \operatorname{cone}(L_f)\right)$. In fact, up to diffeomorphism, the link is independent of the radius of the sphere and of the embedding of $V$ in $\mathbb{R}^n$ (see [17]). This invariant, the link, was introduced by Brauner in [6] and later Kähler took it up in [24], replacing the sphere by a polydisc. In fact, this innovation of using a "square sphere" is now a standard technique in Singularity Theory and it is used in this work in Chapter 3.

The Milnor Fibration Theorem is one of the main results in the study of the topology of singularities. This theorem associates a locally trivial fibration to every complex singularity:

**Milnor Fibration Theorem** ( [33, Th. 4.8])**.** Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a complex analytic germ with an isolated singularity at the origin, and let $L_f = f^{-1}(0) \cap \mathbb{S}_\varepsilon^{2n-1}$ be its link, where $\mathbb{S}_\varepsilon^{2n-1}$ is a $(2n-1)$-sphere, centred at the origin, of radius $\varepsilon$ sufficiently small. Then

$$\phi_f = \frac{f}{|f|}\colon \mathbb{S}_\varepsilon^{2n-1} \setminus L_f \to \mathbb{S}^1$$

is a $C^\infty$ locally trivial fibration.

As we mentioned above, the pair $(\mathbb{S}_\varepsilon^{2n-1}, \phi_f)$ is an open-book; *i.e.*, the sphere $\mathbb{S}_\varepsilon^{2n-1}$ can be seen as an open book where the binding is the link $L_f$ and the pages are the fibres of $\phi_f$.

On the other hand, J. Milnor shows in [33] that some real analytic germs also give rise to fibrations: Given a real analytic germ $f\colon (U \subset \mathbb{R}^{n+k},0) \to (\mathbb{R}^k,0)$ such that it is a submersion on a punctured neighbourhood $U\setminus\{0\}$ of the origin in $\mathbb{R}^{n+k}$, there exists a $C^\infty$ locally trivial fibration

$$\varphi\colon \mathbb{S}_\varepsilon^{n+k-1} \setminus N(L_f) \to \mathbb{S}^{k-1} ,$$

where $N(L_f)$ is a small tubular neighbourhood of the link in the sphere $\mathbb{S}_\varepsilon^{n+k-1}$.

The hypothesis of asking the origin to be an isolated critical point of $f$ is now called the Milnor condition. J. Milnor pointed out in [33, p. 100] that "the major weakness [of the theorem] is that the hypothesis is so strong that examples are very difficult to find" and asked "For which dimensions $n+k \geq k \geq 2$ do non-trivial examples exist?". A classification of the pairs $(n,k)$ for which such examples exists is given by E. Looijenga in [29] and by P. T. Church and K. Lamotke in [11]. In particular they proved that when $k=2$, such examples exist for all $n>0$. Further examples were given later by N. A'Campo [1], B. Perron [47] and others (e.g. E. Rees in [55]).

Let us remark that even when the Milnor condition is satisfied, in general it is not true that the projection $\varphi$ can be given by $\frac{f}{\|f\|}$ as in the complex case; for example, in [33, p. 99], Milnor presents a real polynomial function $f$ with isolated critical point at the origin, such that $\frac{f}{\|f\|}$ cannot be the projection of a smooth locally trivial fibration.

The problem of giving conditions to ensure that the map $\frac{f}{\|f\|}$ is the projection of the Milnor fibration of a function $f$ was first studied by A. Jacquemard in [22], and then by J. Seade in [60] and [61]; M. Ruas, J. Seade and A. Verjovsky in [57]; M. Ruas and R. N. A. dos Santos in [56]; J. L. Cisneros-Molina in [13]; R. N. A. dos Santos in [2], and others (e. g. [3, 12, 53]). In fact the results of [53] and [12] are specially relevant for this work. In [53], A. Pichon and J. Seade prove that given two holomorphic functions $f, g\colon \mathbb{C}^2 \to \mathbb{C}$ such that $f\bar{g}$ has isolated critical point, one has an associated Milnor fibration with projection map given by $\frac{f\bar{g}}{|f\bar{g}|}$. And J. L. Cisneros-Molina, J. Seade and J. Snoussi in [12] define the concept of $d$-regularity and prove that given a real analytic function $f\colon \mathbb{R}^n \to \mathbb{R}^k$ with an isolated critical point is $d$-regular if and only if its Milnor fibration has projection $\frac{f}{\|f\|}$. Actually, the results in [53] and [12] hold in a more general setting that we do not need for this thesis.

Now, one of the main challenges is to find examples which are sufficiently controlled to give rise to a beautiful geometry; *i.e.*, open-book decompositions, but which, at the same time, do not come from the holomorphic context. For example, consider the family of real analytic germs $f\colon (\mathbb{C}^2 \cong \mathbb{R}^4, 0) \to (\mathbb{C} \cong \mathbb{R}^2, 0)$ defined by $f(x, y) = x^p \bar{y} + \bar{x} y^q$ with $p, q \geq 2$. By [61], $f$ has a Milnor fibration with projection $\phi_f = \frac{f}{|f|}$. In [52] A. Pichon and J. Seade prove that the link $L_f$ is isotopic to the link of the holomorphic germ $g(x, y) = xy(x^{p+1} + y^{q+1})$, but the open-book decomposition given by the Milnor fibration of $f$ is not equivalent to the one given by the Milnor fibration of $g$.

In this work we consider real analytic germs of the form:

$$F(x, y, z) = f(x, y)\overline{g(x, y)} + z^r \colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)\ ,$$

where $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ and $g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are holomorphic germs such that $f\bar{g}$ has isolated critical point at the origin. The study of this type of singularities is inspired by [48–50], where A. Pichon studies Milnor fibrations associated to map-germs of the form $f + z^r$ with $f$ a holomorphic germ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}, 0)$. In fact, in a way, this work can be regarded as an extension of A. Pichon's method to the case when the singularities are no longer holomorphic but of the form $f\bar{g}+z^r$.

The main result of this work is that every germ $F$ as above has a Milnor open-book fibration on $\mathbb{S}^5$ and, although we show that $L_F$ is always homeomorphic to the link of a normal complex surface singularity, these open-books are new in Singularity Theory, *i.e.*, we prove that in many cases (at least), these open-books cannot be given by the Milnor fibration of a holomorphic germ $G$ from $(\mathbb{C}^3,0)$ to $(\mathbb{C},0)$.

We first give an explicit description of the link $L_F$ as a graph manifold. This is done by means of the link $L_{f\bar{g}}$ and of the Milnor fibre associated to $f\bar{g}$. In fact, we show that $L_F$ is always homeomorphic to the link of a normal complex surface singularity.

Graph manifolds are a very important class of 3-manifolds, which were introduced and classified by F. Waldhausen in [67]. A graph manifold is a 3-manifold that can be decomposed as a union of Seifert manifolds, glued together along tori $\mathbb{S}^1\times\mathbb{S}^1$. The name comes from the fact that there is a very convenient combinatorial description of a manifold of this type in terms of a graph, whose vertices are the fundamental parts (Seifert manifolds) and (decorated) edges stand for the description of the gluing of these pieces.

Given the aforementioned germ $F$, we also describe the homotopy type of its associated Milnor fibre $\mathscr{F}$ in terms of the Milnor fibre $\mathscr{F}_{f\bar{g}}$ associated to $f\bar{g}$.

We use two methods for showing that these open-books do not come from complex singularities:

**Different Binding:** For some examples it is proved that there does not exist a complex analytic germ $G\colon(\mathbb{C}^3,0)\to(\mathbb{C},0)$ with an isolated singularity at the origin, such that the link $L_G$ is homeomorphic to the link $L_F$, this is, $L_F$ cannot be realised as the link of a singularity in $\mathbb{C}^3$ (although $L_F$ is the link of a normal complex surface singularity). For this we use that the canonical class of a resolution of a hypersurface singularity must be integral, *i.e.*, complex hypersurface singularities are numerically Gorenstein. We exhibit germs $F$ as above for which no complex singularity having $L_F$ as its link can be numerically Gorenstein.

**Different Pages:** For some examples it is proved that if $(X,p)$ is a normal Gorenstein complex surface singularity whose link is homeomorphic to $L_F$, then this singularity cannot be smoothable, and therefore it cannot be realised in $\mathbb{C}^3$. For this we use Laufer's formula, which establishes a relationship amongst numerical invariants that every smoothing of a hypersurface singularity must satisfy. We compute the corresponding invariants for the singularities we envisage in this work, and show that (for certain families) these do not satisfy the conditions imposed by Laufer's theorem.

The thesis is organised as follows:

Chapter 1 presents well-known concepts and techniques used in the following chapters. The first of them is the Milnor Fibration Theorem in the real case, followed by a generalisation for real analytic $d$-regular functions with isolated critical point given by J. L. Cisneros-Molina, J. Seade and J. Snoussi in [12]. In this case the Milnor fibration gives an open-book on a sphere (with binding the link of the singularity) as in the holomorphic case. Then, we recall the notions of open-book and mapping torus, and give a way to construct an open-book from the mapping torus of a diffeomorphism of a surface. We present the Seifert manifolds and the Seifert invariants which determine a manifold of this type. Next we define plumbing of disc-bundles and show how a Seifert manifold can be represented as the boundary of a plumbing graph. Then, we describe the monodromy of the Milnor fibration of a holomorphic function $f\colon \mathbb{C}^2 \to \mathbb{C}$ as a quasi-periodic diffeomorphism beginning with a resolution graph of $f$, regarded as a plumbing tree of the link $L_f$, following the results of F. Michel and P. Du Bois (see for example [15] and [14]). Finally we present classical results about the resolution of complex surface singularities and the topology of their links.

Chapter 2 is devoted to the study of a special class of singularities whose link is a Seifert manifold: Let $p, q \in \mathbb{Z}$ be coprime, let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be defined by $f(x, y) = x^p + y^q$ and let $g\colon \mathbb{C}^2 \to \mathbb{C}$ be defined by $g(x, y) = xy$. We prove that the function $F = f\bar{g} + z^r$, where $r \geq 2$, has an isolated critical point at the origin and that both functions $f\bar{g}$ and $F$ have Milnor fibrations with projections $\frac{f\bar{g}}{|f\bar{g}|}$ and $\frac{F}{|F|}$ respectively. It is shown that the link $L_F$ is a Seifert manifold and the Seifert invariants of $L_F$ are determined explicitly. We give two families of examples of this type of singularities whose corresponding open-books cannot come from complex singularities. In each case we use one of the two reasons aforementioned: either the corresponding link $L_F$ cannot be realised in $\mathbb{C}^3$, or else, even if it does, the corresponding Milnor fibre cannot be homeomorphic to any smoothing of a normal Gorenstein complex surface singularity.

The main objective in Chapter 3 is the explicit description of the link $L_F$ in the general case; that is, $F = f\bar{g} + z^r$, where $f, g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are holomorphic germs such that $f\bar{g}$ has isolated critical point at the origin. Firstly it is proved that there is an open-book fibration on $L_F$ and that the corresponding monodromy is a quasi-periodic diffeomorphism. For this we use a complete invariant of the conjugation class in the mapping-class group of the diffeomorphisms of a surface: the Nielsen graph. Generalising A. Pichon's results (see for example [50] and [49]), it is established the relation between the resolution graph of a function and the Nielsen graph of a quasi-periodic diffeomorphism, weighted by the associated Nielsen invariants. Next, it is recalled the notion of graph manifold and it is presented the Waldhausen graph of a plumbing link $(M, L)$ where $M$ is a graph manifold and $L \subset M$ is a disjoint union of Seifert fibres. Finally, given a quasi-periodic diffeomorphism $h$ of a surface, which is the identity at the boundary of the surface, it is described an isomorphism between the Nielsen graph of $h$ and the Waldhausen graph of the pair $(M, L)$, where $M$ is the open-book constructed from the mapping torus of $h$ and $L$ is the corresponding binding. These results enable us to describe the monodromy of the Milnor fibration of the function $f\bar{g}$ as a quasi-periodic diffeomorphism and then we give explicitly the corresponding Nielsen graph. In this way one gets a description of the monodromy of the open-book $L_F$ in terms of the monodromy of the Milnor fibration of $f\bar{g}$, and a description of the link $L_F$ by its Waldhausen graph. A basic references for this chapter are W. Neumann [39] and A. Pichon [48].

In Chapter 4, following L. H. Kauffman and W. Neumann [25] and W. Neumann [38], it is given a description of the link $L_F$ as a cyclic suspension of the link $L_{f\bar{g}}$, using the notion of fibred knot and the associated open-book: a fibred knot $(\mathbb{S}^n, L)$ gives an structure of open-book for $\mathbb{S}^n$ and vice versa. Then it is proved that the function $F$ has Milnor fibration with projection $\frac{F}{|F|}$ and the homotopy type of the Milnor fibre $\mathscr{F}$ of $F$ is described as the join of the associated Milnor fibre to $f\bar{g}$ and $r$ points. Moreover, the monodromy of the Milnor fibration of $F$ is the join of the monodromies of the Milnor fibrations of $f\bar{g}$ and $z^r$. To end the chapter we show some examples which give new open-books. We do so using the same two types of reasons as for the families in Chapter 2.

# CHAPTER 1 PRELIMINARIES

In this chapter we introduce the basic definitions and results used in the following chapters. We start with the definition of link of a singularity, which is a key concept in this work, later we present one of the main theorems in the study of the topology of a singularity: the Milnor fibration theorem for real functions (polynomials and analytic) which says that under some conditions the complement of the link of a singularity fibres over $\mathbb{S}^1$.

Moreover, the Milnor fibration theorem gives an open-book fibration of the corresponding sphere with the link of the singularity as binding; such concepts are given together with the concept of mapping torus of a diffeomorphism, which allows us to construct open-books.

In order to give a description of the link of some singularities presented in Chapter 2, we present briefly an important class of 3-manifolds: the Seifert manifolds. They also appear as the boundary of some 4-manifolds obtained by plumbing 2-disc bundles over the sphere $\mathbb{S}^2$. In fact, such a 4-manifold can be represented via a plumbing graph.

Later, we will see that the resolution graph corresponding to a resolution of a singularity can be seen as a plumbing graph and it gives a very useful way to describe the monodromy of the Milnor fibration associated to the singularity.

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## 1.1 Link of a singularity

In this section we present a useful tool to study the topology of a singularity: the link and its conic structure. All the results presented here are due to Milnor in [33] for the case of real and complex polynomial functions with isolated singularity and we also give the corresponding versions for real analytic functions in more generality (see for example [8]).

**1.1 Proposition.** *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ be a real analytic germ such that the origin is an isolated critical point of $f$. Then there exists $\varepsilon > 0$ small enough such that $f^{-1}(0)$ intersects transversely $\mathbb{S}_{\varepsilon'}$ for all $0 < \varepsilon' \leq \varepsilon$, where $\mathbb{S}_{\varepsilon'}$ is a sphere centred at the origin with radius $\varepsilon'$.*

In the previos result, "small enough" means that, in the closed ball $\mathbb{B}_\varepsilon$, the origin is the unique critical point of $f$ and of the distance function to the origin.

This result can be proved using the Curve Selection Lemma for real analytic functions (see [8, Prop. 2.2]). The polynomial case of Proposition 1.1 appears in the proof of [33, Cor. 2.9], which is the polynomial version of the following result.

**1.2 Corollary.** *Let $\varepsilon > 0$ and $\mathbb{S}_{\varepsilon'}$ be as above. The intersection $f^{-1}(0) \cap \mathbb{S}_{\varepsilon'}$ is a manifold (possibly empty) of codimension $k+1$, for all $0 < \varepsilon' \leq \varepsilon$.*

**1.3 Definition.** Let $f$ and $\varepsilon > 0$ be as in Proposition 1.1. Let $\mathbb{S}_\varepsilon$ be a sphere centred at the origin with radius $\varepsilon$. Let $L_\varepsilon$ be defined by

$$L_\varepsilon = f^{-1}(0) \cap \mathbb{S}_\varepsilon \, .$$

The manifold $L_\varepsilon$ is called the **link** of the singularity.

**1.4 Definition.** Let $X$ be a topological space. The **cone** $C(X)$ over $X$ is the quotient

$$C(X) = I \times X / (\{0\} \times X) \, .$$

The following result appears as [33, Th. 2.10] for $V$ defined by real or complex polynomial functions with isolated singularity.

**1.5 Theorem** ( [8, Lemma 3.2])**.** *Let $f$ and $\varepsilon > 0$ be as in Proposition 1.1. Then the intersection $f^{-1}(0) \cap \mathbb{B}_\varepsilon$ is homeomorphic to the cone over $L_\varepsilon$. Moreover, the pair $(\mathbb{B}_\varepsilon, \mathbb{B}_\varepsilon \cap f^{-1}(0))$ is homeomorphic to the pair $(C(\mathbb{S}_\varepsilon), C(L_\varepsilon))$.*

**1.6 Corollary.** *Given $\varepsilon > 0$ as in Theorem 1.5, the diffeomorphism type of the link $L_\varepsilon$ is independent of $\varepsilon$.*

**1.7 Definition.** Let $\varepsilon > 0$ be as in Theorem 1.5, the ball $\mathbb{B}_\varepsilon$ centred at the origin is called a **Milnor ball** for $f$.

In the sequel we will denote the link of a singularity by $L$ or $L_f$.

Moreover, the link $L$ is independent of the embedding of $f^{-1}(0)$. This is a fact well known in the folklore of Singularity Theory and usually the reference is [17]. However, in this work, Durfee proves the result for algebraic and semialgebraic sets and it is only mentioned for analytic sets. For completeness, here we reproduce the proof in the case of analytic sets, which is basically the same but using the Curve Selection Lemma for analytic sets (see [8, Prop. 2.2]).

**1.8 Definition.** Let $M$ be an analytic set in $\mathbb{R}^n$ and let $X \subset M$ be a compact analytic set with $M \setminus X$ nonsingular. An (algebraic) **rug function** for $X$ in $M$ is a proper polynomial function $\alpha\colon M \to \mathbb{R}$ such that $\alpha(x) \geq 0$ for $x \in M$ and $\alpha^{-1}(0) = X$.

By [17, Cor. 1.3], any set $X$ as above has a rug function and a rug function has a finite number of critical values (see [17, Lemma 1.4]).

**1.9 Definition.** Let $M$ and $X$ be as in Definition 1.8. A subset $X \subset T \subset M$ is an **algebraic neighbourhood** of $X$ in $M$ if $T = \alpha^{-1}([0,\delta])$ for some rug function $\alpha$ and some $0 < \delta$ such that $\delta$ is smaller than nay critical value of $\alpha$.

**1.10 Example.** Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ an analytic function with isolated singularity at the origin. Let $M = f^{-1}(0)$ and let $X = \{0\}$. Take the rug function $\alpha\colon M \to \mathbb{R}$ defined by

$$\alpha(x) = |x|^2 \,,$$

then the link $L_f$ is the boundary of the algebraic neighbourhood

$$\mathbb{B}_\varepsilon = T_\varepsilon = \alpha^{-1}([0,\varepsilon]) \,.$$

**1.11 Proposition** (Uniqueness of algebraic neighbourhoods)**.** *Let $T_1$ and $T_2$ be algebraic neighbourhoods of $X$ in $M$. Then there is a continuous family of homeomorphisms $h_t\colon M \to M$ for $0 \leq t \leq 1$ such that*

*a) $h_0$ is the identity on $M$;*

*b) for all $t$, $h_t|_X$ is the identity on $X$;*

*c) $h_1(T_1) = T_2$, and $h_1$ is a smooth diffeomorphism of $T_1 \setminus X$ onto $T_2 \setminus X$.*

In the proof it is used the following result, which can be found in the polynomial case as [33, Cor. 3.4].

**1.12 Lemma.** *There is a neighbourhood $U$ of $X$ in $M$ such that* $\operatorname{grad}\alpha_1$ *and* $\operatorname{grad}\alpha_2$ *are nonzero and do not point in opposite directions on $U \setminus X$.*

*Proof.* Let $Y$ be the semianalytic set defined by

$$\begin{aligned} Y &= \{x \in M \setminus X \mid \operatorname{grad}\alpha_1(x) \neq 0, \operatorname{grad}\alpha_2(x) \neq 0 \ \text{ and they point in opposite directions}\} \\ &= \{x \in M \setminus X \mid \langle \operatorname{grad}\alpha_1(x), \operatorname{grad}\alpha_2(x)\rangle < 0\} \\ &\quad \cap \ \{x \in M \setminus X \mid \operatorname{grad}\alpha_1(x) = \gamma \operatorname{grad}\alpha_2(x), \gamma > 0\} \,. \end{aligned}$$

It is sufficient to show $X$ is not in the closure of $Y$, so let us assume it is. Then, by the Curve Selection Lemma (see [8, Prop. 2.2] or [33, Lemma 3.1] for the algebraic case), there is a real analytic curve $\beta\colon [0,\varepsilon) \to M$ with $\beta(0) \in X$ and $\beta(t) \in Y$ for all $t > 0$. Notice $\alpha_i(\beta(0)) = 0$ and $\alpha_i(\beta(t)) > 0$ for all $t > 0$, then near $t = 0$, $\alpha_i(\beta(t))$ is an increasing function of $t$, so

$$\frac{d\alpha_i(\beta(t))}{dt} = \left\langle \operatorname{grad}\alpha_i, \frac{d\beta(t)}{dt} \right\rangle > 0 \,.$$

However, $\operatorname{grad}\alpha_1 = \gamma \operatorname{grad}\alpha_2(x)$ with $\gamma > 0$, which is a contradiction. □

*Proof of Proposition 1.11.* Let $\alpha_1^{-1}([0,\delta_1])$ and $\alpha_2^{-1}([0,\delta_2])$ be two algebraic neighbourhoods of $X$ in $M$ for some rug functions $\alpha_1$ and $\alpha_2$ and without loss of generality let us assume $\delta_1 < \delta_2$. If $\alpha_1 = \alpha_2$, one can "push" $\alpha_2^{-1}([0,\delta_2])$ to $\alpha_1^{-1}([0,\delta_1])$ by a standard technique in Morse Theory (see [31, Th. 3.1]).

Let $\alpha_1 \neq \alpha_2$. Let $U$ be the neighbourhood of $X$ in $M$ given by Lemma 1.12. Let $\delta_2' > 0$ be such that $\alpha_2^{-1}([0,\delta_2']) \subset U$ and let $\delta_1' > 0$ be such that $\alpha_1^{-1}([0,\delta_1']) \subset \alpha_2^{-1}([0,\delta_2'])$.

By the first part of the proof, $\alpha_i^{-1}([0,\delta_i])$ and $\alpha_i^{-1}([0,\delta_i']))$ are isotopic for $i = 1,2$. Hence it is sufficient to prove that $\alpha_1^{-1}([0,\delta_1'])$ and $\alpha_2^{-1}([0,\delta_2'])$ are isotopic. Let $S \subset M$ be the set defined by

$$S = \alpha_2^{-1}([0,\delta_2']) \setminus \alpha_1^{-1}([0,\delta_1'])$$

and let $f\colon S \to [0,1]$ be defined by

$$f(x) = \frac{\alpha_1(x) - \delta_1'}{\big(\alpha_1(x) - \delta_1'\big) + \big(\delta_2' - \alpha_2(x)\big)}\ .$$

Note that $f^{-1}(0) = \alpha_1^{-1}(\delta_1')$, $f^{-1}(1) = \alpha_2^{-1}(\delta_2')$, the denominator of $f$ is never zero in $S$ and $f$ is proper. The gradient of $f$ is given by

$$\operatorname{grad} f = \left( \frac{\operatorname{grad}\alpha_2\big(\alpha_1(x) - \delta_1'\big) + \operatorname{grad}\alpha_1\big(\delta_2' - \alpha_2(x)\big)}{\Big[\big(\alpha_1(x) - \delta_1'\big) + \big(\delta_2' - \alpha_2(x)\big)\Big]^2} \right),$$

then $f$ has no critical value since $\operatorname{grad}\alpha_1$ and $\operatorname{grad}\alpha_2$ are nonzero and never point in opposite directions on $U \setminus X$ by Lemma 1.12. Let $v$ be a vector field on $S$, which is projected under the derivative of $f$ to the vector field $\frac{\partial}{\partial t}$ on $[0,1]$. Integrating $v$ gives the required isotopy $h_t$. □

**1.13 Corollary.** *The algebraic neighbourhood of $X$ in $M$ is independent of the embedding of $M$ in $\mathbb{R}^n$.*

Then, the link $L_f$ is independent of the embedding of $f^{-1}(0)$ in $\mathbb{R}^n$.

## 1.2 Milnor fibration

In this section we will present a main result and starting point of Singularity Theory: Milnor fibration theorem. As we mentioned before, this result states that the complement of a tubular neighbourhood of the link of a singularity fibres over $\mathbb{S}^1$. We restrict our attention to the real case, which was presented by Milnor in [32] and [33] for real polynomials with isolated critical point. Later we present a generalisation by Cisneros-Molina, Seade and Snoussi in [12]for real analytic functions under some conditions.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial function such that $f(0) = 0$ and which satisfies

**1.14 Hypothesis** (Milnor Condition)**.** There exists a neighbourhood $U$ of the origin in $\mathbb{R}^n$ such that the matrix $(\partial f_i/\partial x_j)$ has rank $k$ for all $x \in U$ with $x \neq 0$.

**1.15 Theorem** ( [33, Th. 11.2])**.** *Set $k \geq 2$. The complement of a tubular neighbourhood of $L_f$ in $\mathbb{S}_\epsilon^{n-1}$ is the total space of a smooth fibred bundle over the sphere*

$\mathbb{S}^{k-1}$, *where each fibre is a compact* $(n-k)$*-manifold* $\mathscr{F}$ *such that the boundary* $\partial\mathscr{F}$ *is diffeomorphic to* $L$.

In the proof of this theorem, Milnor uses the following concept and results.

**1.16 Definition.** Let $\varepsilon > 0$ be as in Theorem 1.5 and let $0 < \delta \ll \varepsilon$ be small enough. Let $N(\varepsilon, \delta)$ be the intersection

$$f^{-1}(\partial\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon ,$$

where $\mathbb{B}_\varepsilon$ is the closed ball centred at the origin of radius $\varepsilon$. The manifold $N(\varepsilon, \delta)$ is called the **Milnor tube** of $f$ in $\mathbb{B}_\varepsilon$.

**1.17 Definition.** Let $\varepsilon > 0$ and $0 < \delta \ll \varepsilon$ as in the previous definition. Let $\widehat{N}(\varepsilon, \delta)$ be the intersection

$$f^{-1}(\mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon .$$

The setp $\widehat{N}(\varepsilon, \delta)$ is called the **solid Milnor tube** of $f$.

**1.18 Lemma** ( [33, Proof of Lemma 11.2])**.** *The restriction of* $f$ *to* $\widehat{N}(\varepsilon, \delta) \setminus f^{-1}(0)$ *is the projection map of a locally trivial fibration over* $\mathbb{D}_\delta \setminus \{0\}$.

The main idea is to take the restriction to the Milnor tube $N(\varepsilon, \delta)$ of this locally trivial fibration and "to inflate" it on the sphere $\mathbb{S}_\varepsilon$ using the next result.

**1.19 Theorem** ( [32, Th. 1])**.** *Let* $\varepsilon > 0$ *as in Theorem 1.5 and let* $\widehat{N}(\varepsilon, \delta)$ *be the solid Milnor tube of* $f$ *in* $\mathbb{B}_\varepsilon$. *Then* $\widehat{N}(\varepsilon, \delta)$ *is homeomorphic to* $\mathbb{B}_\varepsilon$ *under a homeomorphism which leaves the intersection*

$$f^{-1}(0) \cap \mathbb{B}_\varepsilon = f^{-1}(0) \cap \widehat{N}(\varepsilon, \delta)$$

*pointwise fixed.*

In his book ( [33, p. 100]), Milnor comments that the major weakness of Theorem 1.15 is that Hypothesis 1.14 is so strong that it is very difficult to find examples, except those that come from holomorphic maps. This raises the problem of finding dimensions $2 \le k \le n$ for which such examples exist (see for example [11] and [29]).

The proof of Theorem 1.15 only ensures the existence of a projection map giving a fibration, but unlike the complex case, it gives no explicit construction. In other words, in the real setting, there is no a priori reason to expect the projection to be the canonical map $f/\|f\|$ as in the complex case (see [57]).

Let us now present a generalisation of Theorem 1.15 for real analytic functions with isolated critical point and satisfying the condition to be $d$-regular (see Definition 1.21). For a function $f$ of this type, the Milnor fibration will be indeed $f/\|f\|$.

Let $U$ be an open neighbourhood of $0 \in \mathbb{R}^n$ with $n > 1$, Let $k \leq n$ and let $f\colon (U,0) \to (\mathbb{R}^k,0)$ be an analytic map defined on $U$ with isolated critical point at 0.

Let $\mathbb{B}_\varepsilon$ be a closed ball in $\mathbb{R}^n$, centred at 0, of sufficiently small radius $\varepsilon$, so that every sphere in this ball, centred at 0, meets transversely $f^{-1}(0)$, if $f^{-1}(0)$ is not an isolated point at the origin. Such a ball exists by Theorem 1.5.

One can define a family of real analytic spaces as follows.

For each $\ell \in \mathbb{RP}^{k-1}$, consider the line $\mathscr{L}_\ell \subset \mathbb{R}^k$ passing through the origin corresponding to $\ell$, let $f|_{\mathbb{B}_\varepsilon}$ be the restriction of $f$ to the ball $\mathbb{B}_\varepsilon$ and set $X_\ell = f|_{\mathbb{B}_\varepsilon}^{-1}(\mathscr{L}_\ell)$.

Let $\mathscr{L}_\ell^\perp$ be the hyperplane orthogonal to $\mathscr{L}_\ell$ and let $\pi_\ell\colon \mathbb{R}^k \to \mathscr{L}_\ell^\perp$ be the orthogonal projection. Set $h_\ell = \pi_\ell \circ f|_{\mathbb{B}_\varepsilon}$, then $X_\ell$ is the vanishing set of $h_\ell$, which is real analytic. Hence $\{X_\ell\}$ is a family of real analytic hypersurfaces parametrised by $\mathbb{RP}^{k-1}$.

The set of critical points of $h_\ell$ is contained in the set of critical points of $f|_{\mathbb{B}_\varepsilon}$ (see [12, Lemma 2.1]). As $f$ has isolated critical point at the origin, $h_\ell$ has the origin as its only critical point.

**1.20 Definition.** The family $\{X_\ell \mid \ell \in \mathbb{RP}^{k-1}\}$ is called the **canonical pencil** of $f$.

The following condition implies that the projection of the Milnor fibration of $f$ is the canonical map $f/|f|$.

**1.21 Definition.** The map $f$ is said to be $d$**-regular** at 0 if there exists a metric $\mu$ induced by some positive definite quadratic form and there exists $\varepsilon' > 0$ such that every sphere (for the metric $\mu$) of radius $\varepsilon \leq \varepsilon'$ centred at 0 meets every $X_\ell \setminus f^{-1}(0)$ transversely whenever the intersection is not empty.

**1.22 Example.** Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a complex analytic germ. By [33, Lemma 4.2] and [8, Th. 2.1], $f$ is $d$-regular for the usual metric.

**1.23 Example.** Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ and $g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be two complex analytic germs such that the real analytic germ $f\bar{g}\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ has an isolated critical value at $0 \in \mathbb{C}$. The map $f\bar{g}$ is $d$-regular at 0 for the usual metric (see [53, Th. 5.3 and Th. 5.8]).

The following result follows from [12, Th. 5.3], [12, Cor. 5.4] and the proof of [33, Th. 11.2].

**1.24 Theorem** (Fibration Theorem)**.** *Let $f\colon (\mathbb{R}^n,0) \to (\mathbb{R}^k,0)$ be a real analytic germ with isolated critical point. Let $\varepsilon \gg \delta > 0$ be such that there exists a locally trivial fibration*

$$f\colon N(\varepsilon,\delta) \to \partial\mathbb{D}_\delta$$

*from the Milnor tube to $\partial\mathbb{D}_\delta$. Then $f$ is $d$-regular if and only if the map*

$$\phi_f = \frac{f}{|f|}\colon \mathbb{S}^{n-1}_\varepsilon \to \mathbb{S}^1$$

*is a locally trivial fibration, which is equivalent to the fibration on the Milnor tube.*

**1.25 Remark.** In fact, Theorem 1.24 appears in [12] in more generality than here. It also holds for $d$-regular real analytic functions with isolated critical value, which satisfy the Thom $a_f$-condition. Then, Hypothesis 1.14 is a sufficient condition in order to have Milnor fibration but no a necessary condition.

## 1.3 Mapping torus and open-book fibrations

The fibration seen in the past section is, in fact, an open-book fibration; concept presented in this section. The formal definition was introduced by Winkelnkemper in [68] and has become an important concept (see for example [54, Appendix]). Open books allow to describe an arbitrary closed manifold in terms of lower dimensional ones.

As a way to construct an open-book, it is presented also the concept of mapping torus of a diffeomorphism.

**1.26 Definition** ( [62, Def. 5.1])**.** Let $M$ be a smooth closed $n$-manifold and let $N$ be a codimension 2 submanifold of $M$ with trivial normal bundle. Let

$$\pi\colon M \setminus N \to \mathbb{S}^1$$

be a map such that

- $\pi$ is a locally trivial fibration and
- given a tubular neighbourhood of $N$ diffeomorphic to $N \times \mathbb{D}^2$, the restriction of $\pi$ to $N \times (\mathbb{D}^2 \setminus \{0\})$ is the map $(x, y) \mapsto y/||y||$.

The map $\pi$ is called an **open-book fibration** of $M$, $N$ is called the **binding** and the fibres of $\pi$ are called the **pages**. The pair $(M,\pi)$ is called an **open-book**.

It follows that the pages are all diffeomorphic and each page $\mathscr{F}$ can be compactified by attaching the binding $N$ as its boundary, thus getting a compact manifold without boundary.

Also, since the base of the fibration is the circle $\mathbb{S}^1$, one can lift a non-zero vector field on $\mathbb{S}^1$ to an integrable vector field $v$ on $M \setminus N$ which is transverse to the fibres. Following the flow lines of the vector field $v$, one can define a "first return map" as follows:

Let $\mathscr{F}_\lambda$ be the fibre $\pi^{-1}(e^{i\lambda})$. Given $p \in \mathscr{F}_0$, let $\alpha_p$ be the flow line of $v$ passing through $p$ and let $p_\lambda \in \mathscr{F}_\lambda$ be the point such that $(\alpha_p)(t) = p_\lambda$ for some $t \in \mathbb{R}^+$. We define a diffeomorphism $h_\lambda \colon \mathscr{F}_0 \to \mathscr{F}_\lambda$ by $h_\lambda(p) = p_\lambda$.

**1.27 Definition.** The map $h = h_{2\pi}$ is called the **monodromy** of the open-book fibration $\pi$ and it is well defined up to isotopy.

One can think of the monodromy of an open-book fibration as the isotopy class $[h]$ of the first return map $h$. Since all the pages have $N$ as boundary, it follows that $h$ extends as the identity on $N$.

One can obtain open-book fibrations in the following way:

Let $V$ be a compact $(n-1)$-manifold with $\partial V \neq \emptyset$ and let $h \colon V \to V$ be a diffeomorphism such that $h|_{\partial V} = id$.

**1.28 Definition.** The **mapping torus** $T(h)$ of $h$ is the quotient of the product $V \times [0,1]$ by the equivalence relation $(x,1) \sim (h(x),0)$.

In the case $V$ is oriented, then $T(h)$ is oriented by the orientation of $V$ followed by the usual orientation of $[0,1]$ (the order is unimportant).

Then, the mapping torus $T(h)$ has boundary $\partial V \times \mathbb{S}^1$. Let $\lambda \in \mathbb{S}^1$, then $\partial V \times \{\lambda\}$ is the boundary of a $(n-1)$-manifold $V_\lambda \subset T(h)$ diffeomorphic to $V$ (See Figure 1.1).

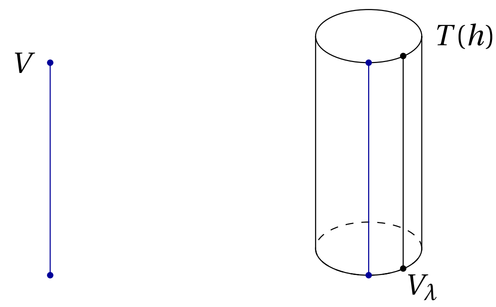



Figure 1.1: The manifold $V = [0,1]$ and the mapping torus $T(h)$ with $h = id_V$.

Now let us "fill" the boundary of $T(h)$: Let

$$\mathcal{O}(h) = T(h) \bigcup_{\partial V \times \mathbb{S}^1} (\partial V \times \mathbb{D}^2)$$

be the union of $T(h)$ and $(\partial V \times \mathbb{D}^2)$ identified with the identity on $(\partial V \times \mathbb{S}^1)$. Let $N = (\partial V \times \{0\}) \subset (\partial V \times \mathbb{D}^2)$. The manifold $\mathcal{O}(h)$ is of dimension $n$ and the manifold $N \subset M$ is of codimension two in $M$ (see Figure 1.2).

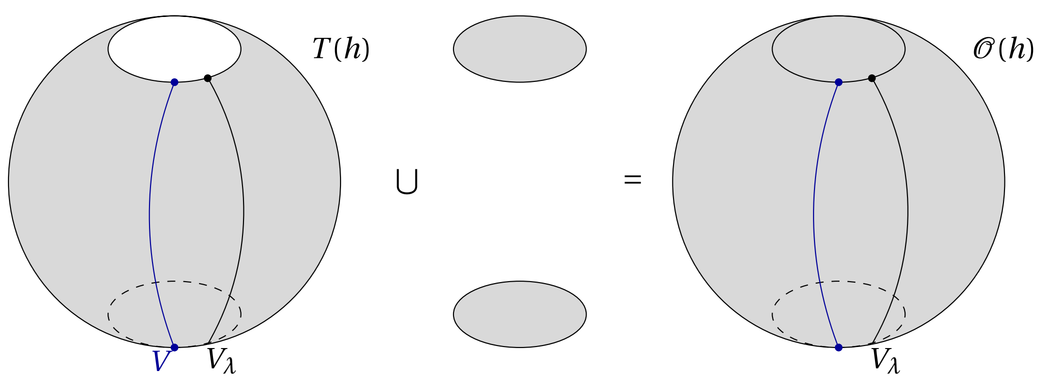



Figure 1.2: The mapping torus $T(h)$ and the manifold $\mathcal{O}(h)$ for $h = id_V$ with $V = [0,1]$.

Let $\lambda \in \mathbb{S}^1$ and let $[0,\lambda] \subset \mathbb{D}^2$ be the ray going from the origin to $\lambda$. Let

$$\mathcal{F}_\lambda = \big(\partial V \times (0,\lambda]\big) \bigcup_{\{\partial V \times \lambda\}} V_\lambda$$

and let

$$\pi\colon (\mathcal{O}(h) \setminus N) \to \mathbb{S}^1$$

be such that $\pi(\mathscr{F}_\lambda) = \lambda$ (see Figure 1.3).

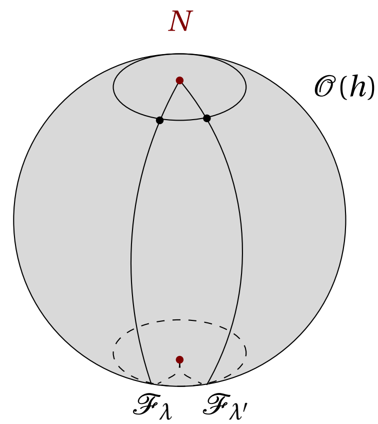



Figure 1.3: The manifolds $\mathscr{O}(h)$, $N$ and $\mathscr{F}_\lambda$.

The map $\pi$ is an open-book fibration of $\mathscr{O}(h)$ with binding $N$ and pages $\mathscr{F}_\lambda$ with $\lambda \in \mathbb{S}^1$.

## 1.4 Seifert manifolds

In this section we present a class of 3-manifolds constructed by Seifert in [64] which are very important for this work. In Chapter 2, we describe the links of some singularities as Seifert manifolds and in Chapter 3, we present a type of 3-manifolds which can be decomposed as union of Seifert manifolds. There are many references in this subject, for example, [23], [45], [36] and [65].

In this work only oriented 3-manifolds will arise. Thus from now on all Seifert manifolds are oriented.

**1.29 Definition.** A **Seifert fibration** is a triplet $(M, B, \pi)$ where $M$ is an oriented compact 3-manifold, $B$ is an orientable surface and $\pi\colon M \to B$ is "almost" a locally trivial fibration with fibre $\mathbb{S}^1$; more precisely, for every $x \in B$, there exists a neighbourhood $\mathbb{D}^2_x$ of $x$ and an orientation preserving diffeomorphism $h\colon D^2_x \times \mathbb{S}^1 \to \pi^{-1}(\mathbb{D}^2_x)$ such that the composition

$$\pi \circ h\colon \mathbb{D}^2_x \times \mathbb{S}^1 \to \mathbb{D}^2_x$$

is defined by

$$\pi \circ h(s\lambda_1, \lambda_2) = s\lambda_1^{\alpha}\lambda_2^{\beta^*}$$

where a point in $\mathbb{D}^2$ is given in its polar coordinates $s \in [0,1]$ and $\lambda_1 \in \mathbb{S}^1$; $\lambda_2 \in \mathbb{S}^1$, $\alpha \in \mathbb{Z}^+$, $\beta^* \in \mathbb{Z}$ and $\gcd(\alpha, \beta^*) = 1$. Also $M$ is called a **Seifert manifold**.

Thanks to a theorem of Epstein (see [19]), a 3-manifold $M$ is a Seifert manifold if and only if $M$ admits a foliation in circles and one can obtain a fixed-point free action of $\mathbb{S}^1$ such that the orbits of the action coincide with the fibres of $\pi$; *i.e.*, for all $x \in M$, there exist $\lambda \in \mathbb{S}^1$ such that

$$\lambda \cdot x \neq x$$

and the orbit of $x$ is the fibre $\pi^{-1}(y)$ for some point $y \in B$; *i.e.*, the surface $B$ can be regarded as the orbit space of the $\mathbb{S}^1$-action.

The $\mathbb{S}^1$-action on $M$ induces an orientation on each orbit. Together with the orientation of $M$, this induces an orientation of $B$ and of each local section of $\pi$, in such a way that the orientation of a local section followed by the orientation of the orbits gives the orientation of $M$.

Given an $\mathbb{S}^1$-action, regarded as a homeomorphism $\mathbb{S}^1 \to Aut(M)$, the kernel can only be either the trivial group, a cyclic group $\mathbb{Z}_\sigma$ with $\sigma \geq 2$ or $\mathbb{S}^1$ itself. In our case, the kernel cannot be $\mathbb{S}^1$ since the action is fixed-point free. If the kernel is the trivial group, then the action is effective and in this case we say that an orbit (*i.e.*, a fibre) is **exceptional** if its isotropy subgroup is non-trivial, which is a finite cyclic subgroup of order $\sigma \geq 2$.

Now, if the kernel is a cyclic group $\mathbb{Z}_\sigma$ with $\sigma \geq 2$, the induced action of the quotient $\mathbb{S}^1 / \mathbb{Z}_\sigma$ (which is homomorphic to $\mathbb{S}^1$) is effective, then we define an exceptional orbit in the same way as above.

As $M$ is compact, the surface $B$ is itself compact and by [23, Prop. 1.3] there are a finite number of exceptional orbits.

Let $p \in B$ such that $O = \pi^{-1}(p)$ is an exceptional orbit, one can choose a small disc $D$ around $p$ such that $\pi^{-1}(D)$, which is an union of orbits, is a tubular neighbourhood of $O$. Let us take the orientation preserving diffeomorphism $h$ from $\pi^{-1}(D)$ to the mapping torus

$$\mathbb{D}^2 \times [0,1] / (\rho(x), 0) \sim (x, 1)$$

of a rotation $\rho$ of order $\alpha$ on the oriented 2-disc $\mathbb{D}^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$, sending orbits to orbits and preserving their orientations.

**1.30 Definition.** A disc $h^{-1}(\mathbb{D}^2 \times \{t\})$ is called a **slice** of $O$, with $t \in [0,1]$ fixed.

As $h^{-1}(\mathbb{D}^2 \times \{t\})$ is a local section of $\pi$, the previous choices of orientation induce an orientation of $\mathbb{D}^2$. Then the angle of the rotation $\rho$ is well defined.

An exceptional fibre is determined by some invariants in the following way:

Suppose that the rotation angle on the 2-disc is equal to $2\pi\beta^*/\alpha$ with $\gcd(\alpha,\beta^*) = 1$. Let $\beta$ be any integer such that $\beta\beta^* \equiv 1 \pmod{\alpha}$. The pair $(\alpha,\beta)$ is an invariant of the exceptional orbit. See [36, pages 135-140] for more details.

Let now $p_1,\dots,p_s \in B$ be the points corresponding to the exceptional orbits $O_1,\dots,O_s \in M$ respectively and let $D_1\dots,D_s$ be disjoint open discs in $B$, such that $D_i$ is a neighbourhood of the point $p_i$. Let $T_i = \pi^{-1}(D_i)$. The choice of $\beta_i$ in its residue class $\pmod{\alpha_i}$ is related to the choice of a section near the exceptional orbit $O_i$ as follows: we can find oriented curves $H_i$ and $Q_i$ in the boundary $\partial T_i$ such that $H_i$ is a fibre in $M$, $Q_i \cdot H_i = 1$ on $\partial T_i$ and $R_i \sim \alpha Q_i + \beta H_i$ on $\mathrm{H}_1(\partial T_i,\mathbb{Z})$, where $R_i$ is a suitably oriented meridian of $T_i$.

Now let $\hat{M} = M \backslash \big(\cup \mathring{T}_i\big)$ where $\mathring{T}_i$ is the interior of $T_i$. Let $V_i$ be a fibred solid torus with $Q_i$ as meridian (see [36, Ex. 4.2.1]) and let

$$M' = \hat{M} \cup V_i$$

where each $V_i$ is attached to $\partial T_i \subset \hat{M}$. Then $M'$ has no exceptional fibres because the meridian of $V_i$ is $Q_i$ which is homologous to $Q_i + 0\cdot H_i$ on $\mathrm{H}_1(\partial V_i,\mathbb{Z})$.

Thus $M'$ is a locally trivial $\mathbb{S}^1$-fibration with Euler class $e$ where the curves $Q_i$ determine a local section for each disc $D_i$ and $e$ can be seen as the obstruction to extend these local sections to a global section $S$ on $M'$. Thus the integers $e$ and $\beta_1,\dots,\beta_s$ depend on the choice of the section $S$ , but the rational number $e_0 = e - \sum \beta_i/\alpha_i$, called the **rational Euler class** of the Seifert fibration, does not.

**1.31 Definition.** The pair $(\alpha,\beta)$ is called **normalised** if a section is chosen in such a way that $0 < \beta < \alpha$.

**1.32 Definition.** The (normalised) **Seifert invariants** of $M$ consists of the data :

$$\Big(g;\, e_0;\, (\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)\Big)$$

where $g \geq 0$ is the genus of the surface $B$, $e_0$ is the rational Euler class and $0 < \beta_i < \alpha_i$.

The rational Euler number has the property of functoriality (see [23, Th. 3.3] or [40, Th. 1.2]):

**1.33 Proposition.** *Let $\pi_1 \colon M_1 \to B_1$ and $\pi_2 \colon M_2 \to B_2$ be Seifert fibrations. Assume there exists a map $f$ such that the diagram*

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "\tilde{f}"] & B_2 \end{tikzcd}$$

*commutes,* $\deg(\tilde{f}) = m$ *and* $\deg(f|_{fibre}) = n$. *Then*

$$e_0(M_1 \xrightarrow{\pi_1} B_1) = \frac{m}{n} e_0(M_2 \xrightarrow{\pi_2} B_2)\ . \tag{1.34}$$

## 1.5 Plumbing

In this section we describe a method to construct manifolds "gluing" disc bundles; in particular we are interested in the construction of 4-manifolds gluing 2-disc bundles over 2-manifolds. We present a result describing a Seifert manifold as the boundary of some manifolds obtained in this way and the use of this concept allows us in Section 1.6 to describe the monodromy of the Milnor fibration of a singularity.

As Bredon in [7, Ch. VI, Sect. 18] and Hirzebruch and Neumann in [21, § 8], we will first describe plumbing in arbitrary dimensions before going into more detail in the case of our interest, namely plumbing of 2-disc bundles over 2-manifolds.

Let $\xi = (E, p, M)$ and $\kappa = (E', p', N)$ be two smooth $n$-disc bundles over smooth $n$-manifolds $M$ and $N$. Around any given point of $M$ there is a neighbourhood $A \cong \mathbb{D}^n$ and a trivialisation

$$\zeta \colon E_A \to (\mathbb{D}^n \times \mathbb{D}^n)$$

where $E_A$ is the total space of the bundle $\xi$ restricted to $A$, such that the following diagram commutes:

$$\begin{tikzcd} E_A \arrow[r, "\cong"] \arrow[d, "p"'] & \mathbb{D}^n \times \mathbb{D}^n \arrow[d, "p_2"] \\ A \arrow[r, "\cong"] & \mathbb{D}^n \end{tikzcd}$$

where $p_2(x, y) = x$ is the projection on the second coordinate.

Similarly, let $B \cong \mathbb{D}^n$ a neighbourhood of a point in $N$ and take a trivialisation

$$\eta\colon E'_B \to (\mathbb{D}^n \times \mathbb{D}^n).$$

Let $\chi\colon (\mathbb{D}^n \times \mathbb{D}^n) \to (\mathbb{D}^n \times \mathbb{D}^n)$ be defined by the change of factors $\chi(x, y) = (y, x)$ and let $\vartheta\colon E'_B \to E_A$ be the composition given by

$$\vartheta\colon E'_B \xrightarrow{\eta} (\mathbb{D}^n \times \mathbb{D}^n) \xrightarrow{\chi} (\mathbb{D}^n \times \mathbb{D}^n) \xrightarrow{\zeta^{-1}} E_A$$

**1.35 Definition.** The **plumbing** of $E$ and $E'$ is defined as the identification

$$P^{2n}\colon E \bigcup_{\vartheta} E' .$$

Note that the identification $\vartheta$ matches the base of one bundle with the fibre of the other (see Figure 1.4).

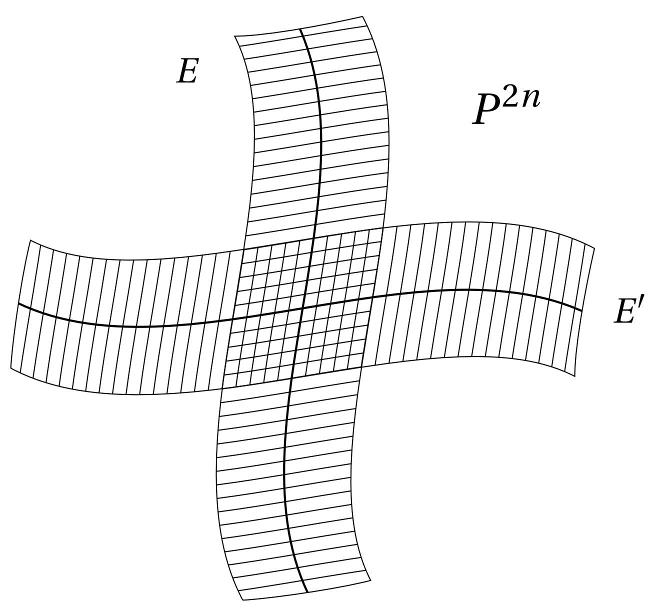



Figure 1.4: Plumbing $P^{2n}$ of $E$ and $E'$.

The space $P^{2n}$ is a topological $2n$-manifold with boundary and is close to being a smooth manifold, but it has "corners". There is a canonical way to smooth these corners and so to produce $P^{2n}$ as a smooth manifold (see [34, p.86-87] and [21, § 8]).

We will describe how to plumb several bundles together according to a finite tree (more generally a connected graph)

Let $T$ be a connected graph. For each vertex in $T$ one takes a $n$-disc bundle over $\mathbb{S}^n$, a plumbing of two bundles is made if and only if there is an edge joining the corresponding vertices.

If several edges of $T$ meet in one vertex $v$, one chooses the corresponding neighbourhoods in $\mathbb{S}^n$ to be disjoint. A theorem of Thom (see [35, Th. 1.1]) assures that the plumbing is independent of the choice of these neighbourhoods.

**1.36 Example.** Given the graph $T$ in Figure 1.5, let us take a trivial 1-disc bundle over $\mathbb{S}^1$ for each vertex in $T$ and make the plumbing of two of them when there is an edge in $T$ joining the corresponding vertices.

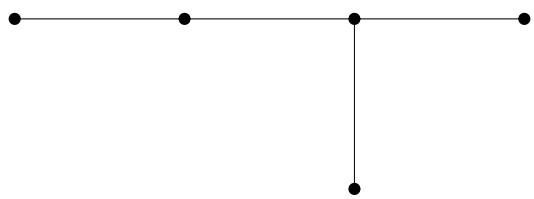


Figure 1.5: Graph $T$, where each vertex represents a trivial 1-disc bundle of $\mathbb{S}^1$.

Let $P^2(T)$ be the result of the plumbing (see Figure 1.6).

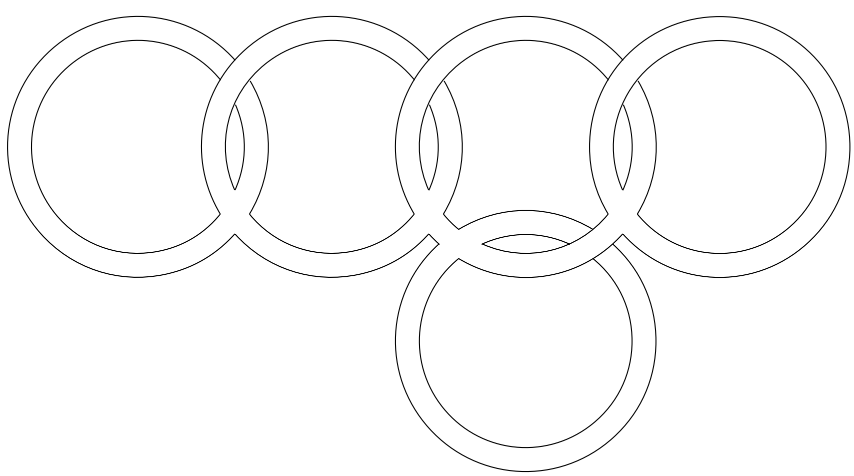


Figure 1.6: Plumbing $P^2(T)$ according to the graph $T$.

We will now restrict to the case in which the bundles are 2-disc bundles over 2-manifolds.

Let $T$ be a connected graph weighted in each vertex $v$ by two integers: $e_v$ and $g_v \geq 0$. Let $\xi_v$ be a 2-disc bundle with Euler class $e_v$ over the surface of genus $g_v$ and let $P^4(T)$ be the 4-manifold with boundary obtained by the plumbing according to $T$; *i.e.*, to plumb two bundles $\xi_v$ and $\xi_{v'}$ when there is an edge in $T$ joining the corresponding vertices $v$ and $v'$.

**1.37 Example.** Let $E_8$ be the graph in Figure 1.7 weighted at each vertex with $-2$ (see [21, p. 61,62]). The other weight is not written because it is equal to zero, then we consider $-2$ as the Euler class of the 2-disc bundles taken for each vertex in $T$; *i.e.*, we take a 2-disc bundle over the sphere $\mathbb{S}^2$ for each vertex in $T$.

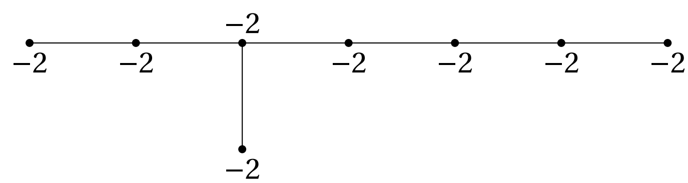



Figure 1.7: Graph $E_8$.

Figure 1.8 presents schematically the plumbing according the graph $E_8$.

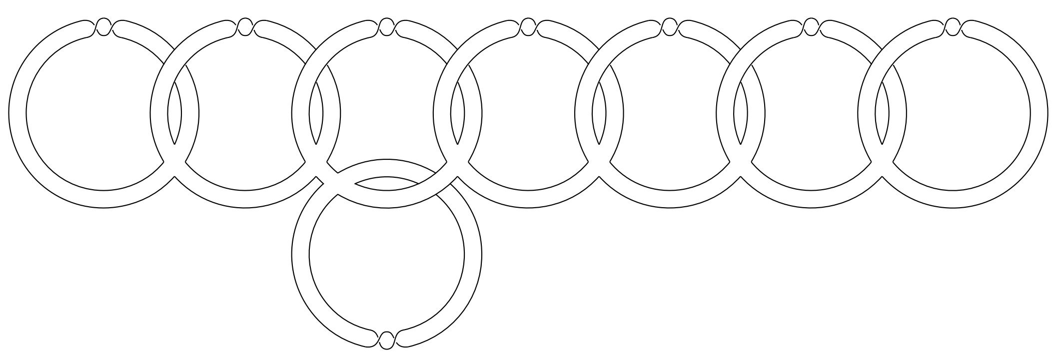


Figure 1.8: Plumbing $P^4(E_8)$ according to the graph $E_8$.

In fact, the boundary of the manifold $P^4(E_8)$ is the Poincaré sphere (see [26, Desc. 1, p.114]).

A **plumbing graph** $\Gamma$ is a connected graph representing a plumbing. In order to be more precise talking about graphs, it is important to have the following concepts:

**1.38 Definition.** Let $\Gamma$ be a plumbing graph and let $v$ be a vertex of $\Gamma$. The **valence** of $v$ is the number of edges ending at $v$.

**1.39 Definition.** Let $\Gamma$ be a plumbing graph. A **rupture vertex** in $\Gamma$ is a vertex with valence $n \geq 3$.

**1.40 Definition.** Let $\Gamma$ be a plumbing graph. A **bamboo ended by a vertex of valence** 1 in $\Gamma$ is a chain of vertices joined by edges such that all the vertices have valence 2 except the first one, which is a rupture vertex, and the last one, which has valence 1 (Figure 1.9).

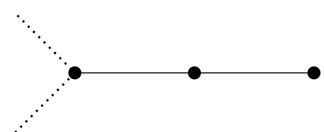


Figure 1.9: A bamboo ended by a vertex of valence 1.

**1.41 Definition.** Let $\Gamma$ be a plumbing graph. A **bamboo joining two rupture vertices** in $\Gamma$ is a chain of vertices joined by edges such that all the vertices have valence 2 except the first and the last one, which are rupture vertices (Figure 1.10).

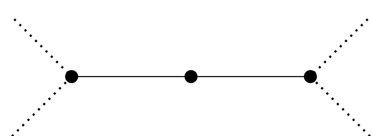
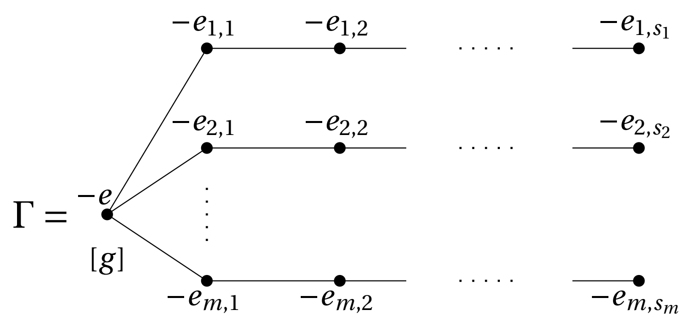


Figure 1.10: A bamboo joining two rupture vertices.

As in [40, Th. 5.1], given $b_1, \ldots, b_k \in \mathbb{Z}$, we use the following notation:

$$[b_1, \ldots, b_k] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \, \cfrac{}{- \cfrac{1}{b_k}}}} \tag{1.42}$$

**1.43 Theorem** ( [40, Th. 5.1])**.** *Let $P(\Gamma)$ be the 4-manifold obtained by plumbing according to the following plumbing graph $\Gamma$:*




*where one takes a 2-disc bundle over a surface of genus $g$ with Euler class $-e$ for the rupture vertex of $\Gamma$ and a 2-disc bundle over $\mathbb{S}^2$ with Euler class $-e_{i,j}$ for the other vertices of $\Gamma$ (with $1 \le i \le m$, $1 \le j \le s_i$).*

*Then $M(\Gamma) = \partial P(\Gamma)$ is a Seifert manifold. Moreover, if $M(\Gamma)$ admits a Seifert structure with orientable base, the Seifert invariants of $M(\Gamma)$ are given by*

$$\Big(g; -e - \sum_{i=1}^{m} \frac{\beta_i}{\alpha_i}; (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\Big)$$

*where $\frac{\alpha_i}{\alpha_i - \beta_i} = [e_{i,1}, \ldots, e_{i,s_i}]$ and $e_{i,j} < 2$ for all $i, j$.*

## 1.6 Resolution of curves and monodromy

In this section we will see the monodromy of the Milnor fibration of a complex analytic function with isolated singularity as a quasi-periodic diffeomorphism using a resolution of the singularity as is made in [15, 1.2 to 1.11]. For this, we will describe the sphere $\mathbb{S}^3$ as the boundary of a plumbing.

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic reduced germ with isolated singularity at the origin and let $V = f^{-1}(0)$. Let $\mathscr{U}$ be a neighbourhood of the origin in $\mathbb{C}^2$ and let $\pi\colon \mathscr{W} \to \mathscr{U}$ be a resolution of $V$ at the origin, given by a finite number of blow-ups in points.

**1.44 Definition.** Let $\widehat{E} = \pi^{-1}(0) \subset \pi^{-1}(\mathscr{U})$, it is called the **exceptional** divisor of $\pi$. Let $\widetilde{E_0} \subset \pi^{-1}(\mathscr{U})$ be the adherence of the complement of $\widehat{E}$ in $\pi^{-1}\big(f^{-1}(0) \cap \mathscr{U}\big)$, $\widetilde{E_0}$ is called the **strict transform** of $f^{-1}(0)$. Let us denote by $E_i$ the irreducible components of $\widehat{E}$, with $i = 1, \ldots, k$.

Let $0 < i \leq k$, at each point $p \in E_i$ of $\widehat{E}$ there exists some local coordinates $(u,v)$ centred at $p$ such that $u = 0$ is a local equation of $E_i$ and, locally,

$$(f \circ \pi)(u,v) = u^{m_i}\iota(u,v) \tag{1.45}$$

where $\iota$ is an unity in the ring of convergent power series $\mathbb{C}\{\{u,v\}\}$. After performing a change of coordinates, one can assume that $\iota = 1$.

Each irreducible component $E_i$ of $\widehat{E}$ is non-singular and $\widehat{E}$ has **normal crossings**; *i.e.*, if $i \neq j$, $E_i$ intersects $E_j$ in at most one point where they meet transversely and no three components of $\widehat{E}$ intersect. Also $\widehat{E} \cap \widetilde{E_0}$ is a normal crossing (see Figure 1.11).

**1.46 Definition.** Let $p_{i,j}$ be the intersection $E_i \cap E_j$ when this intersection is not empty. A point $p \in E_i$ is called **smooth** if $p \notin E_j$ for any $j \neq i$.

**1.47 Definition.** Let $0 < i \leq k$, the order $m_i$ of $f \circ \pi$ in a small neighbourhood of a smooth point of $E_i$ is called the **multiplicity** at $E_i$.

Let $b$ the number of branches of $f$. If the neighbourhood $\mathscr{U}$ is small enough, $\widetilde{E_0}$ consists of $b$ curves transverses to $\widehat{E}$. Let $b_i$ the number of branches of $f$ intersecting $E_i$. As $f$ is reduced, the multiplicity $m_0$ at each component of $\widetilde{E_0}$ is equal to one.

We choose an open neighbourhood of $E_i$ such that it is a fibration of discs with base $E_i$ The fibres of this fibration are called the **fibres** of $E_i$.

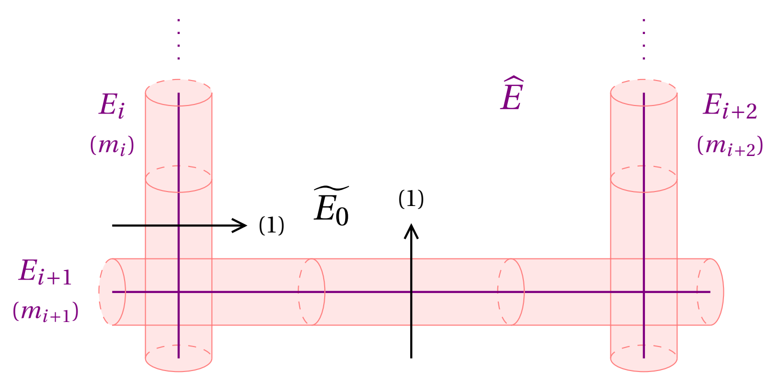



Figure 1.11: The preimage $\pi^{-1}(\mathscr{U})$.

By Lemma 1.18, there exists $0 < \varepsilon$ small enough and $0 < \eta \ll \varepsilon$ also small such that $f$ restricted to the Minor tube $N(\varepsilon,\delta)$ defines a locally trivial fibration over $\mathbb{S}^1_\eta$ (see Figure 1.12).

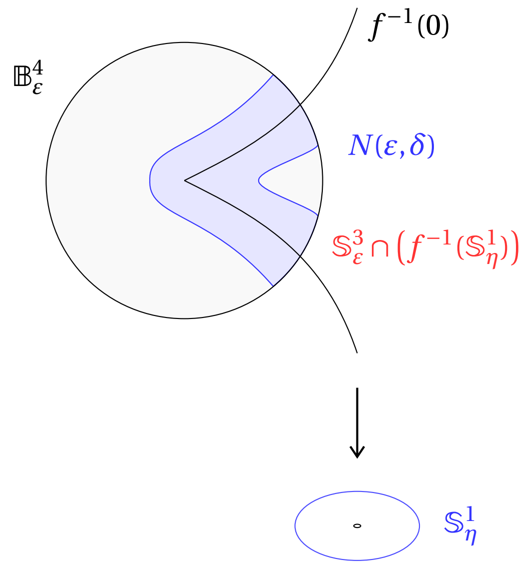



Figure 1.12: Milnor fibration on the Milnor tube.

Let us choose $\eta$ small enough such that if $z \in \mathbb{C}$ with $|z| = \eta$, then the manifold $\mathscr{F}_z = \pi^{-1}\left(f^{-1}(z) \cap \mathbb{B}^4_\varepsilon\right)$ is transverse to the fibres of the irreducible components of $\widehat{E}$ around a small neighbourhood of $\widetilde{E_0}$.

Let

$$\mathscr{F} = \mathscr{F}_z\,, \qquad X = \pi^{-1}\left(f^{-1}(\mathbb{B}^2_\eta) \cap \mathbb{B}^4_\varepsilon\right)\,,$$
$$\mathscr{F}_0 = \pi^{-1}\left(\mathbb{S}^3_\varepsilon \cap \bigcup_{0 \le t \le 1} f^{-1}(tz)\right)\,, \qquad E_0 = \widetilde{E_0} \cap X\,.$$

The manifold $X$ is a closed neighbourhood of $\widehat{E}$ in $W$ and the boundary $\partial X$ is diffeomorphic to the sphere $\mathbb{S}^3_\varepsilon$ (see Figure 1.13). Let $z \in \mathbb{C}$ be fixed, the boundary $\partial\mathscr{F}$ is equal to $\mathscr{F} \cap \mathscr{F}_0$ and there is an isotopy between the identity in $\mathbb{S}^3_\varepsilon$ and the diffeomorphism on the sphere which takes $\partial\mathscr{F}$ to the boundary $L = \partial(\mathscr{F} \cup \mathscr{F}_0)$. Let $\mathscr{U}_0$ be an union of fibres of $E_i$ (with $E_i \cap E_0 \neq \emptyset$) such that

$$\pi\left(\mathscr{U}_0 \cap \partial X\right) \subset \mathbb{S}^3_\varepsilon \cap f^{-1}(\mathbb{B}^2_\eta)\,.$$

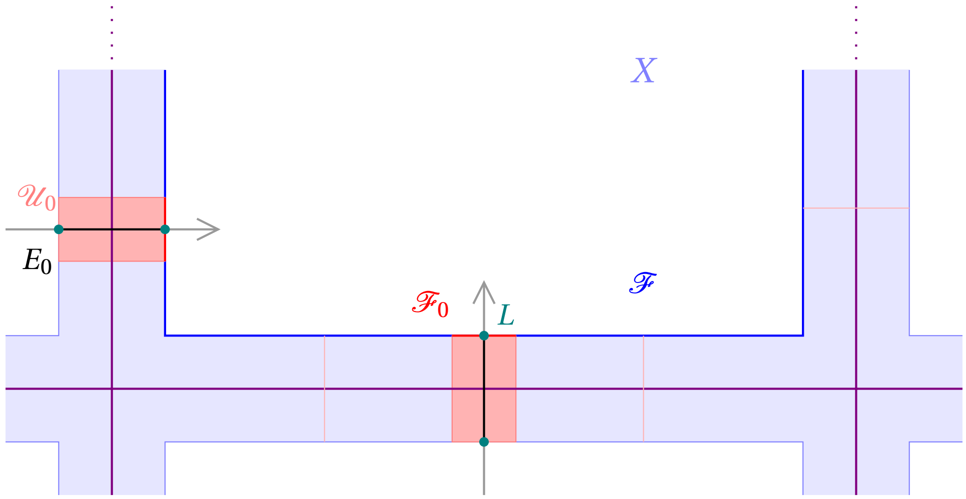



Figure 1.13: Milnor fibration in the resolution of $f$.

Now we construct a manifold $\overline{X}$ in the following way: For all $0 < i \leq k$, let $\overline{X}_i$ be the total space of a fibration of real discs with base $E_i$, $\overline{X}_i$ is isomorphic to a closed neighbourhood of $E_i$ in $X$.

Then $\overline{X}$ will be the manifold obtained after doing plumbing in a neighbourhood of the intersection points $p_{i,j}$, let $\mathbf{B}_{i,j}$ be the corresponding plumbing polydisc and let $\mathbf{T}_{i,j}$ be the plumbing torus defined as the intersection $\mathbf{B}_{i,j} \cap \partial\overline{X}$.

**1.48 Proposition** ( [15, Prop 1.4])**.** *There exists a diffeomorphism with corners $\rho\colon \overline{X} \to X$ such that:*

- *$\rho$ is the identity on $\widehat{E}$,*
- *$\rho^{-1}(E_0)$ is an union of fibres of the $\overline{X}_i$; these fibres are outside of the plumbing polydiscs $\mathbf{B}_{i,j}$ for any $i, j$,*
- *If $\Delta$ is a fibre of $E_i$ outside of $\mathscr{U}_0$, $\rho^{-1}(\Delta)$ is a fibre of $\overline{X}_i$.*

Let $X_i = \rho(\overline{X}_i)$, $B_{i,j} = \rho(\mathbf{B}_{i,j})$ and $T_{i,j} = \rho(\mathbf{T}_{i,j})$ (see Figure 1.14), where $T_{i,j}$ is the image under $\rho$ of the plumbing torus $\mathbf{T}_{i,j}$ with $0 < j < i \leq k$ and $T_{i,0}$ will denote the union of tori in the boundary $\partial X_i$ such that

$$\bigcup_{i>0} T_{i,0} = \pi^{-1}\big(\mathbb{S}^3_\varepsilon \cap f^{-1}(\mathbb{S}^2_\eta)\big)\,.$$

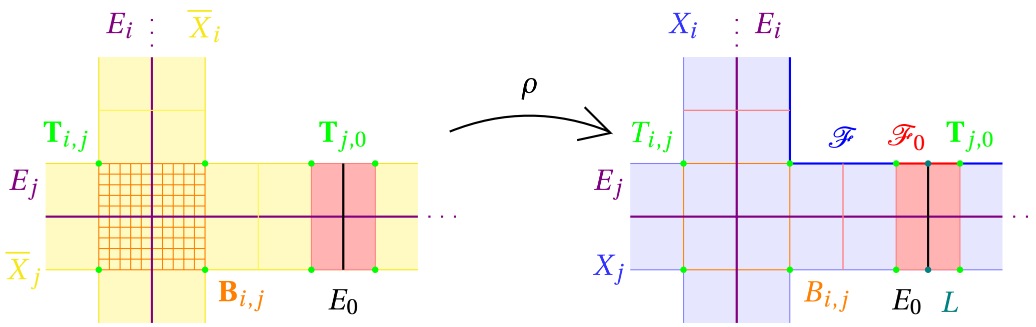



Figure 1.14: The plumbing of the fibred $X_i$'s is diffeomorphic to $X$.

Then we obtain Figure1.15.

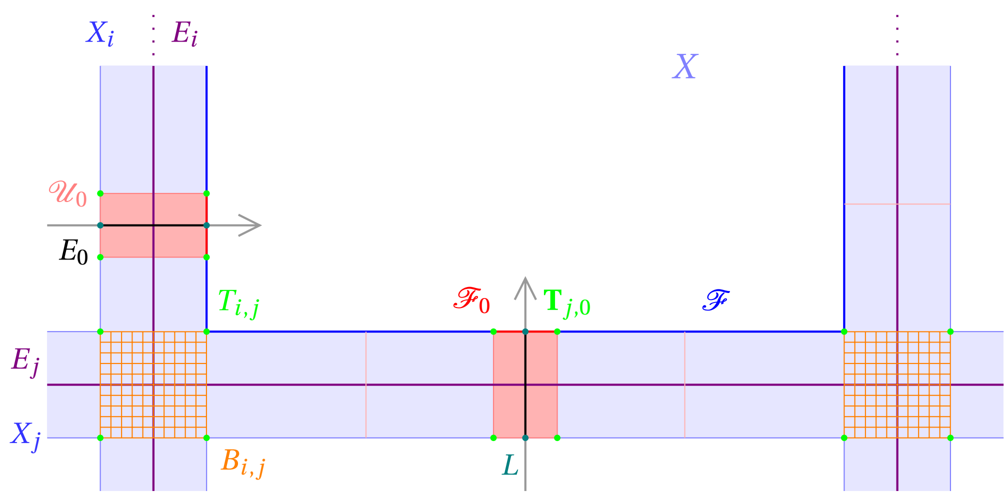



Figure 1.15: The manifold $X$ as plumbing.

**1.49 Definition.** An orientation preserving diffeomorphism $h\colon \mathscr{F} \to \mathscr{F}$ is called **quasi-periodic** if there is a family $\mathscr{C}$ of disjoint simple closed curves in $\mathscr{F}$ and a small neighbourhood $\mathscr{U}(\mathscr{C}) \subset \mathscr{F}$ of $\mathscr{C}$ such that

- for each curve $c \in \mathscr{C}$, $\mathscr{U}(c)$ is a small annulus, neighbourhood of $c$ in $\mathscr{F}$,
- for any pair of curves $c_i, c_j \in \mathscr{C}$, we have that $\mathscr{U}(c_i) \cap \mathscr{U}(c_j) = \emptyset$,
- $h(\mathscr{C}) = \mathscr{C}$
- $h(\mathscr{U}(c)) = \mathscr{U}(c)$,

- the restriction of $h$ to the complement of

$$\mathscr{U}(\mathscr{C}) = \bigcup_{c\in\mathscr{C}} \mathring{\mathscr{U}}(c)$$

is periodic, where $\mathring{\mathscr{U}}(c)$ is the interior of $\mathscr{U}(c)$.

The family $\mathscr{C}$ is called a **reduction system** of curves for the diffeomorphism $h$.

Let $\mathscr{F}_i = \mathscr{F} \cap \partial X_i$, $F = \cup_{i>0} F_i$.

The intersection of the fibres of $E_i$ with the boundary $\partial X_i$ endows $\partial X_i$ with a fibration in circles. Let $h_i \colon \mathscr{F}_i \to \mathscr{F}_i$ be the first return diffeomorphism on $\mathscr{F}_i$ along the fibres of $\partial X_i$.

By equation (1.45), in the local coordinates $(u, v)$, a fibre of $E_i$ over a smooth point is given by the equation $v = c$, where $c$ is a constant. Then the intersection of this fibre with $F_i$ consists of $m_i$ points $(u, c)$, where $c$ is solution of

$$u^{m_i} = c\,.$$

The diffeomorphism $h_i$ permutes these $m_i$ points and $h_i^{m_i}$ is the identity, then the order of $h_i$ is $m_i$ (see Figure1.16).

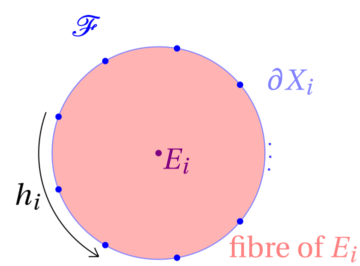



Figure 1.16: The diffeomorphism $h_i$ in a fibre of $E_i$.

Let $h_0 \colon \mathscr{F}_0 \to \mathscr{F}_0$ be the identity on $\mathscr{F}_0$.

Now we take the following family of curves: Let $\mathscr{C} = \cup_{0\le j<i} \mathscr{F}_i \cap \mathscr{F}_j$.

Let $i, j$ be such that $E_i \cap E_j \neq \emptyset$, then let $\widehat{m}_{i,j} = \gcd(m_i, m_j)$. If $0 < j < i$, $\mathscr{C}_{i,j} = \mathscr{F}_i \cap \mathscr{F}_j$ is a collection of $\widehat{m}_{i,j}$ simple closed curves of $\mathscr{F}$.

Now we will construct the neighbourhood $\mathscr{U}(\mathscr{C})$: For each point $p_{i,j}$ (with $0 \le j < i \le k$), we choose closed discs $I(B_{i,j})$ and $J(B_{i,j})$, neighbourhoods of $(B_{i,j} \cap E_i)$ in $E_i$ and $(B_{i,j} \cap E_j)$ in $E_j$ respectively.

Let $E = \widehat{E} \cup E_0$. There exists a deformation retract $R \colon X \to E$ (see Figure1.17) such that

- the plumbing torus $T_{i,j}$ goes to the point $p_{i,j}$,
- if $x \in \big(E_i \setminus \bigcup I(B_{i,j})\big)$, $R^{-1}(x)$ is the fibre of $E_i$ in the point $x$,
- if $x \in I(B_{i,j}) \setminus \{p_{i,j}\}$.

Let

$$V = \bigcup_{0\le j<i} \Big(R^{-1}\big(I(B_{i,j})\big) \cup R^{-1}\big(J(B_{i,j})\big)\Big) \quad \text{and} \quad \mathscr{U}(\mathscr{C}) = V \cap \mathscr{F} \,.$$

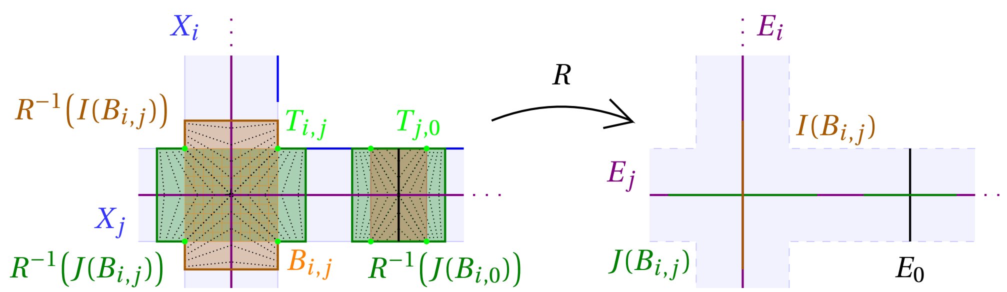



Figure 1.17: The deformation retract $R$ from $X$ to $E$.

The next step is "to glue" the diffeomorphisms $h_i$:

On the $\widehat{m}_{i,j}$ curves in $\mathscr{C}_{i,j}$, $h_i$ is a permutation of these curves and $h_i^{\widehat{m}_{i,j}}$ is the identity. On the boundary of the annuli $\mathscr{U}(\mathscr{C}_{i,j}) \cap \mathscr{F}_i$, the diffeomorphism $h_i^{\widehat{m}_{i,j}}$ is a rotation and $h_i = h$. Then, we extend $h$ from the boundary $\partial\big(\mathscr{U}(\mathscr{C}_{i,j}) \cap \mathscr{F}_i\big)$ to the boundary $\partial F_i$ by an isotopy; this is possible because $\mathscr{U}(\mathscr{C}_{i,j}) \cap \mathscr{F}_i$ is a disjoint union of annuli.

**1.50 Definition** ( [9, p.2])**.** Let $\mathscr{F}$ be a closed, orientable surface, let $c$ be a simple closed curve in $\mathscr{F}$ and let $\mathscr{U}(c) \subset \mathscr{F}$ be a small regular neighbourhood of $c$. Then $\mathscr{U}(c)$ is an annulus; *i.e.*, it is homeomorphic to the Cartesian product $[0,1] \times \mathbb{S}^1$. Then a point in $\mathscr{U}(c)$ is of the form $(s, e^{i\theta})$ with $s \in [0,1]$ and $\theta \in [0,2\pi]$ .

Let $f : \mathscr{F} \to \mathscr{F}$ be a map which is the identity outside $\mathscr{U}(c)$ and inside of $\mathscr{U}(c)$ it is defined by

$$f(s, e^{i\theta}) = (s, e^{i\theta} e^{i2\pi s}) \,.$$

Then $f$ is a **Dehn twist** about the curve $c$ (see Figure 1.18).

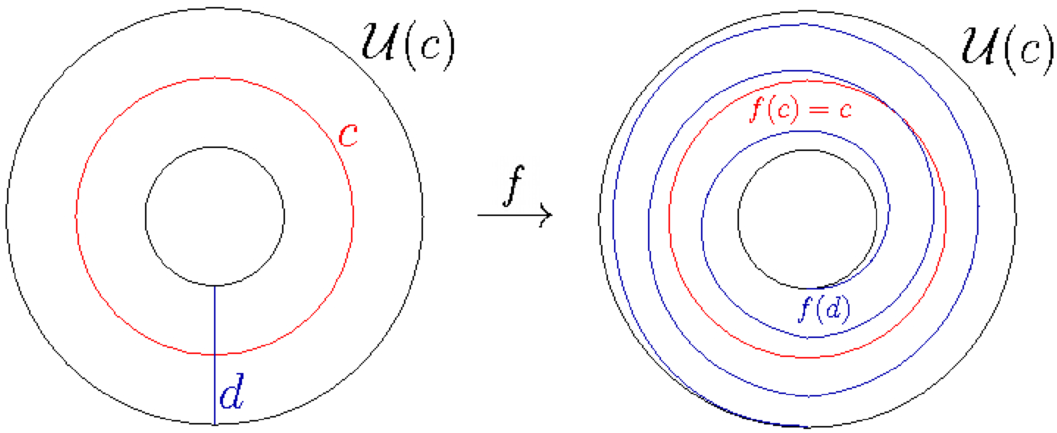



Figure 1.18: Dehn twist

Then we have the following result.

**1.51 Proposition** ( [14, Prop. 1.5])**.** *Let $h$ be the monodromy of the Milnor fibration of $f$. Its restriction $h|_{\mathscr{F}_i}$ is the diffeomorphism $h_i$.*

**1.52 Remark.** The representative of the monodromy found in this way depends on the resolution; in order to obtain a canonical quasi-periodic monodromy, we take the minimal resolution and we proceed in the same way.

Let us finish this section with two results about the topology of $\mathscr{F}$.

For all $0 < i \leq k$, let $r_i = \gcd(\widehat{m}_{i,j})_{E_i \cap E_j \neq \emptyset}$, let $\mathring{\mathscr{F}}_i$ be the interior of $\mathscr{F}_i$ and let $\mathring{E}_i$ be the set of smooth points of $E_i$. Let $v$ be the restriction to $\mathscr{F}$ of the deformation retract $R$ and let $v_i \colon \mathring{\mathscr{F}}_i \to \mathring{E}_i$ be the restriction to $\mathring{\mathscr{F}}_i$.

**1.53 Proposition** ( [14, Prop. 1.6])**.** *The restriction $v_i$ is the finite cyclic covering of order $m_i$, defined by the homomorphism*

$$\begin{aligned} \rho_i \colon \mathrm{H}_1(\mathring{E}_i, \mathbb{Z}) &\longrightarrow \mathbb{Z}_{m_i} \\ ([\mathscr{C}_i, j]) &\longmapsto \omega_j \end{aligned}$$

*where $\omega_j \equiv -m_j \pmod{m_i}$.*

**1.54 Proposition** ( [15, Prop 1.11])**.** *i) The number of connected components of $\mathscr{F}_i$ is $r_i$,*

*ii) $\mathscr{F}_0$ is the disjoint union of $b$ annuli,*

*iii) if $k_i = 1$, $\mathscr{F}_i$ is an union of discs and if $k_i = 2$, $\mathscr{F}_i$ is an union of annuli, where $k_i$ is the number of connected components $E_j$ which intersect $E_i$.*

## 1.7 Resolution of normal surface singularities

We finish this chapter recalling some results and definitions concerning the resolution of normal surface singularities and we will present a result of Neumann saying when a Seifert manifold can be the link of a singularity. The theory presented here will be used in Chapters 2 and 4.

**1.55 Definition.** An isolated surface singularity germ $(V,0)$ is (analytically) **normal** if every bounded holomorphic function on $V^* = V \setminus \{0\}$ extends to a holomorphic function at 0 (see [37, Rmk., p. 114]).

**1.56 Theorem** ( [5, Th. 6.2])**.** *Let* $(V,0)$ *be a normal complex surface singularity such that* $V^* = V \setminus \{0\}$ *is non-singular. Then, there exists a non-singular complex surface* $\widetilde{V}$ *and a proper analytic map* $\pi\colon \widetilde{V} \to V$ *such that*

- *the inverse image of* $0$, $\widehat{E} = \pi^{-1}(0)$, *is a connected, reduced divisor in* $\widetilde{V}$, *i.e., a union of* $1$*-dimensional compact curves in* $\widetilde{V}$, *and*

- *the restriction of* $\pi$ *to* $\widetilde{V} \setminus \widehat{E}$ *is a biholomorphic map between* $\widetilde{V} \setminus \widehat{E}$ *and* $V^*$.

**1.57 Definition.** The surface $\widetilde{V}$ is called a **resolution** of the singularity of $V$ and $\pi$ is the **resolution map**.

**1.58 Definition.** Let $(V,0)$ be a normal complex surface singularity and let $\widetilde{V}$ be a resolution with resolution map $\pi\colon \widetilde{V} \to V$. The resolution is called **good** if

- each irreducible component $E_i$ of $\widehat{E}$ is non-singular, and

- $\widehat{E}$ has normal crossings; *i.e.*, if $i \neq j$, $E_i$ intersects $E_j$ in at most one point where they met transversely and no three components of $\widehat{E}$ intersect,

- $\widehat{E} \cap \widetilde{E_0}$ is a normal crossing, where $\widetilde{E_0}$ is the strict transform of $V$ (see Definition 1.44).

**1.59 Definition.** Let $(V,0)$ be a normal complex surface singularity and let $\widetilde{V}$ a good resolution with resolution map $\pi\colon \widetilde{V} \to V$. The **resolution graph** $A_\pi$ is given in the following way:

- $A_\pi$ has a vertex for each irreducible component $E_i$ of the exceptional divisor $E$,

- two vertices of $A_\pi$ are connected by an edge if and only if the corresponding irreducible components intersect,

- $A_\pi$ has an arrow for each irreducible component of $\pi^{-1}(V)$, attached to the corresponding vertex.

- each vertex $i$ of $A_\pi$ is weighted by the multiplicity $m_i$ at the irreducible component $E_i$ and the Euler class (self-intersection number) $e_i$.

- each arrow is weighted by $+1$ if the functions defining $V$ are reduced.

In the resolution graph $A_\pi$ it is possible to define the concepts of rupture vertex, bamboo ended by a vertex of valence 1 and bamboo joining two rupture vertices as in the plumbing graph $\Gamma$ in Section 1.5. There is also a new concept:

**1.60 Definition.** A **bamboo ended by an arrow** is a chain of vertices joined by edges such that all the vertices have valence 2 except the first one, which is a rupture vertex and where the last vertex has an arrow (Figure 1.19).

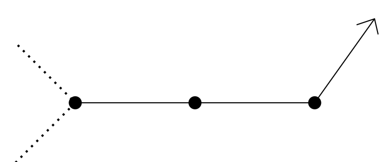


Figure 1.19: A bamboo ended by an arrow.

The following result is well known, see for instance [62, Th. 5.11].

**1.61 Theorem.** *Let $\pi\colon \widetilde{V} \to V$ be a good resolution of $(V,0)$, a normal surface singularity. Then the irreducible components $E_i$ of the exceptional divisor $\widehat{E}$ determine a plumbing graph $\Gamma(\widetilde{V})$, called the* ***dual graph*** *of the resolution. By plumbing according to this graph one obtains a 4-manifold homeomorphic to $\pi^{-1}(V \cap \mathbb{B}_\varepsilon) \subset \widetilde{V}$, whose boundary is the link of the singularity.*

When the link is a Seifert manifold, one has:

**1.62 Theorem** ( [46, Th. 2.6.1]). *Let $M$ be a Seifert manifold. If $M$ is the link of a complex surface singularity, then its resolution graph is star-shaped.*

Finally, the next result gives the condition under which a Seifert manifold is the link of a singularity.

**1.63 Corollary** ( [39, Cor. 6]). *The Seifert manifold $M$ is the link of a complex surface singularity if and only if it has a Seifert fibration with orientable base such that the Euler number of this fibration is negative.*

# CHAPTER 2

# THE SEIFERT CASE

As we mentioned in Chapter 1, the Milnor condition (Hypothesis 1.14) is very stringent; for example, when $k = 2$, the set of critical points of $f$ is, in general, a curve. Therefore it is not easy to find examples of real analytic germs satisfying such condition. Also, even when a function $f$ satisfies this condition, in general it is not true that the projection of the Milnor fibration is given by $\frac{f}{|f|}$ as in the complex case.

For example, consider the family of real analytic germs $f\colon (\mathbb{C}^2 \cong \mathbb{R}^4, 0) \to (\mathbb{C} \cong \mathbb{R}^2, 0)$ defined by $f(x, y) = x^p\bar{y} + \bar{x}y^q$ with $p, q \geq 2$. By [61, Th. 4.8], $f$ has a Milnor fibration with projection $\phi_f = \frac{f}{|f|}$. In [52, Th. 3.1] Pichon and Seade prove that the link $L_f$ is isotopic to the link of the holomorphic germ $g(x, y) = xy(x^{p+1} + y^{q+1})$, but the open-book decomposition given by the Milnor fibration of $f$ is not equivalent to the one given by the Milnor fibration of $g$.

In this chapter we study the family of real germs $F\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by

$$F(x, y, z) = \overline{xy}(x^p + y^q) + z^r$$

with $p, q, r \in \mathbb{N}$, $p, q, r \geq 2$ and $\gcd(p, q) = 1$.

We show that $F$ has an isolated singularity at the origin and the projection of the Milnor fibration is given by $\phi_F = \frac{F}{|F|}$, therefore it gives rise to an open-book decomposition of $\mathbb{S}^5$.

We explicitly describe $L_F$ as a Seifert manifold and show that $L_F$ is homeomorphic to the link of a normal complex surface. Finally, our main results exhibit two families of examples among these germs $F$ whose open-book decompositions of $\mathbb{S}^5$ cannot appear as Milnor fibrations of holomorphic germs from $\mathbb{C}^3$ to $\mathbb{C}$.

## 2.1 Isolated critical point

In this section we show that our family of real germs has isolated critical point at the origin; in order to prove this we use the following result.

**2.1 Proposition.** *Let* $h\colon (\mathbb{R}^n,0) \to (\mathbb{R}^2,0)$ *be a real analytic germ and let* $r \in \mathbb{Z}^+$. *The analytic germ* $H\colon (\mathbb{R}^n \times \mathbb{C},0) \cong (\mathbb{R}^{n+2},0) \to (\mathbb{R}^2,0)$ *defined for* $(x_1,\dots,x_n) \in \mathbb{R}^n$ *and* $z \in \mathbb{C}$ *by*

$$H(x_1,\dots,x_n,z) = h(x_1,\dots,x_n) + z^r$$

*has an isolated singularity at the origin if and only if h has an isolated singularity at the origin.*

*Proof.* The jacobian matrix $M$ of $h(x_1,\dots,x_n) + z^r$ with respect to the coordinates $x_1,\dots,x_n,z,\bar{z}$ is given by

$$\left(\begin{array}{c|cc} Dh(x_1,\dots,x_n) & \frac{1}{2}rz^{r-1} & \frac{1}{2}r\bar{z}^{r-1} \\ & \frac{1}{2i}rz^{r-1} & -\frac{1}{2i}r\bar{z}^{r-1} \end{array}\right)$$

where $Dh(x_1,\dots,x_n)$ is the jacobian matrix of $h$ with respect to the coordinates $x_1,\dots,x_n$.

Let $\mathscr{P}\colon \mathbb{R}^n \times \mathbb{C} \to \mathbb{R}^n$ be the projection defined by $\mathscr{P}(x_1,\dots,x_n,z) = (x_1,\dots,x_n)$.

If $h$ has an isolated singularity at the origin, then $Dh(x_1,\dots,x_n)$ has rank 2 in a neighbourhood $W$ of the origin except at the origin. Let $(x_1,\dots,x_n,z) \in \mathscr{P}^{-1}(W)$. If $(x_1,\dots,x_n) \neq 0$, then $Dh(x_1,\dots,x_n)$ has rank two. Otherwise $z \neq 0$ and the matrix

$$\begin{pmatrix} \frac{1}{2}rz^{r-1} & \frac{1}{2}r\bar{z}^{r-1} \\ \frac{1}{2i}rz^{r-1} & \frac{-1}{2i}r\bar{z}^{r-1}_{n+1} \end{pmatrix}$$

has rank two. Then $h(x_1,\dots,x_n) + z^r$ has rank two at each point of $\mathscr{P}^{-1}(W) \setminus \{0\}$.

If $h + z^r$ has isolated singularity at the origin, then $M$ has rank 2 in a neighbourhood $U$ of the origin except at the origin itself; in particular $M$ has rank 2 at the points $(x_1,\dots,x_n,0) \in U \setminus \{0\}$; then the matrix $Dh(x_1,\dots,x_n,z)$ has rank 2 in the neighbourhood $\mathscr{P}(U)$ of the origin except at the origin. □

**2.2 Corollary.** *Let* $F\colon (\mathbb{C}^3,0) \cong (\mathbb{R}^6,0) \to (\mathbb{C},0) \cong (\mathbb{R}^2,0)$ *be the real analytic function defined by*

$$F(x,y,z) = \overline{xy}(x^p + y^q) + z^r \ ,$$

*with* $p,q,r \in \mathbb{N}$ *and* $p,q,r \geq 2$. *The function* $F$ *has isolated singularity at the origin if and only if* $(p,q) \neq (2,2)$

*Proof.* Set $f(x,y) = xy$ and $g(x,y) = x^p + y^q$. According to Proposition 2.1, to prove the corollary is equivalent to proving that $h(x,y) = f(x,y)\overline{g(x,y)}$ has an isolated singularity at 0 if and only if $(p,q) = (2,2)$. Let us give two proofs of this fact.

First we decompose $f$ in its real and imaginary parts and the corresponding jacobian matrix is:

$$\begin{pmatrix} \frac{p\overline{xy}x^{p-1}+y(\bar{x}^p+\bar{y}^q)}{2} & \frac{\bar{y}(x^p+y^q)+pxy\bar{x}^{p-1}}{2} & \frac{q\overline{xy}y^{q-1}+x(\bar{x}^p+\bar{y}^q)}{2} & \frac{\bar{x}(x^p+y^q)+qxy\bar{y}^{q-1}}{2} \\ \frac{p\overline{xy}x^{p-1}-y(\bar{x}^p+\bar{y}^q)}{2i} & \frac{\bar{y}(x^p+y^q)-pxy\bar{x}^{p-1}}{2i} & \frac{q\overline{xy}y^{q-1}-x(\bar{x}^p+\bar{y}^q)}{2i} & \frac{\bar{x}(x^p+y^q)-qxy\bar{y}^{q-1}}{2i} \end{pmatrix} .$$

Then this matrix has rank less than two in a point $(x,y)$ if and only if the following equations, that are the result of calculating the minors of order 2, are satisfied:

$$p^2|xy|^2|x|^{2(p-1)} = |y|^2|x^p + y^q|^2 \tag{2.2}$$

$$q^2|xy|^2|y|^{2(q-1)} = |x|^2|x^p + y^q|^2 \tag{2.2}$$

$$x\bar{y}|x^p + y^q|^2 = pq|xy|^2\bar{x}^{p-1}y^{q-1} \, . \tag{2.2}$$

From these equations we get that the origin $(0,0)$ is always a critical point, and we also get that if $x = 0$ then $y = 0$ and vice versa.

Therefore, in order to look for another critical points we can suppose $x \neq 0 \neq y$. Simplifying equations (2.2) and (2.2), we get:

$$p^2|x|^{2p} = q^2|y|^{2q} = |x^p + y^q|^2 \, .$$

A direct computation shows that this equation together with (2.2) have non trivial solutions if and only if $p = q = 2$.

An alternative proof consists of using ( [51, Th 5.1]), which states that $f\bar{g}$ has an isolated singularity at 0 if and only if the link $L_{f\bar{g}} = L_f - L_g$ is fibred, where $L_f$ is the link of $f$ and $-L_g$ denotes the link $L_g$ of $g$ with opossite orientation.

Let $\pi : X \to \mathbb{C}^2$ be the minimal resolution of the germ $f g$. Then ( [51, Cor 2.1]) states that $L_f - L_g$ is fibred if and only if $m_f - m_g \neq 0$, where $m_f$ and $m_g$ denote the multiplicities of $f \circ \pi$ and $g \circ \pi$ in the (unique in our case) rupture vertex of the resolution graph $A_\pi(f g)$. As $m_f = pq$ and $m_g = p+q$, the corollary is proved. □

**2.3 Remark.**

- Notice that the arguments we used above to show that $f$ has an isolated critical point are, in this particular case, equivalent to showing that the system given in [51, page 8] has non-trivial solutions if and only if $p = q = 2$.

- We notice as well that Cor 2.2 is consistent with [53, Example 1.1.c] where it is shown that the function

$$\overline{z_1 z_2 \cdots z_n}(z_1^{a_1} + \cdots + z_n^{a_n}) \, , a_i \geq 2 \, ,$$

has 0 as an isolated critical value if and only if the sum of the $\frac{1}{a_i}$ is not 1.

## 2.2 Polar weighted homogeneous polynomials

In this section we recall the definition of a polar weighted homogeneous polynomial and some properties that we will use later. These polynomials were introduced by Cisneros-Molina in [13] following ideas from Ruas, Seade and Verjovsky in [57] and studied by Oka in [43] and [44].

Let $(p_1, \ldots, p_n)$ and $(u_1, \ldots, u_n)$ in $(\mathbb{Z}^+)^n$ be such that $\gcd(p_1, \ldots, p_n) = 1$ and $\gcd(u_1, \ldots, u_n) = 1$, we consider the action of $\mathbb{R}^+ \times \mathbb{S}^1$ on $\mathbb{C}^n$ defined by:

$$(t, \lambda) \cdot (z) = (t^{p_1} \lambda^{u_1} z_1, \ldots, t^{p_n} \lambda^{u_n} z_n) \, ,$$

where $t \in \mathbb{R}^+$ and $\lambda \in \mathbb{S}^1$.

**2.4 Definition.** Let $f : \mathbb{C}^n \to \mathbb{C}$ be a function defined as a polynomial function in the variables $z_i, \bar{z}_i$:

$$f(z_1, \ldots, z_n) = \sum_{\mu,\nu} c_{\mu,\nu} z^\mu \bar{z}^\nu \, ,$$

where $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, with $\mu_i, \nu_i$ not negative integers and $z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$ (same for $\bar{z}$).

The function $f$ is called a **polar weighted homogeneous polynomial** if there exists $(p_1,\dots,p_n)$ and $(u_1,\dots,u_n)$ in $(\mathbb{Z}^+)^n$ and $a,c \in \mathbb{Z}^+$ such that the following functional equality is satisfied:

$$f((t,\lambda)\cdot(z)) = t^a\lambda^c(f(z)) ,$$

where $z=(z_1,\dots,z_n)$.

It is known that any polar weighted homogeneous polynomial has an isolated critical value at the origin (see [13, Prop. 3.2] and [43, Prop. 2]). In the case it has isolated critical point, there is the following result.

**2.5 Proposition.** *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polar weighted homogeneous polynomial with isolated critical point at the origin. Then there exist a fixed-point free action of $\mathbb{S}^1$ on the link $L_f := f^{-1}(0)\cap\mathbb{S}^{2n-1}$ induced by the action of $\mathbb{R}^+\times\mathbb{S}^1$ described above. Hence, if $n=3$, $L_f$ is a Seifert manifold.*

*Proof.* Given a polar weighted homogeneous polynomial $f$, the action $\cdot$ of $\mathbb{R}^+\times\mathbb{S}^1$ on $\mathbb{C}^n$ induces an action of $\mathbb{S}^1$ on $\mathbb{C}^n$ by setting

$$\lambda\star z := (1,\lambda)\cdot z = (\lambda^{u_1}z_1,\dots,\lambda^{u_n}z_n) ,$$

where $\lambda\in\mathbb{S}^1$.

Notice that $f^{-1}(0)$ is invariant under the action $\star$ as well as any sphere $\mathbb{S}^{2n-1}_\varepsilon$. Therefore, the link $L_f$ is invariant under this action.

Moreover, this action is a fixed-point free and effective action. Indeed, let $z\in\mathbb{C}^n$ and suppose $z$ is a fixed point of the action $\star$; *i.e.*, for all $\lambda\in\mathbb{S}^1$ we have

$$\lambda\star z = z ,$$

then

$$f(z) = f(\lambda\star z) = \lambda^c f(z) \text{ with } c\in\mathbb{Z}^+ ,$$

which is a contradiction. When $L_f$ is a 3-manifold, it is a Seifert manifold by a theorem of Epstein in [19] (see also Section 1.4). □

Also a polar weighted homogeneous polynomial has Milnor fibration:

**2.6 Theorem** ( [13, Prop 3.4]). *Let $f\colon\mathbb{C}^n\to\mathbb{C}$ be a polar weighted homogeneous polynomial, then the map $\phi$ defined by*

$$\phi = \frac{f}{|f|}\colon (\mathbb{S}^{2n-1}_\varepsilon\setminus L_f)\to\mathbb{S}^1$$

*is a locally trivial fibration for any $\varepsilon>0$.*

**2.7 Corollary.** *Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a polar weighted homogeneous polynomial with isolated singular point at the origin. Then the Milnor fibration of $f$ is an open-book fibration of $\mathbb{S}^{2n-1}$ with the link $L_f$ as binding and the Milnor fibres as pages.*

Let us return to the case of our interest in this chapter.

**2.8 Proposition.** *Let $p, q, r \in \mathbb{Z}^+$ such that $\gcd(p,q) = 1$. Then the polynomials*

*i)* $\overline{xy}(x^p + y^q)$ *and*

*ii)* $\overline{xy}(x^p + y^q) + z^r$

*are polar weighted homogeneous polynomials.*

*Proof.* Let us consider the following action of $\mathbb{R}^+ \times \mathbb{S}^1$ on $\mathbb{C}^2$:

$$(t,\lambda)\cdot(x,y) = (t^q\lambda^q x, t^p\lambda^p y)\ .$$

let $(x,y) \in \mathbb{C}^2$, for all $(t,\lambda) \in \mathbb{R}^+ \times \mathbb{S}^1$ we have that

$$f((t,\lambda)\cdot(x,y)) = t^{q+p+pq}\lambda^{pq-q-p} f(x,y)\ ,$$

Thence the polynomial in (i) is polar weighted homogeneous.

Now, by [13, Exa. 2.6], the sum of two polar weighted homogeneous polynomials in independent variables is again a polynomial of this type. Then, using that $z^r$ is a polar weighted homogeneous polynomial, we have that $\overline{xy}(x^p + y^q) + z^r$ is also a polar weighted homogeneous polynomial. □

**2.9 Corollary.** *Let $F\colon \mathbb{C}^3 \to \mathbb{C}$ be the function defined by*

$$F(x,y,z) = \overline{xy}(x^p + y^q) + z^r$$

*where $p, q, r \in \mathbb{N}$ with $p, q, r \geq 2$ and $\gcd(p,q) = 1$. The link $L_F = F^{-1}(0) \cap \mathbb{S}^5$ is a Seifert manifold and the Milnor fibration of $F$, $\phi_F = F/|F|$ is an open-book fibration of $\mathbb{S}^5$.*

This result follows from Proposition 2.5, Corollary 2.7 and Proposition 2.8.

## 2.3 The link as a Seifert manifold

In this section we will describe the link $L_F$ as a Seifert manifold giving its Seifert invariants.

**2.10 Theorem.** *Let $(p,q)$ be coprime integers and $r \in \mathbb{N}$ with $p,q,r \geq 2$. Let $F\colon \mathbb{C}^3 \to \mathbb{C}$ be the function defined by $F(x,y,z) = \overline{xy}(x^p + y^q) + z^r$. Set $\delta = \gcd(r, pq-p-q)$. The Seifert invariants of the link $L_F$ are*

$$\left(\frac{\delta-1}{2};\ -\frac{\delta^2}{pqr};\ (qr/\delta, \beta_1), (pr/\delta, \beta_2), (r/\delta, \beta_3)\right),$$

*where*

$$\begin{aligned}
\frac{pq-p-q}{\delta}\beta_1 &\equiv -1 \pmod{qr/\delta}, \\
\frac{pq-p-q}{\delta}\beta_2 &\equiv -1 \pmod{pr/\delta}, \\
\frac{pq-p-q}{\delta}\beta_3 &\equiv 1 \pmod{r/\delta}.
\end{aligned}$$

**2.11 Corollary.** *There exists a normal complex surface singularity $(X,p)$ whose link is homeomorphic to the link $L_F$.*

*Proof.* By Theorem 2.10, the rational Euler class of $L_F$ (seen as a Seifert manifold) is negative. By Corollary 1.63, one has that a Seifert manifold $M$ has rational Euler class $e_0 < 0$ if and only if it is the link of an isolated normal singularity of a complex surface.

We can conclude the existence of a normal complex surface singularity $(X,p)$ whose link is homeomorphic to $L_F$. □

*Proof of Theorem 2.10.* By Corollary 2.9, the link $L_F$ is a Seifert manifold. In order to compute the Seifert invariants of $L_F$, we will consider the intersection of the boundary of a polydisc with $F^{-1}(0)$, we will see that this intersection is diffeomorphic to $L_F$ and then we will compute the corresponding Seifert invariants.

Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be defined by

$$f(x,y) = \overline{xy}(x^p + y^q)$$

with $(p,q) = 1$ and $p,q \geq 2$, and let $F = f(x,y) + z^r$ with $r \in \mathbb{N}$ and $r \geq 2$.

Let $\varepsilon$ be such that the sphere $\mathbb{S}^5_\varepsilon$ is a Milnor ball (see Definition 1.7) for $F$ and let $\varepsilon'$ be such that for all $(x,y,z) \in F^{-1}(0)$ with $(x,y) \in \mathbb{D}^4_{\varepsilon'}$ we have $|f(x,y)|^{1/r} < \varepsilon$.

We consider the polydisc

$$\mathbf{D}^6 = \{(x,y,z) | (x,y) \in \mathbb{D}^4_{\varepsilon'}, |z| \leq \varepsilon\} .$$

For technical reasons we replace the sphere $\mathbb{S}^5$ by the boundary $\partial\mathbf{D}^6$ of the polydisc $\mathbf{D}^6$.

According to [17, Th 3.5], there is a diffeomorphism between the link $L_F$ and the intersection $F^{-1}(0) \cap \partial\mathbf{D}^6$, then in the sequel we will denote this intersection by $L_F$.

First we see that $L_F$ is a Seifert manifold: Let $*$ be the action of $\mathbb{S}^1$ on $\mathbb{C}^3$ given by

$$\lambda * (x,y,z) = (\lambda^{\frac{rq}{\delta}} x, \lambda^{\frac{rp}{\delta}} y, \lambda^{\frac{pq-p-q}{\delta}} z) .$$

Notice that $\partial\mathbf{D}^6$ and $F^{-1}(0)$ are invariant under the action $*$, then so it is the link $L_F$, *i.e.*, by Epstein ( [19]) it is a Seifert manifold (see Section 1.4). From now on we denote by $*$ the restriction of the action to $L_F$.

Let $\bullet$ be the action of $\mathbb{S}^1$ on $\mathbb{C}^2$ given by

$$\lambda \bullet (x,y) = (\lambda^{\frac{rq}{\delta}} x, \lambda^{\frac{rp}{\delta}} y) .$$

As $\mathbb{S}^3$ is invariant under this action, we denote the restriction of the action to $\mathbb{S}^3$ with the same notation. Notice that $L_f$ is invariant by the action $\bullet$ and it consists of three orbits of this action.

Let $\mathscr{P} \colon L_F \to \mathbb{S}^3$ be the projection given by

$$\mathscr{P}(x,y,z) = (x,y) .$$

One can see that the projection $\mathscr{P}$ is a cyclic branched $r$-covering with ramification locus $L_f = f^{-1}(0) \cap \mathbb{S}^3$, and it is equivariant with respect to the actions $*$ and $\bullet$; *i.e.*,

$$\mathscr{P}(\lambda * (x,y,z)) = \lambda \bullet \mathscr{P}(x,y) . \tag{2.12}$$

A consequence of the equivariance of $\mathscr{P}$ is that the preimage of an orbit of the action $\bullet$ is the disjoint union of orbits of the action $*$.

We orient $L_F$ consistently with the orientation of $\mathbb{S}^3$ via the projection $\mathscr{P}$. Let $B$ the orbit space under the action $*$. Let $\pi_*$ and $\pi_\bullet$ the projections of the Seifert fibrations $\pi_* \colon L_F \to B$ and $\pi_\bullet \colon \mathbb{S}^3 \to \mathbb{S}^2$ respectively.

In order to compute the genus of $B$ and the rational Euler class of the Seifert fibration $\pi_*\colon L_F \to B$ we define the induced map $\mathscr{R}\colon B \to \mathbb{S}^2$ in the following way: Let $\mathscr{O} \in L_F$ be an orbit of the action $*$, let $b = \pi_*(\mathscr{O})$ and let $s = \pi_\bullet(\mathscr{P}(\mathscr{O}))$, then

$$\mathscr{R}(b) := s\,.$$

**2.13 Lemma.** *The map $\mathscr{R}$ is a cyclic branched $\delta$-covering with ramification locus $\pi_\bullet(L_f)$.*

*Proof.* Let $s \in \mathbb{S}^2 \setminus \pi_\bullet(L_f)$ and let $(x,y) \in \pi_\bullet^{-1}(s)$. Then $\mathscr{P}^{-1}(x,y)$ consists of the $r$ points

$$(x_0, y_0, w^{1/r} e^{\frac{2\pi k i}{r}})\,,\ k = 0,\dots,r-1\,,$$

where $w = f(x_0,y_0)$. An easy computation shows that these $r$ points are distributed in $\delta$ orbits.

Now, let $s \in L_f$ and let $(x,y) \in \pi_\bullet^{-1}(s)$. Then $\mathscr{P}^{-1}(x,y)$ consists of the single point $(x,y,0)$. □

It follows that we have a commutative diagram:

$$\begin{tikzcd} L_F \arrow[r, "\mathscr{P}"] \arrow[d, "\pi_*"'] & \mathbb{S}^3 \arrow[d, "\pi_\bullet"] \\ B \arrow[r, "\mathscr{R}"] & \mathbb{S}^2 \end{tikzcd}$$

which allows us to compute the rational Euler class $e_0(L_F \to B)$ and the genus $g(B)$. To compute the genus $g(B)$, we apply the Riemann-Hurwitz formula to the branched covering $\mathscr{R}$. The ramification locus consists of 3 points. Therefore,

$$2g(B) - 2 = \delta[2g(\mathbb{S}^2) - 2] + 3(\delta - 1)\,,$$

then $g(B) = \frac{\delta-1}{2}$.

Now we compute the rational Euler class $e_0(L_f \to B)$: by Lemma 2.13, the degree of the restriction of $\mathscr{P}$ to a regular fibre is $r/\delta$ and the degree of $\mathscr{R}$ is $\delta$; since $e_0(\mathbb{S}^3 \to \mathbb{S}^2) = -1/pq$, it follows from (1.34) that

$$e_0(L_F \to B) = -\frac{\delta^2}{pqr}\,.$$

Let us now describe the exceptional orbits. Let $(x_0, y_0, z_0) \in L_F$ such that $x_0 \neq 0$, $y_0 \neq 0$ and $z_0 \neq 0$, then the isotropy subgroup of $(x_0, y_0, z_0)$ by the action $*$ consists of the $\lambda \in \mathbb{S}^1$ such that

$$\lambda^{rq/\delta} = \lambda^{rp/\delta} = \lambda^{(pq-p-q)/\delta} = 1\,.$$

As $\gcd(rq/\delta, rp/\delta, (pq-p-q)/\delta) = 1$, the orbit of $(x_0, y_0, z_0)$ is not exceptional.

If $y_0 = 0$, then $z_0 = 0$ and the isotropy subgroup of the corresponding orbit consists of the $\lambda \in \mathbb{S}^1$ such that $\lambda^{qr/\delta} = 1$; *i.e.*, it is $\mathbb{Z}_{qr/\delta}$.

If $x = 0$, then $z_0 = 0$ and the isotropy subgroup of the corresponding orbit consists of the $\lambda \in \mathbb{S}^1$ such that $\lambda^{pr/\delta} = 1$; *i.e.*, it is $\mathbb{Z}_{pr/\delta}$.

If $x_0 \neq 0$, $y_0 \neq 0$ and $z_0 = 0$, the corresponding isotropy subgroup is $\mathbb{Z}_{r/\delta}$ since it consists of the $\lambda \in \mathbb{S}^1$ such that $\lambda^{r/\delta} = 1$.

Then we have three exceptional orbits:

$$\mathcal{O}_1 = \{(x,y,z) \in L_F | y = 0, z = 0\} ,$$
$$\mathcal{O}_2 = \{(x,y,z) \in L_F | x = 0, z = 0\} ,$$
$$\mathcal{O}_3 = \{(x,y,z) \in L_F | x \neq 0, y \neq 0, z = 0\} .$$

with $\alpha_1 = qr/\delta$, $\alpha_2 = pr/\delta$ and $\alpha_3 = r/\delta$.

Let us now compute each $\beta_i$, following the method that uses the slice representation given by Orlik in [45, pages 57 and 58].

In order to obtain $\beta_1$, one has to compute the angle of the rotation performed by the first return of the orbits of the $\mathbb{S}^1$-action on a slice $D \in L_F$ of $\mathcal{O}_1$, say at $(\epsilon', 0, 0)$; *i.e.*, the angle of the rotation performed on $D$ by the action of $e^{2i\pi/\alpha_1}$.

Instead of dealing with a slice in $L_F$, let us consider a small disk $D'$ close to $D$ in the intersection of $F^{-1}(0) \cap \{x = \epsilon'\}$, *i.e.*,

$$D' = \{\epsilon'^p \bar{y} + \bar{y} y^q + z^r = 0\} .$$

Very close from $(\epsilon', 0, 0)$, $D'$ can be approximated by the disk parametrised by $z$ :

$$D'' = \{(\epsilon', -\bar{z}^r, z), z \in \mathbb{C}, |z| \ll 1)\} ,$$

and the action of $e^{\frac{2i\pi}{\alpha_1}}$ on $D''$ can be approximated by :

$$e^{\frac{2\pi i\delta}{qr}} \cdot (1, -\bar{z}^r, z) = \Big(1, -\overline{\big((e^{\frac{2\pi i\delta}{qr}})^{\frac{pq-p-q}{\delta}} z\big)}^r, (e^{\frac{2\pi i\delta}{qr}})^{\frac{pq-p-q}{\delta}} z\Big) ,$$

where the disk $D''$ is invariant under the action. Therefore, $\beta_1^*$ equals $\dfrac{pq-p-q}{\delta}$ (mod $\dfrac{qr}{\delta}$) up to sign.

In order to get the right sign, the disk $D''$ has to be oriented as a complex slice of $\mathcal{P}(\mathcal{O}_1)$ via $\mathcal{P}$. But

$$\mathcal{P}(D'') = \{(\epsilon', -\bar{z}^r), z \in \mathbb{C}\} ,$$

which is a slice of the orbit $\mathscr{P}(\mathscr{O}_1)$ endowed with the opposite orientation as the complex one. Therefore, we consider $D''$ oriented by $\bar{z}$ (and not by $z$).

We then obtain $\beta_1^* = -\dfrac{pq-p-q}{\delta} \pmod{\dfrac{qr}{\delta}}$, and then $\beta_1$ is defined by

$$-\frac{pq-p-q}{\delta}\beta_1 \equiv 1 \pmod{\frac{qr}{\delta}}\,.$$

A similar computation leads to

$$-\frac{pq-p-q}{\delta}\beta_2 \equiv 1 \pmod{\frac{pr}{\delta}}\,.$$

And for the third orbit $\mathscr{O}_3$, we consider the $z$-plane as the slice, *i.e.*, we parametrise our slice with the coordinate $z$, then we consider the action of $e^{\frac{2\pi i\delta}{r}}$ given by

$$e^{\frac{2\pi i\delta}{r}}\cdot(x,y,z) = (x,y,e^{\frac{2\pi i\delta}{r}\cdot\frac{pq-p-q}{r}}z)\;;$$

this action is the standard action of type $[r/\delta,(pq-p-q)/\delta]$, then $\beta_3$ is defined by the congruence

$$\frac{pq-p-q}{\delta}\beta_3 \equiv 1 \pmod{r/\delta}\,.$$

□

Let $\Gamma_F$ be the plumbing graph such that $L_F \cong \partial P(\Gamma_F)$, where $P(\Gamma_F)$ is the four-manifold obtained by plumbing 2-discs bundles according to $\Gamma_F$.

Then $\Gamma_F$ is given by Figure 2.1, where

$$-e = e_0 + \sum_{i=1}^{3}\frac{\beta_i}{\alpha_i} - 3\,, \qquad \frac{\alpha_i}{\alpha_i-\beta_i} = [e_{i,1},\ldots,e_{i,s_i}],\ \text{for } i = 1,2,3\,.$$

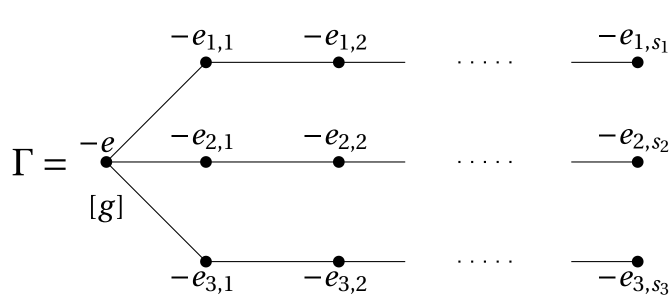



Figure 2.1: Plumbing graph of the Seifert manifold $L_F$.

Now we present some interesting examples of links of real singularities, describing them as Seifert manifolds and showing the associated plumbing graphs; in the next section we will see more of these examples.

Recall the notation $[e_{i,1},\dots,e_{i,s_i}]$ given in equation (1.42).

**2.14 Example.** Let $F$ be the real polynomial defined by

$$F(x,y,z)=\overline{xy}(x^2+y^3)+z^r \text{ with } r>2\,.$$

By Theorem 2.10, the Seifert invariants of $L_F$ are given by

$$\left(0;\,-\frac{1}{6r};\,(3r,3r-1),\,(2r,2r-1),\,(r,1)\right).$$

To compute the weights in the plumbing graph, we have:

$$\begin{aligned}\frac{3r}{3r-(3r-1)}&=[3r]\,,\\ \frac{2r}{2r-(2r-1)}&=[2r]\,,\\ \frac{r}{r-1}&=[\underbrace{2,2,\dots,2}_{(r-1)}]\,;\end{aligned}$$

*i.e.*, the plumbing graph $\Gamma_F$ is given by Figure 2.2.

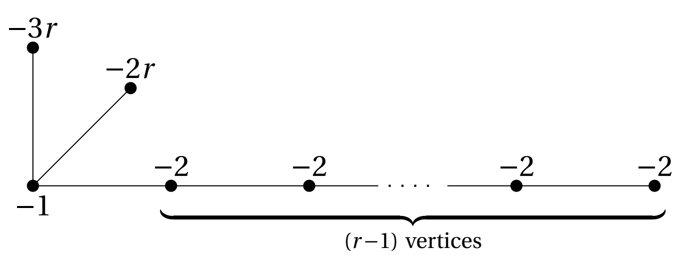



Figure 2.2: Plumbing graph for $L_F$ with $F(x,y,z)=\overline{xy}(x^2+y^3)+z^r$, $r>2$.

**2.15 Example.** Let $F$ be the polynomial defined by

$$F(x,y,z)=\overline{xy}(x^2+y^q)+z^2 \text{ with } q>2\,.$$

We will study two cases.

**Case 1.** Let $q=4a+1$ with $a\in\mathbb{Z}^+$, then by Theorem 2.10, the Seifert invariants of $L_F$ are given by

$$\left(0;\,-\frac{1}{4(4a+1)};\,\big(2(4a+1),2a+1\big),\,(4,1),\,(2,1)\right).$$

Now, the development in continuous fractions for $\alpha_1/(\alpha_1-\beta_1)$ is the following:

$$\frac{8a+2}{6a-1} = [2,2,4] \qquad \text{when } a=1\,,$$
$$\frac{8a+2}{6a-1} = [2,2,3,\underbrace{2,\dots,2}_{(a-2)},3] \qquad \text{when } a\geq 2\,;$$

When $a=1$, the corresponding plumbing graph is presented in Figure 2.3.

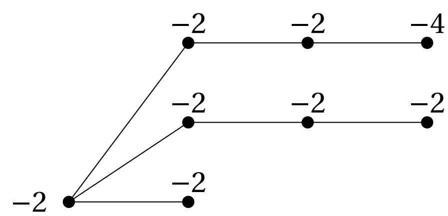


Figure 2.3: Plumbing graph for $L_F$ with $F(x,y,z)=\overline{xy}(x^2+y^5)+z^2$.

In Figure 2.4, there is the corresponding plumbing graph when $a\geq 2$.

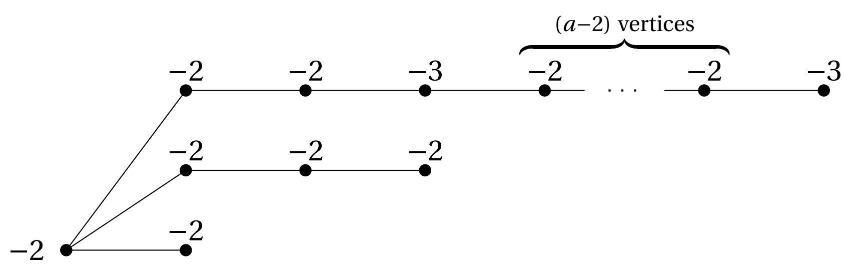



Figure 2.4: Plumbing graph for $L_F$ with $F(x,y,z)=\overline{xy}(x^2+y^{4a+1})+z^2$ and $a\geq 2$.

**Case 2.** Let $q=4a-1$ with $a\in\mathbb{Z}^+$; then by Theorem 2.10, the Seifert invariants of $L_F$ are given by

$$\left(0;\,-\frac{1}{4(4a-1)};\,\big(2(4a-1),6a-1\big),\,(4,3),\,(2,1)\right).$$

The development in continuous fractions for $\alpha_1/(\alpha_1-\beta_1)$ is the following:

$$\frac{8a-2}{2a-1} = [6] \qquad \text{when } a=1\,,$$
$$\frac{8a-2}{2a-1} = [5,3,\underbrace{2,\dots,2}_{(a-2)},3] \qquad \text{when } a\geq 2\,.$$

When $a = 1$, the corresponding plumbing graph is presented in Figure 2.5.

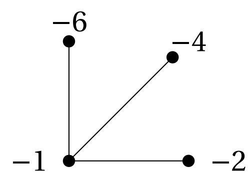



Figure 2.5: Plumbing graph for $L_F$ with $F(x, y, z) = \overline{xy}(x^2 + y^3) + z^2$.

In Figure 2.6, there is the corresponding plumbing graph when $a \geq 2$.

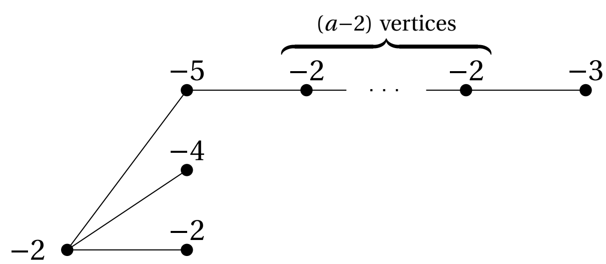



Figure 2.6: Plumbing graph for $L_F$ with $F(x, y, z) = \overline{xy}(x^2 + y^{4a-1}) + z^2$ and $a \geq 2$.

## 2.4 New open-book fibrations

In the previous section we gave, for the singularities we envisage in this chapter, the plumbing graphs corresponding to the Seifert invariants given by the $S^1$-action; we know that this graph has negative definite intersection matrix.

Let us consider now, for a moment, the general setting of an arbitrary plumbing graph $\Gamma$, where each vertex $e_i$ has been assigned a genus $g_i \geq 0$ and a weight $w_i < 0$, so that the corresponding intersection matrix $A_\Gamma$ is negative definite.

The graph determines a 4-dimensional compact manifold $\widetilde{V}_\Gamma$ with boundary $L_\Gamma$ which can be assumed to be an almost-complex manifold and its interior is a complex manifold.

The manifold $\widetilde{V}_\Gamma$ contains in its interior the exceptional divisor $E$, given by the plumbing description, and $\widetilde{V}_\Gamma$ has $E$ as a strong deformation retract (See Section 1.6). In this setting, the intersection matrix corresponds to the intersection product in $\mathrm{H}_2(\widetilde{V}_\Gamma, \mathbb{Q}) \cong \mathbb{Q}^n$, where $n$ is the number of vertices in the graph. The generators are the irreducible components $E_i$ of the divisor $E$; each of these is a compact (non-singular) Riemann surface embedded in $\widetilde{V}_\Gamma$.

The fact that the matrix $A_\Gamma$ is negative definite implies, by Grauert's contractibility criterion (see for instance [5]), that the divisor $E$ can be blown down to a point, and we get a complex surface $V_\Gamma$ with a normal singularity at 0, the image of the divisor $E$. Since $V_\Gamma$ is homeomorphic to the cone over $L_\Gamma$, its topology does not depend on the various choices and it is determined by the topology of $L_\Gamma$. However, in general, its complex structure does depend on the various choices.

In the sequel, we will only consider good resolutions (see Definition 1.58). Since any good resolution has associated a plumbing graph, in order to simplify the notation, we omit the subindex $\Gamma$ and we just write $A$, $\widetilde{V}$, $V$ and $L$.

The manifold $\widetilde{V}$ has an associated *canonical class* $K$:

**2.16 Definition.** Given a normal surface singularity $(V,p)$ and a good resolution $\widetilde{V}$ of $V$, the **canonical class** $K$ of $\widetilde{V}$ is the unique class in $\mathrm{H}_2(\widetilde{V},\mathbb{Q}) \cong \mathbb{Q}^n$ that satisfies the *Adjunction Formula*

$$2g_i - 2 = E_i^2 + K \cdot E_i$$

for each irreducible component $E_i$ of the exceptional divisor.

Note that the canonical class $K$ of a good resolution $\widetilde{V}$ only depends on the topology of the resolution. The existence and uniqueness of this class comes from the fact that the matrix $A$ is non-singular.

Since the canonical class $K$ is by definition a homology class in $\mathrm{H}_2(\widetilde{V},\mathbb{Q})$, it is a rational linear combination of the generators:

$$K = \sum_{i=1}^{n} k_i E_i \text{ ,with } k_i \in \mathbb{Q} \, .$$

**2.17 Definition.** (see for instance [16, Def. 1.2]) A normal surface singularity germ $(V,0)$ is **Gorenstein** if there is a nowhere-zero holomorphic two-form on the regular points of $V$. In other words, its canonical bundle $\mathcal{K} := \bigwedge^2 \big(T^*(V \setminus \{0\})\big)$ is holomorphically trivial in a punctured neighbourhood of 0.

For instance, if $V$ can be defined by a holomorphic map-germ $f$ in $\mathbb{C}^3$, then the gradient $\nabla f$ is never vanishing away from 0 and we can contract the holomorphic 3-form $dz_1 \wedge dz_2 \wedge dz_3$ with respect to $\nabla f$ to get a never vanishing holomorphic 2-form on a neighbourhood of 0 in $V$. So every isolated hypersurface singularity is Gorenstein. More generally, A. Durfee in [16, Def. 1.2, Lemma 1.3] introduced the following concept.

**2.18 Definition.** The singularity germ $(V,0)$ is **numerically Gorenstein** if the canonical class $K$ of some good resolution $\widetilde{V}$ is integral.

**2.19 Remark.** It is easy to show (see [16, Lemma 1.3] or [27, Def. 1.2]) that the condition of $K$ being an integral class is satisfied if and only if the canonical bundle $\mathcal{K}$ is topologically trivial. Therefore numerically-Gorenstein is a condition independent of the choice of resolution (whose dual graph is $\Gamma$) on the germ $(V,0)$ and every Gorenstein germ is numerically Gorenstein.

Notice that the canonical class $K$ is determined by the intersection matrix $A$, and therefore it is independent of the complex structure we put on $\widetilde{V}$. However by definition the class $K$ is associated to the graph that defines the surface $V$, and so does its self-intersection number $K^2$, which is defined in the obvious way. There are many different graphs producing the same singularity germ, and each has a different canonical class, with a different self-intersection number.

Similarly, one has another number, an integer, associated to $\widetilde{V}$: its *Euler-Poincaré characteristic* $\chi(\widetilde{V})$. Of course this number also depends on the graph $\Gamma$ and not only on the topology of the link $L$ of $V$. Yet one has the following result, which extends [62, Remark 7.6.i, p. 125], where this is discussed for Gorenstein singularities.

**2.20 Proposition.** *Assume the germ $(V,0)$ is numerically Gorenstein and let $\widetilde{V}$ be a good resolution. Let L be the link of $(V,0)$. Then the integer*

$$\chi(\widetilde{V}) + K^2$$

*is independent of the choice of resolution of $(V,0)$. Moreover, if the link $L_*$ of another isolated surface singularity germ is orientation preserving homeomorphic to L, then one has:*

$$\chi(\widetilde{V}) + K^2 = \chi(\widetilde{V}_*) + K_*^2\,.$$

This theorem is essentially well-known (see for example [30, 4.2]) and it can be proved in several ways. The first statement can be proved by direct computation, showing that each time we blow up a smooth point of the exceptional divisor, the Euler-Poincaré characteristic of the resolution increases by 1, while the self-intersection of the canonical class diminishes by one. This computation is straight-forward but not easy.

Alternatively, the same statement can be proved by noticing that for a compact complex surface $M$, the invariant $\chi(M)+K(M)^2$ is 12 times the Todd genus $Td(M)$, which is a birational invariant, by Hirzebruch-Riemann-Roch's Theorem for compact complex surfaces. One can then compactify the surface $\widetilde{V}$ by adding a divisor at infinity, whose singularities can be resolved, thus getting a smooth compactification $\widehat{V}$ of $\widetilde{V}$. The fact that the singularity is numerically Gorenstein ensures that the Todd genus $Td(\widehat{V})$ splits in two parts: one of these is $\chi(\widetilde{V})+K^2$. The result then follows from the fact that $Td(\widehat{V})$ remains constant under blowing ups.

We finally remark that for singularities that are Gorenstein and smoothable, the fact that $\chi(\widetilde{V})+K^2$ does not depend on the choice of resolution is also a direct consequence of the Laufer-Steenbrink formula that we explain below, and the fact that the geometric genus is independent of the choice of resolution.

The second statement in this theorem is now an immediate consequence of the previous statement, together with Neumann's Theorem [39, Th. 2], that if two normal surface singularities have orientation preserving homeomorphic links, then their minimal resolutions are homeomorphic.

So, we denote this invariant just by $\chi_L + K_L$, since it depends only on the link $L$.

**2.21 Definition.** A normal surface singularity germ $(V,0)$ of dimension $n \geq 1$ is **smoothable** if there exists a complex analytic space $(W,0)$ of dimension $n+1$ and a proper analytic map:

$$\mathscr{F}\colon W \to \mathbb{D} \subset \mathbb{C}$$

where $\mathbb{D}$ is an open disc with centre at 0, such that:

i) it is not a zero divisor in the local ring of $W$ at 0, *i.e.*, it is not flat;

ii) $\mathscr{F}^{-1}(0)$ is isomorphic to $V$; and

iii) $\mathscr{F}^{-1}(t)$ is non-singular for $t \neq 0$.

Following [16], [66] and [28], we call the manifold $\mathscr{F}^{-1}(t)$ a **smoothing** of $V$. Notice that if $(V,0)$ is a hypersurface germ, then it is always smoothable and the smoothing $\mathscr{F}^{-1}(t)$ is the Milnor fibre.

Now suppose we have a normal Gorenstein surface singularity germ $(V,0)$ and suppose it is smoothable. Let $\widetilde{V}$ be a resolution of $(V,0)$ and let $V^{\#}$ be a smoothing. Then the Laufer-Steenbrink formula (see [28] and [66]) states:

$$\chi(V^{\#}) = \chi(\widetilde{V}) + K^2 + 12\rho_g(V,0)\ ,$$

where $K$ is the canonical class of the resolution and $\rho_g(V,0)$ is the geometric genus, which is an integer, independent of the choice of resolution (see [4]). Then we have the following result.

**2.22 Theorem** ( [59, § 4, Cor 1])**.** *Let $(V,0)$ be a normal Gorenstein complex surface singularity with link $L$. If $(V,0)$ is smoothable, then one has*

$$\chi_L + K_L \equiv \chi(V') \pmod{12}$$

*where $V'$ is a smoothing of $V$ and $\chi_L + K_L$ is the invariant of $L$ previously defined.*

The following results give new open-book fibrations for $\mathbb{S}^5$ from the point of view of Singularity Theory.

**2.23 Theorem.** *Let $F\colon (\mathbb{C}^3,0) \cong (\mathbb{R}^6,0) \to (\mathbb{C},0) \cong (\mathbb{R}^2,0)$ be the real polynomial function defined by*

$$F(x,y,z) = \overline{xy}(x^2+y^3) + z^r \ \textit{with}\ r > 2\ .$$

*Let $(V,p)$ a complex analytic germ such that $L_V \cong L_F$; then $(V,p)$ is not numerically Gorenstein.*

**2.24 Corollary.** *There is not a complex analytic germ $G\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ with isolated singularity at the origin such that the link $L_G$ is isomorphic to the link $L_F$.*

*Proof of Theorem 2.23.* The linear system we need to solve in order to compute the canonical class $K$ of the resolution $\tilde{V}_F$ is of the form:

$$
\begin{aligned}
k - k_{1,1} - k_{2,1} - k_{3,1} &= 1 \ , \\
-k + 3r k_{1,1} &= 2 - 3r \ , \\
-k_{+} 2r k_{2,1} &= 2 - 2r \ , \\
-k + 2k_{3,1} - k_{3,2} &= 0 \ , \\
-k_{3,1} + 2k_{3,2} - k_{3,3} &= 0 \ , \\
&\dots \\
-k_{3,r-3} + 2k_{3,r-2} - k_{3,r-1} &= 0 \ , \\
-k_{3,r-2} + 2k_{3,r-1} &= 0 \ .
\end{aligned}
$$

We solve the system and we obtain:

$$
k = 10 - 6r \ , \quad k_{1,1} = \tfrac{4}{r} - 3 \ , \quad k_{2,1} = \tfrac{6}{r} - 4 \ , \quad k_{3,1} = -\tfrac{10}{r} - 6r + 16 \ ,
$$

$$
\begin{aligned}
&\dots \\
k_{3,r-2} &= \tfrac{20}{r} - 12 \ , \\
k_{3,r-1} &= \tfrac{10}{r} - 6 \ .
\end{aligned}
$$

Then $k_{1,1}$ is an integer only if $r = 4$ (since $r \neq 2$), but then $k_{2,1}$ is not an integer. We can conclude that the canonical class $K$ has not integer coefficients. $\square$

**2.25 Theorem.** *Let* $F \colon (\mathbb{C}^3, 0) \cong (\mathbb{R}^6, 0) \to (\mathbb{C}, 0) \cong (\mathbb{R}^2, 0)$ *be the real polynomial function defined by*

$$
F(x, y, z) = \overline{xy}(x^2 + y^q) + z^2 \ \textit{with} \ q > 2 \ .
$$

*Then the Milnor fibre* $\mathscr{F}$ *of* $F$ *is not the smoothing of a normal Gorenstein complex surface singularity* $(X, p)$.

**2.26 Corollary.** *The open-book fibration of the sphere* $\mathbb{S}^5$ *given by the Milnor fibration of* $F$ *is not given by the Milnor fibration of a normal Gorenstein complex surface singularity* $(X, p)$.

*Proof of Theorem 2.25.* We will consider two cases:

**Case 1.** Let $q = 4a+1$, then we solve the linear system in order to get the canonical class $K$ and we obtain:

$$\begin{array}{llll} k = -4a\,, & k_{1,1} = -3a\,, & k_{2,1} = -3a\,, & k_{3,1} = -2a\,. \\ & k_{1,2} = -2a\,, & k_{2,2} = -2a\,, & \\ & k_{1,j} = -a + (j-3)\,,\ 3 \le j \le a+2 & k_{2,3} = -a\,, & \end{array}$$

Since the canonical class $K$ has integer coefficients, there could exists a complex analytic germ $G\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with isolated singularity with link $L_G$ homeomorphic to the link $L_F$.

Now we proceed to see if the open-book fibrations given by $F$ and $G$ are equivalent.

In order to see if the Milnor fibre $\mathscr{F}$ is diffeomorphic to a smoothing of a normal Gorenstein complex surface singularity we compute the Euler characteristic of $\mathscr{F}$ and the Euler characteristic of the resolution $\tilde{V}_F$ and we apply Theorem 2.22.

We compute the Euler characteristic of $\mathscr{F}$ using the Join Theorem for polar weighted homogeneous polynomials, which is a generalisation of Oka's result (see [42, Th. 1]). Let us recall the definition of a join and then the mentioned Join Theorem.

**2.27 Definition.** Let $X$ and $Y$ be topological spaces. The **join** $X * Y$ is defined as the quotient space

$$X \times [0,1] \times Y \big/ \sim$$

where $\sim$ is the equivalence relation defined by: Given two points $(x, t, y), (x', t', y') \in X \times [0,1] \times Y$, they are equivalent if and only if

i) $(x, t, y) = (x', t', y')$,

ii) $t = t' = 0$ and $y = y'$,

iii) $t = t' = 1$ and $x = x'$.

**2.28 Theorem** ( [13, Th. 4.1])**.** *Let $g\colon \mathbb{C}^n \to \mathbb{C}$ and $h\colon \mathbb{C}^m \to \mathbb{C}$ be polar weighted homogeneous polynomials. Consider the polynomial on $\mathbb{C}^n \times \mathbb{C}^m$ defined by*

$$f(z;w) = g(z) + h(w)\,,$$

*which is also a polar weighted homogeneous polynomial. Let $X = f^{-1}(1)$, $Y = g^{-1}(1)$ and $Z = h^{-1}(1)$. Then there is a homotopy equivalence $\alpha\colon X \to Y * Z$, which is compatible with the monodromy maps and their join.*

Also we have

$$\chi(\tilde{V}) = \sum_{i=1}^{n} \chi(E_i) - \sum_{i<j} E_i \cdot E_j \ ,$$

where $E_i$ is one of the irreducible components of $\tilde{V}$ and

$$K^2 = K^T A K \ ,$$

where $A$ is the intersection matrix of the plumbing graph $\Gamma$.

In our case, the plumbing graph has $a+7$ vertices and $a+6$ edges, then

$$\begin{aligned}\chi(\tilde{V}_F) &= \chi(E) + \sum_{i=1,j=1}^{3,a+2} \chi(E_{i,j}) - \sharp(E \cap E_{i,1})_{i=1,2,3} - \sharp(E_{i,j} \cap E_{i,j'})_{i=1,2,3,j\neq j'} \\ &= (a+7)(2) - (a+6) = a+8 \ .\end{aligned}$$

Also

$$\begin{aligned}K^2 &= K^T A K \\ &= (0,0,0,1,\underbrace{0,\ldots,0}_{(a-2)},1,0,0,0) \\ &= k_{1,3} + k_{1,a+2} = -a + \big(-a + (a+2-3)\big) \\ &= -(a+1) \ .\end{aligned}$$

*i.e.*, we have

$$\chi(\tilde{V}_F) + K^2 = (a+8) - (a+1) = 7 \ .$$

When $a = 1$, a decorated plumbing graph for $f(x,y) = \overline{xy}(x^2 + y^5)$ is given by the graph shown in Figure 2.7.

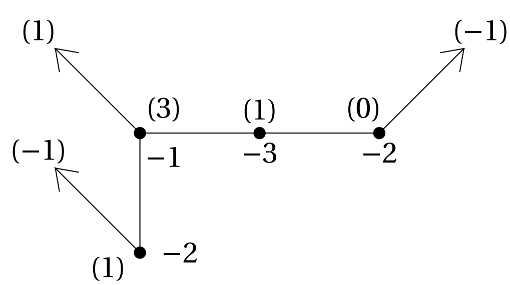



Figure 2.7: A decorated plumbing graph for $f(x,y) = \overline{xy}(x^2 + y^5)$.

When $a \geq 2$, a decorated plumbing graph for $f(x,y) = \overline{xy}(x^2 + y^{4a+1})$ is given by the graph in Figure 2.8.

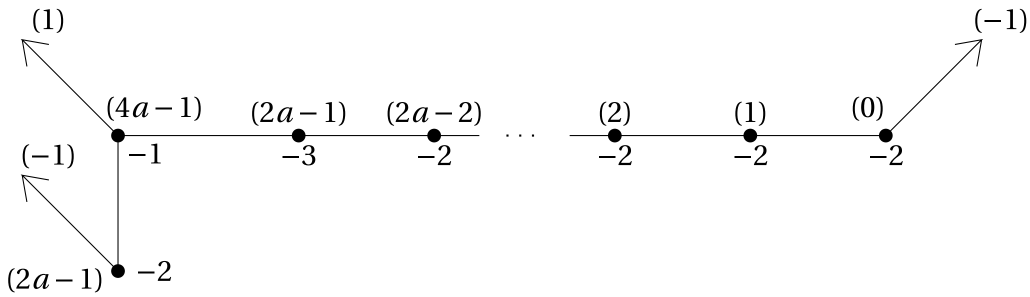



Figure 2.8: A decorated plumbing graph for $f(x,y)=\overline{xy}(x^2+y^{4a+1})$.

Let $\mathscr{F}_f=\bigvee_{i=1}^{k}\mathbb{S}_i^1$ be the Milnor fibre of $f(x,y)=\overline{xy}(x^p+y^q)$ and let $\mathscr{F}_g$ be the Milnor fibre of the function $g(z)=z^r$ with $z\in\mathbb{C}$; then we have

$$\chi(\mathscr{F})=\chi(\mathscr{F}_f*\mathscr{F}_g)=\chi\left(\bigvee_{j=1}^{(r-1)k}\mathbb{S}_j^2\right).$$

Then, for both cases we have

$$\chi(\mathscr{F})=(2-1)\chi(V_f)=(2-1)(4a-1)=1-4a\,,$$

and when

$$\begin{aligned} a&\equiv 1 \pmod 3\,, & 1-4a&\equiv 9 \pmod{12}\,,\\ a&\equiv 2 \pmod 3\,, & 1-4a&\equiv 5 \pmod{12}\,,\\ a&\equiv 0 \pmod 3\,, & 1-4a&\equiv 1 \pmod{12}\,. \end{aligned}$$

Thus, the Milnor fibre $\mathscr{F}$ is not diffeomorphic to a smoothing of a normal Gorenstein complex surface singularity.

**Case 2.** Let $q=4a-1$ with $a\in\mathbb{Z}^+$, then the solutions to the linear system which gives the canonical class $K$ are:

$$\begin{aligned} &k=-2(2a-1)\,, \quad k_{1,1}=-a\,, \qquad k_{2,1}=-a\,, \quad k_{3,1}=-2a+1\,.\\ &k_{1,j}=-a+(j-1)\,,\ \ 2\le j\le a \end{aligned}$$

Since the canonical class $K$ is integral, there could be a complex analytic germ $G\colon(\mathbb{C}^3,0)\to(\mathbb{C},0)$ with isolated singularity with link $K_G$ homeomorphic to the link $L_F$.

Now we proceed to see if the open-book fibrations given by $F$ and $G$ are equivalent as before. Firstly, the plumbing graph has $a+3$ vertices and $a+2$ edges, then

$$\begin{aligned}\chi(\tilde{V}_F) &= \chi(E) + \sum_{i=1,j=1}^{3,a} \chi(E_{i,j}) - \sharp(E\cap E_{i,1})_{i=1,2,3} - \sharp(E_{i,j}\cap E_{i,j'})_{i=1,2,3,j\neq j'} \\ &= (a+3)(2) - (a+2) = a+4\,.\end{aligned}$$

Also

$$\begin{aligned}K^2 &= K^T A K \\ &= (-1,3,\underbrace{0,\ldots,0}_{(a-2)},1,2,0) \\ &= -k+3k_{1,1}+k_{1,a}+2k_{2,1} = 2(2a-1)+3(-a)+(-1)+2(-a) \\ &= -(a+3)\,.\end{aligned}$$

*i.e.*, we have

$$\chi(\tilde{V}_F) + K^2 = (a+4)-(a+3) = 1\,.$$

When $a=1$, a decorated plumbig graph for $f(x,y)=\overline{xy}(x^2+y^3)$ is given by the graph in Figure 2.9.

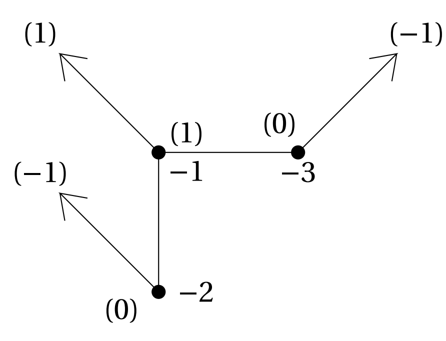



Figure 2.9: A decorated plumbing graph for $f(x,y)=\overline{xy}(x^2+y^3)$.

In Figure 2.10 there is a decorated plumbing graph for $f(x,y)=\overline{xy}(x^2+y^{4a-1})$, when $a\geq 2$.

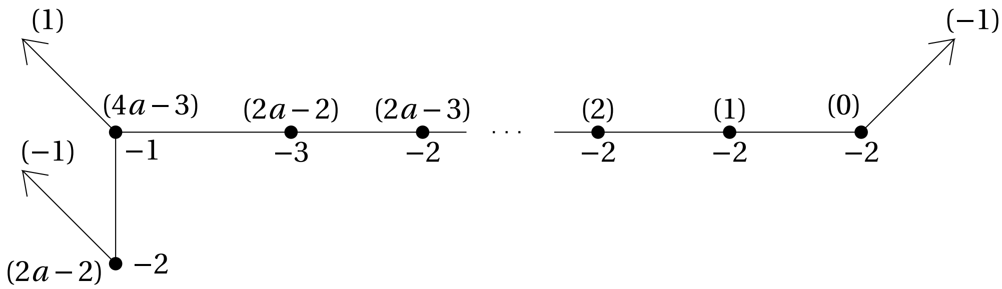



Figure 2.10: A resolution graph for $f(x,y) = \overline{xy}(x^2 + y^{4a-1})$.

In general we have

$$\chi(\mathscr{F}) = (2-1)\chi(V_f) = (2-1)(4a-3) = 3-4a\,,$$

and when

$$\begin{aligned} a &\equiv 1 \pmod 3\,, & 3-4a &\equiv 11 \pmod{12}\,, \\ a &\equiv 2 \pmod 3\,, & 3-4a &\equiv 7 \pmod{12}\,, \\ a &\equiv 0 \pmod 3\,, & 3-4a &\equiv 3 \pmod{12}\,. \end{aligned}$$

Thus, the Milnor fibre $\mathscr{F}$ is not diffeomorphic to a smoothing of a normal Gorenstein complex surface singularity. $\square$

# CHAPTER 3

# COMPUTATION OF THE LINK IN THE GENERAL CASE

In this chapter the main idea is to describe the link $L_F$ of the function $F(x,y,z) = f(x,y)\overline{g(x,y)} + z^r$ with $r \in \mathbb{N}$. In order to achieve this, in the first section it is described as an open-book. Then we recall the theory of periodic diffeomorphisms given by Nielsen in [41], which gives a way to represent a periodic diffeomorphism of a surface by a graph. In the third and fourth section this construction is extended to a quasi-periodic diffeomorphism which is the identity in the boundary of the surface.

In Section 3.5, the concepts of plumbing manifold and plumbing link are presented and in the following section they are related with the open-book of a quasi-periodic diffeomorphism; the last concept is an extension of the construction presented in Section 1.3. It is shown that the link $L_F$ is a plumbing manifold and, moreover, $(L_F, L')$ is a plumbing link, where $L'$ is the preimage of $L_{f\bar{g}}$ under the projection of $\mathbb{C}^3$ on $\mathbb{C}^2$ by projecting on the first two coordinates.

Then, it is shown that the monodromy of a function $f\bar{g}$, with $f, g$ two holomorphic germs from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}, 0)$, is a quasi-periodic diffeomorphism of the associated Milnor fibre. Using the theory of the previous sections, it is given the correspondent Nielsen graph its relation with the graph representing the open-book of a quasi-periodic diffeomorphism.

In Section 3.8 it is described the monodromy of $L_F$ in terms of the monodromy of the Milnor fibration of $f\bar{g}$. In the last section some examples are given an it is presented the algorithm to describe $L_F$ from the information on $L_{f\bar{g}}$.

## 3.1 Description of the link $L_F$ as an open-book

In this section we give an open-book fibration of the link $L_F$, related to the Milnor fibration of the germ $f\overline{g}$, with great importance in the following sections and which allow us to describe the corresponding monodromy in terms of the monodromy of the Milnor fibration of $f\bar{g}$.

Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ and $g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be two complex analytic germs such that the real analytic germ $f\bar{g}\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ has an isolated singularity at the origin.

For $\epsilon > 0$ sufficiently small, let $L_{f\bar{g}} = (f\bar{g})^{-1}(0) \cap \mathbb{S}^3_\epsilon$ be the link of $f\bar{g}$; recall that $L_{f\bar{g}}$ is the oriented link $L_f - L_g$. By Theorem 1.24, the Milnor fibration of $f\bar{g}$ has projection $\Phi_{f\bar{g}} = \frac{f\bar{g}}{|f\bar{g}|}$. It is an open-book fibration of $\mathbb{S}^3_\epsilon$ with binding $L_{f\bar{g}}$.

Let $F\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ be the germ defined by

$$F(x,y,z) = f(x,y)\overline{g(x,y)} + z^r$$

with $r \in \mathbb{Z}^+$. Let $\varepsilon$ be such that $\mathbb{D}^6_\varepsilon$ is a Milnor ball for $F$ (see Definition 1.7) and let $L_F = F^{-1}(0) \cap \mathbb{S}^5_\varepsilon$ be the link of $F$. By Proposition 2.1, $F$ has an isolated singularity at the origin.

Let $\varepsilon'$ be such that for all $(x,y,z) \in F^{-1}(0)$ with $(x,y) \in \mathbb{D}^4_{\varepsilon'}$ we have

$$|f(x,y)\overline{g(x,y)}|^{1/r} < \varepsilon\,.$$

As in the previous chapter, we consider the polydisc

$$\mathbf{D}^6 = \{(x,y,z) \,|\, (x,y) \in \mathbb{D}^4_{\varepsilon'}, |z| \leq \varepsilon\}\,.$$

By Proposition 1.11 and [17, App. 3.8], $L_F$ is homeomorphic to the intersection $F^{-1}(0) \cap \partial\mathbf{D}^6$. In the sequel we will denote again this intersection by $L_F$.

The following proposition is an adaptation of Proposition 1.5 of [49], which deals with the case $f(x,y) + z^k$ where $f$ is holomorphic and reduced.

**3.1 Proposition.** *Let $\mathscr{P}\colon L_F \to \mathbb{S}^3_{\varepsilon'}$ be the projection defined by*

$$\mathscr{P}(x,y,z) = (x,y)$$

*and let $L' = \mathscr{P}^{-1}(L_{f\bar{g}})$. Define $\rho_r\colon \mathbb{C} \to \mathbb{C}$ by $\rho_r(z) = z^r$ and let $\Phi'\colon L_F \setminus L' \to \mathbb{S}^1$ be the map given by $\Phi' = \frac{z}{|z|}$. Then*

*1) the following diagram commutes:*

$$\begin{tikzcd} L_F \setminus L' \arrow[r, "\mathscr{P}"] \arrow[d, "\Phi'"'] & \mathbb{S}^3 \setminus L_{f\bar{g}} \arrow[d, "\Phi_{f\bar{g}}"] \\ \mathbb{S}^1 \arrow[r, "-\rho_r"] & \mathbb{S}^1 \end{tikzcd} \tag{3.2}$$

*2) $\mathscr{P}$ is a cyclic branched $r$-covering with ramification locus $L_{f\bar{g}}$ and the restriction*

$$\mathscr{P} : L' \to L_{f\bar{g}}$$

*is a homeomorphism,*

*3) the projection $\Phi'$ is an open-book fibration with binding $L'$,*

*4) the fibres of $\Phi'$ and $\Phi_{f\bar{g}}$ are diffeomorphic and the monodromy $h'$ of $\Phi'$ is equal to $h^r$ up to conjugacy in the mapping class group of the fibre.*

*Proof.* Let us prove (1). Let $(x, y) \in \mathbb{S}^3 \setminus L_{f\bar{g}}$, then

$$\mathscr{P}^{-1}(x, y) = \{(x, y, z) \in L_F \mid f(x, y)\overline{g(x, y)} \neq 0, z^r = -f(x, y)\overline{g(x, y)}\}$$

Moreover, $\mathscr{P}^{-1}(x, y) \subset L_F \setminus L'$. Then $\mathscr{P}$ is surjective on $\mathbb{S}^3 \setminus L_{f\bar{g}}$.

Let $(x, y, z) \in L_F \setminus L'$, as $z^r = -f(x, y)\overline{g(x, y)}$, then

$$\begin{aligned}(-\rho_r \circ \Phi')(x, y, z) &= -\rho_r\left(\frac{z}{|z|}\right) = -\left(\frac{z}{|z|}\right)^r \\ &= \frac{f(x, y)\overline{g(x, y)}}{|f(x, y)\overline{g(x, y)}|} = \Phi_{f\bar{g}}(x, y) \\ &= (\Phi_{f\bar{g}} \circ \mathscr{P})(x, y, z)\ .\end{aligned}$$

Now, in order to prove (2), first we prove that the diagram (3.2) is a pull-back diagram.

Let $Q$ be the pull-back of $\mathbb{S}^3 \setminus L_{f\bar{g}}$ by $-\rho_r$ defined by

$$\begin{aligned}Q &= \{(x, y, \lambda) \in (\mathbb{S}^3 \setminus L_{f\bar{g}}) \times \mathbb{S}^1 \mid \Phi_{f\bar{g}}(x, y) = -\rho_r(\lambda)\} \\ &= \{(x, y, \lambda) \in (\mathbb{S}^3 \setminus L_{f\bar{g}}) \times \mathbb{S}^1 \mid \frac{f(x, y)\overline{g(x, y)}}{|f(x, y)\overline{g(x, y)}|} = -\lambda^r\}\end{aligned}$$

Then, by the universal property of the pull-back, we have the following diagram:

$$\begin{tikzcd}
L_F \setminus L' \arrow[dr, "p"] \arrow[drr, bend left, "\mathscr{P}"] \arrow[ddr, bend right, "\Phi'"'] & & \\
& Q \arrow[r, "\pi_{1,2}"] \arrow[d, "\pi_3"] & \mathbb{S}^3 \setminus L_{f\bar{g}} \arrow[d, "\Phi_{f\bar{g}}"] \\
& \mathbb{S}^1 \arrow[r, "-\rho_r"] & \mathbb{S}^1
\end{tikzcd}$$

where $\pi_3$ is the projection on the third coordinate, $\pi_{1,2}$ is the projection on the first two coordinates and $p\colon (L_F \setminus L') \to Q$ is defined by

$$p(x,y,z) = \left(x, y, \frac{z}{|z|}\right).$$

Let us define $q\colon Q \to (L_F \setminus L')$ by $q(x,y,\lambda) = (x,y,\lambda|f(x,y)\overline{g(x,y)}|^{1/r})$. Then $q$ is the inverse map of $p$ and $L_F \setminus L'$ is diffeomorphic to $Q$

Thus, as $\mathscr{P}$ is the pull-back of the cyclic covering $-\rho_r$, then $\mathscr{P}$ is itself a cyclic covering of $r$ leaves.

Now, let $(x,y) \in L_{f\bar{g}}$, then $\mathscr{P}^{-1}(x,y) = \{(x,y,0) \in L_F\}$, *i.e.*, $\mathscr{P}^{-1}(x,y)$ consists of only one point. Then $\mathscr{P}$ from $L_F$ to $\mathbb{S}^3$ is a branched cyclic $r$-covering with ramification locus $L_{f\bar{g}}$.

Statement (3) can be proved in the following way: Let $K$ be a connected component of $L_{f\bar{g}}$, since $\Phi_{f\bar{g}}$ is an open-book fibration of $\mathbb{S}^3$, thre exists a small closed tubular neighbourhood $\mathscr{U}$ of $K$ and a trivialization $\delta\colon \mathscr{U} \to (\mathbb{S}^1 \times \mathbb{D}^2)$ such that $\delta(K) = \mathbb{S}^1 \times \{0\}$ and the following diagram commutes:

$$\begin{tikzcd}
\mathscr{U} \setminus K \arrow[rr, "\delta"] \arrow[dr, "\Phi_{f\bar{g}}"'] & & \mathbb{S}^1 \times \left(\mathbb{D}^2 \setminus \{0\}\right) \arrow[dl, "g"] \\
& \mathbb{S}^1 &
\end{tikzcd} \tag{3.3}$$

where $g(\lambda, w) = \frac{w}{|w|}$ with $\lambda \in \mathbb{S}^1$ and $w \in \mathbb{D}^2$.

Let $K' = \mathscr{P}^{-1}(K)$ and $\mathscr{V} = \mathscr{P}^{-1}(\mathscr{U})$. Let $\rho'_r\colon (\mathbb{S}^1 \times \mathbb{D}^2) \to (\mathbb{S}^1 \times \mathbb{D}^2)$ by

$$\rho'_r(\lambda, w) = (-\lambda, w^r),$$

with $\lambda \in \mathbb{S}^1$ and $w \in \mathbb{D}^2$.

The composition $\delta \circ \mathscr{P}$ gives a cyclic branched $r$-covering of $\mathbb{S}^1 \times \mathbb{D}^2$ and in the other hand $-\rho'_r$ gives also a cyclic branched $r$-covering of $\mathbb{S}^1 \times \mathbb{D}^2$, both with the same ramification locus; then there exists an unique diffeomorphism $\delta' \colon \mathscr{V} \to \mathbb{S}^1 \times \mathbb{D}^2$ such that $\delta \circ \mathscr{P} = -\rho'_r \circ \delta'$.

Then the following diagram commutes:

$$\begin{tikzcd} V \setminus K \arrow[rr, "\delta'"] \arrow[dr, "\Phi'"'] & & \mathbb{S}^1 \times (\mathbb{D}^2 \setminus \{0\}) \arrow[dl, "g"] \\ & \mathbb{S}^1 & \end{tikzcd} \tag{3.4}$$

and $\Phi'$ is an open-book fibration of $L_F$.

Now let us prove (4). Let $\mathscr{F}_{f\bar{g}}$ be one Milnor fibre of $\Phi_{f\bar{g}}$ and let $\mathscr{F}' \subset \mathscr{P}^{-1}(\mathscr{F}_{f\bar{g}})$ one fibre of $\Phi'$. Since the diagram (3.2) is a pull-back diagram, the restriction

$$\mathscr{P}|_{\mathscr{F}'} \colon \mathscr{F}' \to \mathscr{F}_{f\bar{g}}$$

is a diffeomorphism. Moreover, the preimage $\mathscr{P}^{-1}(\mathscr{F}_{f\bar{g}})$ is the disjoint union of $r$ fibres of $\Phi'$.

Let

$$\gamma \colon (\mathbb{S}^3 \setminus L_{f\bar{g}}) \times \mathbb{R} \to (\mathbb{S}^3 \setminus L_{f\bar{g}})$$

be the flow of a vector field which is a lifting by $\Phi_{f\bar{g}}$ of the canonical tangent vector field on $\mathbb{S}^1$. Then $\gamma$ is transverse to the fibres of $\Phi_{f\bar{g}}$ and a representative $h$ of the monodromy of $\Phi_{f\bar{g}}$ is defined as the diffeomorphism of first return of $\gamma$ over the fibre $\mathscr{F}_{f\bar{g}}$. Let

$$\gamma' \colon (L_F \setminus L') \times \mathbb{R} \to (L_F \setminus L')$$

be a flow such that for all $\big((x,y,z),t\big) \in (L_F \setminus L') \times \mathbb{R}$,

$$\mathscr{P}(\gamma'\big((x,y,z),t\big)) = \gamma\big(\mathscr{P}(x,y,z),t\big)\ ,$$

and let $h'$ be the representative of the monodromy of $\Phi'$ defined as the diffeomorphism of first return of the flow $\gamma'$ on $\mathscr{F}'$. By construction and since the covering is cyclic, one has the following commutative diagram

$$\begin{tikzcd} \mathscr{F}' \arrow[r, "\mathscr{P}"] \arrow[d, "h'"'] & \mathscr{F}_{f\bar{g}} \arrow[d, "h^r"] \\ \mathscr{F}' \arrow[r, "\mathscr{P}"] & \mathscr{F}_{f\bar{g}} \end{tikzcd}$$

□

## 3.2 Nielsen graph of a periodic diffeomorphism

This section is based in the work of J. Nielsen [41] and is also presented in [50, page 347]. It gives a way to represent a periodic diffeomorphism of a surface by a graph. As an example it is showed the relation with the resolution graph of a complex curve with an only rupture vertex.

Let $\mathscr{F}$ be an oriented, compact, connected surface and let $h\colon \mathscr{F} \to \mathscr{F}$ be an orientation preserving periodic diffeomorphism of order $m \geq 2$. Then $h$ generates an action of the group $\mathbb{Z}_m$ on $\mathscr{F}$.

Let $\mathscr{O}$ be the orbit space of this action, *i.e.*, $\mathscr{O}$ is the quotient of $\mathscr{F}$ under the following equivalence relation: given $x, y \in \mathscr{F}, x \sim y$ if and only if exists $k \in \mathbb{Z}$ such that $h^k(x) = y$. Thus $\mathscr{O}$ is an orbifold of dimension 2 homeomorphic to an orientable, compact, connected surface.

Let $\varpi\colon \mathscr{F} \to \mathscr{O}$ be the projection onto the orbit space of $h$. There exists a finite number of points whose orbits under $h$ are of cardinality $n < m$. Let $p \in F$ be one of them, the orbit $\varpi(p)$ is called an **exceptional orbit** of $h$. Then $\varpi$ is a cyclic branched $m$-covering whose ramification locus is the set of these exceptional orbits in $\mathscr{O}$.

Let $\lambda \in \mathbb{Z}$ be defined as

$$\lambda = \frac{m}{n} \geq 2\,.$$

We first treat the case when $\mathscr{F}$ has empty boundary. Let $p \in \mathscr{O}$ be a point representing an exceptional orbit $O \in \mathscr{F}$ of cardinality $n$ and let $\mathbb{D}$ be a small 2-disc with centre $p$, *i.e.*, such that each $x \in \mathbb{D} \setminus \{p\}$ represents an orbit of cardinality $m$. Then $\varpi^{-1}(\mathbb{D})$ consist of $n$ disjoint discs $D_1, \ldots, D_n$, which are ciclically exchanged by $h$. Let $D_i$ be one of them, the disc $D_i$ being oriented as $\mathscr{F}$, let us endow its boundary $\partial D_i$ by the induced orientation. Then $h|^n_{D_i}: D_i \to D_i$ is conjugate to a rotation of angle $\omega/\lambda$ with $0 < \omega < \lambda$ and $\omega$ prime relative to $\lambda$. The orientation convention for $D_i$ and its boundary is essential to obtain a well-defined angle. Let $\sigma$ be the integer such that $0 < \sigma < \lambda$ and $\omega\sigma \equiv 1 \pmod{\lambda}$.

**3.5 Definition.** The pair $(\lambda, \sigma)$ is the **valency** of $h$ at $p$ (or the valency of $h$ for the orbit $\pi(p)$).

If $\mathscr{F}$ has a non-empty boundary, $\partial\mathscr{O} \neq \emptyset$. Let $\widehat{\mathscr{O}}$ be the closed oriented surface obtained by attaching a 2-disc $D'_i$ on each boundary component of $\mathscr{O}$ and let $\widehat{\mathscr{F}}$ be the surface obtained by attaching a 2-disc on each boundary component of

$\mathscr{F}$. Let $\widehat{h}$ be the conical extension of $h$ to $\widehat{\mathscr{F}}$. It may be that $\widehat{h}$ is not differentiable at the centre of the new discs but this is unimportant.

Then $\widehat{\mathscr{O}}$ is the orbit space of $\widehat{h}$, given by the action of the group $\mathbb{Z}_m$ on the surface $\widehat{\mathscr{F}}$.

Let $p'$ be the centre of one of the discs $D_i'$. We define the valency for the orbit of a boundary component of $\mathscr{O}$ as the valency of $h$ at $p'$ in the same way as for the exceptional orbits.

It is important to notice that the boundary components of $\mathscr{F}$ are oriented as the boundary of the attached discs and not as the boundary of $\mathscr{F}$.

From these valencies, one can construct a graph representing the diffeomorphism $h$:

**3.6 Definition.** Let $\mathscr{G}(h)$ be the graph constructed in the following way:

- The graph $\mathscr{G}(h)$ has one unique vertex representing the surface $\mathscr{O}$. This vertex is weighted by the pair $[m, g]$ where $m$ is the order of $h$ and $g$ is the genus of $\mathscr{O}$,

- we attach to the vertex of $\mathscr{G}(h)$ one **stalk** ($\longrightarrow\!\bullet$) for each exceptional orbit and we weight it by the valency for the corresponding exceptional orbit,

- we attach to the vertex of $\mathscr{G}(h)$ one **boundary-stalk** ($\longrightarrow\!\circ$) for each boundary component of $\mathscr{O}$ and we weight it by the valency for the corresponding boundary component.

We call $\mathscr{G}(h)$ the **Nielsen graph** of the periodic diffeomorphism $h$.

Figure 3.1 shows the Nielsen graph $\mathscr{G}(h)$ of a diffeomorphism $h\colon \mathscr{F} \to \mathscr{F}$ of order $m$ with $s$ exceptional orbits, where $\mathscr{O}$ has genus $g$ and $s' - s$ boundary components.

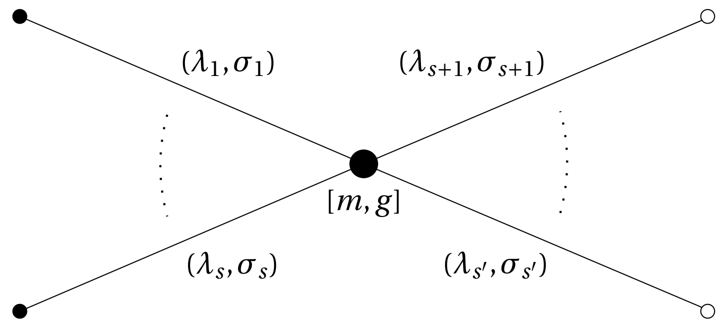



Figure 3.1: A Nielsen graph with $s$ stalks and $s'-s$ boundary stalks.

**3.7 Example.** Let $f\colon (\mathbb{C}^2,0)\to(\mathbb{C},0)$ be a reduced germ of analytic function and let $V=f^{-1}(0)$ . Let $\widetilde{V}$ be a resolution of $V$ with resolution map $\pi\colon \widetilde{V}\to V$. Let us assume that the corresponding resolution graph $A_\pi(f)$ has an unique rupture vertex and let $h$ be the monodromy of the Milnor fibration of $f$.

We consider the graph $A_\pi(f)$ weighted by the multiplicities at the components $E_i$ of the exceptional divisor $E$. By Section 1.6 (see also [15, 1.2 to 1.11]), can be found a periodic representative of the monodromy $h$ and the Nielsen graph $\mathscr{G}(h)$ can be obtained in terms of the multiplicities of $A_\pi(f)$:

- $\mathscr{G}(h)$ has a vertex corresponding to the rupture vertex of $A_\pi(f)$,
- $\mathscr{G}(h)$ has a stalk corresponding to each bamboo of $A_\pi(f)$ ended by a vertex of valence 1 (see Definition 1.40),
- $\mathscr{G}(h)$ has a boundary-stalk corresponding to each bamboo of $A_\pi(f)$ ended by an arrow (see Definition 1.60).

Furthermore, by Proposition 1.53, the numerical information of $\mathscr{G}(h)$ is computed as follows: Let $m$ be the multiplicity of the rupture vertex in $A_\pi(f)$. Let $i$ be a neighbour vertex with multiplicity $m_i$ and set $\hat{m}_i=\gcd(m,m_i)$. The vertex of $\mathscr{G}(h)$ is weighted by the pair $[m,0]$ and the $i$-th stalk ($i$-th boundary-stalk) by

$$\left(\frac{m}{\hat{m}_i},-\frac{m_i}{\hat{m}_i}\right).$$

Let us give an explicit example: Given the function $f\colon \mathbb{C}^2\to\mathbb{C}$ defined by $f(x,y)=x^2+y^3$, we obtain the graph in Figure 3.2 as resolution graph $A_\pi(f)$.

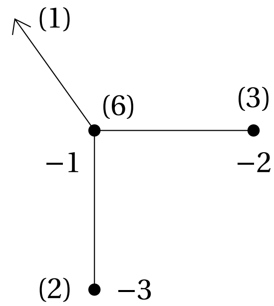



Figure 3.2: Resolution graph of $f(x, y) = x^2 + y^3$.

Then the Nielsen graph $\mathscr{G}(h)$ is the graph in Figure 3.3.

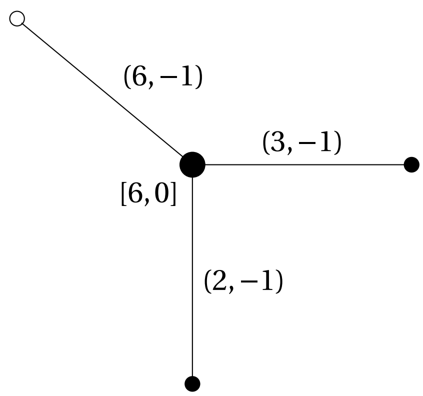



Figure 3.3: Nielsen graph of the monodromy $h$ of the Milnor fibration of $f(x, y) = x^2 + y^3$.

## 3.3 Nielsen graph of a quasi-periodic diffeomorphism

In this section the notion of Nielsen graph is extended for a quasi-periodic diffeomorphism (see [50, page 348]).

Let $\mathscr{F}$ be a compact, connected and oriented surface with Euler characteristic strictly negative.

Let us recall that a quasi-periodic diffeomorphism $h\colon \mathscr{F} \to \mathscr{F}$ (Definition 1.49) is an orientation preserving diffeomorphism with a family $\mathscr{C}$ of disjoint simple closed curves in $\mathscr{F}$ such that for each connected component $c \in \mathscr{C}$ one can choose a small annulus $\mathscr{U}(c) \subset \mathscr{F}$, tubular neighbourhood of $c$ with the following properties:

- for any pair of curves $c_i, c_j \in \mathscr{C}$, we have that $\mathscr{U}(c_i) \cap \mathscr{U}(c_j) = \emptyset$,
- $\bigcup_{c \in \mathscr{C}} = \mathscr{U}(\mathscr{C}) = h(\mathscr{U}(\mathscr{C}))$,
- the restriction of $h$ to the complement of

$$\mathscr{U}(\mathscr{C}) = \bigcup_{c \in \mathscr{C}} \mathring{\mathscr{U}}(c)$$

is periodic, where $\mathring{\mathscr{U}}(c)$ is the interior of $\mathscr{U}(c)$.

Then the family $\mathscr{C}$ is called a reduction system of curves for the diffeomorphism $h$.

Let $h\colon \mathscr{F} \to \mathscr{F}$ be a quasi-periodic diffeomorphism and let $\mathscr{C}$ be a reduction system for $h$. Let $c \in \mathscr{C}$ be a simple closed curve in $\mathscr{F}$ an let $\mathscr{U}(c) \subset \mathscr{F}$ be as before, then there exists an orientation preserving diffeomorphism $\mu\colon [-1,1] \times \mathbb{S}^1 \to \mathscr{U}(c)$ such that $\mu(\{0\} \times \mathbb{S}^1) = c$.

Let $N$ be the smallest integer such that

$$h^N|_{\mathscr{F} \setminus \mathscr{U}(\mathscr{C})} = id_{\mathscr{F} \setminus \mathscr{U}(\mathscr{C})} ,$$

then the restriction $h|_{\mathscr{U}(c)}$ is a Dehn twist characterised by a rational number $t$ in the following way:

Consider the path $\gamma$ in $\mathscr{U}(c)$ defined by $\gamma(s) = \mu(s, e^{i\theta})$ where $\theta$ is fixed and $s \in [-1,1]$. We orient $\gamma$ by $[-1,1]$ and then, we orient $c$ in such a way that $\gamma \cdot c = +1$ in $\mathrm{H}_1(\mathscr{U}(c), \mathbb{Z})$. Then there exists $K \in \mathbb{Z}$ such that the cycles $Kc$ and $h^N(\gamma) - \gamma$ are homologous in $\mathscr{U}(c)$ (see Figure 3.4).

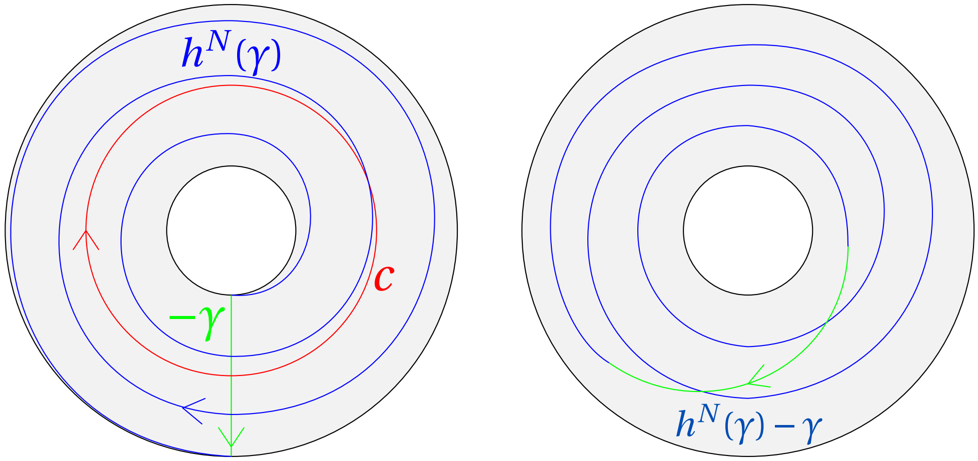



Figure 3.4: The cycles $Kc$ and $h^N(\gamma)-\gamma$ are homologous.

**3.8 Definition.** The rational number $t = \frac{K}{N}$ is called the **twist number** of $h$ along $c$.

**3.9 Definition.** A reduction system $\mathscr{C}$ for the diffeomorphism $h$ is called **Wu-minimal** if given any curve $c \in \mathscr{C}$, we have

- $c$ is not null-homotopic in $\mathscr{F}$,
- $c$ is not homotopic to a boundary component of $\mathscr{F}$,
- $c$ is not homotopic to any other curve $c'$ of $\mathscr{C}$, and
- the twist of $h$ along $c$ is non zero.

**3.10 Theorem** (Wu, [69])**.** *Let $\mathscr{C}$, $\mathscr{C}'$ be two Wu-minimal reduction systems for a quasi-periodic diffeomorphism $h$, then there exists a diffeomorphism $\tau\colon \mathscr{F} \to \mathscr{F}$ isotopic to the identity, such that $\tau(\mathscr{C}) = \mathscr{C}'$.*

**3.11 Definition.** Let $h\colon \mathscr{F} \to \mathscr{F}$ be a quasi-periodic diffeomorphism. A curve $c$ of a reduction system $\mathscr{C}$ for $h$ is called **amphidrome** if there exists $k \in \mathbb{Z}$ such that

$$h^k\left(\overrightarrow{c}\right) = -\overrightarrow{c}\ ,$$

where $\overrightarrow{c}$ is the curve $c$ endowed with an orientation.

Note that for an amphidrome curve $c$, $h^k$ interchanges the boundary components of $\mathscr{U}(c)$.

Let $h\colon \mathscr{F} \to \mathscr{F}$ be a quasi-periodic diffeomorphism and let $\mathscr{C}$ be a Wu-minimal reduction system for $h$ such that $h$ has order $m$ in $\mathscr{F} \setminus \mathscr{U}(\mathscr{C})$. We are interested to have a reduction system without amphidromes curves, for technical reasons. Then, if there is an amphidrome curve $c \in \mathscr{C}$, there exists $k \in \mathbb{Z}$ such that $h^k$ reverses the orientation of $c$.

Let us prove that $k$ is odd: Suppose $k = 2q$, then if $h^q$ interchanges the boundary components of $\mathscr{U}(c)$, $h^k$ leaves them fixed. If $h^q$ fixes the boundary components of $\mathscr{U}(c)$, $h^k$ fixes them as well. Then, as $h^k$ interchanges the boundary components of $\mathscr{U}(c)$, $k$ is odd.

Let $\mu\colon [-1,1] \times \mathbb{S}^1 \to \mathscr{U}(c)$ be a preserving orientation diffeomorphism such that $\mu(\{0\} \times \mathbb{S}^1) = c$. Let us consider the following diagram:

$$\begin{tikzcd}
\mathscr{U}(c) \arrow[r, "h"] & h(\mathscr{U}(c)) \\
{[-1,1] \times \mathbb{S}^1} \arrow[u, "\mu"] \arrow[r, "\mu'^{-1}\circ h\circ\mu"'] & {[-1,1] \times \mathbb{S}^1} \arrow[u, "\mu'"']
\end{tikzcd}$$

where $\mu'$ is an orientation preserving diffeomorphism defined in such a way that

$$\mu'^{-1} \circ h \circ \mu(\{\pm 1\} \times \mathbb{S}^1) = \{\mp 1\} \times \mathbb{S}^1 .$$

Then we obtain a diffeomorphism $\mu'^{-1} \circ h \circ \mu$ which interchanges the boundary components of $[-1,1] \times \mathbb{S}^1$ and preserves orientation. Hence it is isotopic to the diffeomorphism $\eta$ defined by

$$\begin{aligned}
\eta\colon [-1,1] \times \mathbb{S}^1 &\longrightarrow [-1,1] \times \mathbb{S}^1 \\
(s,\lambda) &\longmapsto (-s,\overline{\lambda})
\end{aligned}$$

We modify $h$ with this isotopy in order to obtain

$$\begin{tikzcd}
\mathscr{U}(c) \arrow[r, "h"] & h(\mathscr{U}(c)) \\
{[-1,1] \times \mathbb{S}^1} \arrow[u, "\mu"] \arrow[r, "\eta"'] & {[-1,1] \times \mathbb{S}^1} \arrow[u, "\mu'"']
\end{tikzcd}$$

Let us notice that $h|^k_{\mathscr{U}(c)} = \mu' \circ \eta^k \circ \mu^{-1} = \mu' \circ \eta \circ \mu^{-1}$ since $k$ is odd and $\eta$ has order 2.

This operation allow us to replace the curve $c \in \mathscr{C}$ by the two boundary components of $\mathscr{U}(c)$, eliminating in this way the amphidrome curves in $\mathscr{C}$.

**3.12 Definition.** Let $\mathscr{C}'$ be a Wu-minimal reduction system for $h$, let $\mathscr{C}$ be a reduction system for $h$ without amphidrome curves obtained from $\mathscr{C}'$ by replacing each amphidrome curve by two parallel curves as described above. We call $\mathscr{C}$ the **minimal** reduction system of $h$.

Let $h\colon \mathscr{F} \to \mathscr{F}$ be a quasi-periodic diffeomorphism and let $\mathscr{C}$ be a minimal reduction system for $h$. We construct a Nielsen graph for $h$, which extends the definition of Nielsen graph of a periodic diffeomorphism in Section 3.2, in the following way:

Let $\mathscr{G}_h$ be the following graph:

- $\mathscr{G}_h$ has one vertex for each connected component of $\mathscr{F} \setminus \mathscr{C}$,
- let $\mathscr{F}_i$ and $\mathscr{F}_j$ be connected components of $\mathscr{F} \setminus \mathscr{C}$ such that there is a curve $c \in \mathscr{C}$ with $c \subset \overline{\mathscr{F}_i} \cap \overline{\mathscr{F}_j}$, where $\overline{\mathscr{F}_i}$ and $\overline{\mathscr{F}_j}$ are the closures of $\mathscr{F}_i$ and $\mathscr{F}_j$ respectively. Then $\mathscr{G}_h$ has an edge between the corresponding vertices $u_i$ and $u_j$.

Now, let $\overline{\mathscr{G}_h}$ be the quotient graph of the induced action of $h$ on the graph $\mathscr{G}_h$. Let $i$ be a vertex of $\overline{\mathscr{G}_h}$, then $i$ represents $r_i$ connected components of $\mathscr{F} \setminus \mathscr{U}(\mathscr{C})$ and the diffeomorphism $h$ permutes cyclically these $r_i$ components. Let $\mathscr{F}_i$ be one of them.

Let $h_i$ be the diffeomorphism defined by $h|^{r_i}_{\mathscr{F}_i}\colon \mathscr{F}_i \to \mathscr{F}_i$, then $h_i$ is a periodic diffeomorphism with order $m_i$. Let $g_i$ be the genus of the orbit space $\mathscr{O}_i$ of $\mathscr{F}_i$ by $h_i$.

For each vertex $i$ of $\overline{\mathscr{G}_h}$ we construct the Nielsen graph $\mathscr{G}(h_i)$ and we complete the numerical information by weighting each vertex with the number $r_i$ (see Figure 3.5).

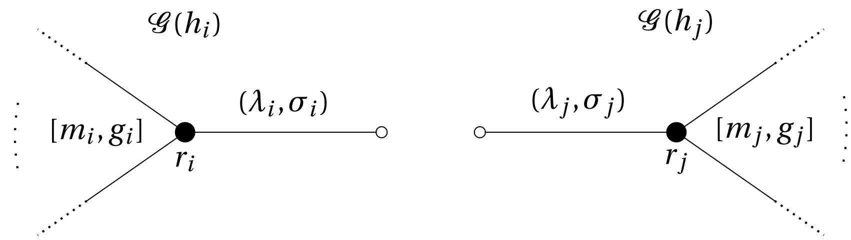



Figure 3.5: Nielsen graphs $\mathscr{G}(h_i)$ and $\mathscr{G}(h_j)$.

We take the disjoint union of the Nielsen graphs $\mathscr{G}(h_i)$. Let $A$ be an edge of the graph $\overline{\mathscr{G}_h}$ connecting the vertices $i$ and $j$, *i.e.*, it represents a curve $c \in \mathscr{C}$ such that $c \subset \overline{\mathscr{F}_i} \cap \overline{\mathscr{F}_j}$. Let $t$ be the twist of $h$ along $c$. The two boundary components of $\mathscr{U}(c)$ are represented by two boundary-stalks in the Nielsen graphs $\mathscr{G}(h_i)$ and $\mathscr{G}(h_j)$ respectively.

We replace these two boundary-stalks by a single edge joining the vertices $i$ and $j$. We weight this edge in the middle with the twist $t$ and the valencies of the eliminated boundary-stalks at its extremes (see Figure 3.6).

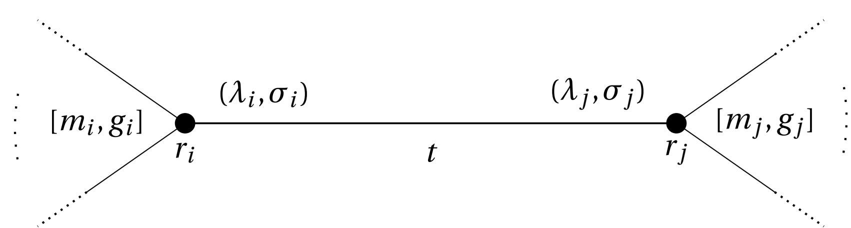



Figure 3.6: Joining the Nielsen graphs $\mathscr{G}(h_i)$ and $\mathscr{G}(h_j)$.

We repeat this process for all the edges in the graph $\overline{\mathscr{G}_h}$ and we obtain a new graph $\mathscr{G}(h)$.

**3.13 Definition.** The graph $\mathscr{G}(h)$ is the **Nielsen graph** of the quasi-periodic diffeomorphism $h$.

**3.14 Example.** Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a reduced germ of analytic function and let $V = f^{-1}(0)$. Let $\widetilde{V}$ be a resolution of $V$ with resolution map $\pi\colon \widetilde{V} \to V$. Let $A_\pi(f)$ be the corresponding resolution graph weighted by the multiplicities $m_i$ of $f \circ \pi$ along the components $E_i$ of the exceptional divisor $E$. Let $h$ be the monodromy of the Milnor fibration of $f$.

As before, according to Section 1.6, it is possible to see $h$ as a quasi-periodic diffeomorphism and the Nielsen graph $\mathscr{G}(h)$ can be obtained in terms of the multiplicities $m_i$.

From $A_\pi(f)$ one can construct the Nielsen graph $\mathscr{G}(h_i)$ corresponding to the periodic diffeomorphism $h_i$ at each rupture vertex $i$ as in the previous section with the following additional condition (see also Section 1.6):

Let $N(i)$ the set of the neighbour vertices of the vertex $i$ and let $k \in N(i)$ such that $k$ is in a bamboo joining $i$ and other rupture vertex (see Definition 1.41), we attach a boundary-stalk to the vertex of the Nielsen graph $\mathscr{G}(h_i)$ weighted by

$$\left(\frac{m_i}{\widehat{m}_{ik}}, -\frac{m_k}{\widehat{m}_{ik}}\right),$$

where $m_i$ and $m_k$ are the multiplicities of $i$ and $k$ respectively and

$$\widehat{m}_{ik} = \gcd(m_i, m_k) .$$

For each bamboo of $A(f)$ joining two rupture vertices $i$ and $j$ we replace the boundary-stalks in the Nielsen graphs $\mathscr{G}(h_i)$ and $\mathscr{G}(h_j)$ by a single edge joining the vertices $i$ and $j$. We weight this edge with the valencies of the eliminated boundary-stalks at its extremes and in the middle with the twist $t$, which is computed in the following way:

Given a bamboo joining two rupture vertices, as in Figure 3.7, let $l$ and $l+1$ be two adjacent vertices on it (including the rupture vertices). Let $m_l$ and $m_{l+1}$ be the multiplicities of $f \circ \pi$ along the irreducible component $E_l$ and $E_{l+1}$ respectively. By Section 1.6, $h^{m_l} = id$ in the component $\mathscr{F}_l$ and $h^{m_{l+1}} = id$ in the component $\mathscr{F}_{l+1}$. Let$c$ be a simple closed curve in the intersection $\mathscr{F}_l \cap \mathscr{F}_{l+1}$ and let $\mathscr{U}(c)$ be the small neighbourhood given in Section 1.6. The smallest integer $k$ such that $h|^k_{\mathscr{U}(c)} = id$ is $k = \operatorname{lcm}(m_l, m_{l+1})$.

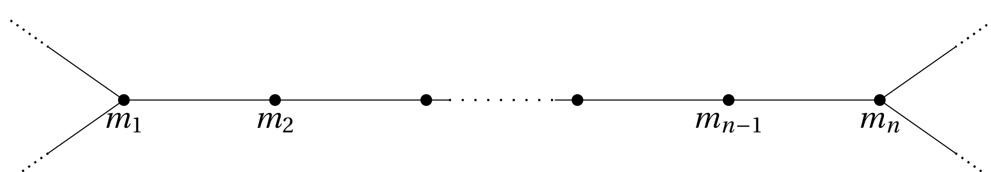



Figure 3.7: Bamboo joining two rupture vertices with twist $t$ given by the multiplicities $m_l$.

According to [18, Paragraph 13], the twist number of $h$ along $c$ can be obtained as

$$t_l = -\frac{1}{k} = -\frac{\gcd(m_l, m_{l+1})}{m_l \cdot m_{l+1}}$$

and the twist number $t$ along the edge joining the vertices $i$ and $j$ in the graph $\mathcal{G}(h)$ is obtained as the sum

$$t = \sum_{l=1}^{n-1} t_l \ .$$

## 3.4 Identity on the boundary

In this section it is given an extra numerical information in order to "complete" the Nilsen graph of a quasi-periodic difeomorphism of a surface with the condition of being the identity on the boundary of the surface.

Let $M$ be a 3-manifold with an open-book fibration $\pi\colon M \setminus L \to \mathbb{S}^1$ with binding $L$. We want to describe the monodromy of this open-book fibration with the invariants we have given for periodic and quasi-periodic diffeomorphisms. In the sequel, we only consider diffeomorphisms whose restriction to the boundary $\partial\mathcal{F}$ of $\mathcal{F}$ is the identity map. Given a diffeomorphism $h$, we consider it up to isotopy in this class of diffeomorphisms.

Let $h\colon \mathcal{F} \to \mathcal{F}$ be such a quasi-periodic diffeomorphism and let $\mathcal{C}$ be a minimal reduction system of $h$. For each boundary component $\zeta$ of $\partial\mathcal{F}$, we consider a small tubular neighbourhood $\mathcal{U}(\zeta)$ of $\zeta$ in $\mathcal{F}$ bounded on one side by $\zeta$ and on the other side by a parallel curve to $\zeta$ in the interior of $\mathcal{F}$.

Then we have a collection of curves in bijection with the boundary components of $\mathcal{F}$ in addition to $\mathcal{C}$. The restriction of $h$ to $\mathcal{U}(\zeta)$ is a Dehn twist (possibly with $t = 0$) such that

$$h|_\zeta = id \ .$$

In the sequel we will complete the Nielsen graph $\mathcal{G}(h)$ of $h$ by weighting the extremity of each boundary-stalk with the corresponding twist $t$.

**3.15 Example.** Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the function defined by $f(x, y) = x^2 + y^3$ and let $h$ the monodromy of the Milnor fibration of $f$.

A boundary stalk in the graph $\mathcal{G}(h)$ is associated to a bamboo ended by an arrow in the resolution graph $A_\pi(f)$ for a resolution $\pi$ of $f$ and the corresponding twist can be computed as sum of partial twists, as was done for a bamboo joining two rupture vertices (see Example 3.14).

The Nielsen graph of $h$ is given in Figure 3.3 and the twist in the boundary-stalk is given by

$$t = \frac{-1}{6} .$$

Then we obtain the Nielsen graph $\mathcal{G}(h)$ (see Figure 3.8).

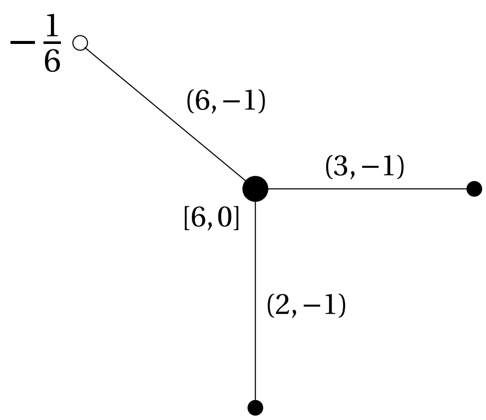



Figure 3.8: Completed Nielsen graph of the monodromy $h$ of the Milnor fibration of $f(x, y) = x^2 + y^3$.

**3.16 Example.** Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the function defined by

$$f(x, y) = x(x^2 + y^3) .$$

Let $V = f^{-1}(0)$ and let $\widetilde{V}$ be a good resolution of $V$ with resolution map $\pi\colon \widetilde{V} \to V$ and resolution graph $A_\pi(f)$ (see Figure 3.9).

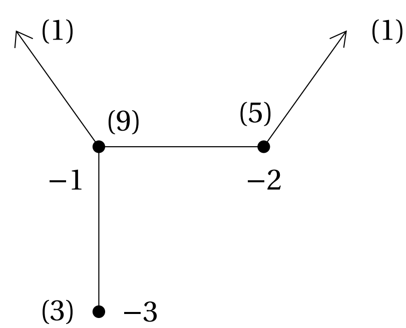



Figure 3.9: Resolution graph $A_\pi(f)$ for $f(x, y) = x(x^2 + y^3)$.

In this case, the twist on the right boundary-stalk can be obtained as it was indicated above, $t$ is the sum of two partial twists:

$$t = -\frac{\gcd(5,9)}{5\cdot 9} - \frac{\gcd(1,5)}{1\cdot 5} = \frac{2}{9}\,.$$

and the Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f$ is given in Figure 3.10.

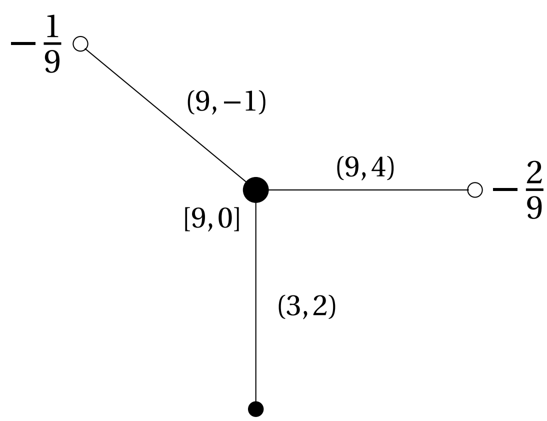



Figure 3.10: Completed Nielsen graph of the monodromy $h$ of the Milnor fibration of $f(x,y) = x(x^2+y^3)$.

## 3.5 Plumbing manifolds and fibred plumbing links

This section presents some results from [10], [14], [50] and [53] on fibred plumbing links and their monodromy, which generalise those presented in Section 1.6. For this, we present the concept of graph manifolds, the graph links and their representation by a graph (see for example [49, § 4]).

**3.17 Definition.** Let $M$ be a 3-manifold. A **Waldhausen decomposition** of $M$ is a decomposition of $M$ as an union of a finite number of 3-manifolds $M_i$, $M = \bigcup M_i$ such that

1. each $M_i$ is a Seifert manifold,
2. if $i \neq j$, the intersection $M_i \cap M_j$ is either empty or it is the union of the common boundary components, *i.e.*, a union of tori.

**3.18 Definition.** Let $M$ be a 3-manifold. We call $M$ a **plumbing manifold** if it admits a Waldhausen decomposition.

**3.19 Definition.** A **plumbing link** is a pair $(M, L)$ where $M$ is a plumbing manifold, boundary of a 4-manifold $P(\Gamma)$ obtained by plumbing according to a plumbing graph $\Gamma$, and $L = K_1 \cup \ldots \cup K_n$ is an oriented link in $M$ which is an union (possibly empty) of $\mathbb{S}^1$-fibres of the plumbed $\mathbb{D}^2$-bundles. Notice that each $K_i$ has a natural orientation as the boundary of a $\mathbb{D}^2$-fibre. We denote by $-K_i$ the knot $K_i$ endowed with the opposite orientation. Then the oriented link $L$ will be denoted by $L = \epsilon_1 K_1 \cup \ldots \cup \epsilon_n K_n$ where $\epsilon_i \in \{-1, +1\}$.

The homeomorphism class of the pair $(M, L)$ is given by the plumbing graph $\Gamma$ decorated with arrows in the following way : for each component $K_i$ of the link $L$, we attach an arrow weighted by the multiplicity $(\epsilon_i)$ to the vertex corresponding to the $\mathbb{D}^2$-bundle of which $K_i$ is a $\mathbb{S}^1$-fibre.

**3.20 Definition.** A **fibred link** $(M, L)$ is a plumbing link $(M, L)$ such that there exists an open-book fibration $\pi : M \setminus L$ with binding $L$.

The following theorem is a generalisation of a result of Eisenbud and Neumann ( [18, Th 11.2, see also p. 136]), reformulated in terms of plumbing links. In [18], this result is proved for multilinks in $\mathbb{Z}$-homology spheres and formulated in terms of splicing diagrams. In terms of graph decompositions it is proved by Chaves ( [10, Th 2.2.10]). For a short survey and proof, see [53, Th. 2.11].

**3.21 Theorem.** *Let $L = \epsilon_1 K_1 \cup \ldots \cup \epsilon_n K_n$ be a plumbing link with plumbing graph $\Gamma$ and intersection matrix $M_\Gamma$. Let $v_1, \ldots, v_s$ be the vertices of $\Gamma$. Let*

$$b(L) = (b_1, \ldots, b_s) \in \mathbb{Z}^s \ ,$$

*where $b_i$ is the sum of the multiplicities $\epsilon_j$ carried by the arrows attached to the vertex $v_i$. Then $L$ is fibred if and only if there exist $(m_1, \ldots, m_s) \in \mathbb{Z}^s$ such that the following two conditions hold:*

*1) $M_\Gamma {}^t(m_1, \ldots, m_s) + {}^t b(L) = 0$, where ${}^t(\cdot)$ means transposition,*

*2) for each rupture vertex $v_j$ of $\Gamma$, the integer $m_j$ is $\neq 0$.*

**3.22 Definition.** The system of equations (1) is called the **monodromical system** of $L$ (see [50, Def 4.2]).

For each vertex $v_i$ of the plumbing graph $\Gamma$, let $V_i$ be the intersection of $M$ and the $\mathbb{D}^2$-fibre bundle corresponding to $v_i$. Then $V_i$ is a $\mathbb{S}^1$-bundle.

If both conditions (1) and (2) hold, then, generalising the arguments of the proof of [18, Th. 4.2], one obtains that any fibration $\phi\colon M\setminus L \to \mathbb{S}^1$ can be modified by an isotopy in such a way that each fibre of $\phi$ is transverse to all the plumbing tori of $M$ and to all the $\mathbb{S}^1$-fibres of any $V_i$ such that $m_i \neq 0$.

**3.23 Remark.** Let $\mathscr{F}$ be a fibre of $\phi$, then the monodromy of the fibration $\phi$ admits a quasi-periodic representant $h\colon \mathscr{F} \to \mathscr{F}$ whose restriction to $\mathscr{F}_i = \mathscr{F} \cap V_i$ coincides with the first return map on $F_i$ of the fibres of $V_i$ endowed with the orientation $\epsilon_i K$, where $K$ is oriented as the boundary of a $\mathbb{D}^2$-fibre of a plumbed bundle, and where $\epsilon_i = \frac{m_i}{|m_i|}$.

In particular, one has:

**3.24 Proposition.** *The order of $h$ on $\mathscr{F}_i$ equals $|m_i|$.*

**3.25 Example.** Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be a holomorphic germ. By Milnor fibration theorem, the link $L_f$ is fibred. Let $\mathscr{U}$ be a neighbourhood of the origin in $\mathbb{C}^2$ and let $\pi\colon \mathscr{W} \to \mathscr{U}$ be a resolution of $f$ at the origin, such that the total transform $(f\circ\pi)^{-1}(0)$ has normal crossings, and let $\Gamma$ be its dual graph.

Let $v_1,\dots,v_s$ be the vertices of $\Gamma$. For each $i = 1,\dots,s$, let $m_i^f$ be the multiplicity of $f\circ\pi$ along the corresponding irreducible component $E_i$ of the exceptional divisor. Let $M_\Gamma$ be the intersection matrix associated with the plumbing graph $\Gamma$. The resolution theory asserts that

$$M_\Gamma\, {}^t(m_1^f,\cdots,m_s^f) + {}^t b(L_f) = 0\,.$$

Therefore, the solution of the monodromical system of $L_f$ is $(m_1^f,\cdots,m_s^f)$, and the restriction of the monodromy of the Milnor fibration has order $m_i^f$ on the intersection $\mathscr{F}_i = V_i \cap \mathscr{F}$. Then is recovered the description given in Section 1.6.

The following is a particular case of [53, Cor. 2.14], which deals with two germs $f$ and $g$ defined on a complex normal surface singularity $(X,p)$. Here we present the case when $(X,p) = (\mathbb{C}^2,0)$.

**3.26 Example.** Let $f,g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be two holomorphic germs. Let $\mathscr{U}$ be a neighbourhood of the origin in $\mathbb{C}^2$ and let $\pi\colon \mathscr{W} \to \mathscr{U}$ be a resolution of the holomorphic germ $fg$ and let $\Gamma$ be its dual graph. Using again the notation of Example 3.25, one has

$$M_\Gamma{}^t(m_1^f,\cdots,m_s^f)+{}^t b(L_f)=0$$

and

$$M_\Gamma{}^t(m_1^g,\cdots,m_s^g)+{}^t b(L_g)=0\,.$$

Now let us consider the link $L=L_f-L_g$, where $-L_g$ denotes $L_g$ endowed with the opposite orientation. Then

$$(M_\Gamma)^{-1\,t}b(L_f\cup -L_g)=(M_\Gamma)^{-1\,t}(b(L_f)-b(L_g))\,.$$

Therefore

$$(M_\Gamma)^{-1\,t}b(L_f\cup -L_g)=-{}^t(m_1^f-m_1^g,\ldots,m_s^f-m_s^g)\,.$$

Hence $(m_1^f-m_1^g,\ldots,m_s^f-m_s^g)$ is the solution of the monodromical system of $L_f\cup -L_g$.

Applying Theorem 3.21, one obtains:

**3.27 Theorem** ( [51, Cor. 2.2]). *The link $L_{f\bar g}=L_f\cup -L_g$ is fibred if and only if for each rupture vertex $v_j$ of $\Gamma$ one has $m_j^f\neq m_j^g$. Moreover, if this condition holds, then the quasi-periodic representant of the monodromy of $L_f-L_g$ has order $|m_j^f-m_j^g|$ on $\mathscr{F}_j=V_j\cap\mathscr{F}$.*

Let $(M,L)$ be a plumbing link. Another way to represent $(M,L)$ according to the Waldhausen decomposition of $M$ is the following: Let $\mathscr{W}(M,L)$ be a graph constructed in the following way:

- the graph $\mathscr{W}(M,L)$ has a vertex $i$ for each Seifert component $M_i$ in the Waldhausen decomposition of $M$. For each exceptional fibre in $M_i\setminus L$ we attach to $i$ a stalk weighted by the corresponding normalised Seifert invariant $(\alpha,\beta)$, *i.e.*, $1\le\beta<\alpha$. For each Seifert fibre in $M_i\cap L$ we attach an arrow weighted by the corresponding normalised Seifert invariant $(\alpha,\beta)$, *i.e.*, $0\le\beta<\alpha$. The vertex is weighted by the genus $g_i$ of the orbit space of $M_i$ and the Euler class $e(M_i)$.

- Let $M_i$ and $M_j$ be two Seifert components in the Waldhausen decomposition of $M$ and let $i$ and $j$ be the corresponding vertices; there is an edge between $i$ and $j$ if and only if the intersection $M_i\cap M_j$ is not empty.

Each edge is oriented by the triplet $(\varepsilon, \alpha, \beta)$ (defined as in [39, § 1]) in the following way: Let $T$ be a separation torus between two Seifert components $M_i$ and $M_j$ (represented by vertices $i$ and $j$ respectively). Let $\mathscr{U}(T)$ be a thickened torus, small neighbourhood of $T$ and let $T_i \subset M_i$ and $T_j \subset M_j$ be its boundary components. Let us orient $T_i$ and $T_j$ as the boundary of $\mathscr{U}(T)$. Let $b_i \subset T_i$ be a Seifert fibre of $M_i$ and $a_i \subset T_i$ a curve such that $a_i \cdot b_i = 1$ in $\mathrm{H}_1(T_i, \mathbb{Z})$. In the same way, we choose $a_j, b_j \subset T_j$ such that $a_j \cdot b_j = 1$ in $\mathrm{H}_1(T_j, \mathbb{Z})$. Let $g\colon T_i \to T_j$ an orientation reversing diffeomorphism, induced by the product structure of $\overline{\mathscr{U}(T)}$. There exists some integers $\varepsilon \in \{-1, 1\}$, $\alpha > 0$ and $\beta, \beta' \in \mathbb{Z}$ such that

$$\varepsilon h^{-1}(b_j) = \alpha a_i + \beta b_i \text{ in } \mathrm{H}_1(T_i, \mathbb{Z})$$

and

$$\varepsilon h^{-1}(b_i) = \alpha a_j + \beta' b_j \text{ in } \mathrm{H}_1(T_j, \mathbb{Z}).$$

Moreover, it is possible to choose the curves $a$ and $a'$ in such a way that the integers $\beta$ and $\beta'$ are normalised, *i.e.*,, $0 \le \beta < \alpha$ and $0 \le \beta' < \alpha$. If $\alpha > 1$, $\beta$ and $\beta'$ satisfy the following relation:

$$\beta\beta' \equiv 1 \pmod{\alpha}.$$

If $\alpha = 1$, then $\beta = \beta' = 0$.

Thus, if the edge is oriented from $i$ to $j$, the corresponding triplet is $(\varepsilon, \alpha, \beta)$; if the orientation is from $j$ to $i$, we write the triplet $(\varepsilon, \alpha, \beta')$.

In Figure 3.11 it is shown the graph $\mathscr{W}(M, L)$ of a graph link $(M, L)$ with $s$ exceptional fibres in $M_i \setminus L$, $s' - s$ Seifert fibres in $M_i \cap L$ and Euler class $e_i = e(M_i)$.

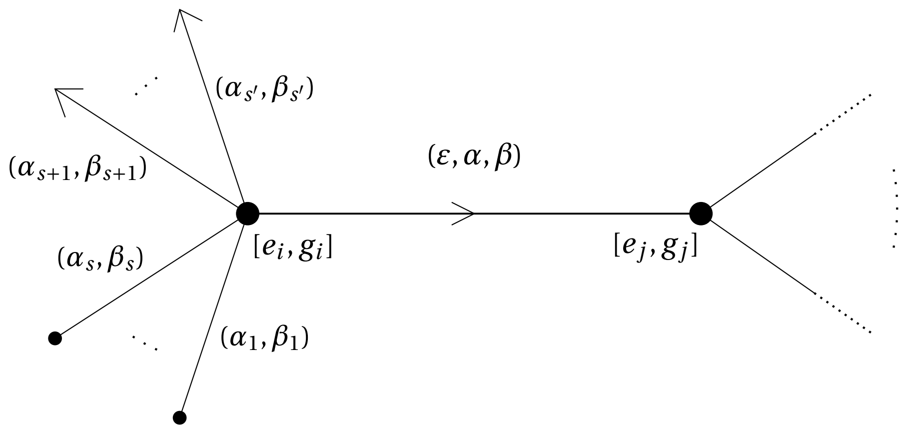



Figure 3.11: Graph $\mathscr{W}(M,L)$ of the graph link $(M,L)$.

## 3.6 Open-book of a quasi-periodic diffeomorphism

Given a quasi-periodic diffeomorphism of a surface, one can construct the associated mapping torus and, using the construction given in Section 1.3, obtain an associated open book; this open book with the corresponding binding is a fibred link. In this section is given a correspondence between the Nielsen graph of the quasi-periodic diffeomorphism and the graph representing the fibred link presented in Section 3.5.

The following result is proved in [36, Section 4.4].

**3.28 Lemma.** *Let $\mathscr{F}$ be a surface without boundary and let $h\colon \mathscr{F} \to \mathscr{F}$ be a periodic diffeomorphism with $s$ exceptional orbits. Let $(\lambda_i, \sigma_i)$ be their valencies with $i = 1, \ldots, s$.*

*Then the mapping torus $T(h)$ is a Seifert manifold whose orbit space is the orbit space of $h$ and whose Seifert invariants are given as follows :*

- *There are $s$ exceptional fibres whose Seifert invariants are $(\alpha_i, \beta_i) = (\lambda_i, \sigma_i)$ with $i = 1 \ldots, s$,*

- *the Euler class $e$ is given by :*
$$e = \sum_{i=1}^{s} \frac{\sigma_i}{\lambda_i} .$$

By Section 1.3, one can construct the open-book $(\mathcal{O}(h), \pi_h)$. The following result gives information on this open-book and it is an adaptation of [50, Lemma 4.4].

**3.29 Theorem.** *Let $h\colon \mathcal{F} \to \mathcal{F}$ be a quasi-periodic diffeomorphism with Nielsen graph $\mathcal{G}(h)$. Then the pair $(\mathcal{O}(h), L(h))$ is a plumbing link (moreover, it is a fibred link) whose corresponding graph $\mathcal{W}(\mathcal{O}(h), L(h))$ is obtained as follows. There exists an isomorphism from $\mathcal{G}(h)$ to $\mathcal{W}(\mathcal{O}(h), L(h))$ sending:*

- *the vertices of $\mathcal{G}(h)$ to the vertices of $\mathcal{W}(\mathcal{O}(h), L(h))$,*
- *the edges of $\mathcal{G}(h)$ to the edges of $\mathcal{W}(\mathcal{O}(h), L(h))$,*
- *the stalks of $\mathcal{G}(h)$ to the stalks of $\mathcal{W}(\mathcal{O}(h), L(h))$,*
- *the boundary-stalks of $\mathcal{G}(h)$ to the arrows of $\mathcal{W}(\mathcal{O}(h), L(h))$.*

*Moreover,*

- *Consider a vertex of $\mathcal{G}(h)$ with genus $g$, order $m$ and with neighbour valencies $(\lambda_i, \sigma_i)$, $i = 1, \ldots, s'$ (taking into account all the incident edges, including those corresponding to stalks and boundary-stalks). Then the corresponding vertex of $\mathcal{W}(\mathcal{O}(h), L(h))$ is weighted by $[g, e]$ where the Euler class $e$ is given by :*
$$e = \sum_{i=1}^{s'} \frac{\sigma_i}{\lambda_i}\,, \tag{3.30}$$
- *for each stalk of $\mathcal{G}(h)$ with valency $(\lambda, \sigma)$, the corresponding exceptional fibre has Seifert invariant*
$$(\alpha, \beta) = (\lambda, \sigma)\,, \tag{3.31}$$
- *for each boundary-stalk of $\mathcal{G}(h)$ with valency $(\lambda, \sigma)$, twist $t$ and order $m$ in the adjacent vertex, the corresponding Seifert invariant is*
$$(\alpha, \beta) = \left(|t\lambda|, -\frac{t}{|t|} \cdot \frac{1 - mt\sigma}{m}\right), \tag{3.32}$$
- *for each edge of $\mathcal{G}(h)$, the triplet of the corresponding edge oriented from left to right is:*
$$(\varepsilon, \alpha, \beta) = \left(-\frac{t}{|t|}, |m_j t\lambda|, -\frac{t}{|t|} \cdot \frac{m_j - m_i m_j t\sigma}{m_i}\right). \tag{3.33}$$

*Notice that for each of the three previous equalities, there exists a choice of $\sigma$ in its class modulo $\lambda$ such that the corresponding pair $(\alpha,\beta)$ is normalised, i.e., $0 \le \beta < \alpha$. Let us fix such integers $\sigma$.*

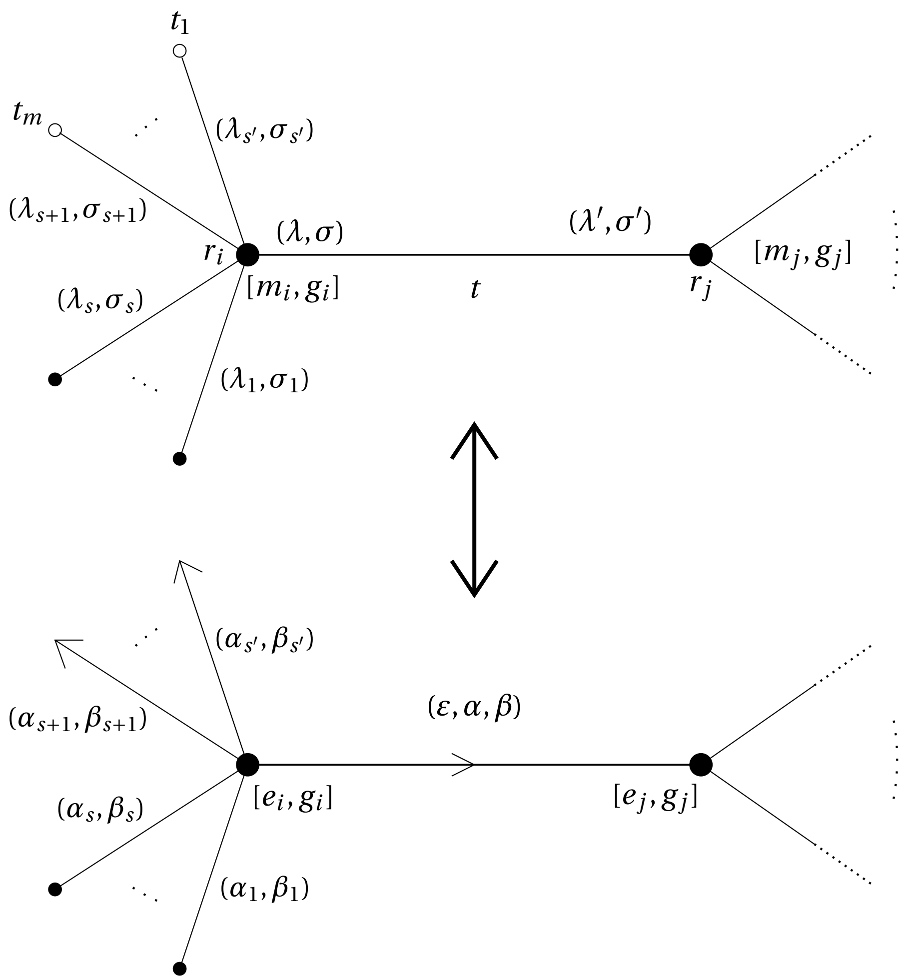



Figure 3.12: Isomorphism between the graphs $\mathscr{G}(h)$ and $\mathscr{W}(\mathscr{O}(h),L(h))$.

The formulae (3.30) and (3.31) are consequences of the Lemma 3.28 and (3.33) is proved in [50, Lemma 4.4] as consequence of the following lemma. This lemma, which is an adaptation of a part of [50, Lemma 4.4], will enable us to prove (3.32).

**3.34 Lemma.** *Let $A = [-1,1]\times\mathbb{S}^1$ be an annulus with boundary components $c = \{-1\}\times\mathbb{S}^1$ and $c' = \{1\}\times\mathbb{S}^1$ and let $h\colon A \to A$ be an orientation preserving diffeomorphism such that*

- *$h(c) = c$ and $h|_c$ is periodic of order $m$,*
- *$h(c') = D'$ and $h|_{c'}$ is periodic of order $m'$.*

*Let $T(h)$ be the mapping torus of $h$ and let $T = c \times \mathbb{S}^1$ and $T' = c' \times \mathbb{S}^1$ be the boundary components of $T(h)$. Let $b \subset T$ be the curve which is the image in $T(h)$ of the union of segments*

$$\bigcup_{i=1}^{m} (-1, h^i(\lambda)) \times [0,1] ,$$

*where $((-1, h^i(\lambda)) \times [0,1]) \subset (A \times [0,1])$ and let $b' \subset T'$ be the curve which is the image in $T(h)$ of the union of segments*

$$\bigcup_{i=1}^{m'} (1, h^i(\lambda)) \times [0,1] .$$

*Then the following equation holds*

$$mb' = -mm't\lambda a + (m' - mm't\sigma)b , \tag{3.35}$$

*where $a$ is a curve on $T$ such that $a \cdot b = 1$ in $\mathrm{H}_1(T, \mathbb{Z})$ and $a'$ is a curve on $T'$ such that $a' \cdot b' = 1$ in $\mathrm{H}_1(T, \mathbb{Z})$.*

*Proof.* Let $d = \{0\} \times \mathbb{S}^1 \subset A$, we orient $d$ in such a way that $d$ and $c$ are homologous in $\mathrm{H}_1(A, \mathbb{Z})$, where $c$ is oriented as boundary of $A$. Let $\gamma = [0,1] \times \{0\} \subset A$ be oriented as $[0,1]$. Then $\gamma$ is transverse to $d$ and we have

$$h^{mm'}(\gamma) - \gamma = mm'td = mm'tc \tag{3.36}$$

in $\mathrm{H}_1(A, \mathbb{Z})$ as in Definition 3.8.

On the other hand, the cycle $m'b - mb' + \gamma - h^{mm'}(\gamma)$ is the boundary of a 2-chain in $T(h)$. Then, from (3.36), we obtain

$$mb' = m'b - mm'td \ \text{ in } \ \mathrm{H}_1(T(h), \mathbb{Z}) . \tag{3.37}$$

For $a$ and $b$, we have the following relation:

$$c = \lambda a + \sigma b \ \text{ in } \ \mathrm{H}_1(T, \mathbb{Z}) . \tag{3.38}$$

Combining (3.36), (3.37) and (3.38) we obtain

$$mb' = -mm't\lambda a + (m' - mm't\sigma)b$$

in $\mathrm{H}_1(T(h), \mathbb{Z})$. $\square$

*Proof of equation* (3.32). Let us take a boundary-stalk of $\mathscr{G}(h)$ with valency $(\lambda, \sigma)$, twist $t$ and order $m$ in the adjacent vertex. As we know, the diffeomorphism $h$ is the identity on the boundary of $\mathscr{F}$, then, applying Lemma 3.34, we obtain

$$b = -t\lambda a' + (1 - mt\sigma)b' \quad \text{in} \quad \mathrm{H}_1(T(h), \mathbb{Z})$$

and on the other hand we have the following equality:

$$\varepsilon b = \alpha a' + \beta b' \quad \text{in} \quad \mathrm{H}_1(T', \mathbb{Z}) .$$

From these last two equations we get

$$\alpha = |t\lambda| \quad \text{and} \quad \beta = -\frac{t}{|t|} \cdot \frac{1 - mt\sigma}{m},$$

where $\varepsilon = -\frac{t}{|t|}$. □

## 3.7 Applications: Explicit computations of monodromies

In this section are presented some applications of Theorem 3.29.

### Application 1: Monodromy of the Milnor fibration of $f$

Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a holomorphic germ. Let us take again the notation used in Example 3.25, writing $m_i$ instead of $m_i^f$. According to Example 3.25, the order of $h$ at the vertex $v_i$ of the Nielsen graph equals $m_i$.

The plumbing representation of the pair $(\mathbb{S}^3, L_f)$ described by the resolution graph $\Gamma$ is a particular Waldhausen decomposition of the pair $(\mathbb{S}^3, L_f)$ (in general non minimal). The formulae of Theorem 3.29 apply to this Waldhausen decomposition.

In particular, if $v_i$ and $v_j$ are two adjacent vertices in the resolution graph, then the plumbing edge between them corresponds to the Waldhausen invariants $(\alpha, \beta) = (1, 0)$ with $\varepsilon = 1$.

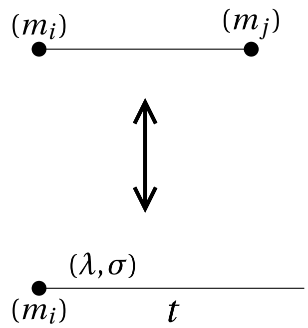



Figure 3.13: Correspondence given by the formulae of Theorem 3.29.

Therefore, the adjacent valency $(\lambda, \sigma)$ in the Nielsen graph (see Figure 3.13) and the "partial" twist $t$ corresponding to this edge is computed using the formula (3.33) with $(\varepsilon, \alpha, \beta) = (1, 1, 0)$ as follows. According to Proposition 3.24, the orders of $h$ on $\mathscr{F} \cap V_i$ and $\mathscr{F} \cap V_j$ equal $m_i$ and $m_j$ respectively. Then $t < 0$, $\alpha = 1$ implies

$$t = -\frac{1}{m_j \lambda} ,$$

and $\beta = 0$ implies

$$\sigma = \frac{1}{m_i t} = -\frac{m_j \lambda}{m_i t} .$$

As $\gcd(\lambda, \sigma) = 1$, one obtains:

$$\lambda = \frac{m_i}{\gcd(m_i, m_j)} \qquad \text{and} \qquad \sigma = -\frac{m_j}{\gcd(m_i, m_j)}$$

Then the result of Du-Bois-Michel stated in Proposition 1.53 is recovered. Moreover, one obtains the formula for the partial twist given in [18, § III.13]:

$$t = -\frac{\gcd(m_i, m_j)}{m_i m_j} .$$

If $j$ is the extremity of an arrow, then the equation (3.32) applied with $(\alpha, \beta) = (1, 0)$ leads to the same expressions.

**3.39 Example.** Let $f : \mathbb{C}^2 \to \mathbb{C}$ be the function defined by

$$f(x, y) = (x^2 + y^3)(x^3 + y^2) .$$

Figure 3.14 shows the resolution graph $A_\pi(f)$ for a resolution $\pi$ of $f$.

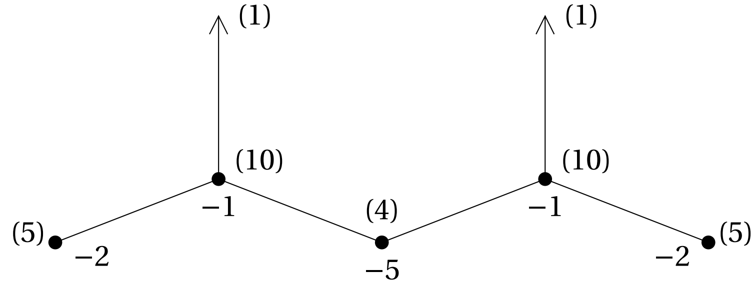



Figure 3.14: Resolution graph for $f(x, y) = (x^2 + y^3)(x^3 + y^2)$.

From $A_\pi(f)$ is obtained the Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f$. First it is constructed the Nielsen graph corresponding to the periodic diffeomorphism at each rupture vertex (see Figure 3.15).

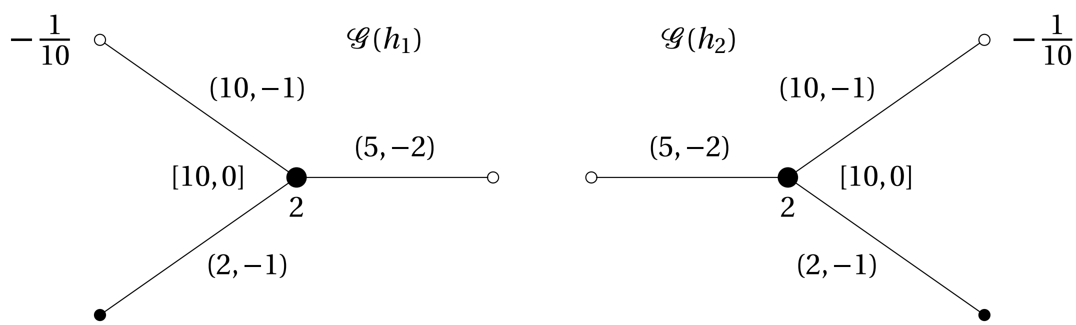



Figure 3.15: Nielsen graph of the diffeomorphisms $h_1$ and $h_2$.

and attaching these graphs one obtains the Nielsen graph $\mathscr{G}(h)$ (see Figure 3.16).

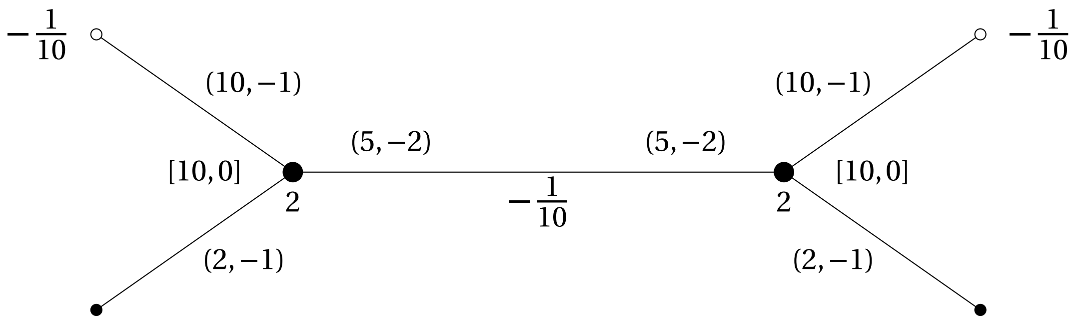



Figure 3.16: Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f(x,y) = (x^2 + y^3)(x^3 + y^2)$.

In this case the twist $t$ is given by

$$t = -\frac{1}{10} = -\frac{\gcd(4,10)}{4 \cdot 10} - \frac{\gcd(4,10)}{4 \cdot 10}\,.$$

Then, $\mathscr{O}(h) = \mathbb{S}^3$ and $L(h) = L_{fg}$. By Theorem 3.29, the graph $\mathscr{W}(\mathbb{S}^3, L_{fg})$ is the graph of Figure 3.17.

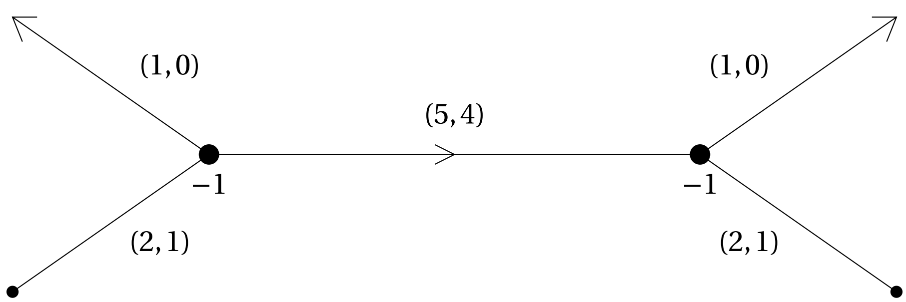



Figure 3.17: Graph $\mathscr{W}(\mathbb{S}^3, L_{fg})$ of $f(x,y)g(x,y) = (x^2 + y^3)(x^3 + y^2)$.

In this case we did not weight the vertices with the genera of the orbits spaces of the corresponding Seifert manifolds because they are zero.

Then the plumbing graph which describes $(\mathbb{S}^3, L_{fg})$ is the resolution graph $A_\pi(f)$ given in Figure 3.14.

## Application 2 : Monodromy of the Milnor fibration of $f\bar{g}$

Let $f, g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs such that $f\bar{g}$ has an isolated singularity at the origin, then the link $L_f - L_g$ is fibred (see [51, Th 5.1]).

By Remark 3.23, when $m_i^f - m_i^g < 0$, the natural orientation of the $\mathbb{S}^1$-fibres of $V_i$ is opposite to the one obtained by lifting the orientation of the circle $\mathbb{S}^1$ with the Milnor fibration. In order to get the right orientation, one must change the orientation of the fibres on each $V_i$ such that $m_i^f - m_i^g < 0$. It implies that the edges joining two vertices $v_i$ and $v_j$ such that $m_i^f - m_i^g < 0$ and $m_j^f - m_j^g \geq 0$ is now weighted by $\varepsilon = -1$.

Also, by Theorem 3.27, the order of the monodromy $h$ of the Milnor fibration of $f\bar{g}$ on $V_i \cap \mathscr{F}$ equals $m_i = |m_i^f - m_i^g|$. We then apply the same process as the one describe in Application 1 to describe the Nielsen graph of the monodromy of the Milnor fibration of $f\bar{g}$, taking into account the $\varepsilon$ on the edges:

- If $v_i$ and $v_j$ are two adjacent vertices in the resolution graph such that $m_i \neq 0$ and $m_j \neq 0$, then the plumbing edge between them corresponds to the Waldhausen invariants $(\alpha, \beta) = (1, 0)$ with $\varepsilon \in \{-1, +1\}$.

  Therefore, the adjacent valency $(\lambda, \sigma)$ in the Nielsen graph (see Figure 3.13) and the "partial" twist $t$ corresponding to this edge are computed using the formula (3.33) with $(\alpha, \beta) = (1, 0)$ as in Application 1. Then,

$$t = -\varepsilon \frac{\gcd(m_i, m_j)}{m_i m_j}$$

  and

$$\lambda = \frac{m_i}{\gcd(m_i, m_j)} \qquad \text{and} \qquad \sigma = -\frac{m_j}{\gcd(m_i, m_j)}.$$

  Notice also that perhaps after performing additional blowing-ups, one can assume that any neighbour vertex $v_j$ of a rupture vertex is such that $m_j \neq 0$. Then the above formula enables one to compute all the valencies in the Nielsen graph.

- if $v_i$ is a vertex such that $m_i = 0$, then by Theorem 3.27, it is a vertex on a string which is not at one of the extremities (*i.e.*, it is a vertex with valence 2 in $\Gamma$). Let $v_j$ and $v_k$ be the two neighbour vertices, say with $m_j = -m_k > 0$.

  Let $e_i$ be the Euler class weighting $v_i$ in the resolution graph. Then the Waldhausen invariant $(\alpha, \beta)$ corresponding to the string consisting of the vertex $v_i$ and the two adjacent edges equals :

$$(\alpha, \beta) = (|e_i|, |e_i| - 1)$$

and one obtains the partial twist corresponding to this edge by equation (3.33):

$$t = +\frac{|e_i|}{m_i}$$

Notice that the $t$ is positive, which never happens for the monodromy of a holomorphic germ (see [50, Th. 5.4]).

**3.40 Example.** Let $f, g$ be as in the Example 3.39. The plumbing tree $\Gamma_{f\bar{g}}$ is given in Figure 3.18.

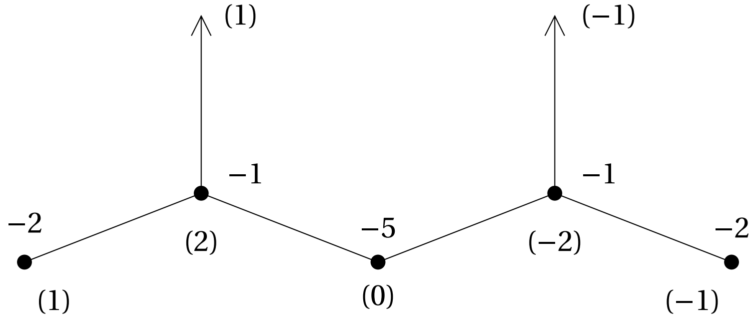



Figure 3.18: Plumbing tree $\Gamma_{f\bar{g}}$ for $f(x,y)\overline{g(x,y)} = (x^2+y^3)\overline{(x^3+y^2)}$.

After the change of orientation of the Seifert fibres as we made above, we obtain the graph in Figure 3.19.

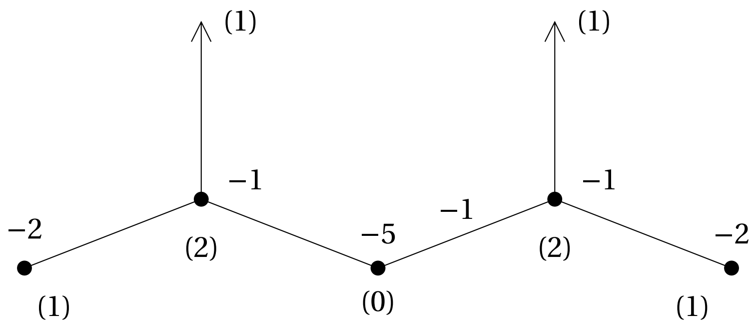



Figure 3.19: After the change of orientation in the plumbing tree $\Gamma_{f\bar{g}}$ for $f(x,y)\overline{g(x,y)} = (x^2+y^3)\overline{(x^3+y^2)}$.

By Theorem 3.29, it is possible to compute directly the Nielsen graph $\mathcal{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f\bar{g}$ (see Figure 3.20).

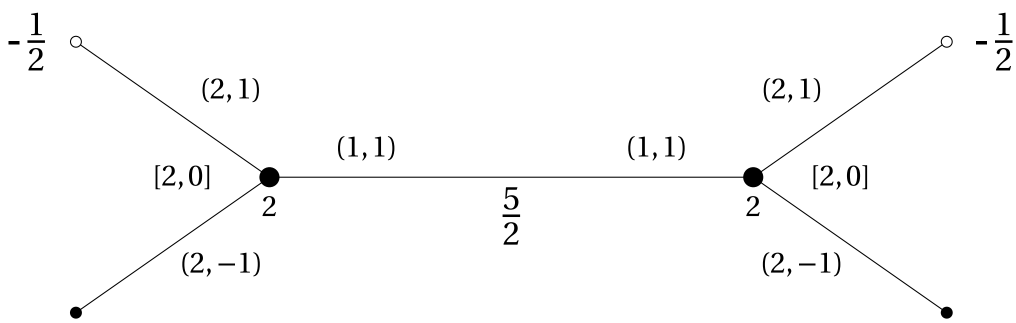



Figure 3.20: Nielsen graph $\mathscr{G}(h)$ for $f(x,y)\overline{g(x,y)} = (x^2+y^3)\overline{(x^3+y^2)}$.

Note that, as we said, the twist is positive as a consequence of the change of sign of $\epsilon$ in the formula (3.33).

## 3.8 The monodromy $h^r$

The following results are Lemma 2.2 and Lemma 2.3 of [49] and their proofs appear there. Given a quasi-periodic diffeomorphism $h$, these lemmas enable us to compute the Nielsen graph $\mathscr{G}(h^r)$ of the quasi-periodic diffeomorphism $h^r$ from the Nielsen graph $\mathscr{G}(h)$.

**3.41 Lemma.** *Let $h\colon \mathscr{F} \to \mathscr{F}$ be a periodic diffeomorphism of order $m$ which preserves the orientation of the surface $\mathscr{F}$ and let $r \geq 2$ be an integer. Let $\pi\colon \mathscr{F} \to \mathscr{O}$ be the projection onto the orbit space $\mathscr{O}$ of $\pi$.*

*Let $\pi^r\colon \mathscr{F} \to \mathscr{O}^{(r)}$ be the projection onto the orbit space $\mathscr{O}^{(r)}$ of $\pi^r$ and let*

$$\rho\colon \mathscr{O}^{(r)} \to \mathscr{O}$$

*be the map defined by $\rho \circ \pi^r = \pi$. Then $\rho$ is a cyclic branched covering of $\gcd(m,r)$ leaves, whose ramification locus is included in the set of exceptional orbits of $\mathscr{O}$. Moreover each exceptional orbit of $\mathscr{O}$ with valency $(\lambda,\sigma)$ is a possible branching point of $\rho$ with order $\frac{\gcd(m,r)}{\gcd(m/\lambda,r)}$.*

*The Nielsen invariants of $h^r$ can be computed from the Nielsen graph $\mathscr{G}(h)$ as follows:*

*Let $\mathscr{G}(h)$ be the Nielsen graph of a diffeomorphism $h\colon \mathscr{F} \to \mathscr{F}$ of order $m$ with $s$ exceptional orbits, where $\mathscr{O}$ has genus $g$ and $s'-s$ boundary components (see Figure 3.21).*

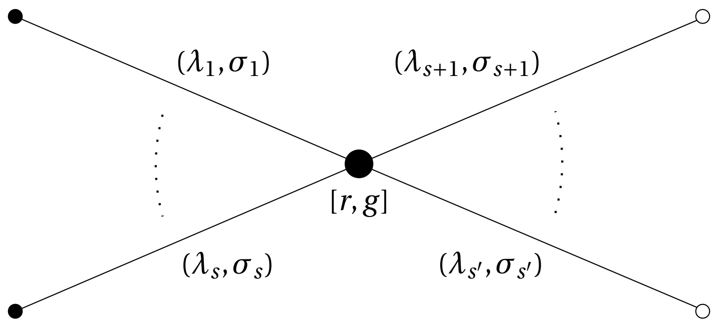



Figure 3.21: A Nielsen graph with $s$ stalks and $s' - s$ boundary stalks.

- *The order of $h^r$ is $n^{(r)} = \frac{m}{\gcd(m,r)}$.*

- *The genus of $\mathscr{O}^{(r)}$ is*

$$g^{(r)} = (g-1)\gcd(m,r) + 1 + \frac{1}{2}\sum_{i=1}^{s'} (\gcd(m,r) - \gcd(m/\lambda_i, r)) \ .$$

- *The orbit space $\mathscr{O}^{(r)}$ has a maximum of $\sum_{i=1}^{s} \gcd(m/\lambda_i, r)$ exceptional orbits and $\sum_{i=s+1}^{s'} \gcd(m/\lambda_i, r)$ boundary curves.*

- *To the $i-th$ exceptional orbit of $\mathscr{O}$ correspond $\gcd(m/\lambda_i, r)$ orbits of $\mathscr{O}^{(r)}$. To the $i-th$ boundary curve of $\mathscr{O}$ correspond $\gcd(m/\lambda_i, r)$ boundary curves of $\mathscr{O}^{(r)}$. In any case, the valency $(\lambda_i^{(r)}, \sigma_i^{(r)})$ is given by:*

$$\lambda_i^{(r)} = \frac{m}{\lambda_i \gcd(m/\lambda_i, r)} \quad \textit{and} \quad \sigma^{(r)} \times \frac{r}{\gcd(m,r)} \equiv \sigma_i \pmod{\lambda^{(r)}} \ . \tag{3.42}$$

*When $\lambda^{(r)} = 1$ the corresponding orbit is regular, in any other case it is a exceptional orbit of $\mathscr{O}^{(r)}$.*

**3.43 Lemma.** *Let $h\colon \mathscr{F} \to \mathscr{F}$ be a quasi-periodic diffeomorphism and let $\mathscr{C}$ be the minimal reduction system for $h$. Then the family $\mathscr{C}$ is a reduction system of $h^r$ and if $t$ and $t^{(r)}$ are the twists of $h$ and $h^r$ respectively near to a curve $c \in \mathscr{C}$, then $t^{(r)} = r\,t$.*

**3.44 Example.** Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the function defined by

$$f(x,y) = (x^2 + y^3)(x^3 + y^2) \ .$$

The Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f$ is given in Figure 3.22.

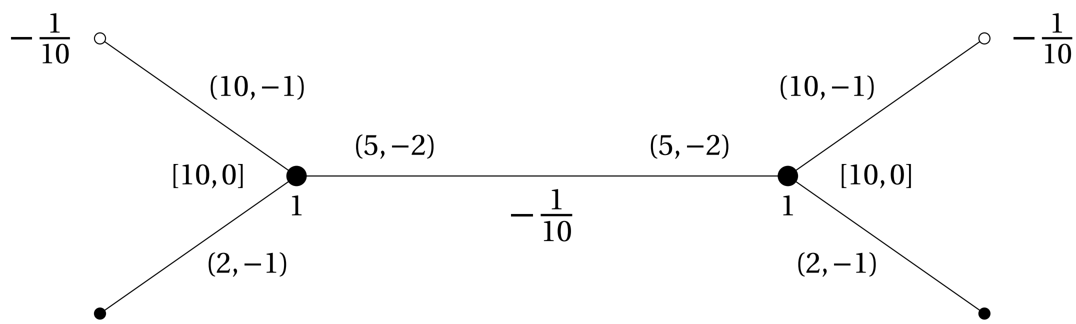



Figure 3.22: Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f(x,y) = (x^2+y^3)(x^3+y^2)$.

Now, let $r = 2$. The Nielsen graph $\mathscr{G}(h^2)$ is given in Figure 3.23.

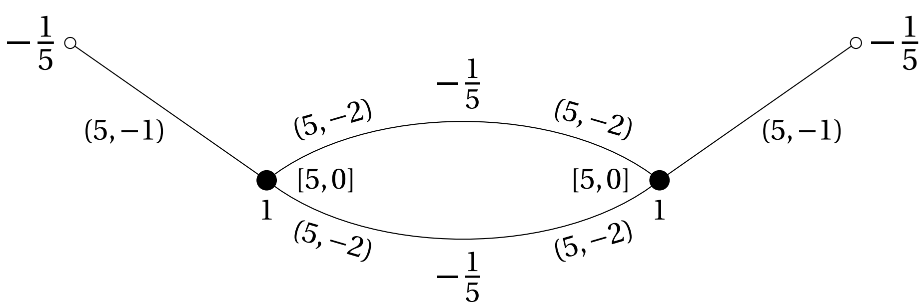



Figure 3.23: Nielsen graph $\mathscr{G}(h^2)$ of the monodromy $h^2$ of the Milnor fibration of $f(x,y) = (x^2+y^3)(x^3+y^2)$.

**3.45 Example.** Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the complex analytic germ defined by $f(x,y) = (x^2+y^3)$ and let $g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the complex analytic germ defined by $g(x,y) = (x^3+y^2)$. The real analytic germ $f\bar{g}$ is given by

$$f(x,y)\overline{g(x,y)} = (x^2+y^3)\overline{(x^3+y^2)}\,.$$

By Section 3.7, the Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration of $f\bar{g}$ is as in Figure 3.24.

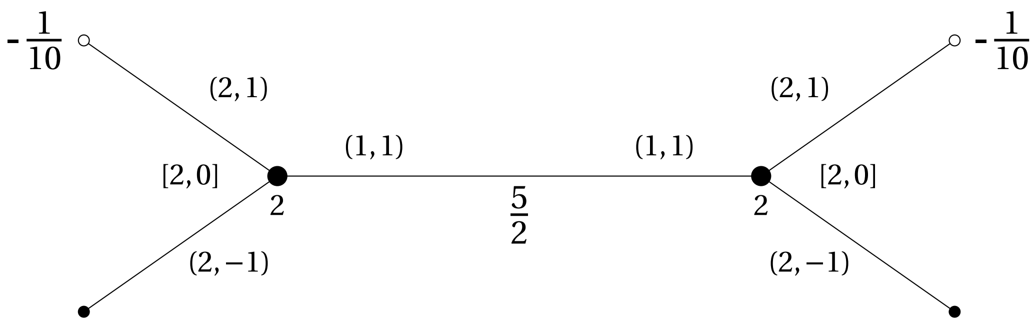



Figure 3.24: Nielsen graph $\mathscr{G}(h)$ for $f(x,y)\overline{g(x,y)} = (x^2+y^3)\overline{(x^3+y^2)}$.

Now, let $r=2$. The Nielsen graph $\mathscr{G}(h^2)$ is given in Figure 3.25.

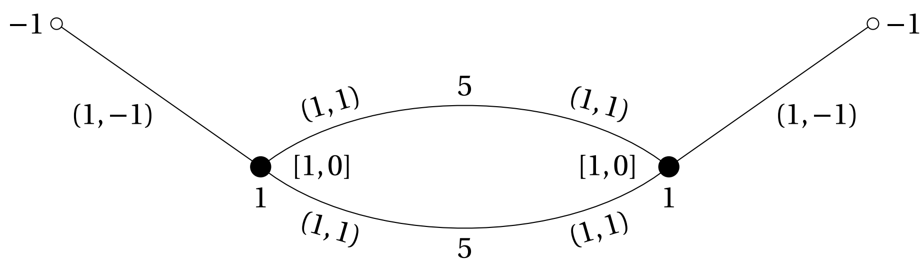



Figure 3.25: Nielsen graph $\mathscr{G}(h^2)$ of the monodromy $h^2$ of the Milnor fibration of $f(x,y)\overline{g(x,y)} = (x^2+y^3)\overline{(x^3+y^2)}$.

## 3.9 Computation of the link of $f\bar{g}+z^r$

Let $f\colon (\mathbb{C}^2,0)\to(\mathbb{C},0)$ and $g\colon (\mathbb{C}^2,0)\to(\mathbb{C},0)$ be two complex analytic germs such that the real analytic germ $f\bar{g}\colon (\mathbb{C}^2,0)\to(\mathbb{C},0)$ has an isolated singularity at the origin. Let $F\colon (\mathbb{C}^3,0)\to(\mathbb{C},0)$ be the germ defined by

$$F(x,y,z) = f(x,y)\overline{g(x,y)} + z^r$$

with $r\in\mathbb{Z}^+$.

As we saw in Section 3.1, the following diagram commutes:

$$\begin{array}{ccc} L_F\setminus L' & \xrightarrow{\mathscr{P}} & \mathbb{S}^3\setminus L_{f\bar{g}} \\ {\scriptstyle\Phi'}\downarrow & & \downarrow{\scriptstyle\Phi_{f\bar{g}}} \\ \mathbb{S}^1 & \xrightarrow{\rho_r} & \mathbb{S}^1 \end{array}$$

where $L_{f\bar{g}}$ is the link of $f\bar{g}$, $L_F$ is the link of $F$, $\mathscr{P}$ is the projection onto the first two coordinates, $L' = \mathscr{P}^{-1}(L_{f\bar{g}})$, $\Phi_{f\bar{g}}$ is the Milnor fibration of $f\bar{g}$, $\Phi' = \frac{z}{|z|}$ and $\rho_r(z) = z^r$.

With the tools given in the previous sections of this chapter, we are able to give a description of the link $L_F$ in terms of the link $L_{f\bar{g}}$ and the monodromy $h$ of the Milnor fibration $\Phi_{f\bar{g}}$ as follows:

**First step:** To compute the plumbing tree $\Gamma_{f\bar{g}}$ where each vertex $i$ is weighted by $m_i = m_i^f - m_i^g$ and change the negative multiplicities into positive ones (Section 3.7).

**Second step:** To compute the Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ (Section 3.7).

**Third step:** To compute the Nielsen graph $\mathscr{G}(h^r)$ of the diffeomorphism $h^r$ (Section 3.8).

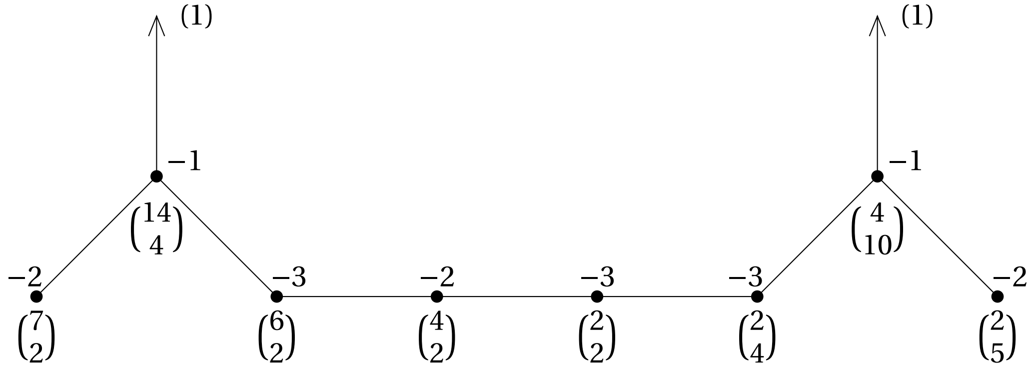
**Fourth step:** Applying Theorem 3.29, to compute the graph $\mathscr{W}(\mathscr{O}(h^r), L(h^r))$, where $\mathscr{O}(h^r) = L_F$ and $L(h^r)) = L'$ (Section 3.6).

**Fifth step:** To compute the plumbing graph corresponding to $L_F$ applying the plumbing calculus given in Section 3.5.

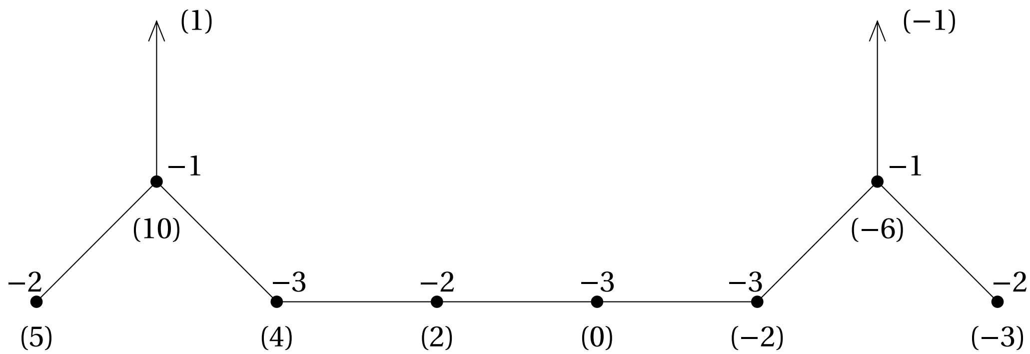
**3.46 Example.** Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the complex analytic germ defined by $f(x,y) = x^2 + y^7$ and let $g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the complex analytic germ defined by $g(x,y) = x^5 + y^2$.

**First step:**

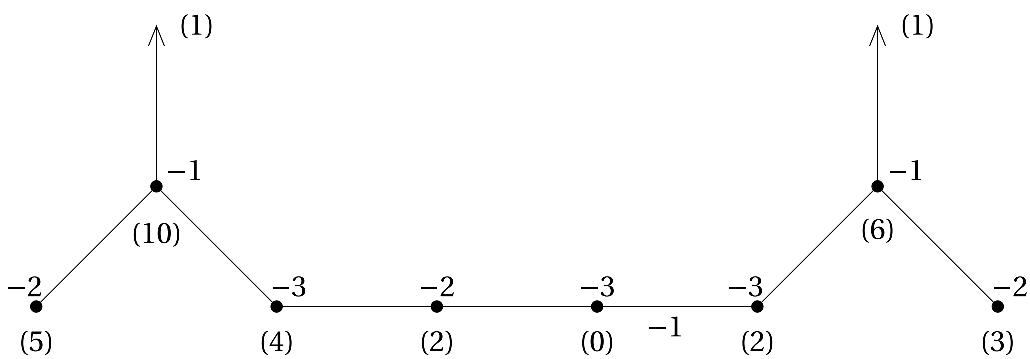
The resolution graph $A(fg)$, where the multiplicity $m_i = m_i^f + m_i^g$ at the vertex $i$ appears as $\binom{m_i^f}{m_i^g}$ is given by




Then, the plumbing tree $\Gamma_{f\tilde{g}}$ is given by

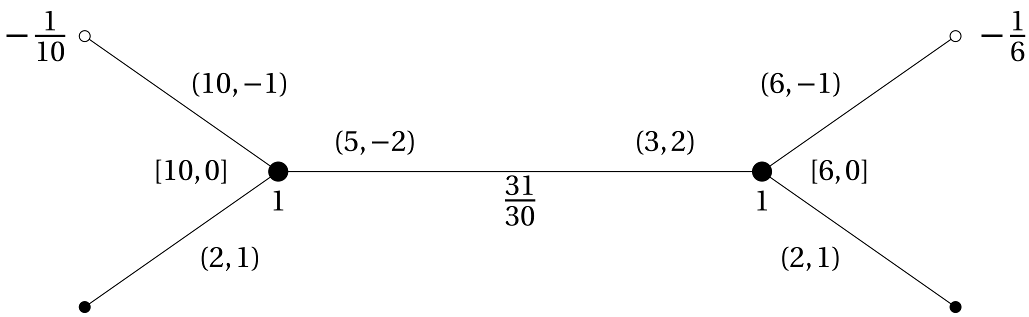



After the change of orientation in the Seifert fibres indicated in Section 3.7, we obtain the following graph:




**Second step:**

From the last graph, we compute the Nielsen graph $\mathcal{G}(h)$ of the monodromy $h$ of the Milnor fibration $\Phi_{f\tilde{g}}$.




Here the twist $\frac{31}{30}$ can be computed in two ways: First one is to consider the partial twists between the rupture vertices and get the twist as the sum of them and the other is to compute the $\alpha$ corresponding to the edge joining the rupture vertices. By Theorem 1.43, we have that

$$\frac{\alpha}{\alpha-\beta} = 3 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{3}}}$$

then $\alpha = 31$. Applying the equation (3.33), one obtains that

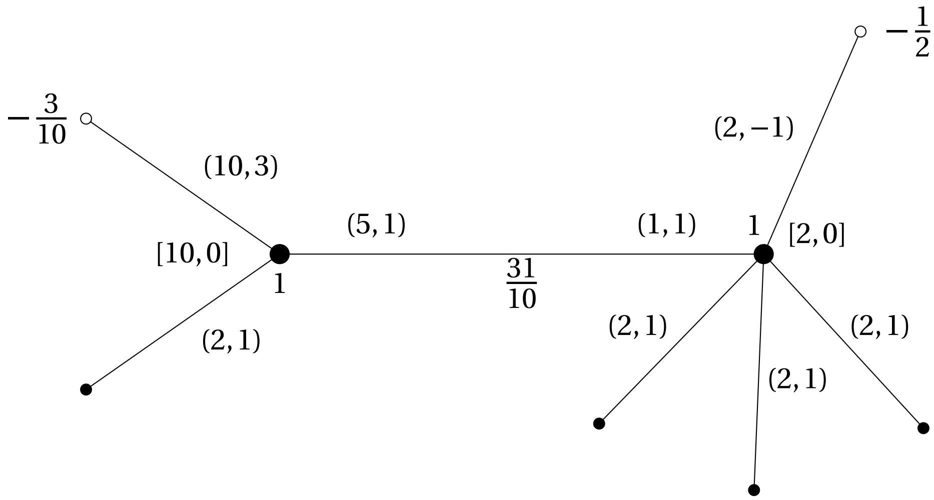
$$t = \frac{\alpha}{m_j \lambda} = \frac{31}{(6)(5)} = \frac{31}{30}\,.$$

**Third step:**

Let $F\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ be the germ defined by

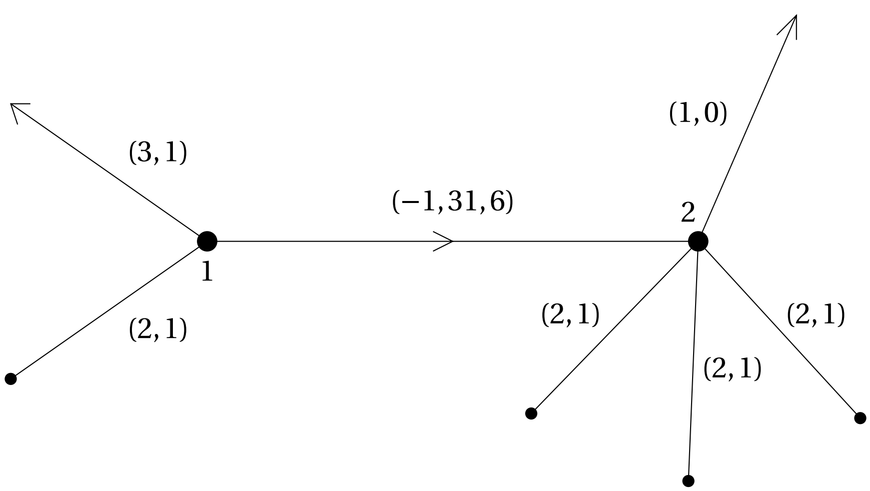
$$F(x,y,z) = (x^2+y^7)\overline{(x^5+y^2)} + z^3\,.$$

The link $L_F$ has an open-book fibration with binding $L'$ and monodromy $h^3$. This monodromy is a quasi-periodic diffeomorphism, then we can compute the Nielsen graph $\mathscr{G}(h^3)$ from the graph $\mathscr{G}(h)$ (as in Section 3.8):




**Fourth step:**

Applying Theorem 3.29, we compute the graph $\mathscr{W}(L_F, L')$ from the Nielsen graph $\mathscr{G}(h^3)$:

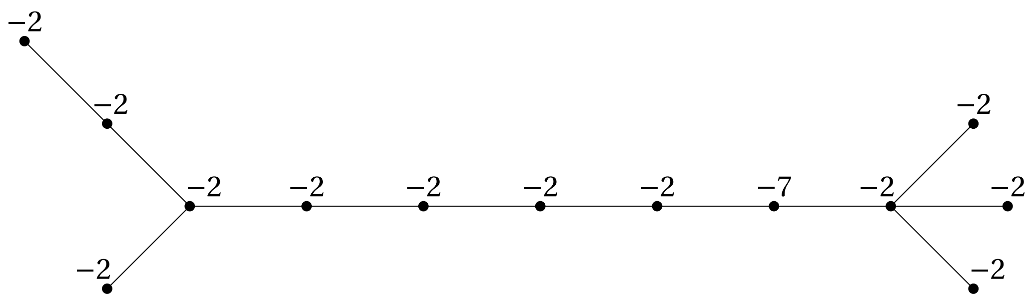



**Fifth step:**

From the graph $\mathscr{W}(L_F, L')$ we can compute the corresponding plumbing graph. First, we have the following equations:

$$\frac{3}{3-1} = 2 - \frac{1}{2} = [2,2]$$
$$\frac{2}{2-1} = 2 = [2]$$
$$\frac{31}{31-6} = [2,2,2,2,7] \,,$$

then the plumbing graph $\Gamma$ is given by

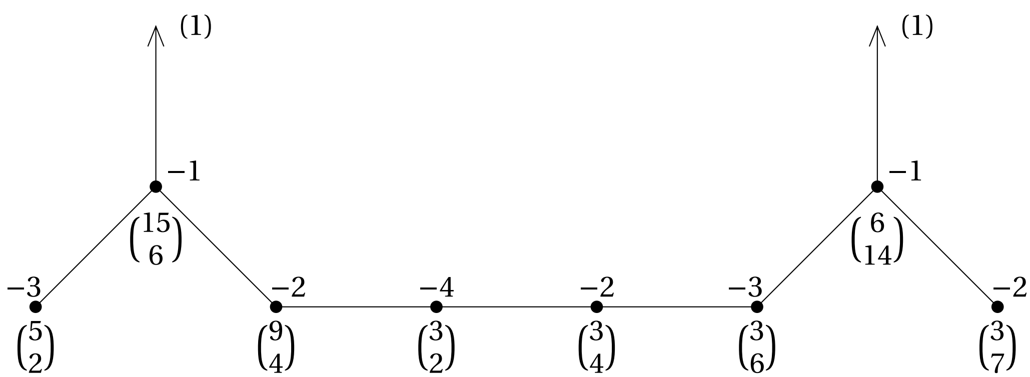



where $L_F \cong \partial P(\Gamma)$ and $P(\Gamma)$ is the four-manifold obtained by plumbing 2-discs bundles according to $\Gamma$.

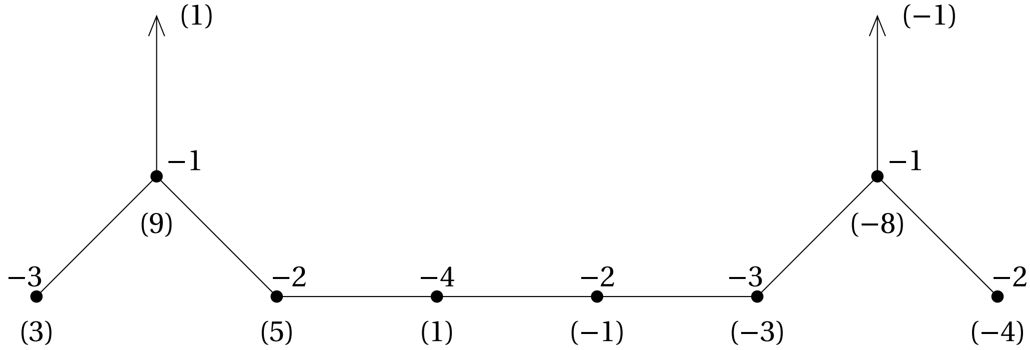
**3.47 Example.** Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the complex analytic germ defined by $f(x, y) = x^3 + y^5$ and let $g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the complex analytic germ defined by $g(x, y) = x^7 + y^2$.

**First step:**

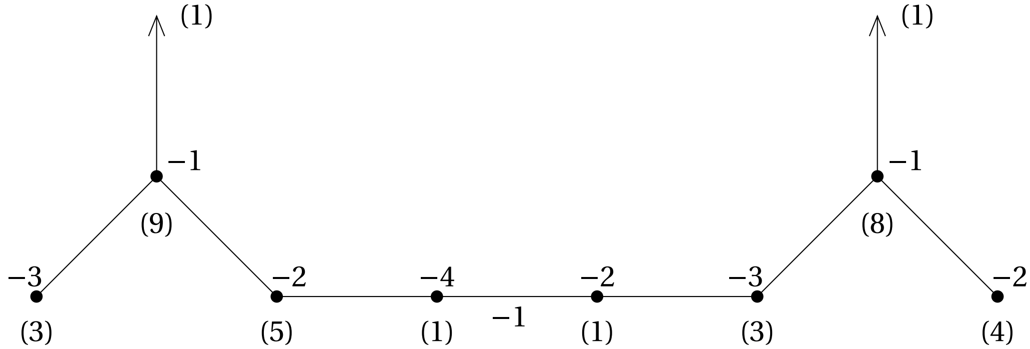
The resolution graph $A(fg)$, where the multiplicity $m_i = m_i^f + m_i^g$ at the vertex $i$ appears as $\binom{m_i^f}{m_i^g}$ is given by




Then, the plumbing tree $\Gamma_{f\bar{g}}$ is given by

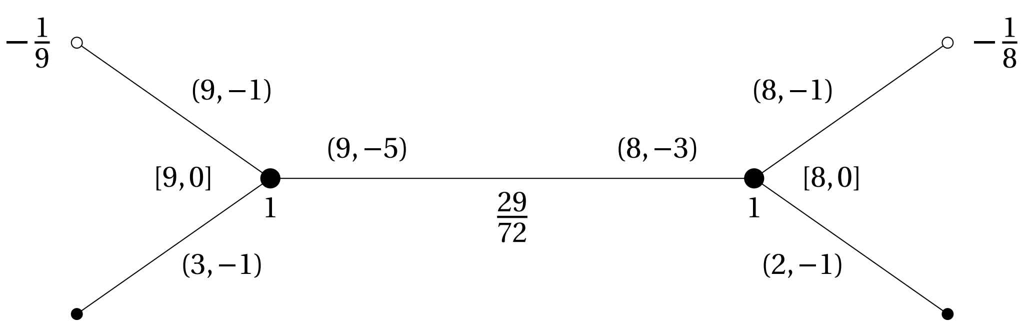



After the change of orientation in the Seifert fibres indicated in Section 3.7, we obtain the following graph:




**Second step:**

From the last graph, we compute the Nielsen graph $\mathscr{G}(h)$ of the monodromy $h$ of the Milnor fibration $\Phi_{f\bar{g}}$.




where the twist $\frac{29}{72}$ is computed by applying the equation (3.33):

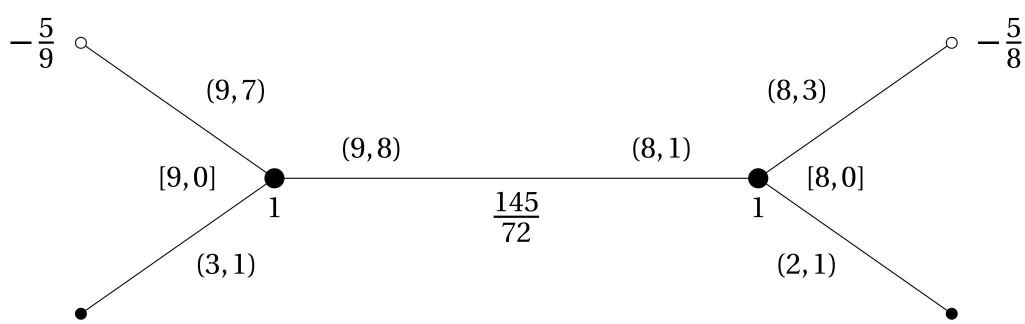
$$t = \frac{\alpha}{m_j \lambda} = \frac{29}{(8)(9)} = \frac{29}{72}\,.$$

**Third step:**

Let $F\colon (\mathbb{C}^3,0)\to(\mathbb{C},0)$ be the germ defined by

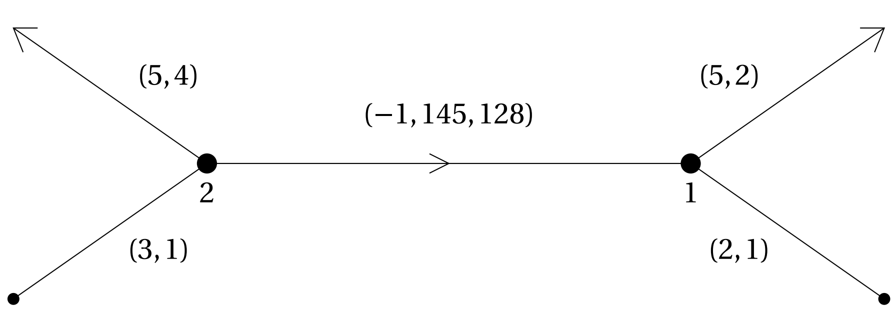
$$F(x,y,z)=(x^3+y^5)\overline{(x^7+y^2)}+z^5 \; .$$

The link $L_F$ has an open-book fibration with binding $L'$ and monodromy $h^5$. This monodromy is a quasi-periodic diffeomorphism, then we can compute the Nielsen graph $\mathscr{G}(h^5)$ from the graph $\mathscr{G}(h)$ (as in Section 3.8):




**Fourth step:**

Applying Theorem 3.29, we compute the graph $\mathscr{W}(L_F,L')$ from the Nielsen graph $\mathscr{G}(h^5)$:

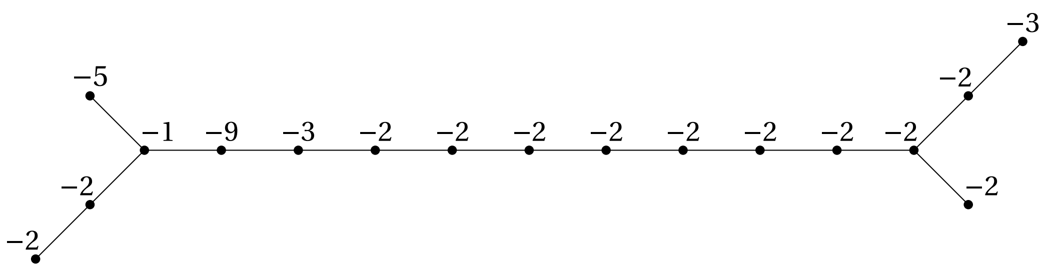


**Fifth step:**

From the graph $\mathscr{W}(L_F, L')$ we can compute the corresponding plumbing graph. First, we have the following equations:

$$\frac{5}{5-4} = 5 = [5]$$
$$\frac{3}{3-1} = 2 - \frac{1}{2} = [2,2]$$
$$\frac{5}{5-2} = 2 - \frac{1}{3} = [2,3]$$
$$\frac{2}{2-1} = 2 = [2]$$
$$\frac{145}{128} = [9,3,2,2,2,2,2,2,2] \ ,$$

then the plumbing graph $\Gamma$ is given by




where $L_F \cong \partial P(\Gamma)$ with $P(\Gamma)$ the four-manifold obtained by plumbing 2-discs bundles according to $\Gamma$.

# CHAPTER 4

# CYCLIC SUSPENSIONS AND MILNOR FIBRATIONS

The theory in this chapter is based in [25]. We first show that if $f$ and $g$ are holomorphic functions from $\mathbb{C}^2$ to $\mathbb{C}$ and $r \geq 2$ is an integer, then the function $F = f\bar{g} + z^r$ has a Milnor fibration with projection map $F/\|F\|$. Then we use [25] to study the topology of this fibration.

For this, in the first section it is given the concept of fibred knot and its relation with open-books. In the second section it is defined the $r$-suspension of a knot and it is shown that the above function $F$ is $d$-regular (see Definition 1.21). This implies that $F$ has Milnor fibration.

In Section 4.3 we apply a join theorem given in [25] to the Milnor fibre of the function $F$ and we obtain the homotopy type of the Milnor fibre of $F$ in terms of the homotopy type of the Milnor fibre of $f$. This join theorem is a generalisation of results as the one proved by Sebastiani and Thom in [63] and Sakamoto's result (see [58, Th. 1]). The last section is devoted to examples.

## 4.1 Open-books and fibred knots

In this section we give concepts related to fibred knots and open-books.

**4.1 Definition.** A **knot** $\mathscr{K} = (\mathbb{S}^k, K)$ is an oriented $k$-sphere with an oriented codimension 2 compact closed submanifold $K \subset \mathbb{S}^k$.

**4.2 Definition.** A knot $\mathscr{L} = (\mathbb{S}^n, L)$ is **fibred** if there is a fibration

$$b\colon \mathbb{S}^n \setminus L \to \mathbb{S}^1$$

such that the closure $\mathscr{F}_t = b^{-1}(t)$ of any fibre of $b$ is a closed submanifold of $\mathbb{S}^n$ with boundary $\partial\mathscr{F}_t = L$.

In Section 1.3 we give a definition of an open-book fibration; in this section we use an equivalent definition, given in [25], which is more convenient for this work.

**4.3 Definition.** Let $M$ be a closed compact manifold. An **open-book structure** for $M$ is a map $b\colon M \to \mathbb{D}^2$ such that zero is a regular value and

$$\phi_b = \frac{b}{\|b\|}\colon M \setminus b^{-1}(0) \to \mathbb{S}^1$$

is a smooth fibration.

This definition is equivalent to Definition 1.26 : Given $b\colon M \to \mathbb{D}^2$ as above, then $N = b^{-1}(0)$ is the binding and the open-book fibration is given by $\phi_b$.

**4.4 Definition.** Given two manifolds $M$ and $M'$ with open-book structures $b$ and $b'$ respectively, we say the open-books $(M,\phi_b)$ $(M',\phi_{b'})$ are **equivalent** if there exists a diffeomorphism $\alpha\colon M \to M'$ such that $b' \circ h$ agrees with $b$ on a neighbourhood of $b^{-1}(0)$ and

$$\frac{b' \circ h}{\|b' \circ h\|} = \frac{b}{\|b\|}$$

on $M \setminus b^{-1}(0) = M \setminus (b' \circ h)^{-1}(0)$.

Note that a fibred knot $\mathscr{L} = (\mathbb{S}^n, L)$ has an open-book structure $b\colon \mathbb{S}^n \to \mathbb{D}^2$ such that $b^{-1}(0) = L$.

**4.5 Definition.** Given a fibred knot $\mathscr{L} = (\mathbb{S}^n, L)$ with open-book structure $b\colon \mathbb{S}^n \to \mathbb{D}^2$, $b$ will also be called the **fibred structure** of $\mathscr{L}$.

**4.6 Definition.** Let $(\mathbb{S}^n, L)$ and $(\mathbb{S}^n, L')$ be two fibred knots with corresponding fibred structures $b\colon \mathbb{S}^n \to \mathbb{D}^2$ and $b'\colon \mathbb{S}^n \to \mathbb{D}^2$; the knots are **isotopic** if $b$ and $b'$ are equivalent in the sense of Definition 4.4 by a diffeomorphism $\alpha\colon \mathbb{S}^n \to \mathbb{S}^n$ which is isotopic to the identity.

## 4.2 Cyclic suspensions

In this section we define the concept of a cyclic suspension of a knot. For that we need some results first. The theory is based in [20], [38] and [25].

Let $(\mathbb{S}^m, K)$ be a fibred knot where $m \geq 3$. Then there exists a fibred structure $b\colon \mathbb{S}^m \to D^2$. Let $\rho_r\colon D^2 \to D^2$ be the $r$-branched cyclic covering of $D^2$ given by $\rho_r(z) = z^r$ with $0 \in D^2$ as ramification locus.

**4.7 Lemma.** *The pull-back $[r](\mathbb{S}^m, K)$ of $\rho_r\colon D^2 \to D^2$ by the map $b$:*

$$\begin{tikzcd} {[r](\mathbb{S}^m, K)} \arrow[r] \arrow[d, "\pi_r"'] & \mathbb{D}^2 \arrow[d, "\rho_r"] \\ \mathbb{S}^m \arrow[r, "b"] & \mathbb{D}^2 \end{tikzcd}$$

*is an $r$-branched cyclic covering of $\mathbb{S}^m$, branched along $K$.*

*Proof.* Consider the pull-back of $\rho_r\colon D^2 \to D^2$ by the map $b$:

$$\begin{tikzcd} {[r](\mathbb{S}^m, K)} \arrow[r] \arrow[d, "\pi_r"'] & \mathbb{D}^2 \arrow[d, "\rho_r"] \\ \mathbb{S}^m \arrow[r, "b"] & \mathbb{D}^2 \end{tikzcd}$$

then

$$[r](\mathbb{S}^m, K) = \{(x, y) \in \mathbb{S}^m \times \mathbb{D}^2 \mid b(x) = \rho_r(y) = y^r\}\ .$$

Let $x \in K$, then $\rho_r^{-1}(b(x)) = 0$ since $b(x) = 0$. Thus $\pi_r^{-1}(x) = \{(x, 0)\}$ and $\pi_r^{-1}(K) = K \times \{0\} \cong K$.

Now consider the restriction of $\rho_r$ to $\mathbb{D}^2 \setminus \{0\}$, which is an $r$-covering. We take the pull-back of it by $b$:

$$\begin{tikzcd} {[r](\mathbb{S}^m, K) \setminus \pi_r^{-1}(K)} \arrow[r] \arrow[d, "\pi_r"'] & \mathbb{D}^2 \setminus \{0\} \arrow[d, "\rho_r"] \\ \mathbb{S}^m \setminus K \arrow[r] & \mathbb{D}^2 \setminus \{0\} \end{tikzcd}$$

and, since an $r$-covering is a fibration with discrete fibre, the pull-back is again a fibration with the same fibre, then $\pi_r|_{[r](\mathbb{S}^m,K)\setminus\pi_r^{-1}(K)}$ is an $r$-covering. Thus we have obtained that $[r](\mathbb{S}^m, K)$ is the branched $r$-covering of $\mathbb{S}^m$ with ramification locus $K$. □

**4.8 Example.** Let $(\mathbb{S}^m, \mathbb{S}^{m-2})$ be a knot, with $m \geq 2$ . Let $p\colon \mathbb{S}^m \subset \mathbb{R}^{m+1} \to \mathbb{D}^2$ defined by $p(x_1, \ldots, x_m, x_{m+1}) = (x_m, x_{m+1})$. Then $(\mathbb{S}^m, \mathbb{S}^{m-2})$ is a fibred knot with fibred structure $p$.

Given the branched cyclic $r$-covering $\rho_r$ and applying Lemma 4.7, we obtain that $[r](\mathbb{S}^m, \mathbb{S}^{m-2})$ is the branched $r$-covering of $\mathbb{S}^m$ with ramification locus $\mathbb{S}^{m-2}$.

Notice that $[r](\mathbb{S}^m, \mathbb{S}^{m-2})$ is diffeomorphic to the sphere $\mathbb{S}^m$ since it is the $(m-2)$-suspension over the branched cyclic $r$-covering $\rho_r$ from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann's sphere.

**4.9 Definition.** Let $M$ be a smooth $m$-manifold and let $V \subset M$ be a submanifold of $M$ of codimension 2. A **proper embedding** of $V$ in $M$ means that $V$ is closed in $M$ and $V$ is transversal to $\partial M$ with $\partial V \subset \partial M$.

**4.10 Lemma** ( [25, Lem. 2.4])**.** *Let $V_1 \subset V_2 \subset M$ be proper embeddings of smooth manifolds of dimension $m-2$, $m$ and $m+2$ respectively. Assume that $M$ and $V_2$ are 2-connected. Let $i\colon V_2 \to M$ be the standard inclusion. Then there exists an embedding of pairs $j\colon (V_2, V_1) \to (M, V_2)$ such that*

*a) the image $j(V_2) \subset M$ is transverse to $V_2$ with $j(V_2) \cap i(V_2) = V_1$,*

*b) the map $j$ is isotopic to $i$ through maps satisfying condition (a),*

*c) the map $j$ is unique up to isotopy through maps which satisfy conditions (a) and (b).*

The proof of this result can be found in [25] in all detail.

**4.11 Corollary.** *Let $(\mathbb{S}^m, K)$ and $(\mathbb{S}^{m+2}, \mathbb{S}^m)$ be knots. Then the following diagram commutes:*

$$\begin{tikzcd}
{[r](\mathbb{S}^m, K)} \arrow[r, "\hat{j}"] \arrow[d, "\pi_r"'] & {[r](\mathbb{S}^{m+2}, \mathbb{S}^m)} \arrow[d, "\pi'_r"] \\
\mathbb{S}^m \arrow[r, "j"] & \mathbb{S}^{m+2}
\end{tikzcd}$$

*where $j$ is the embedding of Lemma 4.10.*

*Proof.* Let $p'\colon \mathbb{S}^{m+2} \to \mathbb{D}^2$ be the map defined by

$$p'(x_1, \ldots, x_{m+2}, x_{m+3}) = (x_{m+2}, x_{m+3}) .$$

Take the pull-back of $\rho_r$ by the composition $p' \circ j$:

$$\begin{tikzcd} N \arrow[r, "\hat{j}"] \arrow[d, "\pi_r"'] & {[r](\mathbb{S}^{m+2},\mathbb{S}^m)} \arrow[r] \arrow[d, "\pi'_r"] & \mathbb{D}^2 \arrow[d, "\rho_r"] \\ \mathbb{S}^m \arrow[r, "j"] & \mathbb{S}^{m+2} \arrow[r, "p'"] & \mathbb{D}^2 \end{tikzcd}$$

where $N = \{(x,y) \in \mathbb{S}^m \times \mathbb{D}^2 \mid (p' \circ j)(x) = \rho_r(y)\}$. Let $\mathbb{S}^m \hookrightarrow \mathbb{S}^{m+2}$ be the standard embedding. By Example 4.8, $p'$ is the fibred structure of the knot $(\mathbb{S}^{m+2},\mathbb{S}^m)$. Then by Lemma 4.10,

$$(p' \circ j)^{-1}(0) = j^{-1}(\mathbb{S}^m) = K\,.$$

Then, given $x \in K$ one has that $\pi_r^{-1}(x) = \{(x,0)\}$. Thus $\pi_r^{-1}(K) = K \times \{0\} \cong K$.

Analogously to the proof of Lemma 4.7, we obtain that $N$ is the an $r$-branched cyclic covering of $\mathbb{S}^m$, branched along $K$. Then $N = [r](\mathbb{S}^m,K)$. □

**4.12 Definition.** Let $K \otimes [r]$ be the image $\hat{j}([r](\mathbb{S}^m,K)) \subset \mathbb{S}^{m+2}$. Then we obtain a new knot $(\mathbb{S}^{m+2}, K \otimes [r])$. This knot is called the $r$**-fold cyclic suspension** or briefly $r$**-cyclic suspension** of the knot $(\mathbb{S}^m,K)$.

**4.13 Theorem.** *Let $f\colon (\mathbb{R}^n,0) \to (\mathbb{R}^2,0)$ be a $d$-regular real analytic germ with isolated critical point at the origin. Let $\varepsilon > 0$ be small enough and let $L_f = f^{-1}(0) \cap \mathbb{S}^{n-1}$ be the link of the singularity at the origin. Let $F\colon (\mathbb{R}^n \times \mathbb{C},0) \cong (\mathbb{R}^{n+2},0) \to (\mathbb{R}^2,0)$ be the map defined by*

$$F(x_1,\dots,x_n,z) = f(x_1,\dots,x_n) + z^r\,,$$

*and denote by $L_F = F^{-1}(0) \cap \mathbb{S}^{n+1}$ its link at the origin. Then the pair $(\mathbb{S}^{n+1},L_F)$ is the $r$-fold cyclic suspension of the knot $(\mathbb{S}^{n-1},L_f)$.*

In the proof of this theorem we need the two following results. The first one gives a characterisation of a $d$-regular function. The second one is a generalisation of [25, Lemma 4.1] and the proof presented here is basically the same presented there.

For more clarity, form now on we denote a point $(x_1,\dots,x_n) \in \mathbb{R}^n$ by $x$ and a tangent vector by $\vec{x}$.

**4.14 Lemma** ( [12, Lemma 5.2])**.** *Let $U$ be an open neighbourhood of $0 \in \mathbb{R}^n$ with $n > 1$, Let $k \le n$ and let $f\colon (U,0) \to (\mathbb{R}^k,0)$ be an analytic map defined on $U$ with isolated critical point at $0$. The map $f$ is $d$-regular, if and only if there exists a smooth vector field $\tilde{v}$ on $\mathbb{B}_\varepsilon \setminus f^{-1}(0)$ which has the following properties:*

*i) It is radial: i.e., it is transverse to all spheres in $\mathbb{B}_\varepsilon$ centred at 0 pointing outwards;*

*ii) it is tangent to each $X_\ell \setminus f^{-1}(0)$, whenever it is not empty;*

*iii) it is transverse to all the tubes $f^{-1}(\partial\mathbb{D}_\delta)$.*

**4.15 Remark.** Notice that in Lemma 4.14, the first condition is equivalent to ask that

$$\langle \tilde{v}(x), \overrightarrow{x} \rangle_{\mathbb{R}^n} > 0 \quad \text{for all} \quad x \in \mathbb{B}_\varepsilon \setminus f^{-1}(0)\ . \tag{4.16}$$

Also, the vector field $\tilde{v}$ can be adjusted, multiplying it by a positive real function and then, the second and third conditions can be interpreted as

$$D_x f(\tilde{v}(x)) = \overrightarrow{f(x)}\ . \tag{4.17}$$

**4.18 Lemma.** *Let $f\colon (\mathbb{R}^n,0) \to (\mathbb{R}^2,0)$ be a $d$-regular real analytic germ with isolated critical point at the origin. For sufficiently small $\varepsilon > 0$, there exists a smooth vector field $v$ on $\mathbb{B}_\varepsilon \setminus \{0\}$ which satisfies:*

*I) $v$ lies over the radial vector field $w(x) = \overrightarrow{x}$ on $\mathbb{C} \cong \mathbb{R}^2$,*

*II) $\|x\|$ increases along trajectories of $v$.*

*Proof.* We will construct $v$ locally and it can be pasted together by a smooth partition of unity. Let $v_1$ be the vector field on $\mathbb{B}_\varepsilon \setminus f^{-1}(0)$ given by Lemma 4.14. By Remark 4.15, we can adjust the length of $v_1$ by a positive real function in order to have condition (4.17).

Let $x_0 \in f^{-1}(0) \cap (\mathbb{B}_\varepsilon \setminus \{0\})$ and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^2 \times \mathbb{R} \cong \mathbb{R}^3$ be the map defined by

$$\varphi(x) = (f(x), \|x\| - \|x_0\|)\ .$$

Let $x \in (\mathbb{B}_\varepsilon \setminus \{0\})$ such that $x \neq x_0$, then $x$ is a regular point of $\varphi$, since $f$ has isolated critical point and $f^{-1}(0)$ is transversal with any sphere contained in $\mathbb{B}_\varepsilon$. Thus $\varphi$ is a submersion in a neighbourhood $U$ of $x_0$ and we can think in it as the canonical submersion; *i.e.*, as the projection in the first three coordinates (with suitable local coordinates):

$$\begin{aligned} \varphi\colon U \subset \mathbb{R}^n &\to \mathbb{R}^3 \\ (a,b,c,\dots) &\mapsto (a,b,c) \end{aligned}$$

In these coordinates, we define $v$ as

$$v_1(a,b,c,\dots) = \overrightarrow{(a,b,c+\|x_0\|,0,\dots,0)},$$

then $v$ is a suitable vector field on the neighbourhood $U$ of $x_0$ and this completes the proof. □

*Proof of Theorem 4.13.* Let

$$\mathbb{S}_\varepsilon^{n+1} = \{(x,z) \in \mathbb{R}^n \times \mathbb{C} \cong \mathbb{R}^{n+2} \mid |(x,z)| = \varepsilon\}\ ,$$

and let

$$\mathbb{S}_\varepsilon(t) = \{(x,z) \in \mathbb{S}_\varepsilon^{n+1} \mid tf(x) + z = 0\}$$

for $0 \le t \le 1$. For $\varepsilon$ small and any $t$, $\mathbb{S}_\varepsilon(t)$ is the intersection of the sphere $\mathbb{S}_\varepsilon^{n+1}$ and the hyperplane $tf(x)+z=0$, then $\mathbb{S}_\varepsilon(t)$ is a $(n-1)$-sphere. Then $\{\mathbb{S}_\varepsilon(t)\}_{0\le t\le 1}$ gives an isotopy between the standard sphere $\mathbb{S}_\varepsilon(0) \cong \mathbb{S}_\varepsilon^{n+1}$ and $\mathbb{S}_\varepsilon(1)$.

Also $\mathbb{S}_\varepsilon(t)$ intersects transversally $\mathbb{S}_\varepsilon(1)$ and $\mathbb{S}_\varepsilon(t) \cap \mathbb{S}_\varepsilon(1) = L_f$ for all $t<1$.

Take $\mathbb{S}_\varepsilon(1) \subset \mathbb{S}_\varepsilon^{n+1}$ as the “standard embedding” and let

$$\big(\mathbb{S}_\varepsilon(0), L_f\big) \subset \big(\mathbb{S}_\varepsilon^{n+1}, \mathbb{S}_\varepsilon(0)\big)$$

be the embedding $j$ of Lemma 4.10. Let

$$\overline{\mathbb{S}}_\varepsilon^{n+1} = \{(x,z) \in \mathbb{R}^n \times \mathbb{C} \cong \mathbb{R}^{n+2} \mid x_1^2 + \cdots + x_n^2 + |z|^{2r} = \varepsilon^2\}$$

and

$$\overline{L}_F = \overline{\mathbb{S}}_\varepsilon^{n+1} \cap F^{-1}(0)\ .$$

Then $\pi\colon \overline{\mathbb{S}}_\varepsilon^{n+1} \to \mathbb{S}_\varepsilon^{n+1}$ given by $\pi(x,z) = (x, z^r)$ gives a branched $r$-covering

$$(\overline{\mathbb{S}}_\varepsilon^{n+1}, \overline{L}_F) \to (\mathbb{S}_\varepsilon^{n+1}, \mathbb{S}_\varepsilon(1))$$

branched along $(\mathbb{S}_\varepsilon(0), L_f)$, and hence identifies $(\overline{\mathbb{S}}_\varepsilon^{n+1}, \overline{L}_F)$ as the $r$-fold cyclic suspension of $(\mathbb{S}_\varepsilon^{n-1}, L_f)$.

Thus, it just remains to show that the knot $(\overline{\mathbb{S}}_\varepsilon^{n+1}, \overline{L}_F)$ is diffeomorphic to the knot $(\mathbb{S}_\varepsilon^{n+1}, L_F)$. This is done by pushing the pair $(\mathbb{S}_\varepsilon^{n+1}, L_F)$ out to the other knot along a vector field defined in a small ball $(\mathbb{B}_{\varepsilon'}^{n+2} \setminus \{0\}) \subset \mathbb{R}^{n+2} \setminus \{0\}$. Such a vector field can be obtained as follows:

By Lemma 4.18, there is a vector field $v$ on a small ball $\mathbb{B}_{\varepsilon'}^n \setminus \{0\}$ such that $v$ lies over the radial vector field on $\mathbb{R}^2$ and $\langle v(x), \overrightarrow{x}\rangle_{\mathbb{R}^n}$ has positive real part (see

Remark 4.15). Then the vector field $v_1(x,z) = (v(x), z/r)$ is suitable on $(\mathbb{R}^n \times \mathbb{C}) \setminus (\{0\} \times \mathbb{C})$ and $v_2(x,z) = (0,z)$ is suitable in a thin neighbourhood of $\{0\} \times (\mathbb{C} \setminus 0)$; so pasting $v_1$ and $v_2$ with a partition of unity we obtain the required vector field $v$. □

## 4.3 Join Theorem

The aim in this section is to describe the homotopy type of the Milnor fibre $\mathscr{F}$ of the germ $F = f + z^r$ in terms of the homotopy type of the Milnor fibre $\mathscr{F}_f$ of $f$, where $f$ is a $d$-regular function. For this, we first show that given a $d$-regular function $f$ as in the previous section, the function $F = f + z^r$ is $d$-regular, which assures that it has Milnor fibration with projection $\frac{F}{\|F\|}$.

**4.19 Proposition.** *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be a $d$-regular real analytic germ with an isolated critical point at the origin. Let $F\colon (\mathbb{R}^n \times \mathbb{C}, 0) \cong (\mathbb{R}^{n+2}) \to (\mathbb{R}^2, 0)$ be the map defined by $F(x,z) = f(x) + z^r$. Then $F$ is $d$-regular.*

*Proof.* Let $\varepsilon_1, \varepsilon_2 > 0$ be such that $\mathbb{S}^{n+1}_{\varepsilon_1}$ is a Milnor ball for $F$ (see Definition 1.7) and $\mathbb{S}^{n-1}_{\varepsilon_2}$ is a Milnor ball for $f$. Let

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$$

and let us consider $\mathbb{S}^{n+1}_{\varepsilon}$ and $\mathbb{S}^{n-1}_{\varepsilon} \subset \mathbb{S}^{n+1}_{\varepsilon}$, where the latter is defined by

$$\mathbb{S}^{n-1}_{\varepsilon} = \left\{(x,0) \in \mathbb{S}^{n+1}_{\varepsilon}\right\} .$$

By Lemma 4.14, there exists a smooth vector field $v_1$ on $\mathbb{B}^n_{\varepsilon} \setminus f^{-1}(0)$ such that

i) $\langle v_1(x), \overrightarrow{x} \rangle_{\mathbb{R}^n} > 0$,

ii) $D_x f(v_1(x)) = \overrightarrow{f(x)}$,

for all $x \in \mathbb{B}^n_{\varepsilon} \setminus f^{-1}(0)$.

On the other hand, let $\rho_r\colon \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C} \cong \mathbb{R}^2$ be the function defined by $\rho_r(z) = z^r$ and let $v_2$ be the radial vector field on $\mathbb{C} \cong \mathbb{R}^2$ defined by $v_2(z) = \frac{\overrightarrow{z}}{r}$, then

i) $\langle v_2(z), \overrightarrow{z} \rangle_{\mathbb{R}^2} > 0$,

ii) $D_z \rho_r(v_2(z)) = \overrightarrow{\rho_r z}$, for all $z \in \mathbb{C} \setminus \{0\}$.

In order to prove that $F$ is $d$-regular, the idea is to find a vector field $v$ on $\mathbb{B}_\varepsilon^{n+2} \setminus F^{-1}(0)$ such that

i) $\langle v(x,z), \overrightarrow{(x,z)} \rangle > 0$ for all $(x,z) \in \mathbb{B}_\varepsilon^{n+2} \setminus F^{-1}(0)$,

ii) $D_{(x,z)}F(v(x,z)) = \overrightarrow{F(x,z)}$ for all $(x,z) \in \mathbb{B}_\varepsilon^{n+2} \setminus F^{-1}(0)$.

Let $v$ be the vector field on $\mathbb{B}_\varepsilon^{n+2} \setminus F^{-1}(0)$ defined by $v(x,z) = (v_1(x), v_2(z))$. Then

$$\langle v(x,z), \overrightarrow{(x,z)} \rangle_{\mathbb{R}^{n+2}} = \langle v_1(x), \overrightarrow{x} \rangle_{\mathbb{R}^n} + \langle v_2(z), \overrightarrow{z} \rangle_{\mathbb{R}^2} > 0\,,$$

since both terms are greater than zero. Then it follows condition (i) (see Figure 4.1). Also

$$\begin{aligned} D_{x,z}F(v(x,z)) &= D_{x,z}F(v_1(x), v_2(z)) \\ &= D_{x,z}F(v_1(x), \overrightarrow{0}\,) + D_{x,z}F(\overrightarrow{0}\,, v_2(z)) \\ &= D_x f(v_1(x)) + D_z \rho_r(v_2(z)) \\ &= \overrightarrow{f(x)} + \overrightarrow{\rho_r(z)} = \overrightarrow{F(x,z)}\,. \end{aligned}$$

And condition (ii) holds. By Lemma 4.14 and Remark 4.15, $F$ is $d$-regular. □

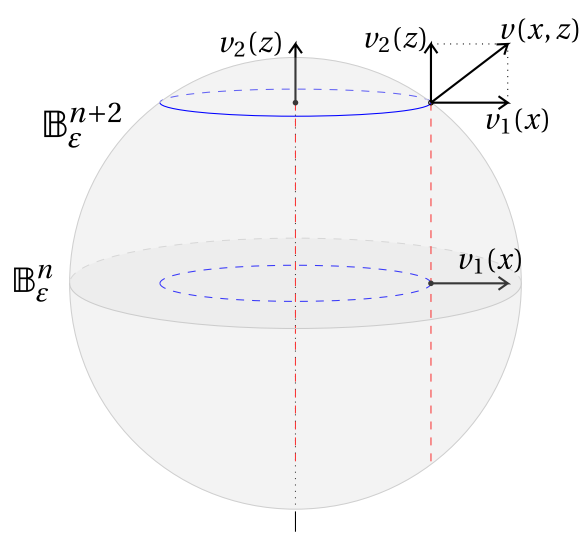



Figure 4.1: The vector field $v$ on $\mathbb{B}_\varepsilon^{n+2} \setminus F^{-1}(0)$.

By Theorem 1.24 and Proposition 4.19, one has the following result.

**4.20 Corollary.** *Given $F$ as in Proposition 4.19, $F$ has Milnor fibration with projection*

$$\phi_F = \frac{F}{|F|} \colon \mathbb{S}^5_\varepsilon \setminus L_F \to \mathbb{S}^1 \, .$$

**4.21 Remark.** Let $\rho_r \colon \mathbb{C} \to \mathbb{C}$ be the function defined by $\rho_r(z) = z^r$. Note that $\rho_r$ has a Milnor fibration with projection

$$\phi_{\rho_r} = \frac{z^r}{|z|^r} \colon \mathbb{S}^1_\varepsilon \to \mathbb{S}^1 \, ,$$

for any $\varepsilon > 0$ and the Milnor fibre $\mathscr{F}_{\rho_r}$ consists of $r$ points in $\mathbb{S}^1_\varepsilon$ and the monodromy $h_{\rho_r}$ is given by a cyclic permutation of this $r$ points.

The following result is an adaptation of [25, Lemma 6.1] to the particular case of cyclic suspensions given by links of singularities and it describes the homotopy type of the Milnor fibre of $F$ in terms of the Milnor fibres of $f$ and $\rho_r$.

**4.22 Theorem.** *Let $f \colon (\mathbb{R}^n, 0) \to (\mathbb{R}^2, 0)$ be a $d$-regular real analytic germ with an isolated critical point at the origin. Consider its Milnor fibration with projection map $\phi_f$ and let $\mathscr{F}_f$ be its fibre and let $h_f$ be its monodromy. Let $F \colon (\mathbb{R}^n \times \mathbb{C}, 0) \cong (\mathbb{R}^{n+2}) \to (\mathbb{R}^2, 0)$ be the map defined by $F(x, z) = f(x) + z^r$ and let $\phi_F = \frac{F}{|F|}$ be the projection map of its Milnor fibration; let $\mathscr{F}$ be its Milnor fibre and $h_F$ its monodromy.*

*Then there exists a homotopy equivalence $\alpha \colon \mathscr{F}_f * \mathscr{F}_{\rho_r} \to \mathscr{F}$ which is compatible with the monodromy maps and their join; i.e., the following diagram commutes*

$$\begin{tikzcd}
\mathscr{F}_f * \mathscr{F}_{\rho_r} \arrow[r, "h_f * h_{\rho_r}"] \arrow[d, "\alpha"'] & \mathscr{F}_f * \mathscr{F}_{\rho_r} \arrow[d, "\alpha"] \\
\mathscr{F} \arrow[r, "h_F"] & \mathscr{F}
\end{tikzcd} \tag{4.22}$$

*where $h_f * h_{\rho_r} \colon \mathscr{F}_f * \mathscr{F}_{\rho_r} \to \mathscr{F}_f * \mathscr{F}_{\rho_r}$ is the map defined by*

$$h_f * h_{\rho_r}([x, t, y]) = [h_f(x), t, h_{\rho_r}(y)] \, .$$

By Example 1.23 and Theorem 4.22, we obtain the following result.

**4.23 Corollary.** *Let $f, g \colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs such that $f\bar{g}$ has isolated singularity. Let $F \colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the germ defined by $F(x, y, z) = f(x, y)\overline{g(x, y)} + z^r$, where $r \in \mathbb{Z}$ and $r \geq 2$. Then:*

*i) The Milnor fibre $\mathscr{F}$ of $F$ is homotopy equivalent to the join of the Milnor fibre $\mathscr{F}_{f\bar{g}}$ and $r$ points; and*

*ii) the above homotopy equivalence is compatible with the monodromy maps and their join.*

## 4.4 Examples of open-books

This section follows what is done in Section 2.4, but here it is treated the general case when $f$ and $g$ are two holomorphic germs from $\mathbb{C}^2$ to $\mathbb{C}$.

Theorem 4.22 enables us to describe the Milnor fibre $F$ in terms of the Milnor fibre of $f$. In the previous chapter was given a way to compute the plumbing graph which describes the link $L_F$.

**4.24 Example.** Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the complex analytic germ defined by $f(x,y) = x^2 + y^7$ and let $g\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be the complex analytic germ defined by $g(x,y) = x^5 + y^2$. Let $F\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ be the germ defined by

$$F(x,y,z) = (x^2 + y^7)\overline{(x^5 + y^2)} + z^3 \,.$$

In Example 3.46, the plumbing graph $\Gamma$ such that $L_F \cong \partial P(\Gamma)$ was computed (see Figure 4.2).

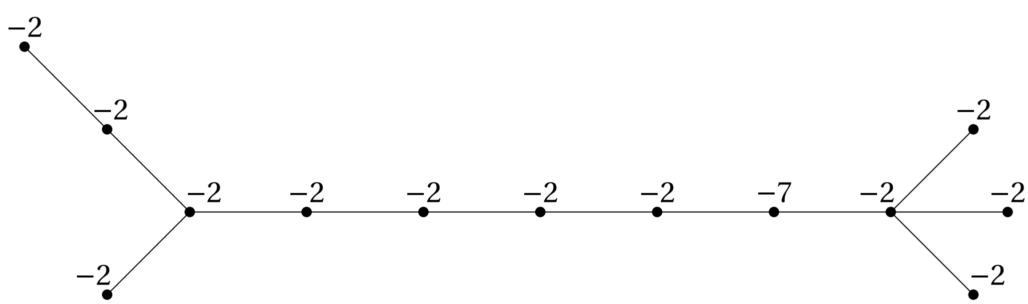



Figure 4.2: Plumbing graph $\Gamma$ such that $L_F \cong \partial P(\Gamma)$.

As it is explained in Section 2.4, the plumbing $P(\Gamma)$ given by the plumbing graph $\Gamma$ contains in its interior the exceptional divisor $E$ as a strong deformation retract. Then the divisor $E$ can be blown down to a point, and we get a complex surface $V_\Gamma$ with a normal singularity at 0. We compute the canonical class $K$ of $V_\Gamma$ and obtain that $K$ has non-integer coefficients.

It follows by Definition 2.18 that the singularity $(V_\Gamma, 0)$ is not numerically Gorenstein and by Remark 2.19, it is not Gorenstein, then there is not a complex analytic germ $G\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with isolated singularity at the origin such that the link $L_G$ is isomorphic to the link $L_F$.

**4.25 Example.** Let $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the complex analytic germ defined by $f(x, y) = (x^2 + y^3)$ and let $g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be the complex analytic germ defined by $g(x, y) = (x^3 + y^2)$. The real analytic germ $f\bar{g}$ is given by

$$f(x, y)\overline{g(x, y)} = (x^2 + y^3)\overline{(x^3 + y^2)} \,.$$

Let $F\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be the germ defined by

$$F(x, y, z) = (x^2 + y^3)\overline{(x^3 + y^2)} + z^2 \,.$$

In example 3.45, it is computed the Nielsen graph of the diffeomorphism $h^2$, where $h$ is the monodromy of the Milnor fibration of $f\bar{g}$ (see Figure 4.3).

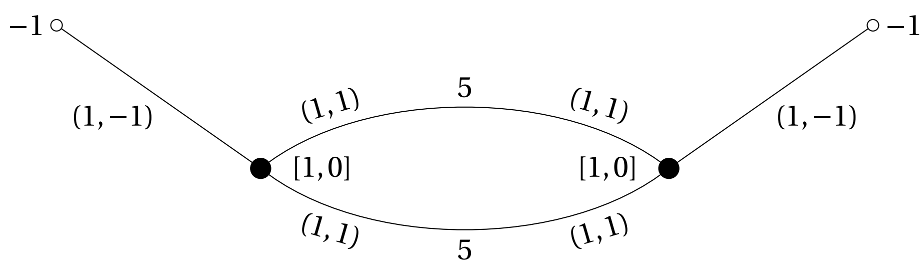



Figure 4.3: Nielsen graph $\mathscr{G}(h^2)$ of the diffeomorphism $h^2$ with $h$ the Milnor fibration of $f(x, y)\overline{g(x, y)} = (x^2 + y^3)\overline{(x^3 + y^2)}$.

From this graph, by Theorem 3.29, the graph $\mathscr{W}(L_F, L')$ is the graph shown in Figure 4.4.

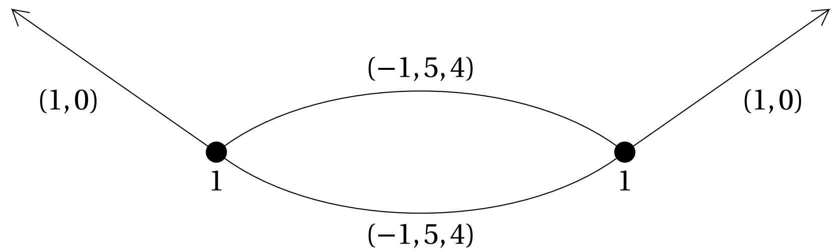



Figure 4.4: Graph $\mathscr{W}(L_F, L')$ for $F(x,y,z) = (x^2+y^3)\overline{(x^3+y^2)} + z^2$.

From the graph $\mathscr{W}(L_F, L')$, the corresponding plumbing graph $\Gamma$ can be computed (see Figure 4.5).

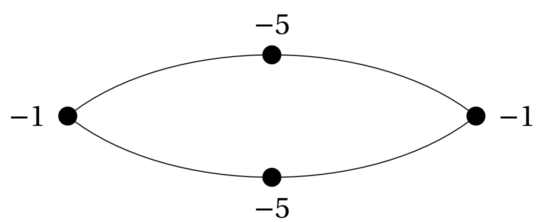



Figure 4.5: Plumbing graph $\Gamma$ corresponding to graph $\mathscr{W}(L_F, L')$ for $F(x,y,z) = (x^2+y^3)\overline{(x^3+y^2)} + z^2$.

As in the previous example, $\Gamma$ represents the exceptional divisor in the interior of the manifold $V_\Gamma$; *i.e.*, it is a graph resolution corresponding to a resolution $\widetilde{V}$ of a normal surface singularity $(V,0)$. Then, blowing down the two vertices with weights $-1$, one obtains the graph in Figure 4.6.

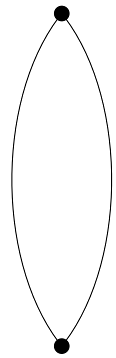



Figure 4.6: Plumbing graph $\Gamma$ corresponding to graph $\mathscr{W}(L_F, L')$ for $F(x,y,z) = (x^2+y^3)\overline{(x^3+y^2)}+z^2$.

This graph is also the plumbing graph for $L_G$, where $G\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ is the germ

$$G(x,y,z) = (x^2+y^3)(x^3+y^2)+z^2$$

as is stated in [49, § 6, Examples]. Then the link $L_F$ is realisable by an holomorphic function from $\mathbb{C}^3$ to $\mathbb{C}$.

As in Section2.4, now we proceed to see if the open-book fibrations given by $F$ and $G$ are equivalent. In order to see if the Milnor fibre $\mathscr{F}$ is diffeomorphic to the Milnor fibre $\mathscr{F}_G$, we compute the genus of $\mathscr{F}$ and the genus of $\mathscr{F}_G$.

By the decorated plumbing graph given in Figure 3.18 and following the construction in Sections 1.6 and 3.7, we obtain that the Milnor fibre $(\mathscr{F}_{f\bar{g}})_i$ is the $m_i$-covering of $V_i$ (see Proposition 3.24), then

$$\chi((\mathscr{F}_{f\bar{g}})_i) = m_i \chi(V_i)\ .$$

As $V_i$ is a cylinder or a disc for the vertices $v_i$ with valence 2 and 1 respectively, our principal interest are the rupture vertices; let $v_i$ a rupture vertex in the graph in Figure 3.18, then

$$\chi((\mathscr{F}_{f\bar{g}})_i) = 2\chi(V_i) \ = 2(-1) = -2.$$

Then, the genus of $(\mathscr{F}_{f\bar{g}})_i$ is 0. “Gluing” the pieces $(\mathscr{F}_{f\bar{g}})_i$ for all $i$, we obtain a surface of genus 1 with two boundary components.

Analogously, for the Milnor fibre $(\mathscr{F}_{fg})_i$ we have

$$\chi((\mathscr{F}_{fg})_i) = 10\chi(V_i) \ = 10(-1) = 10.$$

Then the genus of $(\mathscr{F}_{fg})_i$ is 2. "Gluing" the pieces $(\mathscr{F}_{fg})_i$ for all $i$, we obtain a surface of genus 5 with two boundary components.

Thus, the join of $r$ points with $\mathscr{F}_{f\bar{g}}$ cannot be the same as the join with $\mathscr{F}_{fg}$ and the open-book decompositions given by the Milnor fibrations of $F$ and $G$ are not equivalent.

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