UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO

POSGRADO EN CIENCIAS MATEMÁTICAS

FACULTAD DE CIENCIAS

SELECTIONS AND WEAK ORDERABILITY

T E S I S

QUE PARA OBTENER EL GRADO ACADÉMICO DE

DOCTOR EN CIENCIAS (MATEMATICAS)

P R E S E N T A

IVAN MARTÍNEZ RUIZ

DIRECTOR DE TESIS: DR. MICHAEL HRUSAK

MÉXICO, D.F. JULIO, 2010

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Dedicatoria

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Prólogo

En este prólogo hacemos una descripción general del trabajo, enunciando los resultados más importantes que aquí se presentan.

El estudio de la Teoría de selecciones continuas tiene su origen en un artículo trascendental escrito en 1951 por Ernest Michael [44], donde se estudian propiedades importantes de hiperespacios de espacios topológicos y se establecen los primeros resultados acerca de la existencia de selecciones (continuas) y sus propiedades. En lo que se refiere a ésto último, Michael analiza la siguiente pregunta general:

¿ Cuándo un espacio topológico admite una selección (débil) continua?

Michael mostró que una condición suficiente es que el espacio sea debilmente ordenable. Una pregunta natural es si la condición de ser débilmente ordenable es en realidad una caracterización de aquellos espacios que admiten una selección débil continua. Si bien este problema se presenta de manera implícita en el trabajo de Michael, es en el trabajo de Jan van Mill y Evert Wattel ([46]) donde se expone de forma explícita esta pregunta. Por esta razón, este problema es conocido desde entonces como el *Problema de van Mill y Wattel.*

A partir del artículo de Michael, diversos autores han obtenido soluciones parciales positivas al problema anterior. Sin embargo, el problema general se mantenía abierto:

¿Es todo espacio topolgico que admite una selección débil débilmente ordenable?

En el Capítulo 1 introducimos las nociones básicas en la Teoría de hiperespacios, selecciones y selecciones débiles. Enunciamos tres de los resultados más importantes obtenidos por Michael en la Teoría de Selecciones Continuas y finalmente exponemos los principales resultados obtenidos alrededor del problema de van Mill y Wattel, poniendo especial atención al trabajo realizado por van Mill y Wattel[46].

En el Capítulo 2 presentamos una solución general al problema de van Mill y Wattel. Para ello, estudiamos el comportamiento de selecciones débiles continuas en espacios separables muy especiales, llamados *Espacios de Mrówka-Isbell*. Finalmente, presentamos un ejemplo de un espacio separable, el cual será un espacio de este tipo, que admite una selección débil continua pero no es débilmente ordenable.

En el Capítulo 3 continuamos con el estudio de los espacios separables que admiten una selcciíon débil continua. A partir de las ideas que hay detrás del contraejemplo del Capítulo 1, obtenemos algunas propiedades especiales para estos espacios. Estas propiedades nos permiten resolver parcialmente el siguiente problema de extensión de selecciones débiles establecido por Valentin Gutev y Tsugunori Nogura [24]. En la búsqueda de una solución al problema de van Mill y Wattel, ellos se interesaron por determinar bajo qué condiciones es posible construir un espacio que admite una selección continua para la colección de sus subconjuntos finitos con a lo más n puntos, para algún $n \in \omega$, pero no para la colección de sus subconjuntos con a lo más $n+1$ puntos. Un ejemplo de este tipo sería un contraejemplo para el problema principal. Mostramos en este capítulo que en el caso separable ésto no es posible: Toda selección débil continua definida sobre un espacio separable X puede extenderse continuamente a una selección continua definida sobre el subconjunto de 2^X que consiste de todos los subconjuntos finitos de X.

En el Capítulo 4 realizamos un estudio de selecciones débiles pero desde otro punto de vista. Partiendo de una selección débil en un conjunto dado (no necesariamente dotado de una topología), definimos una topología en el conjunto base, naturalmente determinada por la selección débil. Analizamos sus propiedades básicas, como son axiomas de separación y continuidad de la selección débil que genera la topología del espacio. Definimos y estudiamos clases especiales de espacios topológicos que están muy relacionadas con la existencia de selecciones débiles continuas y sus correspondientes topologías inducidas, interesándonos de nueva cuenta en determinar propiedades de orden satisfechas por ellos.

La mayoría del material presentado en los Capítulos 2 y 3 ha sido publicado en [35]. Los resultados presentados en el Capítulo 4 se presentan en [36].

Introduction

The concept of orderability has been present in Mathematics since its origins, appearing simultaneously with the idea of number. The development of many classical theories was devoted to construct and study the properties of the real line (the most well known ordered set) and its subsets.

When the idea of proximity comes and brings the development of Topology, the study of spaces whose topology is determined by a linear order appears as a natural problem. G. Cantor, [4] who had before contributed to the study of ordered sets with the introduction of ordinal and cardinal numbers and order type for sets as well, stated a result that implicitely involves a topological condition. He proved that any countable ordered set which is densely ordered with no end points is isomorphic, seen as an ordered set, to the rationals with its usual order.

O.Veblen [60] is considered the first who obtained an orderability theorem by proving that every metric continuum with exactly two non-cut points is homeomorphic to the closed interval. Thereafter, the problem of recognizing which topological spaces satisfy an orderability condition has been studied by many mathematicians in various forms. It did not take long to make them aware of the many important properties satisfied by this kind of spaces.

Later, when it was determined that the property of being ordered is not a hereditary condition, the separated study of topological spaces which can be seen as subspaces of ordered spaces also appeared. E. Čech [5] pioneered on the study of these spaces, calling them *generalized ordered spaces* or simply *GO-spaces*. In his PhD dissertation, S. Purish [52] introduced the term *suborderable*, as a translation of "unterordnungsfähig" founded in turn on H. Herrlich's work [33], to refer to such spaces. The **Sorgenfrey line** is perhaps the best known example of a suborderable not ordered space; however, we can find many other interesting examples of these spaces such as the **Michael line** or the set $\{0\} \cup (0, 1]$ with euclidean topology.

Finally, S. Eilenberg [11] further weakens the orderability condition and defines the *weakly orderable spaces*, as those whose base set admits a linear order such that its induced topology is coarser than the original of the space. These are also called *Eilenberg orderable spaces* or *KOTS*.

In all these cases, it was relevant to seek a general result characterizing spaces that meet the corresponding orderability condition. In 1973, J. van Dalen and E. Wattel [8] solved the general problem for the two strongest properties. They first proved that a space is suborderable if and only if X has a subbase consisting of two nests, (i.e. collections of sets that are linearly ordered by inclusion). Going further, they also characterized ordered spaces by adding an extra condition to the corresponding two nests.

The case of weakly orderable spaces is approached separately. As far as we know, unlike orderability and suborderability, there is no general characterization of weakly orderable spaces available so far. However, with the beginning of a new theory, which has turned out to be very important in diverse areas of mathematics, this problem acquired a particular interest.

In 1951, Ernest Michael [44] wrote a seminal paper on the Theory of Hyperspaces. At first, he studied from various viewpoints the hyperspace of a topological space X which consists of all nonempty closed subsets of X equipped with a particular topology. Leopolod Vietoris [62] was the first who introduced a general definition of hyperspace topology. The first deep study of these spaces is due to Michael. The main importance of Michael's paper is the definition and development of a new mathematical theory involving special functions between spaces and hyperspaces, named *Theory of continuous selections.* In the course of our work we will refer in a more detail to the general idea. At the moment we only mention a particular case which will be the one that actually we are more interested in.

One of the most transcendental, and also controversial, principles of modern mathematics is the Axiom of Choice and its various equivalences. The principal form in which it is presented justifies its name: *For every non empty set X, it is possible to define a function(often called choice or selective function) f :*

$\mathcal{P}(X) \to X$ such that $f(A) \in A$ for every $A \in \mathcal{P}(X)$.

Our intention is not to argue on the acceptance of this independent axiom. We will suppose it true and we will also suppose that X is not only a set but a Hausdorff topological space. Moreover, after we have defined a topology for the collection of closed subsets of X, it is valid to question us not only about the existence of selective functions for the hyperspace of X but about their continuity. In fact, we will still consider a more particular case; we can study (continuous) functions defined on the set of pairs of X such that for every pair of points the function chooses one of them. These special functions were named *weak selections* because of their close relation with the weak orderability condition.

Eilenberg characterized connected weakly orderable spaces. On the other hand, Michael noticed that every weakly orderable space admits a continuous weak selection. In order to characterize weakly orderable spaces in terms of the existence of a continuous weak selection, he presented the first result in this direction by proving that every connected space admitting a continuous weak selection is weakly orderable. Since then, in search of a general solution to this fundamental problem, many authors contributed to develop the theory of continuous weak selections, showing that weak orderability is characterized by existence of a continuous weak selection for a rather large collection of spaces. Nevertheless, the general problem remains open:

Question 0.1. *Let X be a space. Is X weakly orderable if and only if it admits a continuous weak selection?*

The aim of the present thesis is to study in detail the above problem and to discuss some others which later appeared. In Chapter 1 we provide basic notions of the theory of hyperspaces, continuous selections and continuous weak selections. We will explicitly enunciate three of the most important Michael's results on the Theory of Continuous Selections and we will finally expose the most important results that have been achieved so far around Question 0.1, paying special attention to the work done by van Mill and Wattel [46] for the compact case.

Chapter 2 provides a negative solution to Question 0.1. To be able to come to it, we will study continuous weak selections in some particular separable spaces, named *Mrówka-Isbell spaces*. Finally, we will present an example of a separable space, namely a Mrówka-Isbell space that admits a continuous weak selection but is not weakly orderable.

In Chapter 3 we continue with the study of separable spaces admitting a continuous weak selection. Starting from the ideas involved in the construction of the counterexample of Chapter 2, we obtain special properties satisfied by these spaces. These properties allow us to approach an extension problem for continuous weak selections stated by Valentin Gutev and Tsugunori Nogura. [24] In search of a solution to the characterization problem, they were interested in knowing under what conditions it is possible to construct an example of a space which admits a continuous selection for the collection of its subsets of at most n points, for some $n \in \omega$, but not for the collection of sets with at most $n+1$ points. Such an example would also be a counterexample for the principal problem. We show that in the separable case it is not possible: for every continuous weak selection defined on a separable space X can be extended to a continuous selection for the subspace of 2^X consisting of all finite subsets of X.

In Chapter 4 we realize a study of weak selections from another point of view. Starting from a weak selection on a given set (not necessarily endowed with a topological structure), we define a topology on the base set, which is naturally determined by the weak selection. We analyze their basic properties, such as separation axioms and continuity of the weak selection generating the topology of the space. We define and study some special classes of topological spaces that are closely related to the existence of continuous weak selections and their corresponding induced topologies, being interested again in determining orderability properties that they satisfy.

Most of the material presented in Chapter 2 and Chapter 3 has been published in [35]. The results presented in Chapter 4 are reproduced from [36].

Chapter 1

2^X and continuous selections

During his study on metric spaces, Hausdorff realized that when (X, d) is a compact metric space, it is possible to state an appropriate way to measure distances between any two closed subsets of X by setting the metric d_H on 2^X, named the *Hausdorff metric*, defined as:

$$d_H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(A, b) : b \in B\}\}.$$

This way, we can now consider closed subsets of X as points of a metric space. Later, this study was generalized by Leopold Vietoris for all topological spaces. He introduced a topology for the collection of all non empty closed subsets, known as the *Vietoris topology*, providing also many properties around it.

Given a topological space X, we define the *hyperspace* 2^X, to be the set of all non empty closed subsets of X equipped with the topology $\tau_{\mathcal{V}}$ generated by sets of the form:

$$\langle U; V_0, \dots, V_n \rangle = \{F \in 2^X : F \subseteq U \text{ and } F \cap V_i \neq \emptyset \text{ for any } i \leq n\},$$

where $U, V_0, \dots, V_n$ are open subsets of X.

It is well known that the collection of sets of the form $\langle V; V_0, \dots, V_n \rangle$, where $V = \bigcup\{V_i : i \leq n\}$, is also a base for $\tau_{\mathcal{V}}$. We will denote this kind of basic sets by $\langle V_0, \dots, V_n \rangle$.

Vietoris introduced this topology in [62], seeking to obtain a suitable notion of manifold on the power set of a space or, more precisely, on the collection of closed subsets, by giving it a spacial structure. It is worth to mention that, in addition to the Vietoris topology, there are many other topologies on 2^X which have been studied, for example the Fell topology, hit-and-miss topologies, etc. We will only be interested in the study of 2^X eqquiped with $\tau_{\mathcal{V}}$, although at the end of this chapter we will pay some attention to Fell topology.

A basic open neighborhood for an element $F \in 2^X$ can be visualized as follows:

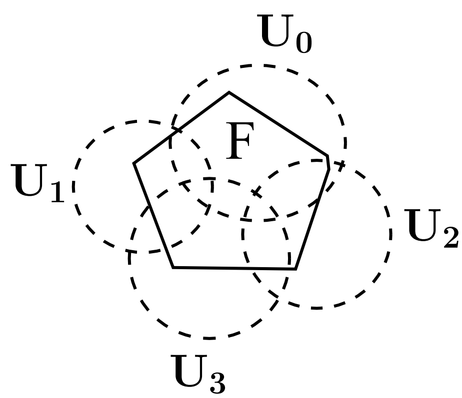

Figure 1.1: $\langle U_0, U_1, U_2, U_3 \rangle$

Many important topological properties are inherited to closed subsets and some of them are carried over to the collection of closed subsets. In the case when a space is metric and compact, the Vietoris topology of its hyperspace coincides with the topology induced by the Hausdorff metric. It follows that if X is compact, then it is metrizable if and only if so is 2^X. Vietoris studied the relation between a space and its corresponding hyperspace with respect to basic topological notions. He proved for instance that 2^X is compact (connected) if and only if X is so. In [44], Ernest Michael presents an extensive study of the Vietoris topology, called finite topology thereby. Among other things, he proves the following results that relate a space and its hyperspace with respect to separation axioms, except Property (6) which was obtained later by Veličko [61] by using results of Keesling ([38] and [39]).

Proposition 1.1 ([44]). *Let X be a topological space. Then*

(1) 2^X is a T_0 space.

(2) If X is T_1 then so is 2^X.

(3) 2^X is Hausdorff if and only if X is regular.

(4) 2^X is regular if and only if it is Tychonoff.

(5) 2^X is regular if and only if X is normal.

(6) 2^X is normal if and only if X is compact. □

He also proved that a space X is locally compact if and only if 2^X is locally compact.

1.1 Some special subsets of 2^X

There are some special subsets of the hyperspace of a given space that are of particular interest in this work.

Definition 1.2. Let X be a space and $n \in \omega \setminus \{0\}$. Define the following sets:

(a) $[X]^n = \{F \in 2^X : |F| = n\}$.

(b) $\mathcal{F}_n(X) = \{F \in 2^X : |F| \leq n\}$.

(c) $\mathfrak{Fin}(X) = \bigcup\{\mathcal{F}_k(X) : k \geq 1\}$.

(d) $\mathcal{K}(X) = \{F \in 2^X : F \text{ is compact}\}$.

It is easy to see that $\mathfrak{Fin}(X)$ is a dense subspace of 2^X. Hence X is separable if and only if 2^X is separable. It also holds that $\mathcal{F}_n(X)$ is a closed subset of 2^X for every $n \in \omega$. It follows that X embeds, via identification of x with $\{x\}$, into 2^X as a closed subspace, i.e. X is *admissible*.

Michael also realized the importance of the subspace $\mathcal{K}(X)$ of 2^X consisting of compact subsets of X.

Proposition 1.3 ([44])**.** *Let X be a space. Then*

(1) X is second countable if and only if $\mathcal{K}(X)$ is second countable.

(2) X is first countable if and only if $\mathcal{K}(X)$ is first countable.

(3) If $\mathfrak{Fin}(X) \subseteq \mathcal{C} \subseteq \mathcal{K}(X)$, then X is locally connected if and only if $\mathcal{C}$ is locally connected.

(4) X is zero-dimensional if and only if $\mathcal{K}(X)$ is zero-dimensional.

(5) X is totally disconnected if and only if $\mathcal{K}(X)$ is totally disconnected.

(6) X is discrete if and only if $\mathcal{K}(X)$ is discrete. □

1.2 Continuous selections

In the study of continuous functions on topological spaces we can find problems related with an extension property: Given X and Y and a continuous function from an special subset of X to Y, when is it possible to extend the given function to a continuous function on the whole space X? A special subset can for instance be a non empty closed subset, as in Extension Theorem for normal spaces, or a dense subset of a compact space, as in the characterization of the Stone-Čech compactification of a space. In some cases, in the search of such an extension we need to add some extra hypotheses for the spaces X and Y; one of these cases involves conditions for each $x \in X$ to be in a certain subset of Y. This last problem motivated Ernest Michael to develop the so called *Selection Theory.*

As mentioned in [55], we can find a more general idea behind Michael's theory, which shows the naturality of the concept of selection and why it goes beyond its application in topology. Many statements in mathematics can be phrased as follows:

$$\forall x \in X \; \exists y \in Y \, P(x, y)$$

To every element $x \in X$ we can associate its corresponding subset $A_x = \{y \in Y : P(x, y)\}$, defining thus a "multivalued function", which can be interpreted as a mapping that associates to every initial data $x \in X$ of a given problem P a nonempty set of solutions of this problem. Following the same idea, we can now question about how and when it is possible to choose a unique solution of the problem under certain initial conditions.

Suppose now that X and Y are topological spaces and let $\phi : X \to Y$ be a multivalued function, i.e. $f(x) \subseteq Y$ for every $x \in X$. Suppose that $\phi(x)$ is closed for every x. After we have fixed a topology on the collection of closed sets of Y, the multivalued function ϕ is actually a single-valued function between the spaces X and 2^Y and thus it makes sense to study topological properties of ϕ, like continuity or extension properties with respect to special subsets. Multivalued functions received a lot of attention in topology. Given a continuous function $f : X \to Y$, the mapping $\phi : Y \to 2^X$ assigning to each $y \in Y$ its preimage with respect to f, i.e. $\phi(y) = f^{-1}\{y\}$, provides a natural example of a multivalued function.

Definition 1.4 ([44]). Let X and Y be spaces and let $\phi : X \to 2^Y$. A *selection* for ϕ is a function $f : X \to Y$ such that

$$f(x) \in \phi(x) \text{ for every } x \in X.$$

As an easy example, if $f : X \to Y$ is a continuous surjective function, a function $g : Y \to X$ is a selection for the function $\phi : Y \to 2^X$ defined above if $g(y) \in f^{-1}(\{y\}) = \phi(y)$ for every $x \in X$.

The definition of selection is due to Michael [44], establishing with this the initial elements on Selection Theory. In the same paper, he stated a fundamental and natural question

Question 1.5 ([44]). *Suppose that $\phi : X \to 2^Y$ is a function. When is it possible to find a continuous selection for ϕ?*

The general characterization of mappings that admit a continuous selection is still unknown. However, many positive cases, and negative as well, can be obtained by adding some extra hypotheses to the involved spaces.

1.3 Michael's Selection Theorems

Michael provides the first fundamental results in what refers to continuous selections on (multivalued) functions. The involved properties in Michael's results show the usefulness of selection theory for applications in several areas. We include here three of his most important results.

Definition 1.6. Let X and Y be topological spaces and let $\phi : X \to Y$ be a multivalued function. We will say that ϕ is

(1) *lower semi-continuous* (abbreviated **l.s.c.**) provided that for every open subset $V \subseteq Y$, the set

$$\{x \in X : \phi(x) \cap V \neq \emptyset\}$$

is open

(2) *upper semicontinuous* (**u.s.c.**) provided that for every open set $V \subseteq X$, the set

$$\{x \in X : \phi(x) \subseteq V\}$$

is open. □

Remark 1.7. *A multivalued function $\phi : X \to Y$ is continuous, with respect to the Vietoris topology, if and only it is **l.s.c.** and **u.s.c.***

Theorem 1.8 (**Convex-valued selection theorem**, [45])**.** *Let X be a paracompact space, B a Banach space and $\phi : X \to Y$ a **l.s.c.** multivalued function with nonempty closed convex values. Then ϕ admits a continuous selection.* □

Theorem 1.9 (**Zero-dimensional selection theorem**, [45])**.** *Let X be a zero-dimensional paracompact space, M a completely metrizable space and $\phi : X \to M$ a **l.s.c.** multivalued function with nonempty closed values. Then ϕ admits a continuous selection.* □

The following theorem corresponds to an extension property.

Theorem 1.10 (**Normed linear selection theorem**, [45])**.** *Let $(L, ||\cdot||)$ be a normed linear space, let X be a space and let $\phi : X \to L$ be a **l.s.c.** multivalued function such that $\phi(x)$ is convex in L and complete with respect to $||\cdot||$. Then for every closed subset A of X and every continuous selection $f : A \to L$ for the function $\phi \restriction A : A \to L$ there exists a continuous selection $g : X \to L$ for ϕ which extends f.* □

We conclude this section by mentioning that from several results on selection theory, included the three theorems presented here, we can obtain a wide collection of results and applications, including well-known results such as the Tietze Extension Theorem, the Uryshon Lemma or the Dugundji Extension Theorem in normed linear spaces. A broad analysis can be found in [48] and [54].

1.4 A particular case

As Michael stated, we can also find a sufficient condition to answer Question 1.5 by reducing the Selection Problem to two problems that are easier to understand and even to solve. The first problem is related to the continuity of the function for which we want to find a selection and the second one is a particular case of the selection problem, but where $X = Y$.

Proposition 1.11 ([44]). *A function $\phi : X \to 2^Y$ admits a continuous selection if:*

(1) ϕ is continuous and

(2) The identity map $i : 2^Y \to 2^Y$ admits a continuous selection. □

Along this work we are interested in the study of these particular selections which, certainly, transmit in a better way the idea of selectivity of this kind of functions.

Definition 1.12. Let $\mathcal{A} \subseteq 2^X$. A function $\psi : \mathcal{A} \to X$ is a *selection on* $\mathcal{A}$ if $\psi(F) \in F$ for every $F \in \mathcal{A}$.

We will say that ψ is *continuous* if it is continuous with respect to the subspace topology on $\mathcal{A}$ and we will denote by $Sel(\mathcal{A})$ the collection of all continuous selections on $\mathcal{A}$.

We can now consider the corresponding question:

Question 1.13. *When does 2^X (or a subspace of it) admit a continuous selection?*

As a first example we present a proof that the real line is a (ordered) space whose hyperspace does not admit a continuous selection.

Proposition 1.14 (Engelking, Heath, Michael, [12]). *There exists no continuous selection on $2^{\mathbb{R}}$.*

Proof. Aiming towards contradiction suppose φ is a continuous selection on $2^{\mathbb{R}}$ and let $F = \{0, 1\}$. Suppose without loss of generality that $\varphi(F) = 1$. Let $f : [0, 1] \to [1, 2]$ be the continuous function $f(t) = 1+t$. Note that, by the continuity of φ and f, $\varphi(\{0, 1+t\}) = 1+t$ for every $t \leq 1$ and thus $\varphi(\{0, 2\} = 2$. Indeed, suppose the contrary and let $l = \sup\{r \in [1, 2] : \varphi(\{0, s\}) = s$ for any $s \in [1, r]\}$. By the continuity of φ we have that $1 < l < 2$ and hence $U \cap [1, l) \neq \emptyset \neq U \cap (1, 2]$ for every open interval U containing l. It follows that there are $u, v \in U$ such that $\varphi(\{0, u\}) = u$ and $\varphi(\{0, v\}) = 0$ and thus φ is not continuous, which is a contradiction.

In the same way, let $g : [0, 1] \to 2^X$ be the continuous function defined by $g(t) = \{0, t, 2\}$. By the continuity of g we obtain that $\varphi(\{0, 1, 2\}) = 2$. Inductively, we can prove that $\varphi(\{0, 1, \ldots, n\}) = n$ for every $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be such that $\varphi(\mathbb{N}) = n_0$. Then there is $\mathcal{U}$, a basic neighborhood of $\mathbb{N}$, with $\varphi(F) \in (n_0 - 1, n_0 + 1)$ for every $F \in \mathcal{U}$. But we can find $n > n_0$ with $\{0, 1, \ldots, n\} \in \mathcal{U}$, which is a contradiction. □

In a similar way, Engelking, Heath and Michael also proved the following result.

Proposition 1.15 ([12]). *There exists no continuous selection on $2^{\mathbb{Q}}$.* □

Later, van Mill, Pelant and Pol obtained a more general result, which specifies a necessary condition for metrizable spaces whose hyperspace admit a continuous selection.

Proposition 1.16 ([49]). *If X is a metric space and $Sel(2^X) \neq \emptyset$ then X is completely metrizable.* □

In case X is strongly zero-dimensional the converse holds as well.

Proposition 1.17 ([12]). *If X is a strongly zero-dimensional complete metric space then $Sel(2^X) \neq \emptyset$.* □

In particular, the hyperspace of $\mathbb{P}$, the set of irrational numbers, admits a continuous selection.

1.5 Continuous weak selections

In search of a general answer to Question 1.13, E. Michael studied properties of continuous selections defined on the collection of pair sets in a given topological space and in particular he asks when the existence of such a continuous selection entails a continuous selection on the whole hyperspace.

Given a space X, a selection $\varphi : \mathcal{F}_2(X) \to X$ is named a *weak selection on* X. If $Sel(\mathcal{F}_2(X)) \neq \emptyset$ we will say that X *admits a continuous weak selection.*

It is not difficult to see that X admits a continuous weak selection if and only if there is a continuous function $\varphi : X^2 \to X$ such that $\varphi(x, y) = \varphi(y, x) \in \{x, y\}$. We will also refer to such a φ as a weak selection and, moreover, since any weak selection on X is continuous for all singletons, we will be interested in selections for $[X]^2$, which will also be named weak selections.

Given a weak selection φ on X and $x, y \in X$, we will denote by $x \to_\varphi y$ (or equivalently $y \leftarrow_\varphi x$) the condition $\varphi(x, y) = y$. In a more general way, if $A, B \subseteq X$ we will write $A \rightrightarrows_\varphi B$ whenever $a \to_\varphi b$ for every $a \in A$ and $b \in B$ and we will say that B *dominates* A *with respect to* φ (or just that B dominates A if φ is clear from context). We will also say that A and B are *aligned* and will write $A||B$ if either $A \rightrightarrows_\varphi B$ or $B \rightrightarrows_\varphi A$. In general, we will suppress the use of subscripts when there is no danger of confusion.

An immediate characterization of continuous weak selections is the following:

Proposition 1.18 (Folklore)**.** *A selection $\varphi : [X]^2 \to X$ is continuous if and only if for every $x, y \in X$ such that $x \to y$, there are open neighborhoods U and V of x and y, respectively, such that $U \rightrightarrows V$.*

Proof. It follows by the fact that

$$\mathcal{B} = \{\langle U, V\rangle \cap [X]^2 : U, V \text{ are open and } U \cap V = \emptyset\}$$

is a base for $[X]^2$. $\square$

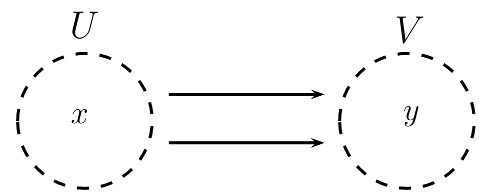

Figure 1.2: Continuity of ψ in $\{x, y\}$.

1.6 Weak selections and orderability properties

Recall that a relation $\leq$ on a set X is a *linear order* if it satisfies the following conditions:

(1) *Reflexivity*: $x \leq x$ for every $x \in X$.

(2) *Transitivity*: If $x \leq y$ and $y \leq z$, then $x \leq z$.

(3) *Antisymmetry*: If $x \leq y$ and $y \leq x$ then $x = y$.

(4) *Linearity*: For every $x, y \in X$ either $x \leq y$ or $y \leq x$.

Recall also that a space is *ordered* if it is possible to find a linear order on it in such a way that the topology determined by this order coincides with that of the space.

Given a linear order $\leq$ on X and $x \in X$, we denote by $(\leftarrow, x)_\leq$ and $(x, \rightarrow)_\leq$ the *initial open segment* and *final open segment* determined by x respectively, i.e. $(\leftarrow, x)_\leq = \{y \in X : y < x\}$ and $(x, \rightarrow)_\leq = \{y \in X : x < y\}$, similarly we denote $(\leftarrow, x]_\leq = X \setminus (x, \rightarrow)_\varphi$ and $[x, \rightarrow)_\leq = X \setminus (\leftarrow, x)_\varphi$.

Suppose that X is an ordered space whose topology is determined by a linear order $\leq$ on X. If x and y are elements of X such that $x < y$, we can find disjoint open intervals I_x and I_y containing x and y, respectively, and such that $x' < y'$ for any $x' \in I_x$ and $y' \in I_y$. Therefore, if we define the weak selection φ on X by $\varphi(\{a, b\}) = \min\{a, b\}$ (or $\varphi(\{a, b\} = \max\{a, b\})$ then clearly φ satisfies the property of Proposition 1.18 and so it is continuous. Therefore, any ordered space admits a continuous weak selection. Indeed, the only required property by the linear order on X to guarantee continuity of φ is that any interval which is open with respect to the order is so in the topological sense. This fact provides a motivation to work with some properties that are weaker than orderability but still preserve the elementary order notions in the topological structure of a space.

Definition 1.19. Let X be a topological space. We will say that X is:

1. *Suborderable* if it is a subspace of an ordered space.

2. *Weakly orderable* if it admits a weaker topology generated by a linear order.

Proposition 1.20 ([44])**.** *Every weakly orderable space admits a continuous weak selection.* □

As a consequence of this proposition, a natural and fundamental problem arises. If a space X admits a continuous weak selection ψ, we can identify the selection ψ as a function that, for any given elements $x, y \in X$, chooses the "smallest" element of both points with respect to the relation $\leq_\psi$, which is defined in the natural way: $x \leq_\psi y$ whenever $x \leftarrow y$. This relation is reflexive, antisymmetric and linear but, unfortunately, it is not transitive since it is possible to find distinct points $x, y, z \in X$ such that $x \to y$ and $y \to z$ but $x \leftarrow z$. Although this last relation does not determine a linear order on X we can still try to find an addecuate linear order induced by properties of the continuous weak selection ψ and to ask, in a more general way, if the existence of a continuous weak selection actually characterizes weakly orderable spaces.

Following with the idea that a weak selection ψ defined on a space X establishes a way to compare any pair of points on X, for every $x \in X$ we can define the sets $L_x = \{y \in X : y \leftarrow x\}$ and $U_x = \{y \in X : x \leftarrow y\}$. Therefore, L_x is the set of points below x and U_x is the set of points above x with respect to $\leq_\psi$. When ψ is continuous, the characterization of continuity in weak selections yields to obtain the following conditions for every $x \in X$:

(i) L_x and U_x are closed,

(ii) $L_x \cup U_x = X$ and

(iii) $L_x \cap U_x = \{x\}$.

Since $L_x \setminus \{x\}$ and $U_x \setminus \{x\}$ are disjoint open sets for any $x \in X$, the set $X \setminus \{x\}$ is not connected when $L_x \setminus \{x\} \neq \emptyset \neq U_x \setminus \{x\}$. This property yields the following result:

Proposition 1.21 ([44]). *If $X = \mathbb{R}^n$ for some $n > 1$ or $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, then $Sel(\mathcal{F}_2(X)) = \emptyset$.* □

1.7 Some history of positive results

As we mentioned above, a fundamental problem in the theory of continuous weak selections is:

Question 1.22. *Is every space admitting a continuous weak selection weakly orderable?*

This problem was implicitly studied by Michael when he introduced the initial notions on Selection Theory. Certainly, the main problem is to find conditions for a space to admit a continuous selection. But trying to solve it, Michael proved that in some particular cases we only need to verify that the space admits a continuous weak selection and, also in some cases, the existence of such selection is closely related with an orderability property of the space. By this reason, Question 1.22 acquired a great importance and, since Michael's paper, there have been many results around the search for an answer to this question.

In this section we present results that, under our consideration, are the most important in this direction and give a partial affirmative answer to Question 1.22 on some topological spaces. They are not presented in chronological order but with respect to the involved properties.

1.7.1 Connected spaces

In 1941, Samuel Eilenberg [11] began the study of weakly orderable spaces, named *ordered topological spaces* there. In particular, he was interested in the connected case. Among other things, he proved that a connected space X is weakly orderable if and only if $P(X) = (X \times X) \setminus \{(x, x) : x \in X\}$ is not connected. He also proved that in this case, $P(X)$ consists of two components, where each one is reflected on the other one by the diagonal of $X \times X$. In fact, these components determine the only two possible linear orders on X whose induced topology is weaker than the (original) topology on X and one of them is an inverse to the other.

Michael first proved by using Eilenberg results that the relation determined by a continuous weak selection on a connected space X is in fact a linear order, and hence the space X is weakly orderable. This also implies that the space X admits only two continuous weak selections.

We will present a different proof of this fact by using properties of particular triplets of points.

Definition 1.23. Let ψ be a weak selection on a space X. A triple $\{x, y, z\} \subseteq X$ is a *3-cycle* with respect to ψ if either $x \to y \to z \to x$ or $x \leftarrow y \leftarrow z \leftarrow x$.

Notice that the abscence of such 3-cycles determines transitivity of the induced relation $\leq_\psi$ and thus weak orderability of the given space. The next result shows that the existence of 3-cycles determines a partition of the space into clopen subsets.

Proposition 1.24. *Let φ be a continuous weak selection on a space X and let $x, y, z \in X$ be such that $\{x, y, z\}$ is a 3-cycle with respect to φ. Then there is a partition $\mathcal{P}$ of X into clopen subsets such that $|\mathcal{P}| \leq 5$ and $|P \cap \{x, y, z\}| \leq 1$ for every $P \in \mathcal{P}$.*

Proof. Suppose without loss of generality that $x \to y \to z \to x$ and consider the following sets:

$$P_0 = (L_z \setminus \{z\}) \cap (U_y \setminus \{y\}),$$

$$P_1 = (L_x \setminus \{x\}) \cap (U_z \setminus \{z\}),$$

$$P_2 = (L_y \setminus \{y\}) \cap (U_x \setminus \{x\}),$$

$$P_3 = (L_x \setminus \{x\}) \cap (L_y \setminus \{y\}) \cap (L_z \setminus \{z\}),$$

$$P_4 = (U_x \setminus \{x\}) \cap (U_y \setminus \{y\}) \cap (U_z \setminus \{z\}).$$

Clearly, $\mathcal{P}$ is a partition of X and, by the continuity of φ, P_i is open (and so clopen) for every $i < 5$. Notice also that $x \in P_0$, $y \in P_1$ and $z \in P_2$. $\square$

Corollary 1.25. *Every connected space admitting a continuous weak selection is weakly orderable.* $\square$

The space $X = \{(0,0)\} \cup \{(x, \sin(\frac{1}{x})) : 0 < x \leq 1\}$, seen as a subspace of $\mathbb{R}^2$, is a connected space that admits a continuous weak selection but is not ordered. On the other hand, Example 1.14 shows that, in general, the existence of a continuous weak selection on a connected space X does not guarantee the existence of a continous selection for the hyperspace 2^X. However, adding the extra condition that every $F \in 2^X$ has a first element with respect to the order determined by a continuous weak selection ψ, Michael proves that the weak selection ψ can be extended to a selection for the whole hyperspace 2^X. The above condition clearly holds when X is compact. Moreover, since for any Hausdorff compact space any nontrivial weaker topology on it coincides with the original one, Michael presents a nice characterization of connected compact spaces that admit a continuous weak selection.

Theorem 1.26 ([44])**.** *Let X be a connected compact space. The following are equivalent:*

(a) X admits a continuous weak selection.

(b) The hyperspace 2^X admits a continuous selection.

(c) X is ordered. $\square$

Finally, Kuratowski, Nadler and Young [41] proved that when the space is also metrizable, it must be an arc, i.e. a space homeomorphic to the closed interval.

1.7.2 Compact spaces

Young claimed in [63], without an explicit proof, that in Theorem 1.26 we can replace connectedness of the space for zero-dimensionality. In 1981, Jan van Mill and Evert Wattel proved that connectedness hypothesis is not necessary in

Michael's result. The technique used by van Mill and Wattel is very original and will be so helpful for future results. Since we will employ it, we decided to include their proof.

Theorem 1.27 ([46]). *Let X be a compact space. The following are equivalent:*

(a) X admits a continuous weak selection.

(b) 2^X admits a continuous selection.

(c) X is ordered.

Proof. We only need to prove $(a) \Rightarrow (c)$ because $(c) \Rightarrow (b) \Rightarrow (a)$ are already proved in [44]. Let ψ be a continuous weak selection on X and for every $x \in X$ let L_x and U_x be defined as above. Let $\prec$ be a well ordering of X. For every $x \in X$, we will recursively construct closed sets $A_x, B_x \subseteq X$ satisfying the following conditions:

(1) $A_x \cup B_x = X$ and $A_x \cap B_x = \{x\}$,

(2) if $y \prec x$ and $x \in A_y$, then $A_x \subseteq A_y \setminus \{y\}$,

(3) if $y \prec x$ and $x \in B_y$, then $B_x \subseteq B_y \setminus \{y\}$,

(4) if $z \in A_x$ and $z \notin \bigcup\{A_y : y \prec x \text{ and } x \in B_y\}$, then $z \in L_x$,

(5) if $z \in B_x$ and $z \notin \bigcup\{B_y : y \prec x \text{ and } x \in A_y\}$, then $z \in U_x$.

For a given $x \in X$, the set A_x will be the initial segment determined by x, including the point x, with respect to the constructed linear order, and B_x will be the final segment. The main idea involved in establishing a relation between a point $x \in X$ and any other element $z \in X$ is to first determine if they are already ordered by a $\prec$-smaller element. In that case, we do not need to do anything else. Otherwise, we just have to let the weak selection decide it.

Let x_0 be the $\prec$-first element of X and define $A_{x_0} = L_{x_0}$, $B_{x_0} = U_{x_0}$. Suppose now that we have defined A_y and B_y for all $y \prec x$ such that they satisfy conditions (1) – (5) and define the sets $E = \{y \prec x : x \notin A_y\}$ and $F = \{y \prec x : x \notin B_y\}$. Let

$$Z = X \setminus (\bigcup\{A_y : y \in E\} \cup \bigcup\{B_y : y \in F\}).$$

The set Z consists of those points for which we cannot decide, from smaller points, how to relate them to the point x. In order to define the sets A_x and B_x, we will require the following special properties. Not all proofs are provided. They are all alike. We only show those that shed some light on the general idea of the proof.

Let $\kappa = |E|$ and for each $\eta < \kappa$ define a point y_η as follows:

(6) $y_0 = \min E$,

(7) $y_\eta = \min\{\{x\} \cup \{y \in E : y_\mu \prec y \text{ for } \mu < \eta \text{ and } y \notin \bigcup\{A_{y_\mu} : \mu < \eta\}\}$.

Let $\xi \leq \kappa$ be the first ordinal for which $y_\xi = x$. We consider only the case of ξ limit, for the case of ξ isolated is similar.

Claim 1: If $\xi_0 \leq \xi$, then $\bigcup\{A_y : y \in E \text{ and } y \prec y_{\xi_0}\} = \bigcup\{A_{y_\mu} : \mu < \xi_0\}$

Claim 2: If $\mu_0 < \mu_1 < \xi$ then $A_{y_{\mu_0}} \subseteq A_{y_{\mu_1}} \setminus \{y_{\mu_1}\}$.

Claim 3: If $\mu_0 < \mu_1 < \xi$ then $A_{y_{\mu_1}} \setminus A_{y_{\mu_0}} \subseteq U_{y_{\mu_0}}$.

Claims (1) – (3) will guarantee that we can approach to the set A_x in an orderly manner by considering a $\prec$-chain of elements from E with their respective initial segments, that are defined above.

Claim 4: If $t \in cl_X(\bigcup\{A_y : y \in E\}) \setminus \bigcup\{A_y : y \in E\}$ then t is a cluster point of the net $\{y_\mu : \mu < \xi\}$.

Suppose the contrary and let C be a closed neighborhood of t such that $C \cap cl_X(\{y_\mu : \mu < \xi\}) = \emptyset$.

From Claim 1 we know that $\bigcup\{A_y : y \in E\} = \bigcup\{A_{y_\mu} : \mu < \xi\}$ and since t is an accumulation point of $\bigcup\{A_y : y \in E\}$, we can find a cofinal set $G \subseteq \xi$ such that for each $\mu \in G$ there exists a point $c_\mu \in C \cap L_{y_\mu}$ such that

$$\mu = \min\{\delta < \xi : c_\mu \in A_{y_\delta}\}.$$

Let $\mu \in G$. We claim that $c_\mu \in L_{y_\mu}$, i.e. $c_\mu \leftarrow y_\mu$. If not, since $c_\mu \in A_{y_\mu}$, by the contrapositive to condition (4) there is a $y \prec y_\mu$ such that $c_\mu \in A_y$ and $y_\mu \in B_y$.

Since $y \prec y_\mu$ and $y_\mu \in B_y$, condition (3) implies that $B_{y_\mu} \subseteq B_y$ or, equivalently, $A_y \subseteq A_{y_\mu} \setminus \{y_\mu\}$. But $y_\mu \in E$ and hence by definition $x \notin A_{y_\mu}$, which implies that $x \notin A_y$. By Claim 1, we can find a $\delta < \mu$ such that $c_\mu \in A_{y_\delta}$. This is because $c_\mu \in A_y$, $y \prec y_\mu$ and $y \in E$. However, this contradicts the condition that μ is the first element with this property. Thus $c_\mu \leftarrow y_\mu$ for every $\mu \in G$.

Let (c, y) be a cluster point of the net $\{(c_\mu, y_\mu) : \mu \in G\}$ (here we use the compactness condition). Then $c \in C$, $y \notin C$ and continuity of ψ yields that $c \leftarrow y$. On the other hand, given $\mu \in G$, we know by Claim (3) that $A_{y_\delta} \setminus A_{y_\mu} \subseteq U_{y_\mu}$ for every $\delta > \mu$ and thus, in particular, $y_\mu \leftarrow c_\delta$. Referring again to continuity of ψ, we obtain $y_\mu \leftarrow c$ and hence $y \leftarrow c$; but $y \neq c$ and $y \leftarrow c \leftarrow y$, a contradiction. Therefore t is a cluster point of $\{y_\mu : \mu < \xi\}$.

Using a similar idea, we can prove the following claim:

Claim 5: If u and t are cluster points of the net $\{y_\mu : \mu < \xi\}$ then $u = t$.

As an immediate consequence of Claims (4) and (5) we get:

Claim 6: $\bigcup\{L_y : y \in E\}$ has at most one boundary point.

Claim 7: If $t \in Z$ and $\mu < \xi$ then $y_\mu \leftarrow t$.

Suppose the contrary, i.e. $t \notin U_{y_\mu}$. Since $t \in Z$ we have $t \in X \setminus (\bigcup A_y : y \in E)$ and in particular $t \notin A_{y_\mu}$. Hence $t \in B_{y_\mu}$ and thus, by condition (5), $t \in B_y$ for some $y \prec y_\mu$ with $y_\mu \in A_y$. First suppose that $x \in A_y \setminus \{y\}$. In this case, $x \notin B_y$ or, equivalently $y \in F$. Therefore $Z \cap B_y = \emptyset$, which is not possible since $t \in Z \cap B_y$. Now suppose that $x \notin A_y$, i.e. $y \in E$. By Claim (1) applied to μ, we know that $\bigcup\{A_z : z \in E \text{ and } z \prec y_\mu\} = \bigcup\{A_{y_\delta} : \delta < \mu\}$. As $y_\mu \in A_y$, we can find a $\delta < \mu$ such that $y_\mu \in A_{y_\delta}$, contradicting again the minimal condition in the definition of y_μ. Therefore, $y_\mu \leftarrow t$.

Notice that $A_{y_\mu} \setminus \{y_\mu\}$ is open for every $\mu < \xi$, $A_{y_{\mu_0}} \subseteq A_{y_{\mu_1}} \setminus \{y_{\mu_1}\}$ for $\mu_0 < \mu_1 < \xi$ and $\bigcup\{A_y : y \in E\} = \bigcup\{A_{y_\mu} : \mu < \xi\}$. Hence, $\bigcup\{A_y : y \in E\}$ has a unique limit point, say a.

By an analogous construction, we can now approximate "from the right", with a collection of elements in F, such that we can find an ordinal η, which we will also suppose limit, and for every $\mu < \eta$ a point $z_\mu \in F$ such that:

(8) $B_\delta \subseteq B_\mu \setminus \{z_\mu\}$ for every $\delta < \mu < \eta$,

(9) $\bigcup\{B_z : z \in F\} = \bigcup\{B_{z_\mu} : \mu < \eta\}$ and

(10) if $t \in Z$ and $\mu < \eta$ then $t \leftarrow z_\mu$.

Again, the set $\bigcup\{B_{z_\mu} : \mu < \eta\}$ has a unique limit point, which is also a cluster point of the net $\{z_\mu : \mu < \eta\}$.

Finally, to define the sets A_x and B_x we consider all possible cases for the points a and b.

Case 1: $a = b$. We assert that $Z = \{a\} = \{b\} = \{x\}$. Indeed, take $t \in Z$. By Claim 7, $y_\mu \leftarrow t$ for every $\mu < \xi$ and thus, by the continuity of ψ and since a is a cluster point of the net $\{y_\mu : \mu < \xi\}$, we have that $a \leftarrow t$. Analogously, by condition (10), $t \leftarrow z_\mu$ for every $\mu < \eta$ and hence $t \leftarrow b$. Therefore $t = a = b$. It follows that $Z = \{x\}$ and $x = a$. Define the sets $A_x = \bigcup\{A_y : y \in E\} \cup \{x\}$ and $B_x = \bigcup\{B_y : y \in F\} \cup \{x\}$. Notice that A_x and B_x are closed because both sets contain the limit point $x = a = b$.

Case 2: $a \neq b$ *and* $x \notin \{a, b\}$. In this case, define $A_x = \bigcup\{A_y : y \in E\} \cup (Z \cap L_x)$ and $B_x = \bigcup\{B_y : y \in F\} \cup (Z \cap U_x)$. By Claim 7 we know that $y_\mu \leftarrow x$ for every $\mu < \xi$ and thus, by the continuity of ψ, $a \leftarrow x$, i.e. $a \in L_x$. Also, since a is an accumulation point of $\bigcup\{A_y : y \in E\}$ and Z closed, we also have that $a \in Z$ and therefore $a \in Z \cap L_x$, which implies that A_x is closed. Analogously, B_x is also closed.

Case 3: $a = x$ and $a \neq b$, In this case, define $A_x = \bigcup\{A_y : y \in E\} \cup \{x\}$ and $B_x = \bigcap\{B_{y_\mu} : \mu < \xi\}$.

Case 4: $b = x$ and $a \neq b$. Define $A_x = \bigcap\{A_{z_\mu} : \mu < \eta\}$ and $B_x = \bigcup\{B_z : z \in F\} \cup \{x\}$.

Let $\leq$ be the relation on X defined by $x \leq y \equiv x \in A_y$. Then $\leq$ is a linear order such that $(\leftarrow, x]_\leq = A_x$ and $[x, \rightarrow)_\leq = B_x$ are closed for every $x \in X$ and thus $\leq$ induces a weaker topology on X. Finally, since X is a Hausdorff compact space, we conclude that X is ordered. □

As van Mill and Wattel pointed out, the technique used in the proof of the compact case cannot be generalized to all topological spaces. By this reason, they explicitly present Question 1.22 and, since then, this problem has been known as the *van Mill-Wattel Problem on continuous selections*.

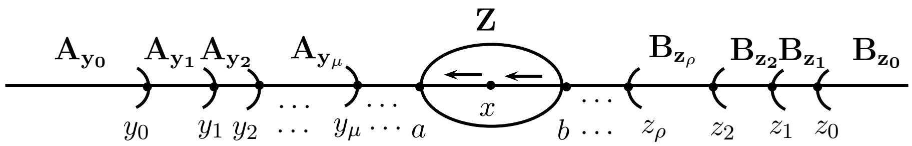

Figure 1.3: The case $a \neq b$ and $x \notin \{a, b\}$

1.7.3 Locally connected spaces

S. Eilenberg proved the following result:

Proposition 1.28 ([11]). *A connected locally connected space X is ordered if and only if $P(X) = (X \times X) \setminus \{(x, x) : x \in X\}$ is not connected.* □

Notice that this proposition, together with Corollary 1.25, implies that every connected locally connected space admitting a continuous weak selection is ordered. Michael [44] also proved that when a space X admits a continuous weak selection and all its connected components are open, then it is weakly orderable. In particular, every locally connected space admitting a continuous weak selection is weakly orderable.

Finally, Nogura and Shakmatov worked around the characterization of locally connected spaces and obtained the most general result in this direction:

Theorem 1.29 ([51]). *A locally connected space which admits a continuous weak selection is ordered.* □

1.7.4 Locally compact spaces

The first result on locally compact spaces is due to Kuratowski, Nadler and Young.

Proposition 1.30 ([41]). *A locally compact separable metric space which admits a continuous weak selection is homeomorphic to a subset of the real line $\mathbb{R}$ (and so it is suborderable).* □

Separability of the space is necessary because any uncountable discrete space satisfies the rest of the hypothesis. As we will see later, metrizability is also a necessary condition in the above result.

In [1], G. Artico, U. Marconi, J. Pelant, L. Rotter and M. Tkachenko also investigated the locally compact case. On one hand, they considered spaces that are also connected and obtained the following result:

Proposition 1.31 ([1]). *Let X be a connected weakly orderable space. If X is locally compact, then it is ordered.* □

As an immediate consequence, every connected locally compact space admitting a continuous weak selection is ordered. On the other hand, they constructed, under the Diamond Principle, a topology τ on ω_1, weaker than the order topology, such that (ω_1, τ) is a locally compact space admitting a continuous selection for 2^{ω_1} (and so for $[\omega_1]^2$) but is not suborderable. Therefore, locally compact spaces behave different than locally connected spaces with respect to being ordered. However, they also proved that that local compactness and local connectedness are equivalent for connected weakly orderable spaces. In order to see if while we weaken the orderability condition we can still obtain a characterization of locally compact spaces admitting a continuous weak selection, Gutev and Nogura posed the following question:

Question 1.32 ([24]). *Let X be a locally compact space, with $Sel(\mathcal{F}_2(X)) \neq \emptyset$. Then, is X weakly orderable?*

Gutev has recently proved that the answer is affirmative for paracompact spaces. In fact, he states the following result:

Theorem 1.33 ([31]). *A locally compact paracompact space is suborderable if and only if it has a continuous weak selection.* □

1.7.5 Pseudocompact spaces

After van Mill and Wattel provided a characterization of compact spaces that admit a continuous weak selection, it is natural to ask how far we can weaken the compactness condition and still have a characterization of spaces admitting a continuous selection in terms of an orderability property. E. van Douwen started the study of this kind and obtained a first partial result.

Proposition 1.34 ([10]). *If X is a countably compact space that admits a continuous weak selection, then X is sequentally compact .* □

J. van Mill and E. Wattel also obtained important results in this direction, which were originally aimed to solve another problem. They wanted to characterize suborderable spaces in terms of certain special continuous weak selections.

Definition 1.35. A weak selection φ on a space X is called *locally uniform* provided that for all $x \in X$ and for each neighborhood U of x there is a neighborhood V with $V \subseteq U$, such that for all $p \in X \setminus U$:

$$V \rightrightarrows \{p\} \text{ if and only if } x \to p.$$

The authors associated the notion of local uniformity of a weak selection with a property involving the Stone-Čech compactification of the space, achieving a characterization of continuous weak selections that are locally uniform. Here we can see the relation between this kind of functions and particular compact spaces.

Proposition 1.36 ([47]). *Let X be a space and let $\varphi : X^2 \to X$ be a weak selection. The following statements are equivalent:*

(1) φ is locally uniform and

(2) for all $p \in \beta X \setminus X$, φ can be extended to a continuous weak selection $\varphi^* : (X \cup \{p\})^2 \times (X \cup \{p\})$. □

Finally, using the last result they obtained a suitable characterization of suborderable spaces.

Theorem 1.37 ([47]). *Let X be a Tychonoff space. The following statements are equivalent.*

(1) X has a locally uniform weak selection and

(2) X is a suborderable space. □

Artico et al. [1] proved that for every space X such that $X \times X$ is pseudocompact, every continuous weak selection is locally uniform and therefore, as a consequence of Proposition 1.34 and Theorem 1.37, X is sequentially compact and suborderable. Working with the relation between locally uniform weak selections and some properties of βX, the authors obtained a more general result.

Theorem 1.38 ([1]). *For a completely regular space X, the following are equivalent:*

(1) βX is ordered,

(2) X is a pseudocompact suborderable space,

(3) X is countably compact and admits a continuous weak selection,

(4) X^2 is pseudocompact and X admits a continuous weak selection. □

Finally, García Ferreira and Sanchis proved that the extra hyphotesis required in condition (4) of Theorem 1.38 is always true in the pseudocompact case under the existence of a continuous weak selection on X.

Theorem 1.39 ([17]). *Let X be a pseudocompact space admitting a continuous weak selection. Then $X \times Y$ is pseudocompact for every pseudocompact space Y.* □

It follows that every pseudocompact space admitting a continuous weak selection is suborderable.

1.7.6 Separable spaces

We first study countable spaces. In this case, García Ferreira et al [16] proved that every countable space X admitting a continuous weak selection is weakly orderable. Concernig the existence of a continuous selection for 2^X, Fuji, Miyazaki and Nogura obtained a characterization for countable regular spaces.

Theorem 1.40 ([15]). *A countable regular space X has a continuous selection if and only if it is scattered.* □

Camillo Costantini considered the case when a separable space admits a countable dense subset consisting of isolated points. Adding an extra hypothesis, he answers Question 1.22 in the affirmative.

Proposition 1.41 ([7]). *Let X be a separable space with a countable dense subset consisting of isolated points. If X admits a continuous weak selection and is second countable, then it is weakly orderable.* □

He also proved that this proposition is true if we replace "second countable" with "collectionwise normal".

Using results related with connected components and disconnectedness properties, like cut points and special sets named *Purish sets*, Gutev has recently generalized Proposition 1.41, by showing that when a space is second countable and admits a continuous weak selection, we do not need to request a special condition for the countable dense set on the space to guarantee its weak orderability.

Theorem 1.42 ([28]). *Every second countable space admitting a continuous weak selections is weakly orderable.* □

In the same way, he also generalized Costantini's result on collectionwise normal spaces.

Proposition 1.43 ([28]). *Let X be a separable space that admits a continuous weak selection and such that $[X]^2$ is collectionwise Hausdorff*[1]*. Then, X is weakly orderable.* □

Finally, Gutev also characterized homogeneous separable metric spaces admitting a continuous selection.

Proposition 1.44 ([26]). *Let X be a homogeneous separable metric space such that $Sel(2^X) \neq \emptyset$. Then, one of the following holds:*

(a) X is a discrete space,

(b) X is a discrete sum of copies of the Cantor set, or

[1] If X is a collectionwise normal space then $[X]^2$ is collectionwise Hausdorff

(c) X is the irrational line[2]*.* □

In Chapter 2 and Chapter 3 we will widely study the separable case.

1.7.7 Spaces of continuous functions

A broad analysis of spaces of continuous functions of the form $C_p(X, E)$ which admit continuous selections is presented by Tamariz Mascarúa [58]. At first, he studied some cases when the hyperspace of a space $C_p(X, E)$ admits a continuous selection.

Proposition 1.45 ([58])**.** *For a countable space X and a metrizable space E the following holf:*

(1) If $Sel(C_p(X, E)) \neq \emptyset$, then X is discrete and E is completely metrizable.

(2) If X is discrete and E is strongly zero-dimensional completely metrizable, then $Sel(C_p(X, E)) \neq \emptyset$. □

In the particular case when E is the real line, Tamariz Mascarúa stated that there is not a continuous selection on $2^{C_p(X)}$ for any space X. The situation concerning continuous selections on $\mathcal{K}(C_p(X))$ (and in particular with respect to continuous weak selections) is not so different from the above.

Proposition 1.46 ([58])**.** *Let $\mathcal{G}$ be a subcollection of $\mathcal{K}(C_p(X))$ containing $\mathcal{F}_2(C_p(X))$. The following statements are equivalent:*

(1) There is a continuous selection on $\mathcal{G}$,

(2) the space $C_p(X)$ is weakly orderable,

(3) the space $C_p(X)$ is suborderable,

(4) the space $C_p(X)$ is ordered,

(5) $|X| = 1$. □

[2]The subspace of irrational numbers on the real line

Tamariz Mascarúa also studied spaces $C_p(X,E)$ where X is zero-dimensional and proved that $C_p(X,E)$ does not admit a continuous weak selection when $c(X) \geq \omega_1$. Gutev has recently characterized the collection of zero-dimensional spaces X for which $C_p(X,E)$ admits a continuous weak selection.

Proposition 1.47 ([32]). *Let X be a zero-dimensional space and let E be a topological space such that $C_p(X)$ admits a continuous weak selection. Then X is separable.* □

Finally, by adding a zero-dimensional property to the space E instead of the space X, Gutev obtained a partial affirmative answer to Question 1.22.

Theorem 1.48 ([32]). *Let E be a strongly zero-dimensional metrizable space. Then for any space X, the following are equivalent:*

(1) $C_p(X,E)$ admits a continuous weak selection and

(b) $C_p(X,E)$ is weakly orderable. □

1.7.8 Topological groups

In [1], Artico et al. obtained a dichotomy theorem for pseudocompact topological groups admitting a continuous weak selection.

Proposition 1.49 ([1]). *A pseudocompact topological group G that admits a continuous weak selection is either finite or topologically homeomorphic to the Cantor set.* □

They also answered the question of van Mill and Wattel for locally pseudocompact topological groups.

Proposition 1.50 ([1]). *If G is a locally pseudocompact topological group admitting a continuous weak selection, then it is locally compact, metrizable and ordered.* □

1.7.9 The Fell topology

Definition 1.51. Let X be a topological space. The *Fell Topology on* 2^X, denoted by $\tau_{\mathcal{F}}$, is the topology generated by the sets of the form $\langle V_0, \dots, V_n \rangle$, as a basis, where $V_0, \dots, V_n$ open subsets of X and $X \setminus \bigcup\{V_i : i \leq n\}$ is compact.

It is clear that the Fell topology is weaker than the Vietoris one and both topologies coincide if X is compact. It turns out that many properties that are distinct for Vietoris topology are equivalent for Fell topology. Some results concerning separation axioms are a sample of this.

Proposition 1.52 ([2]). *Let X be a Hausdorff space. The following are equivalent:*

(1) $(2^X, \tau_{\mathcal{F}})$ *is Hausdorff,*

(2) $(2^X, \tau_{\mathcal{F}})$ *is regular,*

(3) $(2^X, \tau_{\mathcal{F}})$ *is completely regular and*

(4) $(2^X, \tau_{\mathcal{F}})$ *is locally compact.* $\square$

In respect to $\tau_{\mathcal{F}}$-continuous selections, Gutev and Nogura have obtained some nice results which characterize spaces admitting a continuous (weak) selection with respect to Fell topology.

Definition 1.53. An ordered space X is called *topologically well orderable* if every non-empty closed subset of X has a first element.

Theorem 1.54 ([22] and [23]). *Let X be a Hausdorf space. The following are equivalent:*

(1) X *admits a* $\tau_{\mathcal{F}}$*-continuous selection,*

(2) X *admits a* $\tau_{\mathcal{F}}$*-continuous weak selection,*

(3) X *is topologically well-orderable.* $\square$

Chapter 2

Solution to the van Mill-Wattel Problem

In Chapter 1 we mentioned that every countable space admitting a continuous weak selection is weakly orderable. In fact, following van Mill and Wattel's proof of Theorem 1.27 we can convince us that this result is true because, when the space is countable, in every iteration step of the proof we work only with a finite collection of open sets and closed sets and hence we can eventually refine the partial constructions until we obtain a linear order for the countable space, whose induced topology is weaker than the original of the space. This suggests to study the separable case. As a first attempt, we can try to carry out a similar construction: Start with a countable dense subset of a separable space and, using the result for the countable case, obtain an appropriate linear order this dense subset and extend this order to a linear order on the whole space. However, if we try to extend the order from the dense subset to the whole space using the continuous weak selection to separate points, we can run into a problem since, as we will see later, it is not always possible: There are separable spaces containing pairs of points that are indistinguishable by weak selections.

The main goal of this chapter is to analyze this phenomenon in order to show that, contrary to what all the previous results might suggest, the van Mill-Wattel problem on continuous selections has a negative solution. We will present an example of a space which admits a continuous weak selection but is not weakly orderable. This example will also help us to answer a question of Gutev and

Nogura.

2.1 Almost disjoint families and continuous selections

A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (**AD**) if $|A \cap B| < \omega$ for every $A, B \in \mathcal{A}$. A family is called *maximal almost disjoint* (**MAD**), if it is **AD** and it is not a proper subset of an **AD** family.

It is well known that there are **AD** families of size $\mathfrak{c}$. Indeed, identify ω with $2^{<\omega}$ and enumerate 2^ω by $\{f_\alpha : \alpha < \mathfrak{c}\}$. For every $\alpha < \mathfrak{c}$, let $A_\alpha = \{f_\alpha \restriction n : n \in \omega\}$. It is easy to see that the family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ is **AD**.

For a given **AD** family $\mathcal{A}$ we define the ideal $\mathcal{I}(\mathcal{A})$ to consist of sets $F \subseteq \omega$ such that $F \subseteq^* \bigcup \mathcal{A}'$ for some $\mathcal{A}' \in [\mathcal{A}]^{<\omega}$. The dual filter $\langle\{\omega \setminus A : A \in \mathcal{A}\}\rangle$ is denoted $\mathcal{I}^*(\mathcal{A})$ and $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(A)$.

Definition 2.1. Let $\mathcal{A}$ be an **AD** family. The Ψ*-space* or *Mrówka-Isbell space* of the family $\mathcal{A}$, denoted by $\Psi(\mathcal{A})$, is defined as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathcal{A}$ are subsets of the form $\{A\} \cup (A \setminus F)$, where $F \in [\omega]^{<\omega}$.

It is clear that $\Psi(\mathcal{A})$ is a first countable, locally compact and separable space with a countable dense subset consisting of isolated points. Notice also that $\mathcal{A}$ is relatively discrete and thus $\Psi(\mathcal{A})$ is scattered. Another property satisfied by Ψ-spaces, which is closely related to combinatorial properties of the **AD** family, is that $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is **MAD**.

The concept of Ψ-spaces was independently introduced by S. Mrówka [50] and J. Isbell. Psi-spaces have served as counterexamples for many topological problems. As we will see later, the van Mill-Wattel problem turns out to be one of them.

Hrušák, Szeptycki and Tomita [34] worked around Ψ-spaces admitting continuous selections and obtained many results in this direction. In particular, they constructed an example of an almost disjoint family whose Ψ-space admits a continuous weak selection and another example where the corresponding Ψ-

space does not admit it. We will not include these examples here but we will later present examples of both cases.

In order to determine what happens when we work with **MAD** families, Nogura asked whether $\Psi(\mathcal{A})$ admits a continuous weak selection if $\mathcal{A}$ is **MAD**. This question was explicitly answered by Hrušák et al [34]. We will include here the proof of this result to familiarize ourselves with the terms and properties that will be used when we work with selections on Ψ-spaces. We will require the following auxiliar result.

Lemma 2.2 ([43])**.** *For every **MAD** family $\mathcal{A}$ and every decreasing sequence $\{X_i : i \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$ there is $X \in \mathcal{I}^+(\mathcal{A})$ such that $X \setminus i \subseteq \bigcap\{X_j : j < i\}$ for every $i \in X$.* $\square$

Lemma 2.3 ([34])**.** *Let $\mathcal{A}$ be a **MAD** family and let $f : [\omega]^2 \to 2$. Then there exists a set $B \in \mathcal{I}^+(\mathcal{A})$ such that $f''[B]^2 = \{i\}$ for some $i \in 2$.*

Proof. Extend the filter $\mathcal{I}^*(\mathcal{A})$ to a ultrafilter $\mathcal{U}$ and construct a function $g : \omega \to 2$ as follows. For $n \in \omega$ and $i \in 2$ define the set $A^n_i = \{m \in \omega : f(\{n, m\}) = i\}$. Then $\{A^n_0, A^n_1\}$ is a partition of ω and, since $\mathcal{U}$ is ultrafilter, we can choose $g(n) \in 2$ such that $A^n_{g(n)} \in \mathcal{U}$. Notice that for any $\mathcal{A}' \in [\mathcal{A}]^{<\omega}$, the set $\bigcap\{\omega \setminus A : A \in \mathcal{A}'\} \in \mathcal{U}$, which implies that $A^n_{g(n)} \notin \mathcal{I}(\mathcal{A})$ for every $n \in \omega$. Hence, by Lemma 2.3, we can find an $X \in \mathcal{I}^+(\mathcal{A})$ such that $X \setminus n \subseteq \bigcap\{X_i : i < n\}$ for every $n \in \omega$. Finally, define $B_i = \{n \in X : g(n) = i\}$ for every $i \in 2$ and let $j \in 2$ be such that $B_j \in \mathcal{I}^+(\mathcal{A})$. The set $B = B_j$ is as desired. $\square$

This Ramsey-type lemma is related Nogura's question by the following.

From a given weak selection ψ on ω we can define a natural induced function $f : [\omega]^2 \to 2$ by $f(\{n, m\}) = 0$ if and only if $\psi(\{n, m\}) = \min\{n, m\}$. Hence, if we take an arbitrary weak selection on the Mrówka-Isbell space determined by a **MAD** family $\mathcal{A}$ then its restriction to $[\omega]^2$ will be a weak selection on ω and, applying Lemma 2.3, we can find a "big" set where the selection agrees with the usual order on it, i.e. it is either the max weak selection or the min weak selection. However, this fact will actually be a problem to define continuous weak selections on $\Psi(\mathcal{A})$.

Proposition 2.4 ([34]). *The space $\Psi(\mathcal{A})$ does not have a continuous weak selection for any maximal almost disjoint family $\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ is a **MAD** family and let ϕ be a weak selection on $\Psi(\mathcal{A})$. Let $f : [\omega]^2 \to 2$ be the function on $[\omega]^2$ naturally determined by $\phi \restriction [\omega]^2$ and the usual order on ω, defined as in the previous paragraph, and let $B \in \mathcal{I}^+(\mathcal{A})$ be such that $f''[B]^2 = \{i\}$ for some $i \in 2$. Take $A_0, A_1 \in \mathcal{A}$ such that $B \cap A_j$ is infinite for $j \in 2$. We claim that ϕ is not continuous in the element $F = \{A_0, A_1\}$. Indeed, suppose that $\phi(F) = A_0$ and let $\mathcal{U}$ be a basic neighborhood of F. We can find a $k \in \omega$ such that $\mathcal{V} = \langle \{A_0\} \cup (A_0 \setminus k), \{A_1\} \cup (A_1 \setminus k) \rangle \subseteq \mathcal{U}$.

Suppose first that $f''[B]^2 = \{0\}$. Then choose $n > k$ such that $n \in (A_1 \cap B) \setminus A_0$ and also take $m > n$ with $m \in (A_0 \cap B) \setminus A_1$. Note that $\{n, m\} \in \mathcal{V}$ and $\phi(\{n, m\}) = n \notin A_0$. On the other hand, if $f''[B]^2 = \{1\}$, take $n > k$ such that $n \in (A_0 \cap B) \setminus A_1$ and let $m > n$ be with $m \in (A_1 \cap B) \setminus A_0$. Then, $\{n, m\} \in \mathcal{V}$ and $\phi(\{n, m\}) = m \notin A_0$. Therefore, in any case $\phi''(\mathcal{V} \cap [\omega]^2) \not\subseteq A_0$, which implies that ϕ is not continuous on F. $\square$

As Michael et al. indicated in [34], Nogura's question can also be answered by a more general result, due to Artico et al [1], which states that for every pseudocompact scattered space X admitting a continuous weak selection, the space X^2 is pseudocompact. This implies, by Theorem 1.38, that X is a suborderable space. However, every pseudocompact suborderable space is countably compact, which is not the case for $\Psi(\mathcal{A})$.

In general, the study of spaces of the form $\Psi(\mathcal{A})$ becomes more interesting when $\mathcal{A}$ is uncountable because we can then obtain more combinatorial properties than when we only have a finite or a countable almost disjoint family. This, however, is not the case if we want to study the existence of a continuous selection on the hyperspace of $\Psi(\mathcal{A})$ when $\mathcal{A}$ is not countable.

Theorem 2.5 ([34]). *If X is regular, separable and contains an uncountable closed discrete set, then 2^X does not admit a continuous selection.* $\square$

In particular, $\Psi(\mathcal{A})$ does not admit a continuous selection for any uncountable almost disjoint family $\mathcal{A}$.

2.2 Extension of weak selections on ω

We provide ω with the discrete topology. Thus, any weak selection defined on ω is automatically continuous. We investigate first which weak selections on ω can be continuously extended to a Ψ-space. Then we choose an appropriate weak selection to extend. To this end, we will introduce the following version of the alignment conditions between sets with respect to a given weak selection, presented in Chapter 2, but with finite exceptions. It is similar to the "almost containment" and "almost equality" required notions to define the Mrówka-Isbell topology on Ψ-spaces.

Given a weak selection ψ on ω and $A, B \in [\omega]^\omega$, say that B *almost dominates* A with respect to ψ (or simply B *almost dominates* A if ψ is clear from context), and write $A \rightrightarrows^*_\psi B$, whenever there is $k \in \omega$ such that $A \setminus k \rightrightarrows_\psi B \setminus k$. Say that A and B are *almost aligned*, denoted by $A||^*_\psi B$, if there is $k \in \omega$ such that $A \setminus k||_\psi B \setminus k$. If $n \in \omega$, say that A is *almost dominated* by $\{n\}$, and write $A \rightrightarrows^* \{n\}$, if $A \setminus k \rightrightarrows_\psi \{n\}$ for some $k \in \omega$. In a similar way, we can define $\{n\} \rightrightarrows^*_\psi A$ and $A||^*_\psi n$. We will omit the subscript if there is no danger of confusion.

The following result characterizes continuous weak selections on ω that can be extended to a continuous weak selection on a given Mrówka-Isbell space. It is nothing but the translation of Proposition 1.18 using the previous notation.

Lemma 2.6. *Let $\mathcal{A}$ be an **AD** family. A weak selection ψ on ω can be extended to a (unique) continuous weak selection defined on $\Psi(\mathcal{A})$ if and only if the following hold:*

(1) $A||^*B$ for every $A, B \in \mathcal{A}$,

(2) $A||^*\{n\}$ for every $A \in \mathcal{A}$ and $n \in \omega$. □

Example 2.7. Let $\psi : [\omega]^2 \to \omega$ be the weak selection defined by $\psi(\{m, n\}) = \min\{m, n\}$ and let $\mathcal{A}$ be an almost disjoint family with at least two elements. For $A, B \in \mathcal{A}$ and $k \in \omega$, we can find $a_0 \in A$ and $b_0 \in B$ such that $k < a_0 < b_0$. Similarly, there are $a_1 \in A$ and $b_1 \in B$ with $k < b_1 < a_1$. Henc condition (2) of the previous Lemma fails for ψ and $\mathcal{A}$. Analogously, the weak selection max on ω cannot be extended to any $\Psi(\mathcal{A})$ with $|\mathcal{A}| \geq 2$.

The example shows that weak selections extendable to a non trivial Mrówka-Isbell space are nontrivial. We can in fact consider the "most complicated" weak selection defined on ω and try to extend it to a Ψ-space determined by a suitable **AD** family.

2.3 Universal Weak Selection

A *graph* Γ is a couple (X, A), where X is a non empty set and $A \subseteq [X]^2$. Elements of the set X are thought as the *nodes* or *vertices* of the graph and elements of A as the *segments* or *edges* of Γ. If we assign an orientation to every edge of a graph, we obtain a so called *directed graph*. In a more formal way:

Definition 2.8. A *directed graph* is a set $\Gamma = (V, A)$, where V is the set of vertices, and $A \subseteq V^2 \setminus \{(x,x) : x \in V\}$ is such that, for every $x, y \in V$, either $(x,y) \notin A$ or $(y,x) \notin A$. We will say that Γ is *complete* if for any distinct $x, y \in X$, exactly one of the following occurs: $(x,y) \in A$ or $(y,x) \in A$.

A weak selection φ on a set X determines in a natural way a complete directed graph $\Gamma = (V, A)$ on X, where $V = X$ and $A = \{(x,y) \in X^2 : x, y \in X \text{ and } x \to y\}$. Similarly, any complete directed graph with X as its set of vertices induces a weak selection on X. In particular, when X is the countable discrete space ω, any (continuous) weak selection on ω corresponds to an infinite countable directed graph.

In the study of several mathematical structures we can find the existence of a universal structure which is able to explain in a general way the behaviour of any other structure of the same kind. This is the case for infinite graphs. In 1964, R. Rado [53] constructed a countable undirected graph Γ in a simple way as follows: The set of vertices is ω and for $x < y$, the vertices x and y are adjacent if and only if, when y is written in base 2, the x-th digit is 1. He proved that Γ has the following *universal* property: Any finite or countable graph embeds into Γ. R. Fraïssé [14] had done a more general study before Rado. He developed and used the *back-and-forth method* to determine whether two model theoretic structures were elementarily equivalent. The main point of his construction was to show how to approximate an infinite structure by its finitely generated structures in such a way that the infinite structure is a sort of *limit* of its finite substructures. Using this construction it can be proved for instance that $(\mathbb{Q}, <)$, with its usual order, is the Fraïsse limit of the finite linear orderings and it is, up to isomorphism, the only countable dense linear order without endpoints.

It is interesting that Urysohn obtained twenty years before Fraïsse a very similar topological version of his result. In a posthumous paper [59] published in 1927 he proved the existence of a unique homogeneous Polish space, today known now as the *Urysohn space*, where homogeneous means that any isometry defined between finite subsets of the space can be extended to an isometry of the whole space. Separability of the space corresponds to countability in Fraïssé's work.

Analogous to the Universal Rado graph, we can construct a Universal Weak Selection on ω as the Fraïssé limit of the collection of all weak selections defined over a finite subset of ω. Our construction is more combinatorial, using the well know notion of independent families over ω.

A family $\mathcal{I} \subseteq [\omega]^\omega$ is *independent* if $|\bigcap \mathcal{F} \setminus \bigcup \mathcal{F}'| = \omega$ for every $\mathcal{F}, \mathcal{F}' \in [\mathcal{I}]^{<\omega}$. There are independent families of size $\mathfrak{c}$ but, for our construction, we only require a countable independent family.

The following result will be crucial for our study.

Proposition 2.9. *There is a weak selection $\varphi : [\omega]^2 \to \omega$ satisfying the following extension property:*

$(\mathcal{D})$: For every disjoint $F, G \in [\omega]^{\leq\omega}$, there is $n \in \omega \setminus (F \cup G)$ such that $F \rightrightarrows \{n\} \rightrightarrows G$.

Proof. Let $\mathcal{J} = \{J_n : n \in \omega\} \subseteq [\omega]^\omega$ be an independent family. Recursively, we define a family $\mathcal{I} = \{I_n : n \in \omega\} \subseteq [\omega]^\omega$ in the following way:

- $I_0 = J_0$,
- $I_{n+1} = (J_n \setminus \{k \leq n : n+1 \in I_k\}) \cup \{k \leq n : n+1 \notin I_k\}$.

By construction, $|I_n \triangle J_n| < \omega$ for every $n \in \omega$ and so $\mathcal{I}$ is also an independent family. Notice that also by construction, $n \in I_m$ if and only $m \notin I_n$. This property allows us to define $\varphi : [\omega]^2 \to \omega$ by

$$\varphi(\{n, m\}) = n \text{ if and only if } n \in I_m.$$

To verify that φ satisfies the property $\mathcal{D}$, let F and G be disjoint finite subsets of ω. Since $\mathcal{I}$ is independent, $U = \bigcap\{I_n : n \in F\} \setminus \bigcup\{I_m : m \in G\}$ is infinite and, by definition of φ, $F \rightrightarrows \{k\} \rightrightarrows G$ for every $k \in U$. $\square$

Definition 2.10. Let X and Y be sets and let ψ and φ be weak selections on X and Y respectively. Say

(a) ψ and φ are *isomorphic*, and write $\psi \approx \varphi$, if there is a bijection $\rho : X \to Y$ such that $\psi(\{a, b\}) = \varphi(\{\rho(a), \rho(b)\})$ for all $a \neq b \in X$,

(b) the weak selection ψ is *embedded* in φ if $\psi \approx \varphi \restriction [A]^2$ for some $A \subseteq Y$

Property $\mathcal{D}$ of Theorem 2.9 will let us obtain important properties satisfied by the weak selection on ω that will be built. In fact, this extension property characterizes the universal structure that we are looking for.

Proposition 2.11. *Any two weak selections defined on ω satisfying Property $\mathcal{D}$ are isomorphic.*

Proof. We will use the so called "back and forth" argument to define an appropriate bijection on ω. Let ψ and ρ be two weak selections on X satisfying Property $\mathcal{D}$. Before proving the proposition, let us make an easy but important remark.

Suppose that for $F, G \in [\omega]^{<\omega}$ we have defined a bijection f from F to G such that, for every $x, y \in F$, $x \to_\psi y$ if and only if $f(x) \to_\rho f(y)$ and let $z \in \omega \setminus F$. It is possible to extend the bijection f to a bijection g in $F \cup \{z\}$ with the same properties. Indeed, put $A_0 = \{n \in F : n \to_\psi z\}$ and $A_1 = \{n \in F : z \to_\psi n\}$. By property $\mathcal{D}$ we can find $w \in \omega \setminus G$ in such a way that $\{f(k) : k \in A_0\} \rightrightarrows_\rho \{w\} \rightrightarrows_\rho \{f(k) : k \in A_1\}$. Then, it is enough to define $f(z) = w$. In the same way, we can extend f to a bijection g containing a given point $z \in \omega \setminus G$ on its range.

We now construct the desired bijection. In fact, we will construct a collection of finite bijections $\{f_n : n \in \omega\}$, where each f_n is defined on a certain domain

$F_n \in [\omega]^n$ and extends the previous defined functions. Start with $f_0 = \emptyset$ and suppose that we have defined f_m. If m is even, let $k = \min(\omega \setminus F_m)$ and define f_{m+1} to be a bijection defined on $F_m \cup \{k\}$ and extending f_m. Here we used the forth condition. Whenever m is odd we work backwards. Let $r = \min(\omega \setminus f_m[F_m])$ and extend f_m to a bijection f_{m+1} containing r on its range. The function $f = \bigcup\{f_n : n \in \omega\}$ is a bijection on ω, and $n \to_\psi m$ if and only $f(n) \to_\rho f(m)$ for every $n, m \in \omega$. $\square$

We will denote by φ the unique weak selection on ω satisfying Property $\mathcal{D}$, and we will call it the *universal weak selection*. The next propositionn says that φ is universal.

Proposition 2.12. *Every weak selection ψ on ω can be embedded in φ.*

Proof. We will employ the forth property technique used in Proposition 2.11. For every $n \in \omega$, construct a bijection f_n, whose domain is n and such that $l \to_\psi k$ if and only if $f_n(l) \to_\varphi f_n(k)$ for all $l, k \in n$. Define $f_0 = \emptyset$ and suppose that we have defined f_n. Put $U = \{k \leq n : k \to_\psi n+1\}$ and $V = \{k \leq n : n+1 \to_\psi k\}$. By property $\mathcal{D}$, there is $m \in \omega \setminus f_n[n]$ such that $f_n[U] \rightrightarrows_\varphi \{m\} \rightrightarrows_\varphi f_n[V]$. Let f_{n+1} be the bijection extending f_n and such that $f_{n+1}(n) = m$. Finally, let $f = \bigcup\{f_n : n \in \omega\}$. $\square$

2.4 φ-positive sets

The study of a universal structure turns out to be important from distinct points of view. On one hand, the universal structure can help us to explain the behaviour of any substructure with respect to certain properties that itself satisfies. On the other hand, by its own complexity, it satisfies many interesting properties that not any simpler structure carries out. It is the case with the universal weak selection. We can find special properties satisfied by φ which are rarely satisfied by another weak selection. In the following result we present two of these properties which will be useful in subsequent constructions. The first property establishes that φ satisfies a *pigeonhole property* in the sense that when we partition ω into a finite collection of subsets, for at least one of these subsets the restriction of φ on it is as complicated as it was on ω. The second property states that whenever we have found a set that is aligned with two disjoint finite subsets, we still have a complicated selection when we restrict φ on this set.

We will denote by $\mathcal{R}$ the collection of all infinite subsets of ω such that the restriction of φ to each of them is as complicated as the original one:

$$\mathcal{R} = \{X \subseteq \omega : \varphi \restriction [X]^2 \approx \varphi\}.$$

Elements of $\mathcal{R}$ will be called *φ-positive sets.*

Proposition 2.13. *Let φ be the universal weak selection. Then*

(1) If $\{P_0, P_1\}$ is a partition of ω then either $P_0 \in \mathcal{R}$ or $P_1 \in \mathcal{R}$.

(2) If $F, G \in [\omega]^{<\omega}$ are disjoint then

$$\{k \in \omega \setminus (F \cup G) : F \rightrightarrows \{k\} \rightrightarrows G\} \in \mathcal{R}$$

.

Proof. To prove (1), let us suppose the contrary and let $\{P_0, P_1\}$ be a partition of ω such that neither $P_0 \in \mathcal{R}$ nor $P_1 \in \mathcal{R}$. For $i \in 2$, we can find F_i, G_i finite disjoint subsets of P_i such that every $n \in P_i$ does not dominate the set F_i or is not dominated by G_i with respect to φ. By Property $\mathcal{D}$, we can find an $n \in \omega$ such that $F \rightrightarrows \{n\} \rightrightarrows G$, where $F = F_0 \cup F_1$ and $G = G_0 \cup G_1$. As $\{P_0, P_1\}$ is a partition of ω, $n \in P_i$ for $i \in 2$ but in both cases we have a contradiction.

To verify the property (2), again aiming towards contradiction, suppose that we can find $F, G \in [\omega]^{\leq\omega}$ such that

$$A = \{k \in \omega \setminus (F \cup G) : F \rightrightarrows \{k\} \rightrightarrows G\} \notin \mathcal{R}$$

. By property (1), $\omega \setminus A \in \mathcal{R}$. Since F and G are finite subsets of $\omega \setminus A$, there is $n \notin A$ dominating F and dominated by G; but n must be an element of A, which clearly is not possible. □

As a consequence of the above result, if we make finite changes to the universal weak selection φ, we still have a weak selection as complex as φ. We can for instance delete a finite number of vertices or change the direction of some of the arrows and get a weak selection that is isomorphic to φ.

2.5 φ-positive sets and linear orders on ω

In order to obtain an extension of the universal weak selection to an appropriate Ψ-space, we will simultaneously study linear orders on ω and special properties determined by φ-positive sets. Our plan is, given a linear order on ω, to carefully construct two disjoint infinite sets that are aligned with respect to the weak selection φ but whose elements are alternating with respect to the given order.

Definition 2.14. Let $\leq$ be a linear order on a set X. Then

(1) A subset $S \subseteq X$ is *downward closed* if $x \leq y$ and $y \in S$ implies $x \in S$.

(2) A subset $S \subseteq X$ is *upward closed* if $x \geq y$ and $y \in S$ implies $x \in S$.

(3) An infinite subset $Y \subseteq X$ is *monotone* if either there is a downward closed set $S \subseteq X$ containing Y such that, for every $s \in S$, $Y \cap (\leftarrow, s)_{\leq}$ is finite, or there is an upward closed set $T \subseteq X$ with $Y \subseteq T$ such that $Y \cap (t, \rightarrow)_{\leq}$ is finite for every $t \in T$.

Proposition 2.15. *Let φ be the universal weak selection and let $\preceq$ be a linear order on ω. If $X \in [\omega]^{\omega}$ is a φ-positive set, then there are $X_0, X_1 \in [X]^{\omega}$ satisfying:*

(1) $X_0 \cap X_1 = \emptyset$,

(2) $X_0 \rightrightarrows X_1$,

(3) $X_0 \cup X_1$ is monotone with respect to $\preceq$.

Proof. We will recursively construct the sets X_0 and X_1 as follows. If $X \cap (\leftarrow, 0)_{\preceq} \in \mathcal{R}$, define $M_0 = X \cap (\leftarrow, 0)_{\preceq} \in \mathcal{R}$ and define $M_0 = X \cap [0, \rightarrow)_{\preceq}$ otherwise. Notice that, by Proposition 2.13 (1), in both cases M_0 is a φ-positive set and thus we can find distinct $a_0, b_0, c_0 \in M_0$ such that $\{a_0, b_0, c_0\}$ is a 3-cycle in M_0. Choose $x_0, y_0 \in \{a_0, b_0, c_0\}$ in such a way that $x_0 \prec y_0$ and $x_0 \rightarrow y_0$ and define then the set $D_1 = \{n \in M_0 : x_0 \rightarrow n \rightarrow y_0\} \setminus \{x_0, y_0\}$. By Proposition 2.13 (2), the set D_1 is also positive. As before, let $M_1 = D_1 \cap (\leftarrow, 1)_{\preceq}$ be if $D_1 \cap (\leftarrow, 1)_{\preceq} \in \mathcal{R}$, and $M_1 = D_1 \cap [1, \rightarrow)_{\preceq}$ otherwise. Choose now $a_1, b_1, c_1 \in M_1$

such that $\{a_1, b_1, c_1\}$ is a 3-cycle in M_1 and let $x_1, y_1 \in \{a_1, b_1, c_1\}$ be such that $x_1 \to y_1$ and $y_1 \prec x_1$. By construction, $\{x_0, x_1\} \rightrightarrows \{y_0, y_1\}$.

Following this procedure, recursively form a collection $\{M_n : n \in \omega\}$ of φ-positive sets and disjoint sets $W_0 = \{x_n : n \in \omega\}$, $W_1 = \{y_n : n \in \omega\} \in [X]^\omega$ such that for every $n \in \omega$, $M_{n+1} \subseteq M_n$, $\{x_0, \ldots, x_n\} \rightrightarrows \{y_0, \ldots, y_n\}$, $x_n \prec y_n$ whenever n is even, and $y_n \prec x_n$ if n is odd. Also by construction, when the set $S = \{n \in \omega : M_n \subseteq (n, \to)_\preceq$ is infinite, then it is $\preceq$-downward closed, and $T = \{n \in \omega : M_n \subseteq (\leftarrow, n)_\preceq\}$ is $\preceq$-upward closed if T is infinite.

We refine now the sets W_0 and W_1 to obtain, after joining both refinements, a $\preceq$-monotone subset. We claim that either $W_0 \cap S$ and $W_1 \cap S$ are infinite or $W_0 \cap T$ and $W_1 \cap T$ are infinite. To see this, suppose e.g. that $W_0 \cap S$ is finite. Since S and T form a partition of ω, we can find a $k \in \omega$ such that for all $n \geq k$, $x_n \in T$. Otherwise, $x_m \prec y_m$ whenever $m \geq k$ is even and since T is $\preceq$-upward closed, $y_n \in T$. This proves that, in this case, $|W_0 \cap T| = \omega = |W_1 \cap T|$. When $W_0 \cap S$ and $W_1 \cap S$ are both infinite, define $X_0 = W_0 \cap S$ and $X_1 = W_1 \cap S$. In this case, the recursion guarantees that, for every $n \in S$, whenever $k \geq n$ we have that $M_k \subseteq M_n \subseteq (n, \to)_\preceq$, and then $x_k, y_k \in (n, \to)_\preceq$. Therefore, $(X_0 \cup X_1) \cap (\leftarrow, n)_\preceq \subseteq n$ and so $X_0 \cup X_1$ is monotone. In the other case, when either $W_0 \cap S$ is finite or $W_1 \cap S$ is finite, define $X_0 = W_0 \cap T$ and $X_1 = W_1 \cap T$. Using the same reasoning, it can be proved that $X_0 \cup X_1$ is monotone. $\square$

2.6 The counterexample

In this section we provide a negative answer to the question of van Mill and Wattel. The main idea is to define an almost disjoint family of size $\mathfrak{c}$ such that we can simultaneously extend the universal weak selection φ to the corresponding Mrówka-Isbell space and, using the elements of the **AD** family, to "destroy" all possible linear orders on the constructed Ψ space. In fact, by density of ω we will only be interested in linear orders on ω. This is the reason why in Proposition 2.6 we just restrict our attention to ω and its orders.

We first find a large almost disjoint family and a weak selection on the induced Ψ-space that extends the universal weak selection.

Lemma 2.16. *There is an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ satisfying the following conditions:*

(1) $|\mathcal{A}| = \mathfrak{c}$,

(2) $\mathcal{A} \subseteq \mathcal{R}$,

(3) $A||^* B$ *for every* $A, B \in \mathcal{A}$.

Proof. Identify ω with $2^{<\omega}$. We construct the almost disjoint family on $2^{<\omega}$ using the branches determined by elements of 2^ω. Given $f \in 2^\omega$, put $A_f = \{f \restriction n : n \in \omega\}$ and define $\mathcal{A} = \{A_f : f \in 2^\omega\}$. Clearly $\mathcal{A} = \mathfrak{c}$. If $f, g \in 2^\omega$ are distinct functions, there is $n \in \omega$ such that $f(k) = g(k)$ for $k < n$ and $f(n) \neq g(n)$, which guarantees that $|A_f \cap A_g| < \omega$.

If $f, g \in 2^{<\omega}$, we will write $f \perp g$ whenever there is an $n \in \omega$ such that $f(n) \neq g(n)$ and $f \not\perp g$ if $f \subseteq g$ or $g \subseteq f$. Notice that $f \not\perp g$ if and only if there is $h \in 2^\omega$ such that $f, g \in A_h$. Define the weak selection ψ on $2^{<\omega}$ by $\psi(f, g) = g$ if and only if either $f \not\perp g$ and $\varphi(|f|, |g|) = |g|$ or $f \perp g$ and $f(f \triangle g) = 0$, where $f \triangle g = \min\{k \in \omega : f(k) \neq g(k)\}$.

Without loss of generality we may suppose, by universality of φ, that $\psi \subseteq \varphi$. Moreover, by definition of ψ if $f \in 2^\omega$, then $\psi \restriction [A_f]^2 \approx \varphi$. It turns out that the branch determined by f is φ-positive for every $f \in 2^\omega$. Finally, if $f, g \in 2^\omega$ are distinct functions and $f(f \triangle g) = 0$ then, again by definition of ψ, $A_f \setminus 2^{f \triangle g} \rightrightarrows A_g \setminus 2^{f \triangle g}$, which implies that $A_f||^* A_g$. $\square$

We require elements of $\mathcal{A}$ to belong to $\mathcal{R}$ to independently work with each of them and, taking advantage of the properties satisfied by φ-positive sets, to take care of each of the possible orders on ω. To start, enumerate by $\{\leq_\alpha : \alpha < \mathfrak{c}\}$ the set of all linear orders on ω. Our plan is to refine the obtained family $\mathcal{A}$ by replacing each element of $\mathcal{A}$ by two disjoint infinite subsets which are almost aligned among themselves and with any element of ω. This condition will allow us to extend the universal weak selection to the given Mrówka-Isbell space. We will also require that, for each $\alpha < \mathfrak{c}$, the two chosen subsets to replace A_α must have the same behaviour with all elements of ω with respect to the linear order $\leq_\alpha$, except possibly a finite set. In fact, both sets together, seen as subspaces of

ω, are like a Dedekind cut for $\leq_\alpha$. The latter will imply that the corresponding linear order on $\Psi(X)$ which extends $\leq_\alpha$ may not determine a topology weaker than that of the Ψ-space.

Lemma 2.17. *For every $\alpha < \mathfrak{c}$, there are $X_0^\alpha, X_1^\alpha \in [A_\alpha]^\omega$ such that:*

(1) $X_0^\alpha \cap X_1^\alpha = \emptyset$*,*

(2) $X_0^\alpha ||^* X_1^\alpha$*,*

(3) $X_i^\alpha ||^* \{n\}$ *for every* $n \in \omega$ *and* $i \in 2$*,*

(4) $X_0^\alpha \cup X_1^\alpha$ *is* $\leq_\alpha$*-monotone.*

Proof. Fix $\alpha < \mathfrak{c}$. By Lemma 2.16, A_α is φ-positive and, by Proposition 2.15, we can also find $X_0, X_1 \in [A_\alpha]^\omega$ such that $X_0 || X_1$ and $X_0 \cup X_1$ is $\leq_\alpha$-monotone. We will recursively refine the sets X_0 and X_1 to satisfy condition (3).

For every $x \in X$, either $x \to 0$ or $0 \to x$ and therefore, $C_0 \in [X_0]^\omega$ such that $C_0 || \{0\}$. Recursively, we can construct a family of infinite subsets $\mathcal{C} = \{C_n : n \in \omega\}$ such that, for every $n \in \omega$, $C_{n+1} \subseteq C_n$ and $C_n || \{n\}$. Let X_0^α be a pseudointersection of the family $\mathcal{C}$. For every $n \in \omega$, the set $X_0^\alpha \setminus C_n$ is finite, wich guarantees that $X_0^\alpha || \{n\}$. Similarly, we construct a family $\mathcal{E} = \{E_n : n \in \omega\} \subseteq [X_1]^\omega$ to satisfy $E_{n+1} \subseteq E_n$ and $E_n || \{n\}$ for every $n \in \omega$. Therefore, if X_1^α is a pseudointersection of the family $\mathcal{E}$ then X_0^α, X_1^α satisfy, by construction, properties $(1) - (3)$, and (4) is also true, because $X_0^\alpha \cup X_1^\alpha$ is an infinite subset of $X_0 \cup X_1$, which satisfies 2.16(4). □

We are now ready to present the main result of this chapter. It answers the question of van Mill and Wattel in the negative.

Theorem 2.18. *There is a separable, first countable, locally compact space admitting a continuous weak selection but that is not weakly orderable.*

Proof. Let $\mathcal{B} = \{X_0^\alpha, X_1^\alpha : \alpha < \mathfrak{c}\}$, where X_0^α and X_1^α are as in Lemma 2.17 for every $\alpha < \mathfrak{c}$ and $i \in 2$, and let $X = \Psi(\mathcal{B})$ be the Mrówka-Isbell space associated to $\mathcal{B}$. By Lemma 2.17, the space X and the universal weak selection φ satisfy all conditions on Lemma 2.6 and therefore, φ can be continuously extended to a weak selection $\bar{\varphi}$ on X.

To conclude, we will prove that X is not weakly orderable by showing that non linear order on X induces a topology that is weaker than the original of the space. Aiming towards a contradiction, suppose that $\preceq$ is a linear order on X whose induced topology is coarser than the topology on X. Let $\alpha < \mathfrak{c}$ be such that $\preceq\restriction \omega^2 =\leq_\alpha$. We can suppose without loss of generality that there are points $X_0^\alpha, X_1^\alpha \in X$ such that $X_0^\alpha \preceq X_1^\alpha$. By Lemma 2.17(4), the set $X' = X_0^\alpha \cup X_1^\alpha$ is $\leq_\alpha$-monotone. We may assume that there is a downward closed subset $S \in [\omega]^\omega$ containing X' and such that, for every $s \in S$, $X' \cap (\leftarrow, s)_\preceq$ is finite. If there is $s \in S$ such that $X_0^\alpha \preceq s$ then, since $(\leftarrow, s)_\preceq$ is open, we can find a finite set $F \subseteq \omega$ such that $X_0^\alpha \in \{X_0^\alpha\} \cup (X_0^\alpha \setminus F) \subseteq (\leftarrow, s)_\preceq$. But $|X_0^\alpha \cap (\leftarrow, s)_\preceq| < \omega$, which is a contradiction. On the other hand, if $s \in (\leftarrow, X_0^\alpha)_\preceq$ for every $s \in S$, i.e. $S \subseteq (\leftarrow, X_0^\alpha)_\preceq$, then also $X_1^\alpha \subseteq (\leftarrow, X_0^\alpha)_\preceq$ and thus $X_1^\alpha \cap (X_0^\alpha, \rightarrow)_\preceq = \emptyset$. However, $X_0^\alpha \preceq X_1^\alpha$ implies that for some $F \in [\omega]^{\leq\omega}$, $\{X_1^\alpha\} \cup (X_1^\alpha \setminus F) \subseteq (X_0^\alpha, \rightarrow)_\preceq$ which, again, is not possible.

In a similar way it can be proved that when X' is contained in an upward directed set T we obtain a contradiction of the same kind. Therefore, the topology determined by the order $\preceq$ cannot be coarser than the original of X and consequently X is not weakly orderable. $\square$

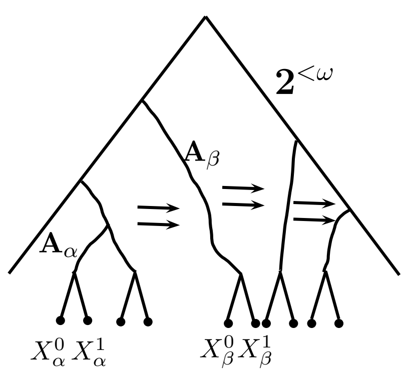

Figure 2.1: The space $\Psi(\mathcal{B})$

The above result also allows us to provide a negative answer to Question 1.32. Hence when we restrict ourselves to the collection of locally compact spaces, the existence of a continuous weak selection is not equivalent to be weakly orderable.

2.7 An example on $\beta\omega$

The first idea when we started to work on the van Mill-Wattel problem was to prove that for the collection of separable spaces, the existence of a continuous weak selection was equivalent to being weakly orderable. However, we encountered the difficulty of extending the linear order determined by the continuous weak selection from the countable dense subset to the entire space. One reason is that for certain separable spaces we can find points that cannot be distinguished by the corresponding continuous weak selection. In the example presented in Theorem 2.18, we can find uncountable many of these points. This follows from results that will be presented in Chapter 3, but we can explicitly obtain it by adjusting the construction of points X_0^α and X_1^α in the proof of Proposition 2.6. Indeed, for every $n \in \omega$, we can refine M_n to $M_n' = M_n \cap (\leftarrow, n)_\varphi$ if it is φ-positive or $M_n' = M_n \cap (n, \rightarrow)_\varphi$ otherwise. Later, we can continue with the proof.

In order to provide a negative answer to Question 1.5, we started to work with $\beta\omega$ index$\beta\omega$ and the universal weak selection. It was possible to achieve here the first step of the construction of the counterexample, which involved to find two appropriate disjoint infinite subsets of ω to take care of a given linear order just as in the proof of Proposition 2.6. At the end, it was easier to work with a Ψ-space instead of a space containing free ultrafilters as elements. We can now return to the original idea and make the construction of a counterexample as a subspace of the Čech-Stone compactification of ω. Actually, it follows immediately from the previous results by identifying every element of the almost disjoint family with an appropriate ultrafilter on ω.

Proposition 2.19. *There is a space $X \subseteq \beta\omega$ which admits a continuous weak selection but is not weakly orderable.*

Proof. Let $\mathcal{B}$ be the almost disjoint family presented in Theorem 2.18. For every $B \in \mathcal{B}$, let p_B be an ultrafilter on ω containing B, i.e. $p_B \in B^*$, and define the space $X = \omega \cup \{p_B : B \in \mathcal{B}\}$. The universal selection φ determines a continuous selection $\psi : [X]^2 \to X$ in the natural way and thus X admits a continuous weak selection. Analogously, for every linear order $\leq$ on X we can find $p_A, p_B \in X$

which guarantee, exactly as in the proof of Theorem 2.18, that $\leq$ does not induce a weaker topology on X. $\square$

Chapter 3

Selections on separable spaces

Once we know that the existence of a continuous weak selection on a space does not guarantee it to be weakly orderable, we can still try to understand why the constructed space in Theorem 2.18 does not satisfy the weak orderability condition and, in a more general way, how close separable spaces are to being weakly orderable.

To start, we should go back to the idea of defining an appropriate linear order by considering the natural relation determined by a continuous weak selection. Recall that for a given space X and a continuous weak selection ψ on X, the relation $\leq_\psi$ defined by $x \leq_\psi y$ if and only if either $\psi(x,y) = x$ or $x = y$ is a reflexive, transitive and total relation. However, $\leq_\psi$ cannot be in general a linear order on X because it is not transitive: We can find distinct points x, y, z in X such that $x \to y$ and $y \to z$ but $x \not\to z$, i.e. $\{x, y, z\}$ is a 3-cycle with respect to ψ. In the connected case we do not have this situation because, as it was shown in Proposition 1.24, the existence of a 3-cycle implies the existence of a finite clopen partition of the space. On other hand, in the separable case it is clear that we cannot avoid the existence of 3-cycles but, as we will see in this chapter, these special triples can help us to obtain special properties in the collection of separable spaces with respect to orderability conditions and extension properties of continuous weak selections.

3.1 2-to-1 functions on linear orders

If X is a space that admits a continuous 1-to-1 function f from X into a linear order $(L, \leq)$ then we can define a linear order $\sqsubseteq$ on X by $x \sqsubseteq y$ if and only if $f(x) \leq f(y)$. It implies, by the continuity of f, that the order $\sqsubseteq$ induces a weaker topology on X, i.e. the space is weakly orderable. It turns out that for separable spaces admitting a continuous weak selection we cannot always find an injective function to a linear space. However, as we will show in Section 3.2, it will be possible to define a 2-to-1 continuous function.

A relation R on a set X is said to be *total* if for every $x, y \in X$, either $(x, y) \in R$ or $(y, x) \in R$. If X is equipped with a topology, the relation R is said to be *closed* if it is closed with respect to the product topology on $X \times X$. We will also say that R *separates* the points $x, y \in X$ if either $(x, y) \notin R$ or $(y, x) \notin R$. Finally, recall that for a given weak selection ψ on a space X, any point $x \in X$ determines in a natural way two closed subsets: $L_x = \{z \in X : z \leftarrow x\}$ and $U_x = \{z \in X : x \leftarrow z\}$.

Proposition 3.1. *Let X be a separable space that admits a continuous weak selection ψ. Then there is a closed, reflexive, total and transitive relation $R \subseteq X \times X$ such that $|\{z \in X : (x, z) \in R$ and $(z, x) \in R\}| \leq 2$ for all $x \in X$.*

Proof. Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of X. Enumerate by $\mathcal{T} = \{T_n : n \in \omega\}$ the set of all triples $T \in [D]^3$ that are 3-cycles with respect to ψ. For every $n \in \omega$, let $\mathcal{E}_n$ be the canonical partition determined by T_n, as in Proposition 1.24. We will recursively construct for $n \in \omega$ closed relations $R_n \subseteq X \times X$, where each R_n will refine the previously defined relations. We will carry out the construction in such a way that, for every $n \in \omega$, R_n will separate all points x and y in $X \setminus \{d_n\}$ that have not been separated by the relations defined before and such that $\{d_n\} \not\| \{x, y\}$. We will also require R_n to refine the partition $\mathcal{E}_{n-1}$ such that x and y will be separated by R_n if they belong to distinct elements of $\mathcal{E}_{n-1}$.

Let $R_0 = X \times X$ and suppose that we have defined the relation R_n which is closed, reflexive, total and satisfies the following condition:

There is a unique finite family $\mathcal{C}_n = \{C_0, \ldots, C_{k_n}\}$ of closed subsets of X such that for every $x, y \in X$ and $i < j \leq k_n$:

(1) $X = \bigcup\{C_l : l \leq k_n\}$,

(2) $\mathcal{C}_n$ *is a refinement of the partition* $\mathcal{E}_{n-1}$*, where* $\mathcal{E}_{-1} = \{X\}$,

(3) If $x, y \in C_i$ *then* $(x, y) \in R_n \cap R_n^{-1}$,

(4) If $x \in C_i$ *and* $y \in C_j \setminus C_i$ *then* $(y, x) \notin R_n$,

(5) If $C_i \cap C_j \neq \emptyset$ *then* $C_i \cap C_j = \{d_l\}$ *for some* $l < n$ *and* $C_i \cap C_j \neq \emptyset$ *only when* $j - i = 1$,

(6) If $d \in C_j \cap \{d_l : l < n\}$ *and* $z \to d$ *for some* $z \in C_j \setminus \{d\}$ *then* $C_j \rightrightarrows \{d\}$,

(7) If $d \in C_j \cap \{d_l : l < n\}$ *and* $d \to z$ *for some* $z \in C_j \setminus \{d\}$ *then* $\{d\} \rightrightarrows C_j$,

(8) If $d \in C_i \cap C_{i+1}$*, then* $C_{i+1} \rightrightarrows \{d\} \rightrightarrows C_i$.

Conditions (3) and (4) guarantee uniqueness of the family $\mathcal{C}_n$. These properties certainly state an equivalence between C_n and R_n: $(x, y) \notin R_n$ if and only if there are $i < j \leq k_n$ such that $y \in C_i$ and $x \in C_j \setminus C_i$. The rest of conditions settles how we will recursively extend the closed relations.

Step 1: The first approximation of the relation R_{n+1}

Consider the point $d_n \in D$ and let $i \leq k_n$ be such that $d_n \in C_i$. If d_n is isolated then define the sets $C_{i,0} = C_i \cap L_{d_n} \setminus \{d_n\}$, $C_{i,1} = \{d_n\}$ and $C_{i,1} = C_i \cap U_n \setminus \{d_n\}$. Define also the set

$$S_n = R_n \setminus \{(x, y) : x \in C_{i,l}, y \in C_{i,s} \text{ and } 0 \leq s < l \leq 2\}.$$

Otherwise, if d_n is not isolated, put $C_{i,0} = C_i \cap L_{d_n}$ and $C_{i,1} = C_i \cap U_{d_n}$. In this case, let

$$S_n = R_n \setminus \{(x, y) : x \in C_{i,1} \setminus \{d_n\}, y \in C_{i,0} \setminus \{d_n\}\}.$$

In any case, the relation S_n is total and reflexive since so is R_n and we are only separating points x and y in C_i that were not separated before. We claim that S_n is also closed. Indeed, suppose that $(x, y) \notin S_n$. Since R_n is closed and $S_n \subseteq R_n$ we can also suppose that $(x, y) \in R_n$, which yields that $x, y \in C_i$, $x \to d_n$ and $d_n \to y$. By conditions (5) and (8) we get that $x \notin C_{i-1}$ since otherwise this would imply $x \in C_{i-1} \cap C_i$ and thus, in particular, $C_i \rightrightarrows \{x\}$.

But $d_n \in C_i$ and $x \to d_n$. Similarly, we can prove that $y \notin C_{i+1}$. Therefore, there are disjoint neighborhoods U_x and U_y of x and y respectively such that $U_x \subseteq (C_i \cup C_{i+1}) \setminus (C_{i-1} \cup C_{i+2})$ and $U_y \subseteq (C_{i-1} \cup C_i) \setminus (C_{i-2} \cup C_{i+1})$. In the case when $x = d_n$ is isolated, we can take $U_x = \{d_n\}$ and analogously for U_y. Define $U_x^{'} = U_x \cap (L_{d_n} \setminus \{d_n\})$ and $U_y^{'} = U_y \cap (U_{d_n} \setminus \{d_n\})$. The open subset $V = U_x^{'} \times U_y^{'} \subseteq X \times X$ contains the ordered pair (x, y) and, by condition (4) and the definition of S_n, is disjoint from S_n. We proved that S_n is closed.

Define now the collection $\mathcal{C}^{'} = \{C_0^{'}, \ldots, C_{k_n+2}^{'}\}$, where $C_j^{'} = C_j$ for $0 \leq j < i$, $C_i^{'} = C_{i,0}$, $C_{i+1}^{'} = C_{i,1}$, $C_{i+2}^{'} = C_{i,2}$ and $C_j^{'} = C_{j-2}$ if $i + 2 < j \leq k_n + 2$. When d_n is not isolated, take $C_{i+2}^{'} = \emptyset$.

Notice that we are only refining the element $C_i \in \mathcal{C}$ and we do it in such a way that the closed relation S_n together with $\mathcal{C}^{'}$ satisfies conditions $(1) - (8)$, except possibly condition (2). We only require to refine the relation S_n so as to obtain a refinement of the partition $\mathcal{E}_n$.

Step 2: A refinement of $\mathcal{E}_n$ and definition of R_{n+1}

Define $k_n^{'} = k_n + 2$ if d_n is isolated and $k_n^{'} = k_n + 1$ otherwise and fix $j \leq k_n^{'}$. We will find a partition $\mathcal{D}_j$ of $C_j^{'}$ consisting of closed subsets which refines $\mathcal{E}_n$. We have to consider separately five possible cases:

Case 1: There is an $E \in \mathcal{E}_n$ such that $C_j^{'} \subseteq E$. In this case we are done; simply define the trivial partition $\mathcal{D}_j = \{D_{j,0}\}$, where $D_{j,0} = C_j^{'}$.

Case 2: $\{E \in \mathcal{E}_n : E \cap C_j^{'} \neq \emptyset\} = \{E_0, \ldots, E_t\}$, where $0 < t \leq 5$, $C_j^{'} \cap \{d_l : l \leq n\} = \{d\}$, $\{d\} \rightrightarrows C_j^{'}$ and $d \in E_t$. In this case, we only need to guarantee that the set $E_t \cap C_j^{'}$ is above any other element of the desired partition with respect to the refinement of S_n, where "above" with respect to a relation S means that $(x, y) \in S$ for every $x \in C_j^{'}$ and $y \in E_t \cap C_j^{'}$. We obtain this by defining $\mathcal{D}_j = \{D_{j,l} : 0 \leq l \leq t\}$, where $D_{j,l} = E_l \cap C_j^{'}$ for every l.

Case 3: $\{E \in \mathcal{E}_n : E \cap C_j^{'} \neq \emptyset\} = \{E_0, \ldots, E_t\}$, where $0 < t \leq 5$, $C_j^{'} \cap \{d_l : l \leq n\} = \{d\}$, $C_j^{'} \rightrightarrows \{d\}$ and $d \in E_0$. We want to define our partition such that $E_0 \cap C_j^{'}$ is below any other element of the desired partition. "Below" has an analogous meaning as "above" in the previous case. Again, define $\mathcal{D}_j = \{D_{j,l} : 0 \leq l \leq t\}$, where $D_{j,l} = E_l \cap C_j^{'}$ for every l.

Case 4: $\{E \in \mathcal{E}_n : E \cap C_j^{'} \neq \emptyset\} = \{E_0, \ldots, E_t\}$, with $0 < t \leq 5$, $C_j^{'} \cap \{d_l : l \leq n\} = \{d, d^{'}\}$, $d^{'} \to d$, $d \in E_0$ and $d^{'} \in E_t$. This is a combination of Cases

2 and 3. We need to simultaneously guarantee the two conditions presented in cases 4 and 5. But the partition $\mathcal{D}_j$ defined as before will do.

Case 5: $\{E \in \mathcal{E}_n : E \cap C_j' \neq \emptyset\} = \{E_0, \ldots, E_t\}$ with $0 < t \leq 5$, $C_j' \cap \{d_l : l \leq n\} = \{d, d'\}$, $d' \to d$ and $d, d' \in E_0\}$. In this case, we need to split the element E_0 to obtain a subset above any element of C_j' and another subset below. Take $x \in C_j' \setminus E_0$. By conditions (6) and (7), $d' \to x$ and $x \to d$. Let $\mathcal{D}_j = \{D_{j,l} : 0 \leq l \leq t+1\}$, where $D_{j,0} = C_j' \cap E_0 \cap \{y \in X : y \leftarrow x\}$, $D_{j,l} = C_j' \cap E_l$ for $1 \leq l \leq t$ and $D_{j,t+1} = C_j' = C_j' \cap E_0 \cap \{y \in X : x \leftarrow y\}$.

Define now $\mathcal{C}_{n+1} = \bigcup\{\mathcal{D}_j : j \leq k_n'\}$. By construction, we can enumerate $\mathcal{C}_{n+1}$ as $\{D_j : j \leq k_{n+1}\}$ in such a way that, for every $i \leq j \leq k_{n+1}$, if $x \in D_i$ and $y \in D_j$ then $(x, y) \in S_n$. Therefore, $\mathcal{C}_{n+1}$ is the refinement we are looking for. Finally, define the relation R_{n+1} as follows:

$$R_{n+1} = S_n \setminus \{(x, y) : y \in D_i, x \in D_j \text{ and } i < j\}.$$

We have obtained a transitive, reflexive and total relation, contained in R_n and refining $\mathcal{E}_n$. In fact, it can be proved, in the same way as for S_n, that the relation R_{n+1} is also closed and, together with $\mathcal{C}_{n+1}$, it satisfies conditions $(1)-(8)$. This concludes the recursive construction.

Let $R = \bigcap\{R_n : n \in \omega\}$. The relation R is closed, reflexive and total because so is each R_n is. We only need to prove that R is transitive. To show this we will use two auxiliary properties of R.

Fact 1: If $x, y \in X$ and there is a $d \in D$ such that $x \to d \to y$ and d belongs to a 3-cycle with respect to ψ, then $(x, y) \notin R \cap R^{-1}$.

Let $n \in \omega$ be such that $d \in T_n$. Since $\{x, y\} \not\parallel \{d\}$, the points x and y do not belong to the same element of the partition $\mathcal{E}_n$ and thus either $(x, y) \notin R_{n+1}$ or $(y, x) \notin R_{n+1}$

By density of D, we can generalize this property to get: If $x, y \in X$ and there is a $z \in X$ such that $\{x, y, z\}$ forms a 3-cycle, then $(x, y) \notin R \cap R^{-1}$.

Fact 2: For any $x \in X$, the set $P_x = \{z \in X : (x, z) \in R \cap R^{-1}\}$ contains at most two points.

Suppose the contrary. Let y and z be distinct points of $P_x \setminus \{x\}$. We can also suppose, without loss of generality, that $x \to y \to z$ (the other cases are treated analogously). Since D is dense, we can find a $d \in D$ such that $x \to d \to y$. Let

$k = \min\{l \in \omega : \{x, y\} \not\parallel \{d_l\}\}$ and let $\mathcal{C}_k$ be the collection that together with the relation R_k satisfies $(1) - (7)$. Since $y \in P_x$, there is a $C \in \mathcal{C}_k$ such that $x, y \in C$. We also have that $d_k \notin C$ because otherwise the relation R_{k+1} (and so R) would separate the points x and y, contradicting our first assumption. The construction of R_k implies that there is an $l < k$ such that either $\{x, y, d_k\} \not\parallel \{d_l\}$ or x, y and d_k do not belong to the same element of the canonical partition $\mathcal{E}_l$ associated to the 3-cycle T_l or there are $E \in \mathcal{E}_l$ and $w \in X \setminus E$ with $x, y, d_k \in E$, $\{x, y\} \| \{w\}$ and $\{x, y, d_k\} \not\parallel \{w\}$. In the first case, either $\{x, y\} \rightrightarrows \{d_l\}$ and $d_k \leftarrow d_l$ or $\{d_l\} \rightrightarrows \{x, y\}$ and $d_l \leftarrow d_k$. In any of these subcases, we can build a 3-cycle containing d_k. In an analogous way, we can prove the same in the third case. Finally, when the second possibility occurs, we can also find a 3-cycle containing the point d_k.

To conclude the proof, we need to show that R is transitive. Let $x, y, z \in X$ be such that $(x, y) \in R$ and $(y, z) \in R$. Aiming at a contradiction, suppose that $(x, z) \notin R$ and $(z, x) \in R$. Let $n \in \omega$ be such that $(x, z) \notin R_n$, and let $\mathcal{C}_n$ be the corresponding family determined by R_n. There are three possible cases:

Case 1: $(x, y) \in R \cap R^{-1}$. Since $(x, z) \notin R_n$, there are $i < j \leq k_n$ such that $z \in C_i$ and $x \in C_j \setminus C_i$. But (x, y), (y, x) and (y, z) belong to R_n, which implies that $j = i + 1$ and $y \in C_i \cap C_{i+1}$. Notice that $(y, x) \in R_n \cap R_n^{-1}$, i.e.˜the points y and z are indistinguishable until the n-th stage. But, by Fact 2 they must eventually be separated. However, since $y = d_l$ for some $l < n$ and $C_n \rightrightarrows \{y\}$, the construction of the relation R yields that $(z, y) \in R$, which is not possible.

Case 2: $(y, z) \in R \cap R^{-1}$ is analogous to Case 1.

Case 3: $(y, x) \notin R$ and $(z, y) \notin R$. As in the previous cases, there is $n \in \omega$ such that, if $\mathcal{C}_n$ is as before, there are $i < j < k$ such that $z \in C_i$, $x \in C_j \setminus C_i$ and $y \in C_k \setminus C_j$. But $k > i + 1$, which implies that $C_i \cap C_k = \emptyset$ and thus $y \in C_k \setminus C_i$. We conclude, by condition (4), that $(y, z) \notin R_n$, which is a contradiction. □

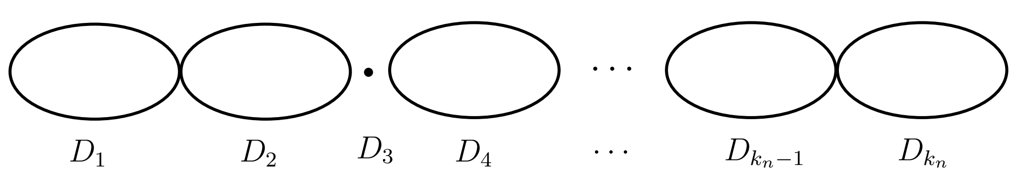

Figure 3.1: The relation R_n

3.2 Almost weakly orderable spaces

The relation R defined above is determined by the dense subset, the weak selection and the set of 3-cycles in the space X. Therefore, if we work with another countable dense subset, or even the same dense set but with another enumeration, we can obtain different closed relations. In any case, the obtained relation fails to be a linear order if there are points x and y in X that are "indistinguishable" with respect to the weak selection ψ in the sense that for any point z distinct from x and y, $x \to z$ if and only if $y \to z$. Certainly, as we will shall see later, the problem occurs if the collection of pairs consisting of indistinguisable points is uncountable. This is the case with the counterexample presented in Theorem 2.18, where the Ψ-space was constructed in such a way that for any element of the **AD** family we can find another element indistinguishable with this.

The next proposition states that for a separable space, we are close to separate points under the existence of a continuous weak selection and thus, we are also close to define a topology on the space, weaker than the original and determined by a linear order.

Corollary 3.2. *Let X be a separable space that admits a continuous weak selection ψ. Then there are an ordered space L and a continuous function $f : X \to L$ satisfying:*

(i) $|f^{-1}[\{y\}]| \leq 2$ *for every* $y \in L$ *(i.e., f is $\leq$ 2-to-1),*

(ii) if $\{x, y, z\}$ *is a 3-cycle with respect to* ψ*, then* $f \restriction \{x, y, z\}$ *is injective.*

Proof. Let R be the closed, transitive, reflexive and total relation constructed in Proposition 3.1. Let D be the countable dense subset of X required to define R and let $\mathcal{T} = \{T \in [D]^3 : T$ is a 3-cycle with respect to $\psi\}$. Define the relation $\sim_R$ on X as follows:

$$x \sim_R y \text{ if and only if } P_x = P_y,$$

where $P_z = \{w \in X : (z, w) \in R \cap R^{-1}\}$.

By Proposition 3.1 we know that P_x contains at most two points for every $x \in X$. In particular, when x is an isolated element of D then $P_x = \{x\}$. Hence, the relation $\sim_R$ is in fact an equivalence relation. Let $L = X/\sim_R$ be the set of equivalence classes determined by $\sim_R$. Define the relation $\leq$ on L as follows:

$$[x]_{\sim_R} \leq [y]_{\sim_R} \text{ if and only if either } x = y \text{ or } (x,y) \in R \text{ and } (y,x) \notin R.$$

After we have identified every pair of indistinguishable points on X using the relation $\sim_R$, we can guarantee the antisymmetry of $\leq$. The rest of order properties also hold because the relation R satisfies them. With all this, $\leq$ is a linear order.

Define the function $f : X \to L$ by $f(x) = [x]_{\sim_R}$. Clearly, $|f^{-1}[\{y\}]| \leq 2$ for every $y \in L$. Also, if $\{x, y, z\}$ is a 3-cycle then, by density of D, we can find a $T \in \mathcal{T}$ such that the points x, y and z belong to distinct elements of the canonical partition determined by T, which implies that $f \restriction \{x, y, z\}$ is injective. To verify continuity of f, let F be a closed subset of L. Let us prove that $f^{-1}[F]$ is closed. Take $x \in X \setminus f^{-1}[F]$, i.e. $f(x) \notin F$. Since F is closed, there are $a, b \in X$ such that $[a]_{\sim_R} < [x]_{\sim_R} < [b]_{\sim_R}$ and $([a]_{\sim_R}, [b]_{\sim_R})_{\leq} \cap F = \emptyset$. Hence, we can find $n \in \omega$ such that $(x, a) \notin R_n$ and $(b, x) \notin R_n$. Therefore, we can find open subsets $W_0 \subseteq X$ and $V_0 \subseteq X$ such that $(x, a) \in W_0 \times V_0 \subseteq (X \times X) \setminus R_n$. The same way, let W_1 and V_1 be neighborhoods of b and x such that $W_1 \times V_1 \subseteq (X \times X) \setminus R_n$. Finally, the open subset $V = W_0 \cap V_1$ contains the point x and $f[V] \cap F = \emptyset$, which guarantees the continuity of f. $\square$

If we enumerate by $\{(x_\alpha, y_\alpha) : \alpha < \kappa\}$ the collection of all pairs $(x, y) \in X \times X$ such that $x \leftarrow y$ and $x \sim_R y$, then $X = X_0 \cup X_1$, where $X_0 = X \setminus \{y_\alpha : \alpha < \kappa\}$ and $X_1 = X \setminus \{x_\alpha : \alpha < \kappa\}$. Notice that X_i is weakly orderable for $i \in 2$. Therefore, X can be written as the union of two weakly orderable spaces.

3.3 Second countable spaces

As mentioned in Chapter 1, Costantini [7] proved that every second countable space that admits a continuous weak selection is weakly orderable if it has a countable dense subset consisting of isolated points. Gutev [28] generalized this result by suppresing the extra condition of the dense subset. In this section we provide another proof of the result due to Gutev. For this purpose, we apply some of the ideas involved in Costantini's proof.

Proposition 3.3. *([28]) Every second countable space which admits a continuous weak selection is weakly orderable.*

Proof. Let ψ be a continuous weak selection on X, $D = \{d_n : n \in \omega\}$ a countable dense subset of X and let $\mathcal{T} = \{T \in [D]^3 : T \text{ is a 3-cycle}\}$. Consider the closed relation R determined by D and $\mathcal{T}$ as in Proposition 3.1. In order to obtain an adequate linear order on X whose induced topology is weaker than the (original) topology of X, we will take care of all pair of points of X that are indistinguishable with respect to R.

Define the set $\mathcal{P} = \{(x, y) \in X \times X : x \sim_R y \text{ and } x \leftarrow y\}$.

Fact 1: If $(x, y) \in \mathcal{P}$, then U_x and L_y are disjoint clopen sets.

To prove that U_x is open we only need to prove that x is an interior point of U_x. Since $y \to x$, we can find two disjoint neighborhoods W_x and W_y of x and y respectively such that $W_y \rightrightarrows W_x$. We know that x and y are indistinguishable with respect to R (and also with respect to ψ) and thus, for all $z \in X$, $y \to z$ if and only if $x \to z$. Hence, $\{x\} \rightrightarrows W_x$ and $x \in W_x \subseteq U_x$, which implies that U_x is open. By the same argument, $W_y \rightrightarrows \{y\}$ and so L_y is clopen.

Fact 2: The set $\mathcal{P}$ is relatively discrete in $X \times X$.

Take $(x, y) \in \mathcal{P}$ and let W_x and W_y be as in Fact 1. We claim that the open set $W = W_x \times W_y$ satisfies $W \cap \mathcal{P} = \{(x, y)\}$. Indeed, consider any $(x^{'}, y^{'}) \in W$. Since $W_y \rightrightarrows W_x$ and $y^{'} \in W_y$, we have that $y^{'} \to x$, which also implies that $y^{'} \to y$. Similarly, given that $x^{'} \in W_x$ we have $y \to x^{'}$. Therefore, $y^{'} \to y \to x^{'}$ and hence $(x^{'}, y^{'}) \notin \mathcal{P}$.

Note that these two properties do not require the space to be second countable.

Fact 3: The set $\mathcal{P}$ is countable.

Let $\mathcal{B}$ be a countable base for $X \times X$ and, for every $(x, y) \in \mathcal{P}$, choose $B_{x,y} \in \mathcal{B}$ such that $B_{x,y} \cap \mathcal{P} = \{(x, y)\}$. Clearly, $B_{x,y} \neq B_{x^{'},y^{'}}$ when $(x, y) \neq (x^{'}, y^{'})$. This guarantees that $|\mathcal{P}| \leq \omega$.

Enumerate the set $\mathcal{P}$ as $\{(x_k, y_k) : k \in \omega\}$ and fix $n \in \omega$. In the proof of Proposition 3.1 we can refine the relation R_n to a closed relation Q_n such that whenever $(x, y) \in R_n \cap R_n^{-1}$, $x \in U_{x_n}$ and $y \in L_{y_n}$ then $(y, x) \notin Q_n$. Indeed, put $Q_n = R_n \setminus \{(y, x) : (x, y) \in R_n, x \in U_{x_n} \text{ and } y \in L_{y_n}\}$.

Notice that if $\mathcal{C}_n = \{C_i : i \leq k_n\}$ is the collection of closed sets associated to the relation R_n then, since $x_n \sim_R y_n$, we can find an $r \leq k_n$ such that $x_n, y_n \in C_r$ and, again because x_n and y_n are indistinguishable, for every $i \neq r$ either $C_i \subseteq U_{x_n}$ or $C_i \subseteq L_{y_n}$. Thus, the collection $C_n' = \{C_0, \ldots, C_{r-1}, C_{r,0}, C_{r,1}, C_{r+1}, \ldots, C_{k_n}\}$ satisfies properties $(1)-(7)$ in the proof of Proposition 3.1, where $C_{r,0} = C_r \cap U_x$ and $C_{r,1} = C_r \cap L_x$. Hence, we can continue the construction as before. Finally, let $Q = \bigcap\{Q_n : n \in \omega\}$. If $(x, y) \in Q$ then $(x, y) \notin R$, $(x_n, y_n) \in Q$ and $(y_n, x_n) \notin Q$ for every $n \in \omega$. Therefore, the relation Q is in fact a linear order and thus X is weakly orderable. $\square$

Another consequence of the last result is: Every separable space that admits a continuous weak selection ψ is weakly orderable whenever the set of indistinguishable pairs, with respect to ψ, is at most countable. This is behind the motivation to add an extra condition in the van Mill-Wattel problem and to state the following question.

Question 3.4. *Let ψ be a continuous weak selection on a space X such that for every $x, y \in X$ there is $z \in X \setminus \{x, y\}$ which satisfies $\{x, y\} \not\parallel \{z\}$. Is X a weakly orderable space?*

3.4 From $\mathcal{F}_2(X)$ to $\mathcal{F}_3(X)$

Recall that every (weakly) orderable space X admits a natural continuous weak selection determined by its compatible linear order. Moreover, for any $n \in \omega$ the selection $\min : \mathcal{F}_n(X) \to X$ is continuous. Indeed, suppose that $\preceq$ is a linear order on X whose generated topology is weaker than the topology of X and let $F = \{x_0, \ldots, x_k\}$, with $k \leq n$ and such that $x_i \prec x_j$ if $i < j$. Take disjoint open sets W_i, for $i \leq k$, containing x_i and such that $x \prec y$ if $x \in W_i$, $y \in W_j$ and $i < j$. Then the open set (in the subspace $\mathcal{F}_n(X)$) $\mathcal{W} = \langle W_0, W_1, \ldots, W_k \rangle$ contains F and $\min[\mathcal{W}] \subseteq W_0$, which guarantees the continuity of min.

Michael [44] in fact obtained a more general result and proved that every weakly orderable space admits a continuous selection for the collection of its compact subsets, i.e. $Sel(\mathcal{K}(X)) \neq \emptyset$ (the selection min also works in this case). However, we cannot always extend the given selection from $\mathcal{K}(X)$ to the hyperspace 2^X since we can find ordered spaces, for instance the real line, such that $Sel(2^X) = \emptyset$.

In the same spirit and also trying to obtain an answer to the van Mill-Wattel problem, the following question was stated by Gutev and Nogura:

Question 3.5 ([24]). *Is there exist a space X which admits a continuous weak selection, but $Sel(\mathcal{F}_n(X)) = \emptyset$ for some $n > 2$.*

A positive solution to the van Mill-Wattel problem would entail a negative answer to Gutev and Nogura's question. Thus, now when we know that Question 1.22 has a negative answer, the previous problem acquires a greater importance. It should also be mentioned that Question 3.5 is still open even when $n = 3$. We present in this section some extension results for the first case $n = 3$.

The first known result in this direction is due to J. Steprans.

Proposition 3.6 ([57]). *Let X be a separable space with a countable dense subset of isolated points. If X admits a continuous weak selection then $Sel([X]^3) \neq \emptyset$.* $\square$

García Ferreira, Gutev and Nogura found the following result. It states that if we want to extend a continuous selection from X to $\mathcal{F}_3(X)$, we just need to know how to continuously choose elements from triples. We present here a proof of this result, somewhat different from the original.

Proposition 3.7 ([18]). *Let ψ be a continuous weak selection and let $\rho : [X]^3 \to X$ be a continuous selection. Then $\mathcal{F}_3(X)$ admits a continuous selection ϕ such that $\phi \restriction \mathcal{F}_2(X) = \psi$.*

Proof. We will define the selection ϕ by cases. Let $F \in \mathcal{F}_3(X)$.

Case 1: $F \in \mathcal{F}_2(X)$. In this case, let $\phi(F) = \psi(F)$.

Case 2: $F \in [X]^3$ and F is not a 3-cycle with respect to ψ. Hence, we can find a unique $x \in F$ such that $F \rightrightarrows \{x\}$. In this case, define $\phi(F) = x$.

Case 3: $F \in [X]^3$ and F is a 3-cycle with respect to ψ. In this case define $\phi(F) = \rho(F)$.

To prove the continuity of ϕ, suppose first that $F = \{x, y\}$ and $\phi(F) = \psi(F) = x$. By the continuity of ψ, we can find disjoint neighbourhoods U and

V of x and y, respectively, such that $V \rightrightarrows U$. Let $\mathcal{U} = \langle U, V\rangle$. If $G \in \mathcal{U} \cap \mathcal{F}_3(X)$ then $|G| = 2$ or $|G| = 3$ but, in either case, we can find $w \in G \cap U$ such that $G \rightrightarrows \{w\}$, which implies that $\phi(G) = w \in U$.

Suppose now that $F = \{x, y, z\}$ is not a 3-cycle and $\phi(F) = x$. Hence $F \rightrightarrows \{x\}$ and, again by the continuity of ψ, we can find disjoint open sets U and V, where $x \in U$ and $y, z \in V$, such that $V \rightrightarrows U$, which guarantees continuity of ϕ in F.

Finally, suppose that $F = \{x, y, z\}$, $x \to y \to z \to x$ and $\phi(F) = \rho(F) = x$. By the continuity of ρ and of ψ, we can find pairwise disjoint open sets U, V and W such that $x \in U$, $y \in V$, $z \in W$, $U \rightrightarrows V \rightrightarrows W \rightrightarrows U$ and, if $\mathcal{U} = \langle U, V, W\rangle$, then $\rho[\mathcal{U}] \subseteq U$. Notice that any $G \in \mathcal{U} \cap [X]^3$ is a 3-cycle and therefore $\phi(G) = \rho(G) \in U$. We conclude that ϕ is continuous. □

In the same paper, they also stated the following extension result.

Proposition 3.8 ([18]). *Let X be a space, $p \in X$, and let $f : \mathcal{F}_2(X) \to X$ and $g : \mathcal{F}_3(X \setminus \{p\}) \to X$ be continuous selections. Then f can be extended to a continuous selection for $\mathcal{F}_3(X)$.* □

Finally, the next proposition, due to Gutev and Nogura, sets an extension property from triples to quadruples.

Proposition 3.9 ([30]). *If X is a space such that $Sel(\mathcal{F}_3(X)) \neq \emptyset$, then $Sel(\mathcal{F}_4(X)) \neq \emptyset$.* □

3.5 Extension of continuous selections

Recall that, given a weak selection ψ on a space X, the sets $A, B \subseteq X$ are said to be aligned (with respect to ψ), denoted by $A||B$, if either $A \rightrightarrows B$ or $B \rightrightarrows A$. We can go beyond and consider a more general situation, working not only with pairs but with families of aligned sets. The discussion of this case is presented in [18], where the notion of the so called *decisive sets* is introduced.

Definition 3.10. Let ψ be a weak selection on a space X. A set $\mathcal{C} \subseteq \wp(X)$ is *ψ-decisive* if $A||_\psi B$ for any $A, B \in \mathcal{C}$.

When there is no danger of confusion we shall write *decisive* instead of ψ-decisive. Similarly, a *decisive partition* of a set $F \subseteq X$ is a partition of F where any pair of elements is aligned with respect to the weak selection.

For any space X and any weak selection on it, X admits the trivial decisive partitions $\mathcal{C} = \{\{x\} : x \in X\}$ and $\mathcal{E} = \{X\}$. When X is an ordered set, we can find many distinct decisive partitions of size 2 with respect to the min selection. Indeed, if $\leq$ is a linear order on X, then for any $x \in X$ the partition $\mathcal{P} = \{(\leftarrow, x)_{\leq}, [x, \rightarrow)_{\leq}\}$ is decisive. In the general case, we can study how sets behave aligned with respect to a weak selection by defining the *decisive index*.

Definition 3.11 ([27])**.** Let ψ be a weak selection on X and let $F \subseteq X$. The *decisive index* of F with respect to ψ, denoted by $di(F, \psi)$, is 1 if $|F| = 1$ and

$$di(F, \psi) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a decisive partition and } |\mathcal{P}| \geq 2\}$$

otherwise.

Our aim will be to answer Question 3.5 in the case of separable spaces and, more generally, to work with the collection of finite subsets $\mathfrak{Fin}(X)$ of an arbitrary separable space X. Therefore, we are interested in the study of the decisive index of finite subsets. The following uniqueness result will be useful for extending continuous selections in a way more general than in the previous section.

Proposition 3.12 ([27])**.** *Let ψ be a weak selection on a space X and let $F \in \mathfrak{Fin}(X)$. If $di(F, \psi) > 2$ then there is a unique decisive partition $\mathcal{P}$ on F with $|\mathcal{P}| = di(F, \psi)$. Moreover, if $\mathcal{M}$ is another decisive partition of F then for every $M \in \mathcal{M}$ there is a $P \in \mathcal{P}$ such that $M \subseteq P$ (i.e. $\mathcal{M}$ refines $\mathcal{P}$).* $\square$

The above result is not true when $di(F, \psi) = 2$. For instance, any ordered set does not satisfy it when $\psi = min$. However, although a finite subset can admit more than one decisive partition of size 2, we can define a sort of minimal element between all these decisive partitions.

Definition 3.13. Let ψ be a weak selection on a space X and let $B \subseteq X$. We will say that B is ψ-minimal if it satisfies the following conditions:

(1) $(X \setminus B) \rightrightarrows B$,

(2) $B \subseteq C$ whenever $C \subseteq X$ and $(X \setminus C) \rightrightarrows C$.

Clearly the ψ-minimal subset, if it exists, is unique. For the compact case (and in particular for the finite case), García Ferreira et al. [18] proved that the minimal set exists. We include here the proof of their result.

Proposition 3.14 ([18]). *Let ψ be a weak selection on a space X. Then every non-empty compact set F has a unique ψ-minimal set.*

Proof. Consider the family

$$\mathcal{D} = \{B \in 2^F : (F \setminus B) \rightrightarrows B\}.$$

Notice that the family $\mathcal{D}$ is non empty because $F \in \mathcal{D}$. We claim that any pair of elements of $\mathcal{D}$ are $\subseteq$-comparable. To show this, suppose that we can find B and C in $\mathcal{D}$ such that $B \setminus C \neq \emptyset \neq C \setminus B$. Take $x \in B \setminus C$ and $y \in C \setminus B$. Since $y \in F \setminus B$, we have that $y \to x$. The same way $x \in F \setminus C$, which implies that $x \to y$. The two conditions imply $x = y$, a contradiction.

Since $\mathcal{D}$ is a subset of $\mathcal{K}(X)$ and, by this property, has the finite intersection property, it yields $D = \bigcap \mathcal{D} \neq \emptyset$. Finally, if $x \in D$ and $y \notin D$ then there is a $B \in \mathcal{D}$ such that $x \in B$ and $y \notin B$, which implies that $y \to x$. Therefore, D is ψ-minimal. □

The next result is due to Gutev.

Proposition 3.15 ([27]). *Let X be a space and let $\psi \in Sel(\mathcal{F}_2(X))$. The function $di_\psi : (\mathfrak{Fin}(X) \setminus [X]^1) \to \omega$ defined by $di_\psi(F) = di(F, \psi)$ is continuous.* □

Applying the above proposition, he obtained a generalization of Proposition 3.7 for every $n \in \omega$. For completeness we outline the proof.

Theorem 3.16 ([27]). *Let X be a topological space. If $Sel(\mathcal{F}_n(X)) \neq \emptyset$ and $Sel([X]^{n+1}) \neq \emptyset$ then $Sel(\mathcal{F}_{n+1}(X)) \neq \emptyset$.*

Proof. Let $\psi : \mathcal{F}_n(X) \to X$ and $\rho : [X]^{n+1} \to X$ be continuous selections. Since $[X]^1$ admits a trivial continuous selection, to define a function on $\mathcal{F}_{n+1}(X)$ we will only consider the set $\mathcal{D} = \mathcal{F}_{n+1} \setminus [X]^1$. Put $\psi' = \psi \restriction [X]^2$. The continuous function $di_{\psi'}$, defined above, determines a clopen partition $\{\mathcal{C}_k : 1 \leq k \leq n+1\}$ of $\mathcal{D}$, where $\mathcal{C}_k = \{F \in \mathcal{D} : di(F, \psi') = k\}$ for every $1 \leq k \leq n+1$. Hence, we can separately define a continuous selection ϕ_k on $\mathcal{C}_k$.

Notice that $\mathcal{C}_{n+1} \subseteq [X]^{n+1}$ so that we can define $\phi_{n+1}(F) = \rho(F)$ for every $F \in \mathcal{C}_{n+1}$. Suppose now that $F \in \mathcal{C}_2$ and let $\mathcal{P} = \{P_0, P_1\}$ be the decisive partition containing the ψ-minimal set P_0 of F. In this case, observe that $|P_i| \leq n$ for $i \in 2$, which allows us to define $\phi_2(F) = \psi(P_0)$. Finally, suppose that $F \in \mathcal{C}_k$ for some $3 \leq k \leq n$ and let $\mathcal{P} = \{P_i : i < k\}$ be the decisive partition of F of cardinality k. Again, for every $i < k$ we have $|P_i| \leq k$. In this case, define $\phi_{n+1}(F) = \psi(\{\psi(P_i) : i < k\})$. □

Finally, using graph theory, in particular special functions termed *flows*, Gutev also generalized Proposition 3.9.

Theorem 3.17 ([27]). *Let X be a space such that $Sel(\mathcal{F}_{2n+1}(X)) \neq \emptyset$ for some $n \geq 1$. Then, $Sel([X]^{2n+2}) \neq \emptyset$.* □

3.6 Selections for finite subsets

The goal of this section is to provide a partial answer to Question 3.5 by showing that any separable space that admits a continuous weak selection in fact admits a continuous selection on the collection of its finite subsets.

As an antecedent, we should mention that Jiang has two results that also allow, under suitable conditions, to answer Question 3.5 in the negative.

Proposition 3.18 ([37]). *Let X be a Hausdorff space with a single non-isolated point $p \in X$, and let f be a continuous weak selection. Then f can be extended to a continuous selection on $\mathfrak{Fin}(X)$.* □

Proposition 3.19 ([37]). *If a scattered hereditarily paracompact Hausdorff space X admits a continuous weak selection, then it also admits a continuous selection on $\mathfrak{Fin}(X)$.* □

We first prove the following auxiliary result that gives a partial answer to Question 3.5.

Proposition 3.20. *Let X be a space that admits a continuous weak selection ψ. If there are an ordered space $(Y, \preceq)$ and a continuous function $f : X \to Y$ such that:*

(i) $|f^{-1}[\{y\}]| \leq 2$ *for every* $y \in Y$,

(ii) If $\{x_0, x_1, x_2\}$ *is a 3-cycle with respect to* ψ *then* $f \restriction \{x_0, x_1, x_2\}$ *is injective,*

then there is a sequence $\{\psi_n : n \geq 2\}$ *of compatible continuous selections such that* $\psi_n \in Sel(\mathcal{F}_n(X))$ *for every* $n \in \omega$.

Proof. We will inductively construct the family $\{\psi_n : n \geq 2\}$ of continuous selections. Start with $\psi_2 = \psi$ and suppose that we have defined continuous selections $\psi_k : \mathcal{F}_k(X) \to X$, for $k \leq n$, such that $\psi_{s+1} \restriction \mathcal{F}_s(X) = \psi_s$ for every $s < n$.

We will define the continuous selection $\psi_{n+1} : \mathcal{F}_{n+1}(X) \to X$ by cases:

Case 1: Suppose that $F \in \mathcal{F}_{n+1}$ and $di(F, \psi) \leq n$. Define $\psi_{n+1}(F)$ exactly as in the proof of Theorem 3.16. Indeed, let $\mathcal{P}$ be the decisive partition of F (ψ-minimal partition if $di(F, \psi) = 2$) and define $\psi_{n+1}(F) = \psi_n(\{\psi_n(P) : P \in \mathcal{P}\})$.

Case 2: Suppose that $F \in [X]^{n+1}$ and $di(F, \psi) = n + 1$. If x and y are elements of F and $x \to y$, we claim that there must be an element $z \in F \setminus \{x, y\}$ such that $\{x, y\} \not\parallel \{z\}$ because, otherwise, the partition $\mathcal{P} = \{\{z\} : z \in F \setminus \{x, y\}\} \cup \{x, y\}$ would be a decisive partition of size $n - 1$. Hence, any two elements of F are comparable and the function f restricted to the set F is injective. Therefore, we can find a unique $x \in F$ such that $f(x) = \min\{f(z) : z \in F\}$. Define $\psi_{n+1}(F) = x$.

Case 1 is defined exactly the same way as in the proof of Theorem 3.16, so we only need to prove continuity of ψ_{n+1} in Case 2. Let $F = \{x_0, x_1, \ldots, x_n\} \in [X]^{n+1}$ be such that $x_0 \prec x_1 \prec \cdots < x_n$. Then $\psi_{n+1}(F) = x_0$. By the continuity of ψ, for every $i \leq n$ we can find an open neighbourhood V_i containing x_i and such that $V_i \rightrightarrows V_j$ if and only if $x_i \rightrightarrows x_j$ for $i \neq j$. On the other hand, by the continuity of f, we can find open pairwise disjoint intervals $I_0, \ldots, I_n$ in Y, where each I_i contains $f(x_i)$ and such that, for every $i < j \leq n$, $z \prec w$ for every $z \in I_i$ and $w \in I_j$. For every $i \leq n$ define $W_i = V_i \cap f^{-1}[I_i]$. Finally, if we consider the open set $\mathcal{W} = \langle W_0, \ldots, W_n \rangle$, then $\mathcal{W} \subseteq [X]^{n+1}$, $F \in \mathcal{W}$ and $\psi_{n+1}[\mathcal{W}] \subseteq W_0$, which guarantees the continuity on F.

To conclude the inductive step, we need to show that ψ_{n+1} extends the selection ψ_n. Let $F \subseteq X$ be such that $|F| \leq n$. If $di(F, \psi) = n$ then $\psi_{n+1}(F) = \psi_n\{\{\psi_n(\{x\}) : x \in F\}\} = \psi_n(F)$. Otherwise, if $di(F, \psi) < n$, let $\mathcal{P}$ be the decisive partition of F with $|\mathcal{P}| = di(F, \psi)$. Since $\mathcal{P} < n$ and $|P| < n$ for every $P \in \mathcal{P}$, the inductive hypothesis yields:

$$\psi_{n+1}(F) = \psi_n(\{\psi_n(P) : P \in \mathcal{P}\}) = \psi_{n-1}(\{\psi_{n-1}(P) : P \in \mathcal{P}\}) = \psi_n(F).$$

$\square$

Theorem 3.21. *Let X be a space that admits a continuous weak selection ψ, let Y be an ordered space and $f : X \to Y$ a continuous function as in Proposition 3.20. Then $Sel(\mathfrak{Fin}(X)) \neq \emptyset$.*

Proof. Let $\{\psi_n : n \geq 2\}$ be a sequence of compatible continuous selections, as in Proposition 2.6. For every $n \geq 2$ we have that $\psi_n \in Sel(\mathcal{F}_n(X))$ and $\psi_{n+1} \restriction \mathcal{F}_n(X) = \psi_n$. Define $\Phi = \bigcup\{\psi_n : n \geq 2\}$. It is clear that Φ is a selection on $\mathfrak{Fin}(X)$. To prove the continuity of Φ we will use the following property:

Claim: Let $F \in \mathfrak{Fin}(X)$ and let $\mathcal{M}$ be a decisive partition of F. Then $\Phi(F) = \Phi(\{\Phi(M) : M \in \mathcal{M}\})$.

We argue by induction on $|F|$. Clearly the result is true when $|F| = 2$. Assume that this result is true for every $E \subseteq X$ with $|E| \leq n$ and let $F \in [X]^{n+1}$. Let $\mathcal{M}$ be an arbitrary decisive family of F and put $G = \{\Phi(M) : M \in \mathcal{M}\}$. We will show that $\Phi(F) = \Phi(G)$.

The result is evidently true if $di(F, \psi) = n + 1$ because in this case $\mathcal{M} = \{\{x\} : x \in F\}$. Hence, we can suppose that $2 \leq di(F, \psi) \leq n$. We will consider separately the cases $di(F, \psi) = 2$ and $di(F, \psi) > 2$.

Case 1: Suppose first that $di(F, \psi) = 2$ and let $\mathcal{P} = \{P_0, P_1\}$ be the partition of F, where P_1 is ψ-minimal. In particular, $P_0 \rightrightarrows P_1$ and thus $\Phi(F) = \Phi(P_1)$. For the decisive partition $\mathcal{M}$, we must study separately the cases when $|\mathcal{M}| = 2$ and when $2 < |\mathcal{M}| \leq n$.

Subcase 1: If $\mathcal{M} = \{M_0, M_1\}$ and $M_0 \rightrightarrows M_1$ then $\Phi(G) = \Phi(M_1)$. Since P_1 is minimal, we have $P_1 \subseteq M_1$. Hence $\{M_1 \setminus P_1, P_1\}$ is a decisive partition of M_1 and $|M_1| \leq n$. Therefore, by inductive hypothesis and because $M_1 \setminus P_1 \rightrightarrows P_1$, $\Phi(M_1) = \Phi(\{\Phi(M_1 \setminus P_1), \Phi(P_1)\}) = \Psi(P_1)$ and thus $\Phi(G) = \Phi(P_1)$.

Subcase 2: Suppose now that $\mathcal{M} = \{M_0, \ldots, M_{k-1}\}$ and $2 < k < n$. Define the sets $\mathcal{M}_0 = \{M \in M : M \cap P_0 \neq \emptyset\}$ and $\mathcal{M}_1 = \{M \in \mathcal{M} : M \cap P_1 \neq \emptyset\}$. We assert that $|\mathcal{M}_0 \cap \mathcal{M}_1| \leq 1$. Aiming at a contradiction, suppose that $M, M' \in \mathcal{M}_0 \cap \mathcal{M}_1$. Hence, $M \cap P_0 \rightrightarrows M' \cap P_1$ and also $M' \cap P_0 \rightrightarrows M \cap P_1$, which contradicts that $\mathcal{M}$ is decisive. Again, we will separately consider the cases $|\mathcal{M}_0 \cap \mathcal{M}_1| = 0$ and $|\mathcal{M}_0 \cap \mathcal{M}_1| = 1$.

If $\mathcal{M}_0$ and $\mathcal{M}_1$ are disjoint then $\mathcal{M}_i$ is a decisive partition of P_i. For $i \in 2$, let $N_i = \{\Phi(M) : M \in \mathcal{M}_i\}$. Notice that $\mathcal{N} = \{N_0, N_1\}$ is a decisive partition of $G = \{\Phi(M) : M \in \mathcal{M}\}$ such that $N_0 \rightrightarrows N_1$. Hence, by inductive hypothesis, $\Phi(G) = \Phi(N_1)$. But $\mathcal{M}_1$ is a decisive partition of P_1 and $|M_1| \leq n$, which implies that $\Phi(G) = \Phi(N_1) = \Phi(P_1)$.

On the other hand, suppose that $\mathcal{M}_1 \cap \mathcal{M}_2 = \{M^*\}$. Then, $\{M^* \cap P_1\} \cup (\mathcal{M}_0 \cup \mathcal{M}_1) \setminus \{M^*\}$ is a decisive partition of $F' = F \setminus (M^* \cap P_0)$. Applying the inductive assumption to the set M^* and its partition $\{M^* \cap P_0, M^* \cap P_1\}$, we obtain $\Phi(M^*) = \Phi(M^* \cap P_1\}$. Hence:

$$\Phi(G) = \Phi\{\{\Phi(M^* \cap P_1)\} \cup \{\Phi(M) : M \in (\mathcal{M}_1 \cup \mathcal{M}_2) \setminus \{M^*\}\}\} = \Phi(F').$$

But $\{P_1, F' \setminus P_1\}$ is also a decisive partition of F' with $(F' \setminus P_1) \rightrightarrows P_1$, which implies that $\Phi(G) = \Phi(F') = \Phi(P_1)$.

Case 2: Suppose that $2 < di(F, \psi) \leq n$ and let $\mathcal{P}$ be the decisive partition of F of cardinality $di(F, \psi)$. For every $P \in \mathcal{P}$, we have $|P| \leq n$ and $\mathcal{M}_P = \{M \in \mathcal{M} : M \subseteq P\}$ is a decisive partition of P. Therefore, $\Phi(P) = \Phi(\{\Phi(M) : M \in \mathcal{M}_P\})$. Notice also that $\mathcal{P}' = \{G \cap P : P \in \mathcal{P}\}$ is a decisive partition of G and $|G| = |\mathcal{M}| \leq n$. By the inductive assumption, $\Phi(G) = \Phi(\{\Phi(G \cap P) : P \in \mathcal{P}\})$. Furthermore, because of the way G is defined, for every $P \in \mathcal{P}$ we have:

$$\Phi(G \cap P) = \Phi(\{\Phi(M) : M \subseteq P\}) = \Phi(\{\Phi(M) : M \in \mathcal{M}_P\}) = \Phi(P).$$

Therefore $\Phi(G) = \Phi(\{\Phi(P) : P \in \mathcal{P}\})$ and thus $\Phi(F) = \Phi(G)$. The inductive proof is complete.

To prove the continuity of Φ, let $F = \{x_i : i < n\} \subseteq X$ and let U be an open set containig $\Phi(F)$. By the continuity of ψ_n, there is a pairwise disjoint decisive family $\{V_i : i < n\}$ of open subsets of X such that $x_i \in V_i$ for every $i < n$ and $\psi_n(G) \subseteq \mathcal{V}$ for every $G \in \mathcal{V} \cap \mathcal{F}_n(X)$, where $\mathcal{V} = \langle V_0, \ldots, V_{n-1} \rangle$. Moreover, if $G \in \mathfrak{Fin}(X) \cap \mathcal{V}$ then $\{V_i \cap G : i < n\}$ is a decisive partition for G and thus, by the previous claim, we obtain that:

$$\Phi(G) = \Phi(\{\Phi(V_i \cap G) : i < n\}) = \psi_n(\{\Phi(V_i \cap G) : i < n\}) \subseteq U,$$

which guarantees the continuity of Φ. $\square$

As an immediate consequence of Corollary 3.2 together with Theorem 3.21, we get the following result.

Corollary 3.22. *Let X be a separable space that admits a continuous weak selection. Then $Sel(\mathfrak{Fin}(X)) \neq \emptyset$.* $\square$

3.7 A selection for $\mathcal{K}(X)$

The counterexample X constructed in Theorem 2.18, being separable, not only admits a continuous weak selection but a continuous selection on $\mathfrak{Fin}(X)$. By Theorem 2.5 we also know that 2^X does not admit a continuous selection. However, we can still ask what happens with the collection of the compact subsets of X. In order to analyze the compact case, let us prove an easy and well known result which characterizes compact subsets of Ψ-spaces.

Proposition 3.23 (Folklore)**.** *Let $\mathcal{B}$ be an* ***AD*** *family and let X be its induced Mrówka Isbell space. A set $K \subseteq X$ is compact if and only if $|K \cap \mathcal{B}| < \omega$ and $|K \setminus \bigcup\{B : B \in K \cap \mathcal{B}\}| < \omega$.*

Proof. Suppose first that $|K \cap \mathcal{B}| < \omega$ and $|K \setminus \bigcup\{B : B \in F \cap \mathcal{B}\}| < \omega$. Let $\mathcal{C}$ be a cover for F consisting of basic open sets. For every $B \in K \cap \mathcal{B}$, choose $C_B \in \mathcal{C}$ such that $B \in C_B$. Note that, by hypothesis and since each C_B is a basic open set, $(K \cap \omega) \subseteq^* \bigcup\{C_B : B \in K \cap \mathcal{B}\}$. Hence,

$$\mathcal{C}' = \{C_B : B \in K \cap \mathcal{B}\} \cup \{\{n\} : n \in (K \cap \omega) \setminus \bigcup\{C_B : B \in K \cap \mathcal{B}\}\}$$

is a finite subcover for K.

On the other hand, if $K \cap \mathcal{B}$ is infinite then, as $|U \cap K \cap \mathcal{B}| \leq 1$ for any basic open set U, K is not compact. Suppose that $K \cap \omega \setminus \bigcup\{B : B \in K \cap \mathcal{B}\}$ is infinite and for every $B \in K \cap \mathcal{B}$ let $C_B = \{B\} \cup B$. The cover

$$\mathcal{C} = \{C_B : B \in K \cap \mathcal{B}\} \cup \{\{n\} : n \in K \cap \omega \setminus \bigcup\{B : B \in K \cap \mathcal{B}\}\}$$

does not contain a finite subcover for K and thus K is not compact. $\square$

Utilizing the idea from the the beginning of Chapter 2, let us make the following remark. Given a weak selection ψ on ω we can define the function $f : [\omega]^2 \to 2$ as follows:

$$f(\{n, m\}) = \begin{cases} 0 & \text{if } \psi(\{n, m\}) = \min\{n, m\} \\ 1 & \text{if } \psi(\{n, m\}) = \max\{m, n\} \end{cases}.$$

Now if we consider in particular the universal weak selection φ and the almost disjoint family $\mathcal{B}$ constructed in the proof of Theorem 2.18, by Ramsey's Theorem we can find for every $B \in \mathcal{B}$ an infinite set $B^* \subseteq B$ such that either $\varphi \restriction [B^*]^2 = \min$ or $\varphi \restriction [B^*]^2 = \max$. In fact, we can slightly modify the construction of the **AD** family $\mathcal{B}$ to include this additional property. From this remark and the results obtained in the previous section we obtain a generalization of Theorem 2.18, which states that even the existence of a continuous selection on the collection of compact subets of a space X is not sufficient to guarantee the weak orderability of X.

Proposition 3.24. *There is a separable space which admits a continuous selection on $\mathcal{K}(X)$ but is not weakly orderable.*

Proof. Let φ be the universal weak selection and let $\mathcal{B}$ be the almost disjoint family introduced in Theorem 2.18, with the extra property that either $\varphi \restriction [B]^2 = \min$ or $\varphi \restriction [B]^2 = \max$ for every $B \in \mathcal{B}$. Consider the space $X = \Psi(\mathcal{B})$ and consider also the continuous selection $\Phi \in Sel(\mathfrak{Fin}(X))$ determined by Corollary 3.22. We will define a continuous selection Θ on $\mathcal{K}(X)$ point by point. Let K be a compact subset of X. By Proposition 3.23, we can find integers

$q \leq s$, a finite set $\mathcal{B}_0 = \{B_0, \ldots, B_n\} \subseteq \mathcal{B}$, a family $\mathcal{A}_0 = \{A_i \in [B_i]^{\leq\omega} : i \leq q\}$ and a finite set $F \subseteq \omega \setminus \bigcup\{B_i : i \leq n\}$ such that $K = F \cup \mathcal{B}_0 \cup \bigcup \mathcal{A}_0$. Let

$$k = \min\{n \in \omega : \{B_i \setminus n : i \leq s\} \cup \{\{x\} : x \in F \cup G_n\} \text{ is decisive}\},$$

where $G_n : \bigcup\{A_j \cap n : j \leq q\}$ for every $n \in \omega$. The last is possible because of the extra condition added to $\mathcal{B}$ and the fact that $B||^*C$ and $B||^*\{n\}$ for every $B, C \in \mathcal{B}$ and $n \in \omega$.

Enumerate the set $F \cup G_k$ as $\{m_0, \ldots, m_t\}$ and, for every $i \leq s$, set $x_i = \min(A_i \setminus k)$ when $(A : i \setminus k) \cap K \neq \emptyset$ and $\varphi \restriction [B_i]^2 = \min$ or $x_i = B_i$ otherwise. Define

$$\Theta(K) = \Phi(\{x_i : i \leq s\} \cup \{m_j : j \leq t\}).$$

To prove continuity of Θ in K, let V be a neighborhood of $\Theta(K)$. By continuity of Φ, we can find an $r \in \omega$ such that $r \geq k$ and if for every $i \leq s$, $V_i = \{x_i\}$ when $x_i \in \omega$, and $V_i = \{B_i\} \cup B_i \setminus r$ when $x_i = B_i$, then for $\mathcal{V} = \langle V_0, \ldots, V_s, \{m_0\}, \ldots, \{m_t\}\rangle \cap \mathfrak{Fin}(X)$ it holds that $\Phi[\mathcal{V}] \subseteq V$.

Enumerate the set $\bigcup\{(A_j \cap r) \setminus k : j \leq q\}$ as $\{m_{t+1}, m_{t+2}, \ldots, m_v\}$ and define

$$\mathcal{W} = \langle \{B_0\} \cup B_0 \setminus r, \ldots, \{B_s\} \cup B_s \setminus r, V_0, \ldots, V_s, \{m_0\}, \ldots, \{m_v\}\rangle \cap \mathcal{K}(X).$$

Notice that $\mathcal{W}$ is a neighborhood of K. It only remains to prove that $\Theta[\mathcal{W}] \subseteq V$. Let $K' \in \mathcal{W}$. Hence, we can find integers $u, w \in j+1$ with $u \leq w$, $z_0, z_1, \ldots, z_w \in j+1$, a family $\mathcal{A}' = \{A'_{z_j} \subseteq B_{z_j} : j \leq u\}$ and a finite subset $F' \subseteq \omega \setminus \bigcup\{B_{z_j} : l \leq w\}$ such that $K' = F' \cup \bigcup\{A'_{z_j} : j \leq u\} \cup \{B_{z_i : i \leq w}$. As before, define

$$l = \min\{n \in \omega : \{B_{z_j} \setminus n : j \leq w\} \cup \{\{x\} : x \in F' \cup G'_n\} \text{ is decisive}\},$$

where $G'_n = \bigcup\{A_{z_j} \cap n : j \leq u\}$ for every $n \in \omega$.

Notice that $F' \subseteq \{m_0, \ldots, m_v\} \cup \bigcup\{B_j \setminus r : j \leq s\}$, which implies that $l \leq k$. For every $j \leq w$, let $y_j = \min(A_{z_j} \cap l)$ if $\varphi \restriction [B_{z_j}]^2 = \min$ and $(B_{z_j} \cap l) \cap K' \neq \emptyset$ and let $y_j = B_{z_j}$ otherwise. Notice also that in the particular case when $\Theta(K') = B_{z_j}$ for some $j \leq w$ and $\varphi \restriction [B_{z_j}]^2 = \min$, we have $(K' \cap k) \cap (A_{z_j} \setminus l) = \emptyset$ because, if $M_j = (K' \cap k) \cap (B_{z_j} \setminus l)$ were nonempty then $(\{M_j \cup \{\Phi(K)\}\}) \cup (\{\{x_i : i \leq s\}\} \setminus \Phi(K)) \cup \{\{m\} : m \in \{m_0, \ldots, m_v\} \setminus M_j\}$ would be a decisive partition of $\{x_i : i \leq s\} \cup \{m_i : i \leq v\}$ and $\Phi(M_j \cup \{\Theta(K)\}) = \min M_j$, which is not possible. Therefore, $y_j = \Theta(K)$. Hence $\Phi(\{y_j : j \leq w\} \cup (F' \cup G_l')) = \Phi(\{y_j : j \leq w\} \cup (F' \cup G_l') \cup \{x_i : i \leq s\} \cup \{m_j : j \leq v\} \subseteq V$, which implies that $\Theta(K') \subseteq V$. We conclude that $Sel(\mathcal{K}(X)) \neq \emptyset$. $\square$

A property useful for obtaining the above result was that for every element of the almost disjoint family, the defined weak selection agrees with the usual order of its elements (with respect to min or max). We do not know if the extension condition holds for every Ψ-space and every continuous weak selection defined on it. This is the reason to state the following questions:

Question 3.25. *Does $Sel(\mathcal{K}(X)) \neq \emptyset$ for every Mrówka-Isbell space X admitting a continuous weak selection?*

Question 3.26. *Does every (separable) space which admits a continuous weak selection admit a continuous selection for all compact sets?*

Remark 3.27. *The example X presented in Proposition 2.19 also admits a continuous selection for the collection of its compact subsets, as $\mathcal{K}(X) = \mathfrak{Fin}(X)$.*

3.8 A curious example

We conclude the chapter with the following result, which will serve us for two purposes. On one hand, it presents an example of a Ψ-space that does not admit a continuous weak selection. On the other hand, it proves that for a given space X, the existence of a selection for $[X]^3$ does not imply the existence of a continuous weak selection.

Proposition 3.28. *There is a separable space X that admits a continuous selection for $[X]^3$ but $Sel(F_2(X)) \neq \emptyset$.*

Proof. Identify ω with $2^{<\omega}$. For every $f \in 2^\omega$ let $A_f = \{f \restriction n : n \in \omega\}$ be the branch determined by f and define $\mathcal{A} = \{A_f : f \in 2^\omega\}$. Notice that $\mathcal{A}$ is an almost disjoint family of size $\mathfrak{c}$ and enumerate it by $\{A_\alpha : \alpha < \mathfrak{c}\}$. Enumerate also the collection of weak selections on $2^{<\omega}$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$.

For every $\alpha < \mathfrak{c}$ we will define a function $g_\alpha : [A_\alpha]^2 \to 2$ as follows:

$$g_\alpha(\{f \restriction n, f \restriction m\}) = \begin{cases} 0 & \text{if } f_\alpha(\{f \restriction n, f \restriction m\}) = f \restriction \min\{n, m\}, \\ 1 & \text{if } f_\alpha(\{f \restriction n, f \restriction m\}) = f \restriction \max\{n, m\}, \end{cases}$$

where $f \in 2^\omega$ is such that $A_\alpha = A_f$. By Ramsey's Theorem , we can find a g_α-homogeneous set $B_\alpha \in [A_\alpha]^\omega$ such that $g_\alpha \restriction [B_\alpha] = i$ for some $i \in 2$. Split now the set B_α as $\{B^0_\alpha, B^1_\alpha\}$, so that $|B^0_\alpha| = |B^0_\alpha| = \omega$. Define the **AD** family $\mathcal{B} = \{B^0_\alpha, B^1_\alpha : \alpha < \mathfrak{c}\}$ and, finally, let $X = \Psi(\mathcal{B})$ be the Mrówka Isbell space associated with $\mathcal{B}$.

Define the relation $\preceq$ on X as follows:

$$x \preceq y \text{ if } \begin{cases} x = y \\ x, y \in 2^{<\omega} \text{ and } x \subseteq y \\ x = f \restriction n \in 2^{<\omega} \text{ and } y = B^i_\alpha \text{ , where } A_\alpha = A_f \text{ and } i \in 2. \end{cases}$$

The relation $\preceq$ is clearly reflexive, transitive and antisymmetric. It is not total because elements belonging to distinct branches are incomparable. In general, when $x \not\preceq y$ and $y \not\preceq x$, we will write $x \perp y$. In this case, we will associate to the pair $\{x, y\}$ an element $\Delta_{x,y}$ of $\omega \cup \{\omega\}$ thus:

$$\Delta_{x,y} = \begin{cases} \min\{n : x(n) \neq y(n)\} & \text{if } x, y \in 2^{<\omega}, \\ \min\{n : x(n) \neq f(n)\} & \text{if } x \in 2^{<\omega} \text{ and } y = B^i_\alpha \in \mathcal{B}, \\ \min\{n : f(n) \neq g(n)\} & \text{if } x = B^i_\alpha, y = B^j_\beta \text{ with } i, j \in 2 \text{ and } \alpha \neq \beta, \\ \omega & \text{if } \{x, y\} = \{B^0_\alpha, B^1_\alpha\} \text{ for some } \alpha < \mathfrak{c}. \end{cases}.$$

Define the function $\rho : [X]^3 \to X$ by $\rho(\{x, y, z\}) = x$ if either $x \preceq y$ and $x \preceq z$, or $x \perp y$, $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.

To prove that ρ is well defined, take $F = \{x, y, z\} \in [X]^3$. Notice that F can contain at most an element that is $\preceq$-less than any other element of F and, hence, the function is well defined. Suppose now that $x \perp y$ and $x \perp z$. Notice that if either $y \leq z$ or $z \leq y$, then $\Delta_{x,y} = \Delta_{x,z}$ and so $\rho(\{x, y, z\}) = x$. Therefore we can also suppose that $y \perp z$. If $\Delta_{x,y} = \Delta_{x,z}$ then $x \restriction \Delta_{x,y} = y \restriction \Delta_{x,y} = z \restriction \Delta_{x,y}$, $x(\Delta_{x,y}) \neq y(\Delta_{x,y})$ and $x(\Delta_{x,y}) \neq z(\Delta_{x,y})$, which implies that $y(\Delta_{x,y}) = z(\Delta_{x,y})$ and then $\Delta_{x,y} < \Delta_{y,z}$. In this case $\rho(F) = x$. Now suppose that $\Delta_{x,y} < \Delta_{x,z}$.

Then $x(\Delta_{x,y}) = z(\Delta_{x,y})$, which shows that $\Delta_{y,z} = \Delta_{x,y}$ and thus $\rho(F) = y$. Otherwise, if $\Delta_{x,y} > \Delta_{x,z}$ we have that $\Delta_{y,z} = \Delta_{x,z}$ and so $\rho(F) = z$.

Let us prove now the continuity of ρ. Let $F = \{x, y, z\} \in [X]^3$ and suppose that $\rho(F) = x$. We study separately both possible cases.

Case 1: $x \preceq y$ and $x \preceq z$. There are $f \in 2^\omega$ and $n \in \omega$ such that $x = f \upharpoonright n$. In this case, set $W_x = \{x\}$. If $y \in 2^{<\omega}$, set $W_y = \{y\}$. Otherwise, if $y = B^i_\alpha$ for some $i \in 2$, where $A_\alpha = A_f$, let $W_y = \{B^y_\alpha\} \cup (\omega \setminus \{f \upharpoonright k : k \leq n\})$. In the same manner, let $W_z = \{z\}$ if $z \in 2^{<\omega}$ and let $W_z = \{z\} \cup \{f \upharpoonright k : k \leq n\}$ otherwise. Hence $\mathcal{W} = \langle W_x, W_y, W_z\rangle$ is a neighborhood of F and by construction $\rho[\mathcal{W}] \subseteq W_x$, which guarantees continuity of ρ in F.

Case 2: $x \perp y$, $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$. Suppose first that $x \in 2^{<\omega}$ and let $W_x = \{x\}$. If also $y \in 2^{<\omega}$, let $W_y = \{y\}$ and let $W_y = \{y\} \cup B^i_\alpha \setminus \{g \upharpoonright n : n \leq \Delta_{x,y}\}$ if $y = B^i_\alpha$ for some $\alpha < \mathfrak{c}$ and $i \in 2$. Define the set W_z exactly the same way and consider the open set $\mathcal{W} = \langle W_x, W_y, W_z\rangle$. Note that for any $y_0 \in W_y$ and $z_0 \in W_z$ we have $x \perp y_0$, $x \perp z_0$ and $\Delta_{x,y_0} = \Delta_{x,z_0} = \Delta_{x,y}$. Therefore $\rho[\mathcal{W}] = \{x\}$.

Suppose now that $x = B^i_\alpha$ for some $\alpha < \mathfrak{c}$ and $i \in 2$. Let $A_\alpha = A_f$ and let V be an open set (in X) containing x. Choose $n \in \omega$ in such a way that $\{B^i_\alpha\} \cup B^i_\alpha \setminus \{f \upharpoonright k : k \leq n\} \subseteq V$. Let $m = \max\{n, \Delta_{x,y}\}$ and define $W_x = \{x\} \cup B^i_\alpha \setminus \{f \upharpoonright k : k \leq m\}$. In an analogous way to the Case 1, if $y \in 2^{<\omega}$ define $W_y = \{y\}$ and if $y = B^j_\beta$ for some $\beta < \mathfrak{c}$ and $j \in 2$, let $W_y = \{y\} \cup B^j_\beta \setminus \{g \upharpoonright k : k \leq m\}$, where $A_\beta = A_g$. In a similar way, define the open set W_z containing z. As before, define the open set $\mathcal{W} = \langle W_0, W_1, W_2\rangle$, which satisfies $\rho[\mathcal{W}] \subseteq W_x \subseteq V$. This proves that $\rho \in Sel([X]^3)$.

To conclude the proof, we will show that X does not admit a continuous weak selection. Let h be any weak selection on X and let $h \upharpoonright 2^{<\omega} = f_\alpha$. Let $A_\alpha = A_f$ and assume, without loss of generality, that $h(\{B^0_\alpha, B^1_\alpha\}) = B^0_\alpha$. Let $\mathcal{V}$ be a basic neigborhood of $\{B^0_\alpha, B^1_\alpha\}$. Moreover, suppose that there is $k \in \omega$ such that $\mathcal{V} = \langle\{B^0_\alpha\} \cup B^0_\alpha \setminus \{f \upharpoonright l : l < k\}, \{B^1_\alpha\} \cup B^1_\alpha \setminus \{f \upharpoonright l : l < k\}\rangle$. If $f_\alpha \upharpoonright [B_\alpha]^2 = f \upharpoonright \min\{n, m\}$, choose $n, m \in \omega$ such that $n > m > k$, $f \upharpoonright n \in B^0_\alpha$ and $f \upharpoonright m \in B^1_\alpha$. Then $\{f \upharpoonright n, f \upharpoonright m\} \in \mathcal{V}$ and $h(f \upharpoonright n, f \upharpoonright m) = f \upharpoonright m \notin B^0_\alpha$. In the other case, if $f_\alpha[B_\alpha]^2 = f \max\{n, m\}$, choose $n, m \in \omega$ such that $n > m > k$, $f \upharpoonright m \in B^0_\alpha$ and $f \upharpoonright n \in B^1_\alpha$. Again, $h(f \upharpoonright n, f \upharpoonright m) \notin B^0_\alpha$, which shows that h

is not continuous in $\{B_\alpha^0, B_\alpha^1\}$. $\square$

Chapter 4

Spaces determined by selections

Our analysis of (continuous) weak selections has been developed so far in two directions. On one hand, we have been interested in the search of required conditions to set up an equivalence between the existence of a continuous weak selection on a topological space and an order property satisfied by it. On the other hand, we have studied the extension property of continuous weak selections in certain spaces, in such a way that we cannot only continuously choose points from sets of pairs but from finite (or even compact) sets. In both cases, we start from a given topology for the space and we work with selections and weak selections that are continuous with respect to the hyperspace topology determined by the topology of the spaces.

Our goal in this chapter will be somewhat different. Starting from a weak selection on an arbitrary set, we will define a topology on the base set from the given weak selection or, more precisely, from the natural relation determined by it, already mentioned in the preceding chapters. Thus, the given weak selection will acquire a preponderant role in the corresponding space as topological properties will be related one or another way to properties satisfied by the given function.

4.1 The topology τ_ψ

Let ψ be a weak selection on a set X and consider the relation $\leq_\psi$ on X. Throughout this work we have highlighted the order properties satisfied by this relation and its proximity to define a linear order on X. Because of this, some authors agree to name it an *order-like relation.* Following this idea, it is possible to induce an *order-like topology* for the set X, where open sets are determined by sets that correspond to open intervals in the ordered case.

Given a weak selection on a set X, we will work with a notation different from that introduced in Section 1.6 to represent the sets L_x and U_x for every $x \in X$. The reason behind this is to stage in a better way the "almost" orderability condition of the relation $\leq_\psi$. At first, we define the "*open intervals*" with respect to $\leq_\psi$ as follows:

$$(\leftarrow, x)_\psi = \{y \in X \setminus \{x\} : y \leftarrow x\},$$

$$(x, \rightarrow)_\psi = \{y \in X \setminus \{x\} : x \leftarrow y\},$$

$$(x, y)_\psi = (x, \rightarrow)_\psi \cap (\leftarrow, y)_\psi.$$

Similarly, we define the "*closed intervals*" $(\leftarrow, x]_\psi = (\leftarrow, x)_\psi \cup \{x\}$, $[x, \rightarrow)_\psi = (x, \rightarrow)_\psi \cup \{x\}$ and $[x, y]_\psi = [x, \rightarrow)_\psi \cap (\leftarrow, y]_\psi$. Notice that since $\leq_\psi$ is not a transitive relation, the open sets $(x, y)_\psi$ and (y, x) can be simultaneously nonempty, and likewise for the closed case.

Now that we have defined the corresponding open and closed intervals in X, we construct the induced topology on X exactly in the same way as when we start from a linear order on X.

Definition 4.1. Let ψ be a weak selection on a set X. The *topology generated by the weak selection* ψ, denoted by τ_ψ, is the one having as subbase the set

$$\mathcal{B} = \{(\leftarrow, x)_\psi : x \in X\} \cup \{(x, \rightarrow)_\psi : x \in X\}.$$

Therefore, any basic neighborhood of a point $x \in X$ will be determined by a finite set of points, all of them distinct from x, and their respective arrow directions.

4.2 Regularity of τ_ψ

Let ψ be a weak selection on a set X and let x and y be points in X such that $x \leftarrow y$. Suppose first that there is a point $z \in X \setminus \{x, y\}$ so that $\{x, y\} \not\parallel \{z\}$.

Without loss of generality we can also suppose that $x \leftarrow z \leftarrow y$. Then $V_x = (\leftarrow, z)_\psi$ and $V_y = (z, \rightarrow)_\psi$ are disjoint open sets such that $x \in V_x$ and $y \in V_y$. Now suppose that $\{x, y\}||\{z\}$ for every $z \in X \setminus \{x, y\}$. By proof of Proposition 3.3, the sets $V_x = [x, \rightarrow)_\psi$ and $V_y = (\leftarrow, y]_\psi$ are clopen, disjoint and, clearly, $x \in V_x$ and $y \in V_y$. Thus, in any case there are disjoint open sets that, respectively, contain the points x and y, which ensures that (X, τ_ψ) is a Hausdorff space.

In fact, Gutev and Nogura [25] obtained a more general result and proved that every space of the form (X, τ_ψ) is a regular space. In order to determine whether it is possible to strengthen this result even more, they state the following natural question.

Question 4.2 ([25])**.** *Is (X, τ_ψ) a normal space?*

García Ferreira and Tomita have recently answered this question in the negative.

Proposition 4.3 ([19])**.** *There is a weak selection ψ defined on $\mathbb{P}$, the set of irrational numbers, such that $(\mathbb{P}, \tau_\psi)$ is not normal.* □

The constructed example is Tychonoff but not normal. This observation inspired the authors to reformulate the previous question by weakening the separation condition.

Question 4.4 (García Ferreira - Tomita)**.** *Are there a set X and a weak selection ψ on X such that the space (X, τ_ψ) is not Tychonoff?*

They also provided a partial negative answer in a certain collection of spaces.

Proposition 4.5 ([19])**.** *If ψ is a weak selection on X such that τ_ψ has countable pseudocharacter, then (X, τ_ψ) is Tychonoff.* □

4.3 Components and quasicomponents in τ_ψ

In this section we review a well-known property of spaces of the form (X, τ_ψ), which is interesting in its own right and will help us to provide a general answer to Question 4.4. The proof is included for completeness.

Definition 4.6. Let X be a topological space. For every $x \in X$ we define the following sets:

(1) The *(connected) component* of x, denoted by C_x, is

$$C_x = \bigcup \{C \subseteq X : x \in C \text{ and } C \text{ is connected}\}.$$

(2) The *quasi-component* of x, denoted by C_x^*, is defined as:

$$C_x^* = \bigcap \{C \subseteq X : x \in C \text{ and } C \text{ is clopen}\}.$$

For any given space X and any point $x \in X$ it turns out that $C_x \subseteq C_x^*$. On other hand, there are examples where these sets are not the same. However, it is not the case in the situation that concerns us. We prepare an auxiliary result.

Lemma 4.7 ([20])**.** *Let ψ be a weak selection on a set X. If $x \in X$ and $y, z \in C_x^*$, where C_x^* is the τ_ψ-quasicomponent of x, then $[y, z]_\psi \subseteq C_x^*$.*

Proof. Aiming towards a contradiction, suppose that we can find $y, z \in C_x^*[X]$ and $t \in X \setminus C_x^*$ with $y \leftarrow t \leftarrow z$. Since $t \notin C_x^*$ there is a clopen set $V \subseteq X$ such that $C_x^* \subseteq V$ and $y \notin V$. Define the set

$$W = (\leftarrow, t] \cap V = (\leftarrow, t) \cap V.$$

Notice that W is a clopen set and $z \notin W$. It yields a contradiction since $z \in C_x^* = C_y^* \subseteq W$. □

Theorem 4.8 ([20])**.** *Let ψ be a weak selection on X and let $x \in X$. Then $C_x = C_x^*$.*

Proof. We need to prove that $C_x^* \subseteq C_x$. It will be sufficient to show that C_x^* is connected. By Lemma 4.7 we have

$$C_x^* = \bigcup \{[y, z]_\psi : y, z \in C_x^*, y \leftarrow z \text{ and } y \leftarrow x \leftarrow z\}.$$

Hence, we are done if we prove that $[y, z]_\psi$ is connected, where y, z are as above.

Suppose to the contrary that $[y, z]_\psi$ is not connected for some points $y, z \in C_x^*$ with $y \leftarrow z$. Thus, there is a clopen (in $[y, z]_\psi$) set $W \subseteq [y, z]_\psi$ such that $z \in W$

and $[y, z]_\psi \setminus W \neq \emptyset$. Take $t \in [y, z]_\psi \setminus W$ and define $T = W \cap [t, z]_\psi$. Note that T is clopen in $W \cap [t, z]_\psi$, $z \in T$ and $t \notin T$.

Claim: $G = T \cup [z, \rightarrow)_\psi$ is clopen in X. Actually, the set G is closed because T and $[z, \rightarrow)_\psi$ are closed. To prove that G is also open notice first that, since W is open in $[y, z]_\psi$, we can find an open set (in X) V such that $W = V \cap [y, z]_\psi$, $z \in V$ and $t \notin V$. Therefore:

$$\begin{aligned} T &= W \cap [y, z]_\psi \\ &= V \cap [y, z]_\psi \cap [t, z]_\psi \\ &= (V \cap [y, z]_\psi \cap [t, z]_\psi) \setminus (\leftarrow, t)_\psi \\ &= V \setminus (\leftarrow, t)_\psi \cap [y, z]_\psi \cap [t, z]_\psi \\ &= V \setminus (\leftarrow, t]_\psi \cap [t, z]_\psi \\ &= E \cap [t, z]_\psi, \end{aligned}$$

where $E = V \setminus (\leftarrow, t]_\psi$ is open. Therefore

$$\begin{aligned} G &= T \cup [z, \rightarrow)_\psi \\ &= (E \cap [t, z]_\psi) \cup [z, \rightarrow)_\psi \\ &= E \cup (z, \rightarrow). \end{aligned}$$

Hence G is a clopen set in X such that $z \in G$ and $t \notin G$. However, this contradicts $t, z \in C_x^*$. Therefore, we conclude that C_x^* is connected and it turns out that $C_x = C_x^*$. $\square$

4.4 (X, τ_ψ) is Tychonoff

Start with a set X and a weak selection ψ on X. Given a point $x \in X$, consider the sets $V_0 = (\leftarrow, x)_\psi \setminus C_x$ and $V_1 = (x, \rightarrow)_\psi \setminus C_x$. Take $y \in C_x$ and let $z \in V_0$. Suppose first that $y \leftarrow x$. By Lemma 4.7, $[y, x]_\psi \subseteq C_x$. Therefore $z \leftarrow y$ because otherwise we would obtain $z \in [y, x]_\psi$, which is not possible. On the other hand, suppose that $x \leftarrow y$. If $y \leftarrow z$ then $T = \{x, y, z\}$ would be a 3-cycle and then x and y would belong to distinct elements of the canonical partition determined by the 3-cycle T which is not possible. Thus, in this case we also obtain that $z \leftarrow y$. We conclude that $C_x \rightrightarrows V_0$. Analogously it can be proved that $V_1 \rightrightarrows C_x$.

Therefore, any $x \in X$ determines a finite decisive partition $\mathcal{P}$ of X, consisting of two open sets V_0 and V_1 together with the closed connected set C_x, which satisfies $V_1 \rightrightarrows C_x \rightrightarrows V_0$. With all this, we obtain a strange way to represent the space (X, τ_ψ).

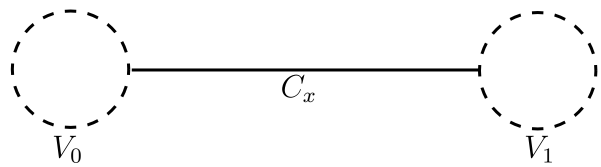

Figure 4.1: The space (X, τ_ψ)

Using the last figure, we can arrive to an idea of how to prove the following result.

Lemma 4.9. *Let $x \neq y \in X$ and let ψ be a weak selection on X such that $x \leftarrow y$. Then there are τ_ψ-continuous functions $f : X \to [0,1]$ and $g : X \to [0,1]$ such that:*

(1) $f(x) = 1$ and $f''[y, \to)_\psi = \{0\}$,

(2) $g(y) = 1$ and $g''(\leftarrow, x]_\psi = \{0\}$.

Proof. We will prove (1). The proof of (2) can be done in the same manner. There are two possibles cases:

Case 1: There is a clopen set $C \subseteq X$ such that $x \in C$ and $y \notin C$.

Let $U = (\leftarrow, y)_\psi \cap C$. Note that $U = (\leftarrow, y] \cap C$, which yields U is a clopen set such that $x \in C$ and $[y, \to)_\psi \subseteq (X \setminus U)$. In this case, In this case, let $f : X \to [0,1]$ be the function defined by $f(z) = 1$ if $z \in U$ and $f(z) = 0$ otherwise.

Case 2: For every $C \subseteq X$ clopen, $x \in C$ if and only if $y \in C$.

Then $y \in C_x$. By Theorem 4.8, C_x is connected. Moreover, the relation $\leq_\psi$ restricted to $C_x \times C_x$ is transitive and hence the component C_x, considered a subspace of X, is ordered. In particular, it is a normal subspace and thus $[x, y]_\psi$, being closed in X (and hence in C_x), is also normal. This condition lets us find a continuous function $h : [x, y]_\psi \to [0,1]$ such that $h(x) = 1$ and

$h(y) = 0$. The next step in the proof will be to extend the function h to X. Define $f : X \to [0,1]$ by

$$f(z) = \begin{cases} 1 & \text{if } z \in (\leftarrow, x]_\psi, \\ h(z) & \text{if } z \in [x,y]_\psi, \\ 0 & \text{if } z \in [y, \to)_\psi. \end{cases}.$$

Notice that, by Lemma 4.7, $(\leftarrow, x]_\psi \cap [x,y]_\psi = \{x\}$, $[x,y]_\psi \cap [y,\to)_\psi = \{y\}$ and $(\leftarrow, x]_\psi \cap [y,\to)_\psi = \emptyset$. Hence f is well defined. Moreover, since f is continuous on the closed sets $(\leftarrow, x]_\psi$, $[x,y]_\psi$ and $(y,\to)_\psi$ we have that f is τ_ψ-continuous. □

We finally give a negative answer to Question 4.4.

Theorem 4.10. *Let ψ be a weak selection on a set X. Then (X, τ_ψ) is Tychonoff.*

Proof. Let $x \in X$ and let U be a basic neighborhood of x. There are $z_0, z_1, \ldots, z_n \in X \setminus \{x\}$ such that $U = \bigcap\{U_i : i \leq n\}$, where $U_i = (\leftarrow, z_i)_\psi$ if $x \leftarrow z_i$ and $U_i = (z_i, \to)_\psi$ if $z_i \leftarrow x$. By Lemma 4.9, for every $i \leq n$ we can find a continuous function $f_i : X \to [0,1]$ with $f(z_i) = 1$ and $f''(X \setminus U_i) = \{0\}$. Define the function $f : X \to [0,1]$ given by $f = \prod\{f_i : i \leq n\}$.

The function f is continuous and, by construction, $f(x) = 1$. If $z \notin U$ then $z \notin U_i$ for some $i \leq n$ and so $f_i(z) = 0$, which implies that $f(z) = 0$. Therefore, $f''(X \setminus U) \subseteq \{0\}$. This proves that (X, τ_ψ) is Tychonoff. □

4.5 Topologies determined by selections

we can try to analyze how this weak selection behaves with respect to the topology determined by itself. more generally, we can start from a topological space that admits a continuous weak selection and investigate the relationship between the original topology on the space and that induced by the continuous weak selection. Certainly, at the end of Section 1.6 we have implicitly mentioned a result in this direction.

Proposition 4.11 ([21]). *Let ψ be a continuous weak selection on a Hausdorff space (X, τ). Then $\tau_\psi \subseteq \tau$.* □

Therefore a weak selection, when continuous, induces a weaker topology. This result motivates studying the case where the inclusion in Proposition 4.11 is actually an equality and hence the space not only admits a continuous weak selection but is topologically determined by it.

4.5.1 wDS spaces

Definition 4.12. A topological space (X, τ) is *weakly determined by selections* (**wDS**) if there is a weak selection ψ on X such that $\tau = \tau_\psi$.

Every ordered space is clearly **wDS** since its topology is determined by the weak selection $\psi = \min$. In this case, the weak selection is even τ_ψ-continuous. The converse is not true because, as will be shown in the next subsection, not even weak orderability of a **wDS** can be guaranteed.

In [25] the authors present a wide collection of spaces of the form (X, τ), which admit a τ-continuous weak selection ψ such that $\tau_\psi \neq \tau$. As we will see later, there are spaces for which not only $\tau_\psi \neq \tau$ but ψ is not τ_ψ-continuous, even when ψ is τ-continuous. In all these cases ψ is trivially continuous when X is endowed with the discrete topology. In order to determine wether a space X admits a minimal topology that makes the weak selection ψ continuous, Gutev and Nogura [21] ask if it is always possible to find the coarsest topology $\tau^* \subseteq \tau$ such that ψ is τ^*-continuous. The following result lets us provide a negative answer to this question.

Proposition 4.13. *Let ψ be a weak selection on a set X. Then τ_ψ is the intersection of all Hausdorff topologies τ on X such that ψ is τ-continuous.*

Proof. Define the topology τ^* on X by:

$$\tau^* = \bigcap \{\tau : \tau \text{ is a Hausdorff topology on } X \text{ and } \psi \text{ is } \tau\text{-continuous}\}.$$

By Proposition 4.11 it follows that $\tau_\psi \subseteq \tau^*$. We only need to prove that $\tau^* \subseteq \tau_\psi$.

For every $x \in X$ define

$$\mathfrak{N}_x = \{U \subseteq X : x \in U \text{ and } U \text{ is } \tau_\psi\text{-open}\}.$$

For every $x \in X$, let τ_x be the topology generated by $\mathfrak{N}_x \cup \{\{y\} : y \in X \setminus \{x\}\}$. Thus any point in $X \setminus \{x\}$ is isolated and every τ_ψ-open neighborhood of x is τ_x-open. Let $y \in X \setminus \{x\}$ and suppose, without loss of generality, that $x \leftarrow y$; the other case can be treated likewise. Then $U_x = (\leftarrow, y)_\psi$ and $U_y = \{y\}$ are disjoint τ_x-open sets which respectively contain x and y and $U_y \rightrightarrows U_x$. This proves that ψ is τ_x-continuous for every $x \in X$ and thus $\tau^* \subseteq \bigcap\{\tau_x : x \in X\}$, which yields that $\tau^* \subseteq \tau_\psi$. $\square$

We conclude that there exists the coarsest topology on τ^* on X such that ψ is τ^*-continuous if and only if ψ is τ_ψ-continuous. In this case, $\tau^* = \tau_\psi$.

4.5.2 The counterexample again

We study once again the counterexample $X = \Psi(\mathcal{B})$ presented in Chapter 2, together with the weak selection $\overline{\varphi}$ extending the universal selection φ, but now we focus on the topology generated by $\overline{\varphi}$.

We first consider the universal selection φ and ask if it is continuous when we endow ω with the topology τ_φ. Recall that the main property of φ states that for any pair of disjoint sets A and B in $[\omega]^{<\omega}$, the set $\{n \in \omega : A \rightrightarrows \{n\} \rightrightarrows B\}$ is infinite. On the other hand, τ_φ-open sets are determined by a finite set, assigning an arrow direction to each one of its elements. The continuity of φ would mean that for any $n, m \in \omega$ such that $n \to m$ we can find finite sets $F = F_0 \cup F_1$ and $G = G_0 \cup G_1$ such that $\mathcal{U}_0 = \bigcap\{(\leftarrow, x)_\varphi : x \in F_0\} \cap \bigcap\{(x, \to)_\varphi : x \in F_1\}$ and $\mathcal{U}_1 = \bigcap\{(\leftarrow, y)_\varphi : y \in G_0\} \cap \bigcap\{(y, \to)_\varphi : y \in G_1\}$ are disjoint open sets, containing n and m, respectively, with $\mathcal{U}_0 \rightrightarrows \mathcal{U}_1$. However, this implicitly entails that two finite sets, or more precisely the arrow direction of two finite sets, determine a unique way to select point from pairs, which would contradict the extension property of φ.

The next proposition shows that not only the universal selection but its extending weak selection $\overline{\varphi}$ is not continuous with respect to the topology it generates.

Proposition 4.14. *Let $X = \Psi(\mathcal{B})$ and $\overline{\varphi} \in Sel(\mathcal{F}_2(X))$ be as in Theorem 2.18. Then $\overline{\varphi}$ is not a $\tau_{\overline{\varphi}}$ -continuous.*

Proof. Aiming towards a contradiction, let us suppose that $\overline{\varphi}$ is continuous with respect to the topology $\tau_{\overline{\varphi}}$ defined on X.

Recall that $\mathcal{B} = \{X_0^\alpha, X_1^\alpha : \alpha < \mathfrak{c}\}$ is an almost disjoint family, $X_0^\alpha \to X_1^\alpha$ for every $\alpha < \mathfrak{c}$ and, for every branch A in $2^{<\omega}$, $\overline{\varphi} \restriction [A]^2 \approx \varphi$, where φ is the universal selection. Choose $\alpha < \mathfrak{c}$ so that $\{X_0^\alpha, X_1^\alpha\}||\{z\}$ for every $z \in X \setminus \{X_0^\alpha, X_1^\alpha\}$, i.e. X_0^α and X_1^α are indistingushable with respect to φ. This is possible because otherwise X would be weakly orderable. By the $\tau_{\overline{\varphi}}$-continuity of the weak selection $\overline{\varphi}$, we can find two disjoint $\tau_{\overline{\varphi}}$ - open neighbourhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of X_0^α and X_0^α, respectively, such that $\mathcal{U}_0 \rightrightarrows \mathcal{U}_1$. In fact, we can suppose that the open sets $\mathcal{U}_0$ and $\mathcal{U}_1$ are determined by the same finite set $F = \{y_i : i < n\} \in [X]^{<\omega} \setminus \{X_0^\alpha, X_1^\alpha\}$, with their corresponding arrow directions, in such a way that $\mathcal{U}_0 = \bigcap\{U_i : i < n\} \cap (X_1^\alpha, \to)_{\overline{\varphi}}$ and $\mathcal{U}_1 = \bigcap\{U_i : i < n\} \cap (\leftarrow, X_0^\alpha)_{\overline{\varphi}}$, where $U_i = (y_i, \to)_{\overline{\varphi}}$ if $\{X_0^\alpha, X_1^\alpha\} \rightrightarrows \{y_i\}$ and $U_i = (\leftarrow, y_i)_{\overline{\varphi}}$ if $\{y_i\} \rightrightarrows \{X_0^\alpha, X_1^\alpha\}$.

Let A_α be the branch containing $X_0^\alpha \cup X_1^\alpha$ and, without loss of generality, let us suppose that $\overline{\varphi} \restriction [A_\alpha]^2 = \varphi \restriction [A_\alpha]^2$. We can find $k \in \omega$ such that $\{x\}||(A_\alpha \setminus k)$ for every $x \in F \setminus A_\alpha$. Let $A_\alpha^{'} = (A_\alpha \setminus k) \cup (F \cap k)$. Since $|A_\alpha \setminus A_\alpha^{'}| < \omega$ and A_α is a φ-positive set, it turns out that $A_\alpha^{'}$ is also φ-positive. Now, as $X_0^i||^*\{w\}$ for every $w \in F \cap k$ and $i \in 2$, we can find $x_0 \in X_0^\alpha \setminus k$ and $x_1 \in X_1^\alpha \setminus k$ satisfying:

(1) $x_0 \to X_1^\alpha$ and $X_0^\alpha \to x_1$,

(2) $\{x_i, X_i^\alpha\}||\{w\}$ for every $w \in F \cap k$ and $i \in 2$.

Notice that $x_0 \in \mathcal{U}_0$ and $x_1 \in \mathcal{U}_1$. Define the finite sets $G_0 = \{w \in F \cap k : w \to X_0^\alpha\} \cup \{x_1\}$ and $G_1 = \{w \in F \cap k : X_0^\alpha \to w\} \cup \{x_0\}$. Given that $A_\alpha^{'}$ is φ-positive and $G_0, G_1 \in [A_\alpha]^{<\omega}$ are disjoint, we can find $z \in A_\alpha^{'} \setminus (G_0 \cup G_1)$ such that $G_0 \rightrightarrows \{z\} \rightrightarrows G_1$. Hence $\{z, X_0^\alpha, X_1^\alpha\}||\{w\}$ for every $w \in F$ and $x_1 \to z \to x_0$. There are two possible cases.

Case 1: $X_0^\alpha \to z$. Then $z \in \mathcal{U}_1$. On the other hand, $x_0 \in \mathcal{U}_0$, which should imply that $x_0 \to z$, which is a contradiction.

Case 2: $z \to X_1^\alpha$. Then $z \in \mathcal{U}_0$. Since $x_1 \in \mathcal{U}_1$, it occurs that $z \to x_1$ which, again, is not possible. □

Remark 4.15. *If τ is the Mrówka-Isbell topology associated with the space $X = \Psi(\mathcal{B})$ then, since $\overline{\varphi}$ is τ-continuous, we have that $\tau_{\overline{\varphi}} \subseteq \tau$. Hence $(X, \tau_{\overline{\varphi}})$ is not* *weakly orderable because otherwise (X, τ) would also be. We conclude that there are **wDS** spaces that are not weakly orderable.*

As consequence of Proposition 4.14 we can restate the problem of van Mill and Wattel, starting now from the fact that a given space has a property stronger than only admitting a continuous weak selection.

Question 4.16. *Is every space X that admits a weak selection ψ which is τ_ψ-continuous a weakly orderable space?*

This is a motivation to study and to identify topological spaces determined by weak selections that are also continuous with respect to their induced topology.

4.5.3 DS spaces

Definition 4.17. A topological space (X, τ) is *determined by selections* (**DS**) if there is a continuous weak selection ψ on X such that $\tau = \tau_\psi$.

In this case, the weak selection ψ is clearly τ_ψ-continuous. In order to try to relate the **DS** property with an orderability condition, in this section we will present some examples of spaces, most of them well-known suborderable spaces,.

The double arrow space

Consider the set $\mathbb{R} \times \mathbb{R}$ and define the order $\prec$ on it by letting $(x, y) \prec (z, w)$ whenever $x < z$ or $x = z$ and $y < w$. This relation is called the *lexicographic order*. The space is a classical example of an ordered space that is studied in a basic course in topology.

We will be interested in studying this order relation but restricting ourselves to the set $X = \mathbb{R} \times \{0, 1\}$. This space is often called the *Alexandroff double arrow space*. The first reason to present this space is since it provides an example of a separable ordered space (and hence **DS**) such that for every $x \in X$ we can find a unique point which is indistinguishable from x with respect to the min weak selection. That is, there is $y \in X \setminus \{x\}$ such that $\{x, y\} || X \setminus \{x, y\}$. Indeed, given the point $(x, 0) \in X$, we have that for every $z \in \mathbb{R} \setminus \{x\}$ and every $i \in 2$, $(z, i) \prec (x, 0)$ if and only if $(z, i) \prec (x, 1)$. Therefore this separable space has uncountably many indistinguishable points, similar to our counterexample, but turns out to be even ordered.

On the other hand, we will study a well-known subspace of X. To do this, we should identify how the basic open set $((a,0),(b,1)$ looks when $a<b$:

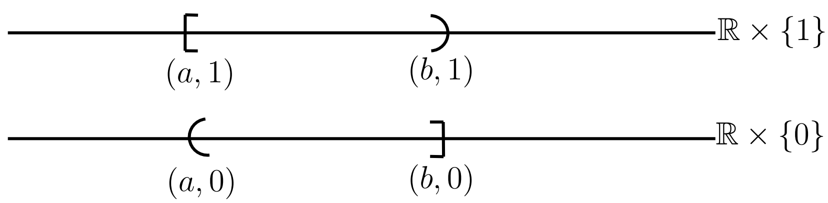

Figure 4.2: The open set $((a,0),(b,1))_{\prec}$

The Sorgenfrey line

We will denote by $\mathbb{R}_l$ the set of the real numbers $\mathbb{R}$ equipped with the topology τ_l having as a base the set $\mathcal{B}=\{[a,b):a,b\in\mathbb{R}$ and $a<b\}$. This space is named the *Sorgenfrey line* after R. Sorgenfrey [56], who first presented this example. $\mathbb{R}_l$ is a classical example of a separable, first countable normal space which is not second countable (and hence not metrizable).

An interesting property satisfied by $\mathbb{R}_l$ is that it is homeomorphic to a countable disjoint sum of copies of itself. Indeed, for every $z\in\mathbb{Z}$, partition the interval [z,z+1) as $\bigcup\{[z+1-\frac{1}{n},z+1-\frac{1}{n+1}):n\in\omega\setminus 1\}$ and, for every $n\in\omega$, let $Z_n=\bigcup\{[z+1-\frac{1}{n},z+1-\frac{1}{n+1}):z\in\omega\}$, endowed with the subspace topology with respect to τ_l. It is not difficult to see that $Z_n\approx\mathbb{R}_l$ for every $z\in\omega$ and hence it can be proved that $(\mathbb{R}_l,\tau_l)\approx(Z,\tau)=\bigcup\{Z_n\times\{n\}:n\in\omega\}$, where τ is the topology on Z corresponding to the disjoint sum of Z, i.e. Z_n is τ-open for every $n\in\omega$ and, fixing $n\in\omega$, if U is open in Z_n then $U\times\{n\}$ is also τ-open.

Notice that $\mathbb{R}_l=X\cap(\mathbb{R}\times\{1\})$, where X is the double arrow space, and hence it is a suborderable space. On the other hand, it can be proved that the Sorgenfrey line is not ordered by proving that none linear order on it determines its topology. However, the same result can be obtained easier as a consequence of the following classical result.

Recall that the diagonal of a space Y is $\Delta(Y)=\{(y,y)\in Y\times Y:y\in Y\}$

Theorem 4.18 (Lutzer, [42]). *An ordered space with a G_δ diagonal is metrizable.* $\square$

Since $\Delta(\mathbb{R}_l) = \bigcap\{\Delta_n(\mathbb{R}_l) : n \in \omega\}$, where $\Delta_n(\mathbb{R}_l) = \{(z, w) \in \mathbb{R}_l \times \mathbb{R}_l : n + z \geq w\}$ is open for every $n \in \omega$, we have that $\Delta(\mathbb{R}_l)$ is G_δ and thus, the Sorgenfrey line is a suborderable space that is not ordered.

In a similar way, we can define the space $\mathbb{R}_l^* = X \cap (\mathbb{R} \times \{0\})$, whose base of open subsets consists of sets of the form $(a, b]$, where $a < b$. It can be proved that $\mathbb{R}_l$ and $\mathbb{R}_l^*$ are homeomorphic spaces.

Our interest in the Sorgenfrey line stems from the following: Trying to determine the strongest orderability condition satisfied by **DS** spaces, we required to first establish if it was possible that every **DS** space might be ordered. The next result solved our initial question by showing that orderability is not a necessary condition for a space to be determined by selections.

Proposition 4.19. *$\mathbb{R}_l$ is a suborderable **DS** space which is not ordered.*

Proof. Let $(X, \tau) = \bigcup\{X_n \times \{n\} : n \in \omega\}$. where $X_n = \mathbb{R}_l$ if n is odd, $X_n = \mathbb{R}_l^*$ if n is even and τ is the sum topology on X. Notice that $(\mathbb{R}_l, \tau_l) \approx (X, \tau)$. Hence we can work in X. By construction, basic open sets in $X_n \times \{n\}$ are of the form $[x, y) \times \{n\}$ if n is odd and $(x, y] \times \{n\}$ if n is even, where $x, y \in \mathbb{R}$ are such that $x < y$. To simplify our notation, for every $n \in \omega$ write $Y_n = X_n \times \{n\}$.

Define $\psi : [X]^2 \to X$ as follows: $\psi(\{(x, n), (y, m)\}) = (x, n)$ if and only if one of the following conditions is met:

(1) $x < y$ and $|n - m| \leq 1$,

(2) $x = y$, $n = 2k + 1$ for some $n \in \omega$ and $|n - m| = 1$,

(3) $m - n > 2$,

(4) $n - m = 2$.

We will prove that ψ is continuous and $\tau_\psi = \tau$. Let us first show that $\tau \subseteq \tau_\psi$.

Fix $n \in \omega$ and let $x, y \in \mathbb{R}$ be such that $x < y$. Define

$$\mathcal{U} = ((x, n + 1), \to)_\psi \cap (\leftarrow, (y, n + 1))_\psi \cap (\leftarrow, (x, n + 3))_\psi.$$

By (3) and (4) on the definition of ψ, $Y_k || \{(x, n+1), (y, n+1)\}$ for every $k \in \omega$ with $|k-(n+1)| \geq 2$. Since $(x, n+1) \leftarrow (z, w) \leftarrow (y, n+1)$ for every $(z, w) \in \mathcal{U}$, we can start from $\mathcal{U} \subseteq (Y_{n-1} \cup Y_n \cup Y_{n+1})$. Moreover, since we also know by (3) that $Y_{n+1} \rightrightarrows Y_{n+3}$ and we have that $(z, w) \leftarrow (x, n+3)$ for $(z, w) \in \mathcal{U}$, it turns out that $\mathcal{U} \subseteq (Y_{n-1} \cup Y_n)$. Finally, since $Y_{n-1} \rightrightarrows Y_{n+1}$ and $(z, w) \leftarrow (y, n+1)$ for any $(z, w) \in \mathcal{U}$, we conclude that $\mathcal{U} \subseteq Y_n$. With all this, it is not difficult to verify that $\mathcal{U} = [(x, n), (y, n)) \times \{n\}$ if n is odd and $\mathcal{U} = ((x, n), (y, n)] \times \{n\}$ if n is even, which proves that $\tau \subseteq \tau_\psi$ and, in particular, $X_n \times \{n\}$ is τ_ψ-clopen for every $n \in \omega$.

To prove that $\tau_\psi \subseteq \tau$, by Proposition 4.11 it is enough to show that ψ is τ-continuous. To that end, let $(x, n), (y, m) \in X$ be such that $(x, n) \leftarrow (y, m)$. There are three possible cases:

Case 1: $n = m$. Then $x < y$. Let $z \in \mathbb{R}$ be such that $x < z < y$ and define the τ-open sets $\mathcal{U} = (x-1, z) \times \{n\}$ and $\mathcal{V} = (z, y+1) \times \{n\}$. We have that $\mathcal{U}$ and $\mathcal{V}$ are disjoint open sets such that $(x, n) \in \mathcal{U}$, $(y, n) \in \mathcal{V}$ and $\mathcal{V} \rightrightarrows \mathcal{U}$.

Case 2: $|n - m| = 1$. If $x < y$ then the continuity of ψ in $\{(x, n), (y, n)\}$ is verified as in Case 1. If $x = y$ then n is odd and m is even. In this case, define the τ-open sets $\mathcal{U} = (x-1, x] \times \{n\}$ and $\mathcal{V} = [x, x+1) \times \{m\}$. It holds that $(x, n) \in \mathcal{U}$, $(y, m) \in \mathcal{V}$ and, by construction, $\mathcal{V} \rightrightarrows \mathcal{U}$.

Case 3: $|n-m| > 1$. As $Y_n || Y_m$ and both Y_n and Y_m are τ-open, it is enough to define $\mathcal{U} = Y_n$ and $\mathcal{V} = Y_m$. $\square$

A suborderable space which is not DS

In this section we will present an easy example of a suborderable space whose topology cannot be determined by a continuous weak selection. It will follow that suborderability is not necessary for **DS**.

Proposition 4.20. $X = (0, 1) \cup \{2\}$*, as a subspace of* $\mathbb{R}$*, is suborderable but not* ***DS****.*

Proof. Let ψ be a continuous weak selection on X. Using an idea similar to that presented in the proof of Proposition 1.14, we can deduce that either $\psi \restriction [(0, 1)]^2 = \min$ or $\psi \restriction [(0, 1)]^2 = \max$.

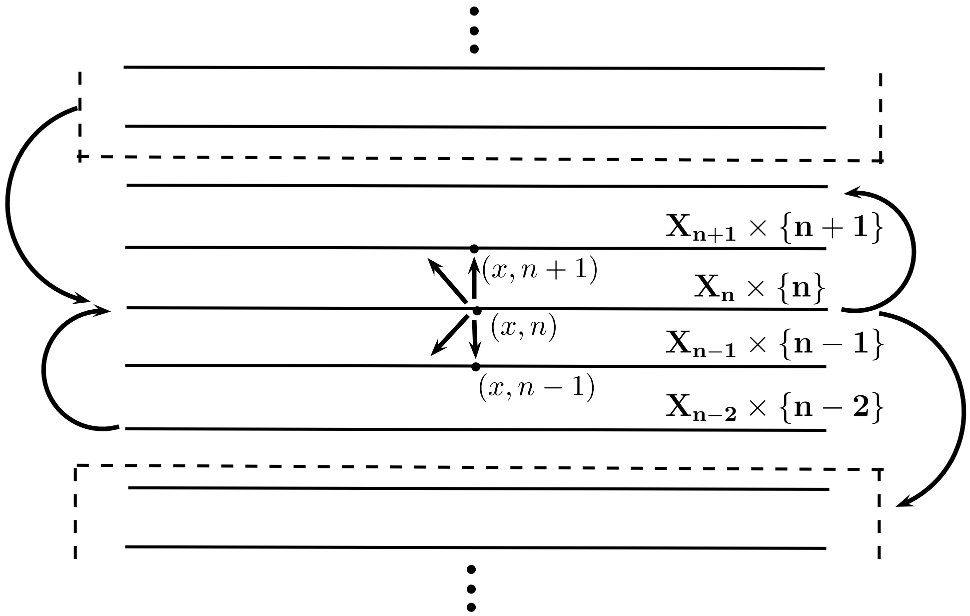

Figure 4.3: Case when (x, n) is odd

Without loss of generality, suppose that $\psi \restriction [(0,1)]^2 = \min$. If there is a point $z \in (0,1)$ such that $z \leftarrow 2$ then, by connectedness of $(0,1)$ and the continuity of ψ, it occurs that $\{2\} \rightrightarrows (0,1)$. In this case, $(X, \tau_\psi) \approxeq (0,1]$.

On the other hand, if $2 \leftarrow z$ for some $z \in (0,1)$, then $(0,1) \rightrightarrows \{2\}$ and hence $(X, \tau_\psi) \approxeq [0,1)$. In any case, (X, τ_ψ) is not homeomorphic to the space $(0,1) \cup \{2\}$. $\square$

All the **DS** spaces we studied were suborderable spaces. In order to determine if all **DS** spaces are suborderable, we state the following question.

Question 4.21. *Is every* ***DS*** *space a suborderable space?*

We do not even know the solution to the van Mill-Wattel problem if we restrict to the collection of **DS** spaces.

Question 4.22. *Is every* ***DS*** *space a weakly orderable space?*

u k

4.5.4 sDS spaces

Definition 4.23. A topological space (X, τ) is *strongly determined by selections* (**sDS**) if X is **DS** and $\tau_\psi = \tau$ for every continuous weak selection τ on X.

An immediate result is that every weakly orderable **sDS** is ordered. The real line is an example of a **sDS** space and even finitely many copies of $\mathbb{R}$ are so.

Generally, by Eilenberg and Michael's results, every connected space can only admit two continuous weak selectm,m,ions, namely the min and max selections. Henceb a connected space is **DS** (and thus **sDS**) only if it is ordered. By Proposition 1.29, this happens when X is also locally connected. Therefore every connected locally connected **DS** space is **sDS**. The next example states that local connectedness is not sufficient.

Example 4.24 ([25])**.** There exists a locally connected **DS** space (X, τ) and $\psi \in Sel(X, \tau)$ such that $\tau \neq \tau_\psi$.

Proof. Let $X = \{0\} \cup \{2^{-n} : n \in \omega\}$ and let τ be the discrete topology on X. As the space (X, τ) is discrete (and hence ordered), it is **DS**. To prove that it is not **sDS**, define $\psi : [X]^2 \to X$ by $\psi(\{x, y\}) = \min\{x, y\}$. Then, τ_ψ is the euclidean subspace topology on X and hence $\tau_\psi \neq \tau$. □

On the other hand, every compact space (X, τ) which is **DS** is also **sDS** simply because if ψ is a continuous weak selection on X, then $\tau_\psi \subseteq \tau$ and the identity function $i : (X, \tau) \to (X, \tau_\psi)$ is a continuous closed bijection. Hence in order to determine if the above properties characterize **sDS** spaces the next question is posed in [25]:

Question 4.25 (Gutev-Nogura)**.** *Is there a noncompact* ***sDS*** *space that is neither connected nor locally connected?*

The following result gives an affirmative answer to Question 4.25.

Proposition 4.26. *There is a* ***sDS*** *space which is neither compact nor connected nor locally connected.*

Proof. Let $X = \bigcup\{U_n : n \in \omega\}$, where $U_0 = (-1, 0]$ and $U_n = (\frac{1}{n+1}, \frac{1}{n})$ for every $n \geq 1$, endowed with the subspace topology τ. Clearly, the space X is neither compact nor connected. Moreover, the space X is not locally connected at the point 0. Since the min selection induces the subspace topology on X, we have that X is a **DS** space. To prove that it is **sDS**, let ψ be any continuous weak selection ψ on X.

We already know that either $\psi \restriction [U_n]^2 = \min$ or $\psi \restriction [U_n]^2 = \max$ for every $n \in \omega$ and, availing from the fact that each U_k is connected and ψ is continuous, we also have that $U_n || U_m$ for every $n, m \in \omega$. In any case, notice that for every interior point $x \in X \setminus 0$ and any open set U containing it, we can find $a, b \in X$ such that $a \leftarrow x \leftarrow b$ and $(a, \rightarrow)_\psi \cap (\leftarrow, b)_\psi \subseteq U$. Therefore, we only need to show that any basic open neighbourhood of 0 in X contains a τ_ψ-open neigbourhood of 0.

Let $U = (a, b) \cap X$ an open set containing 0 and suppose, without loss of generality, that $a \leftarrow 0$ (the another case is analogous). We can also suppose that $b = \frac{1}{n}$ for some $n \in \omega$, i.e. $U = (a, 0] \cup \bigcup\{U_k : k \geq n\}$. By the continuity of ψ, we can also suppose that $U \setminus U_0 \rightrightarrows U_0$. Define the set $F = \{0 < k < n : U_k \rightrightarrows U_0\}$. If F is empty then, in particular, $\{a\} \rightrightarrows U_k$ for every $0 < k < n$. This last implies that $(a, \rightarrow)_\psi \subseteq (a, 0] \cup \bigcup\{U_k : k \geq n\} = U$ and hence $(a, \rightarrow)_\psi \cap (\leftarrow, b)_\psi$ is a τ_ψ-open neighbourhood of 0 contained in U. Therefore, we can suppose that F is non empty. It should also be noted that $U_k \rightrightarrows \{0\}$ for every $k \in F$.

By the continuity of ψ and since $|F| < \omega$, we can find an $m > n$ in such a way that $\{U_k : k \in F\} \rightrightarrows \bigcup\{U_s : s > m\}$. Take $z \in U_{m+1}$ and consider the τ_ψ open set $V = (a, \rightarrow)_\psi \cap (\leftarrow, z)_\psi$. Notice that $0 \in V$. On the other hand, if $x \in X \setminus U$ then either $x \in (0, a]$ or $x \in U_k$ for some $k < n$. If $x \in (0, a]$ then $0 \leftarrow x \leftarrow a$ and hence $x \notin V$. On the other hand, if $x \in U_k$ for some $k < n$, there are two possible cases. In the first case, if $k \notin F$ then, since $U_0 \rightrightarrows U_k$, $x \leftarrow 0$ and again $x \notin V$. Finally, if $k \in F$ then $z \leftarrow x$, which implies that $x \notin V$. We conclude that $V \subseteq U$ and therefore $\tau \subseteq \tau_\psi$ $\square$

Most of the non compact **sDS** examples we studied were spaces homeomorphic to copies of $\mathbb{R}$ or copies of disjoint open intervals on an ordered set, which occur to be locally connected. The example presented in Proposition 4.26 is not locally connected but is very close to, since 0 is the only point that does not admit a connected local base. This is behind the motivation to present the following question:

Question 4.27. *Let X be a non-compact **sDS** space. Is the set*

$$\{x \in X : x \text{ is locally connected}\}$$

dense in X?.

Chapter 5

Open questions

In this chapter we pose some open questions oirginated from the resuts of this work. Some of these questions have been presented in previous chapters.

Question 5.1. *Let ψ a continuous weak selection on X such that every pair of points in X are distinguishable with respect to ψ. Is X a weakly orderable space?*

Question 5.2. *Is every space which admits a continuous weak selection a continuous $\leq 2-to-1$ preimage of an ordered space?*

Question 5.3. *Is there a space X such that $Sel(\mathcal{F}_2(X)) \neq \emptyset$ but $Sel(\mathcal{K}(X)) = \emptyset$?*

Question 5.4. *Does $Sel(\mathcal{K}(X)) \neq \emptyset$ for every Mrówka-Isbell space X admitting a continuous weak selection?*

Question 5.5. *Let ψ be a τ_ψ-continuous weak selection on X. Is (X, τ_ψ) weakly orderable?*

Question 5.6. *Is every **DS** space a normal space?*

Question 5.7. *Is every **DS** space a weakly orderable space?*

Question 5.8. *Let ψ a continuous weak selection on X such that every pair of points in X are distinguishable with respect to ψ. Is X a weakly orderable space?*

Question 5.9. *Is every space which admits a continuous weak selection a continuous $\leq 2-to-1$ preimage of an ordered space?*

Glossary

ω The first infinite countable ordinal. It is the set $\{0, 1, 2, \dots\}$. It is identical to $\mathbb{N}$.

ω_1 The first uncountable ordinal.

$\mathfrak{c}$ The cardinality of $\mathbb{R}$.

$\mathbf{f''A}$ The image of A under the function f.

$\mathbb{Q}$ The set of rational numbers.

$\mathbb{P}$ The set $\mathbb{R} \setminus \mathbb{P}$ of irrational numbers.

$\mathbf{C_p(X, E)}$ For topological spaces X and E, $C_p(X, E)$ denotes the space which consists of all continuos maps $f : X \to Y$, endowed with the subspace topology with respect to the product space E^X. This topology is named the pointwise convergence topology.

Banach A Banach space is a vector space over the real or complex numbers with a norm $\|\cdot\|$ such that every Cauchy sequence (with respect to the metric $d(x, y) = \|x - y\|$ has a limit.

collectionwise Hausdorff A topological space X is collectionwise Hausdorff if for any closed discrete $D \subseteq X$ there are disjoint open sets separating points of D.

complete (metric space) A metric space X is complete if every Cauchy sequence of points in X has a limit point that is also in X.

completely metrizable A topological space X is completely metrizable if it admits a metric d such that the space (X, d) is a complete metric space and d induces the topology on X.

convex A set C in a linear space X is convex if for all $x, y \in C$ and all $t \in [0,1]$, the point $tx + (1-t)y$ is in C.

conv(A) For a set A on a linear space X, $conv(A)$ denotes the smallest convex subset of X containing A.

diamond Principle The diamond principle is a combinatorial principle which is an independent statement of **ZFC**. It says that there exists a collection $\{A_\alpha : \alpha < \omega_1\}$ such that for every $\alpha \in \omega_1$ $A_\alpha \subseteq \alpha$ and for any subset A of ω_1 the set $A \cap A_\alpha$ is stationary.

Dugundji Extension Theorem Let L be a normed linear space. For every space X, for every closed $A \subseteq X$ and for every continuous function $f : A \to L$ there exists a continuous extension $\overline{f} : X \to L$ such that $\overline{f}(X) \subseteq conv(f(A))$.

filter A collection $\mathcal{F}$ of subsets of ω is a filter if it does not contain the empty set, $A \cap B \in \mathcal{F}$ whenever A and B are in $\mathcal{F}$ and if $C \in \mathcal{F}$ and $C \subseteq A$ then $A \in \mathcal{F}$.

flow network Given a finite directed graph $\Gamma = (X, A)$ with a source vertex s and a sink vertex t in which for every edge $\{u, v\}$ it has been asigned a non-negative real value $c(u, v)$, named the capacity of u and v, a flow network is a function $f : V \times V \to \mathbb{R}$ with the following conditions:

(1) $f(u, v) \leq c(\{u, v\})$ for all $(u, v) \in V \times V$, where $c(\{u, v\}) = 0$ when $\{u, v\} \notin A$.

(2) $f(u, v) = -f(v, u)$ for all $(u, v) \in V \times V$.

(3) $\sum\{f(u, v) : v \in V\} = 0$ for all $u \in V \setminus \{s, t\}$.

In this case, the flow of the network is $\sum\{f(s, v) : v \in V\}$. **zero-dimensional** A topological space is zero-dimensional if it admits a base consisting of clopen sets.

homogeneous A topological space X is homogeneous if for any $x, y \in X$ there is an homeomorphism $f : X \to X$ such that $f(x) = y$.

linear space A topological space is linear if it is a real vector space endowed with a (separable metrizable) topology with the properties that the basic vector functions are continuous.

locally pseudocompact (group) A topological group is locally pseudocompact if it contains a nonempty open set with pseudocompact closure.

Michael line The Michael line is the space (X, τ), where $X = \mathbb{R}$ and τ is the topology with base $\{(a, b) : a, b \in \mathbb{R}\} \cup \{\{x\} : x \in \mathbb{P}\}$.

net normed A linear space X is normed if it admits a norm $\|\cdot\|$ such that the topology of X is induced by the metric d defined by $(x, y) = \|x - y\|$.

order topology If $(X, \leq)$ is an ordered set, the order topology on X is the topology generated by sets of the form $(x, \rightarrow)_{\leq} = \{y \in X : x < y\}$ and $(\leftarrow, x)_{\leq} = \{y \in X : y < x\}$, where $x \in X$.

paracompact A topological space is paracompact if every open cover has a locally-finite open refinement.

Polish space A topological space X is a Polish space if the topology on X is compatible with a complete seaparable metric.

pseudocharacter of a point For a space X and $x \in X$, the pseudocharacter of x, denoted by $\psi(x, X)$, is defined as

$$\psi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a family of open sets such that } \bigcap \mathcal{B} = \{x\}\}.$$

pseudocharacter of a space The pseudocharacter of a topological space X, denoted by $\psi(X)$ is defined as follows:

$$\psi(X) = \sup\{\psi(x, X) : x \in X\}.$$

pseudocompact A topological space X is pseudocompact if every continuous function $f : X \to \mathbb{R}$ is bounded.

Purish set For a topological space X such that $|\{x \in X : X \setminus \{x\} \text{ is connected }\}| \leq 2$, a subset Z of X is Purish if for every $x \in X$ the following hold:

(a) $C[x] \subseteq Z$ provided $C[x]$ is a singleton, where $C[x]$ is the component of x.

(b) $|C[x] \cap Z| = 1$ provided $\{x \in X : X \setminus \{x\} \text{ is connected }\} = \emptyset$.

(c) $|C[x] \cap Z| = 2$ and $\{x \in X : X \setminus \{x\} \text{ is connected }\} \subseteq Z$ otherwise.

Ramsey's theorem This result says that for any mapping $f : [\omega]^2 \to 2$ there is a set $A \in [\omega]^\omega$ such that $f \restriction [A]^2 = \{i\}$ for some $i \in 2$.

scattered A topological space X is scattered if for every closed subset C of X the set of isolated points of C is dense in C.

sequentally compact A topological space is sequentally compact if every sequence has a convergent subsequence.

strongly zero-dimensional A topological space is strongly zero-dimensional if βX is zero-dimensional.

topological group A group G is a topological group if it is endowed with a topology such that the group's binary operation and the group's inverse function are continuous.

totally disconnected A topological space is totally disconnected if all its connected componentes are points.

ultrafilter A filter $\mathcal{F}$ is an ultrafilter if it is not properly contained in any other filter or, equivalently, if for every $A \subseteq \omega$ either $A \in \mathcal{F}$ or $\omega \setminus A \in \mathcal{F}$.

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