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MODELOS PROBABILISTAS DISCONTINUOS: PARTE 1.- MODELOS ASOCIADOS A PROCESOS ESTABLES. PARTE 2.- MODELOS ASOCIADOS A PROCESOS EVOLUTIVOS.

TESIS

QUE PARA OBTENER EL GRADO ACADÉMICO DE:

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PRESENTA:

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Modelos Probabilistas Discontinuos:

Parte 1.- Modelos asociados a procesos estables.
Parte 2.- Modelos asociados a procesos evolutivos.

José Luis Ángel Pérez Garmendia

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Contents

Introducción

Este trabajo esta dividido en dos partes, en la primera construimos y estudiamos propiedades de 2 tipos de procesos estocásticos: los procesos estable temperados, vistos como integrales estocásticas de ciertas funciones deterministas con respecto a medidas aleatorias estable temperadas, y procesos de Lévy relacionados mediante la representación de Lamperti (ver [46]) para procesos de Markov autosimilares positivos con algunos procesos estables. En la segunda parte se construye un modelo probabilista microscópico para describir la dinámica de una población sexual. Daremos un breve panorama del contenido de este trabajo.

En el primer capítulo damos una construcción de la integral estable temperada como una forma de construir procesos estocásticos cuyas distribuciones finito dimensionales son estable temperadas. Definimos esta integral como la integral de una función determinista con respecto a una medida aleatoria estable temperada. También estudiamos ciertas propiedades que satisface la integral estable temperada en términos de sus integrandos deterministas como: linealidad, independencia, cambio de variable y algunos teoremas de convergencia. Y por último damos ejemplos de procesos estable temperados, como el movimiento de Lévy estable temperado, el proceso de Orstein-Uhlenbeck, etc.

Posteriormente en el capítulo 2 utilizamos la construcción de la integral estable temperada para construir una nueva clase de procesos que les llamamos procesos de promedios móviles pesados temperados o WTMAP. A diferencia de los procesos de promedios móviles temperados que se definen como la integral estocática con respecto a una medida aleatoria estable temperada con la medida de Lebesgue como medida de control, para definir un WTMAP utilizamos como medida de control cualquier medida absolutamente continua con respecto a la medida de Lebesgue; y su derivada de Radon-Nikodým es el peso al cual deben su nombre. Estos procesos permiten mas libertad para aplicaciones en series de tiempo que los procesos de promedios móviles usuales. Probamos que estos procesos, a pesar de no ser estacionarios en general, para $\alpha \in (0,1)$ tienen la propiedad de mixing, lo cual implica ergodicidad. Y por último probamos que a tiempos cortos se comportan como procesos de promedios móviles pesados estables y a tiempos cortos como procesos de promedios móviles pesados Gaussianos.

En el capítulo 3 estudiamos una nueva clase de procesos de Lévy a la cual llamamos procesos de Lamperti estables. Estos procesos surgen como una generalización de una familia de procesos de Lévy estudiados por Lamperti [46] y recientemente por Caballero y Chaumont [14] que están relacionados mediante la representación de Lamperti para procesos de Markov autosimilares positivos con el subordinador estable y algunos procesos estables condicionados. Esta clase de procesos tienen la ventaja de que se pueden calcular muchas expresiones matemáticas, como por ejemplo: funcionales exponenciales, funciones de escala, descomposición de Weiner Hopf, etc.

Comenzamos estudiando las distribuciones Lamperti estables en $\mathbb{R}^d$ y obtenemos algunas de sus propiedades. Entre estas propiedades podemos mencionar brevemente que tienen momentos de todos los ordenes, y tienen densidad en C^∞.

Posteriormente estudiamos los procesos de Lévy Lamperti estables con énfasis en el caso unidimensional y obtenemos una forma cerrada explícita para su exponente característico, y probamos que a tiempos cortos se comportan como procesos de Lévy estables mientras que a tiempos largos como un movimiento Browniano.

Finalmente proporcionamos una representación en series para estos procesos que permite realizar simulaciones de sus trayectorias y damos algunos ejemplos de ellas.

Motivados por el trabajo desarrollado en el capítulo 3 nos enfocamos a estudiar una nueva clase de procesos de Lévy. Dado que la norma de un proceso estable simétrico en $\mathbb{R}^d$ es un proceso de Markov autosimilar positivo, obtenemos esta nueva clase de procesos mediante la representación de Lamperti que se mencionó anteriormente.

Calculamos el generador infinitesimal de esta nueva clase y por ende, siguiendo a Lamperti (ver [46]), sus características, es decir: su medida de Lévy y su coeficiente lineal. También obtenemos en el caso $d = 1$ una descomposición de esta clase de procesos como la suma de dos procesos de Lévy independientes: un proceso de Lévy Lamperti estable y un proceso de Poisson compuesto.

También obtenemos identidades explicitas para el problema de salida de un intervalo, la distribución de ínfimo, y exploramos el problema de que el proceso toque dos puntos.

Por último obtenemos la descomposición de Weiner Hopf para esta clase lo cual nos permite obtener una forma cerrada explícita para su exponente característico.

La segunda parte de este trabajo esta enfocada a encontrar un modelo matemático probabilista para describir la dinámica de una población biológica haploide donde los individuos se reproducen sexualmente. En este modelo consideramos los fenómenos de selección natural y recombinación genética para describir la evolución de la población, a través de: la competencia entre los individuos por recursos y el intercambio génetico entre ellos al dar descendencia. Uno de los objetivos de este modelo es estudiar el fenónemo de especiación simpátrica, es decir la formación de dos o mas especies a partir de una sola especie ancestral sin obstáculos geográficos.

En el capítulo 5 damos preliminares biólogicos sobre los factores que nos interesa incluir y estudiar en el modelo: como los fenómenos de recombinación genética, selección natural, especiación simpátrica, etc.

Dado que nos interesaba obtener un modelo a tiempo continuo realista desde el punto de vista biológico decidimos estudiar modelos discretos que se encontraban ya publicados en la literatura.

El primero, que se estudia en el capítulo 6, fue un modelo propuesto por Bürger [11] para estudiar la evolución de la densidad de individuos con determinadas características en una población asexual. Encontramos una relación entre su modelo y los propuestos por Del Moral [22] a tiempo discreto, que satisfacen la fórmula de Feynman-Kac. Posteriormente estudiamos una aproximación a tiempo continuo del modelo de Bürger, discretizando el tiempo y considerando una renormalización adecuada de los factores de selección y mutación. Finalmente estudiamos este modelo a tiempo continuo y la relación que guarda con los modelos que satisfacen la fórmula de Feynman-Kac a tiempo continuo.

En el capítulo 7, estudiamos el modelo determinista propuesto por Nagylaki [54] a tiempo discreto, para describir la dinámica de la densidad de individuos con determinado genotipo en una población sexual haploide. En este modelo intervienen dos factores principales: selección natural, y recombinación genética. Posteriormente motivados por la idea desarrollada en el capítulo anterior obtenemos una aproximación a tiempo continuo del modelo a tiempo discreto de Nagylaki, recuperando asi el modelo propuesto por Shashahani [72].

Finalmente en el capítulo 8, inspirados por el trabajo de Méléard y Champagnat [19], desarrollamos un modelo probabilista para estudiar la dinámica de una población sexual. Asignando a cada individuo relojes aleatorios de nacimiento y muerte podemos describir su evolución en la población.

En este capítulo damos la construcción del modelo de población como la solución a cierta ecuación diferencial estocástica; asimismo demostramos propiedades de momentos y de martingala para este proceso.

Luego obtenemos una aproximación cuando el tamaño de la población es grande del proceso bajo cierta renormalización, la cual resulta ser la solución a una ecuación diferencial determinista. También vemos como esta aproximación se reduce bajo ciertas condiciones a los modelos deterministas macroscópicos estudiados por Shashahani [72] y Doebeli [25].

Finalmente damos un algoritmo de simulación para este proceso, y lo ilustramos con algunos ejemplos orientados en estudiar el fenómeno de especiación simpátrica.

Part I

Probabilistic models associated with stable stochastic processes.

Chapter 1

On tempered stable stochastic integrals

1.1 Introduction

Tempered stable processes have been known for quite some time as the truncated Lévy flight model used to model turbulence. The Tempered stable Lévy processes and the Tempered fractional motion have been recently studied by Rosiński and Houdré respectively in [67] and [35]. The importance of these processes is that in a certain way they mix both α-stable and Gaussian trends, and so they have many applications for example in mathematical finance to model volatility or in option pricing.
In this paper we use the parametrization for tempered stable distributions given in [67] to construct the tempered α-stable stochastic integral.
To do this (Section 3), we first prove the existence of a stochastic process by means of its finite dimensional distributions; this is used to construct a tempered α-stable random measure (Section 4). Then we define the tempered α-stable stochastic integral for a deterministic function in a certain class F with respect to a tempered α-stable random measure. We first define it for simple deterministic functions, and then the definition for $f \in F$ is obtained as a limit in probability of a sequence of tempered α-stable integrals of simple functions (Section 5).
The difference between this approach and the one given in [67] is that in the later one they construct the stochastic tempered integral by means of the shot noise representation. Each method has advantages, the shot noise method allows numerical approximations among other things. With our method we can study some properties that would be difficult to explore with the shot noise representation.
These properties are the main results of this paper and are given in Section 6. in Theorem 1, we prove that under suitable conditions on the deterministic functions (f_n) the corresponding integrals converge in probability.

Theorem 2 gives conditions under which two tempered α-stable integrals are independent, and in Therorem 3 we explore the effect of a change of variable in the integral.
Another advantage of our method is that it allows us to construct, with a different approach, examples that have already been studied: the tempered α-stable Lévy motion, and the tempered Ornstein-Uhlenbeck process in [67], and the tempered stable fractional motion in [35]; as well as new ones, like the tempered moving averages process. All of which are given in Section 7.

1.2 Preliminaries and notations

We set some notations that will be used throughout the paper.
$\mathbb{R}^d$ is the d-dimensionl euclidean space with the norm $\|\cdot\|$.
$\mathbb{R}^d_0 = \mathbb{R}^d - \{0\}$, and $\mathbb{B}(\mathbb{R}^d_0)$ is the Borel σ-field of $\mathbb{R}^d_0$. We will write $\stackrel{d}{=}$ to denote equality in law, and $\stackrel{f.d.d.}{\rightarrow}$ is used for convergence in the sense of the finite dimensional distributions.
Now we recall the definition of a tempered stable distribution as well as some of their properties, which will be used later.

Definition 1. *An infinitely divisible probability measure, without Gaussian part on $\mathbb{R}^d_0$ is called tempered stable if its Lévy measure ν has the following form*

$$\nu(B) = \int_{\mathbb{R}^d_0}\int_0^\infty 1_B(sx)s^{-\alpha-1}e^{-s}dsR(dx), \quad B \in \mathbb{B}(\mathbb{R}^d).$$

Where $\alpha \in (0,2)$ and where R is a measure defined in $\mathbb{B}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty.$$

In a similar way as in the stable case the two parameters α and R determine a tempered stable distribution in $\mathbb{R}^d$.
The characteristic function of a tempered stable distribution in $\mathbb{R}^d$, is given by (see [67]).

$$\widehat{\mu}(y) = \exp\{\int_{\mathbb{R}^d_0} \psi_\alpha(\langle y,x\rangle)R(dx) + i\langle y,b\rangle\}. \tag{1.2.1}$$

Under the assumption that when $\alpha = 1$

$$\int_{\mathbb{R}^d_0} \|x\|(1+\log^+\|x\|)R(dx) < \infty,$$

where

$$\psi_\alpha(s) = \begin{cases} k_\alpha[1-(1-is)^\alpha], & 0<\alpha<1 \\ (1-is)\log(1-is)+is, & \alpha = 1 \\ k_\alpha[(1-is)^\alpha - 1 + i\alpha s], & 1<\alpha<2. \end{cases} \tag{1.2.2}$$

Here $k_\alpha = |\Gamma(1-\alpha)|, \alpha \neq 1$. And is important to note that (1.2.1) determines the pair (α, R) uniquely.
We write $\mu \sim TS(\alpha, R; b)$ if the characteristic function of μ is given by (1.2.1).
Finally let (E, ε, m) be a a measure space and let us choose when $\alpha \neq 1$

$$F = L^\alpha(E, \varepsilon, m),$$

where

$$L^\alpha(E, \varepsilon, m) = \{f : E \to \mathbb{R} \text{ measurable}, \int_E |f(x)|^\alpha m(dx) < \infty\};$$

when $\alpha = 1$ we take

$$F = F(E, \varepsilon, m),$$

where

$$F(E, \varepsilon, m) = \{f : f \in L^1(E, \varepsilon, m) \quad \text{and} \int_E |f(s) \log |f(s)||m(ds) < \infty\}.$$

Without loss of generality we will suppose that m is σ-finite, because if $f \in F$ it implies that the support of f is contained in a region of E where m is σ-finite.

1.3 Construction of the tempered stable integral process

An α-tempered stable integral is defined by

$$I(f) = \int_E f(s)M(ds),$$

where M is an independently scattered α tempered stable random measure on E with control measure $m(dx)$ and $0 < \alpha \leq 2$, (E, m) is a σ-finite complete measure space, $f : E \to \mathbb{R}$ is measurable and such that $f \in F$.
The distribution of the process $\{I(f) : f \in F\}$ is determined by its finite dimensional distributions, given in terms of its characteristic functions. We will construct the stable tempered integral as a stochastic processes $\{I(f), f \in F\}$ indexed by a set of functions F, the idea follows the construction of the stable integral in [69].

1.3.1 Specification of the finite-dimensional distributions

Given $f_1, \ldots, f_d \in F$, we will define a probability measure $P_{f_1,\ldots,f_d}$ in $\mathbb{R}^d$ by its characteristic function as follows:

$$\phi_{f_1,\ldots,f_d}(y_1, \ldots, y_d) = \exp\left\{k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha(x\langle f(s), y\rangle)m(ds)R(dx)\right\}. \quad (1.3.3)$$

where

$$\begin{aligned} f(s) &= (f_1(s), \ldots, f_d(s)), \\ y &= (y_1, \ldots, y_d). \end{aligned}$$

Where the function ψ_α is defined in (1.2.2) and R is a measure in $\mathbb{R}_0$ satisfying

$$\int_{\mathbb{R}_0} |x|^\alpha R(dx) < \infty, \tag{1.3.4}$$

and when $\alpha = 1$ we ask additionally that

$$\int_{\mathbb{R}_0} |x|| \log |x||R(dx) < \infty,$$

Next, in order to prove that $\phi_{f_1,\ldots,f_d}$ is the characteristic function of a probability measure $\mathbb{R}_0^d$, we make the change of variables $z_j = f_j(s)$ in (1.3.3) to obtain

$$\begin{aligned} \phi_{f_1,\ldots,f_d}(y_1, \ldots, y_d) &= \exp\left\{ k_\alpha \int_{\mathbb{R}_0} \int_{E_+} \psi_\alpha(x\langle f(s), y\rangle) m(ds) R(dx) \right\} \\ &= \exp\left\{ k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}^d} \psi_\alpha(x\langle z, y\rangle) \rho(dz) R(dx) \right\}, \end{aligned} \tag{1.3.5}$$

where

$$E_+ = \{s \in E : \sum_{j=1}^d f_j(s)^2 > 0\},$$

and

$$\rho(A) = \int_{E_+} 1_A(f(s)) m(ds) \qquad \text{for every } A \in B(\mathbb{R}_0^d).$$

Finally we will construct another measure in $\mathbb{R}_0^d$ given by

$$\mu(A) = \int_{\mathbb{R}_0} \int_{\mathbb{R}_0^d} 1_A(xz) \rho(dz) R(dx) \qquad \text{for every } A \in B(\mathbb{R}_0^d),$$

In terms of μ, (1.3.5) can be written as:

$$\begin{aligned} \phi_{f_1,\ldots,f_d}(y_1, \ldots, y_d) &= \exp\left\{ k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}_0^d} \psi_\alpha(\langle xz, y\rangle) \rho(dz) R(dx) \right\} \\ &= \exp\left\{ k_\alpha \int_{\mathbb{R}_0^d} \psi_\alpha(\langle \omega, y\rangle) \mu(d\omega) \right\}. \end{aligned} \tag{1.3.6}$$

Assume for the moment that $\alpha \neq 1$. We claim that $\int_{\mathbb{R}_0^d} \|\omega\|^\alpha \mu(d\omega) < \infty$. Indeed observe that

$$\begin{aligned}
\int_{\mathbb{R}_0^d} \|\omega\|^\alpha \mu(d\omega) &= \int_{\mathbb{R}_0} \int_{\mathbb{R}_0^d} \|xz\|^\alpha \rho(dz) R(dx) \\
&= \left(\int_{\mathbb{R}_0} |x|^\alpha R(dx) \right) \left(\int_{\mathbb{R}_0^d} \|z\|^\alpha \rho(dz) \right) \\
&= \left(\int_{\mathbb{R}_0} |x|^\alpha R(dx) \right) \left(\int_E \|f(s)\|^\alpha m(ds) \right) \\
&\leq \left(\int_{\mathbb{R}_0} |x|^\alpha R(dx) \right) \left(\sum_{i=1}^d \int_E f_j(s)^\alpha m(ds) \right).
\end{aligned} \tag{1.3.7}$$

Which together with (1.3.4) and the fact that $f_j \in F$ for $j = 1, \ldots, d$, proves true the claimed fact. And so it follows from (1.3.6) that in the case $\alpha \neq 1$, $\phi_{f_1,\ldots,f_d}$ is the characteristic function of a tempered stable law in $\mathbb{R}^d$. The case $\alpha = 1$ is done in a similar way.

So we have a family of probability measures $\{P_{f_1,\ldots,f_d} : f_1, \ldots, f_d \in F\}$ in $\mathbb{R}^d$ that play the role of the finite dimensional distributions. We will see that this family is consistent in order to apply Kolmogorov's existence theorem, for any permutation $(\pi(1), \ldots, \pi(d))$ of $(1, \ldots, d)$ we have

$$\phi_{f_{\pi(1)},\ldots,f_{\pi(d)}}(y_{\pi(1)}, \ldots, y_{\pi(d)}) = \phi_{f_1,\ldots,f_d}(y_1, \ldots, y_d),$$

and for any $n \leq d$

$$\phi_{f_1,\ldots,f_n}(y_1, \ldots, y_n) = \phi_{f_1,\ldots,f_n,\ldots,f_d}(y_1, \ldots, y_n, 0, \ldots, 0),$$

which proves consistency.

So there is a stochastic process $\{I(f), f \in F\}$ whose finite-dimensional distributions are defined by its Fourier transform in (1.3.3). We will call this process *the tempered stable integral of* f, and m is its control measure.

As expected $I(f)$ has a tempered stable distribution and $I(\cdot)$ is linear:

Property 1. *Let $f \in F$. Then $I(f) \sim TS(\alpha, \mu; 0)$ where*

$$\mu(A) = \int_{\mathbb{R}_0} \int_{E_+} 1_A(xf(s)) m(ds) R(dx) \quad \textit{for } A \in B(\mathbb{R}_0). \tag{1.3.8}$$

Proof. To obtain the characteristic function of $I(f)$ we take $y_2 = \cdots = y_d = 0$ and $f_1 = f$ in (1.3.3), then

$$\phi_{f_1}(y) = \exp\left\{ k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha(x f_1(s) y) m(ds) R(dx) \right\}.$$

Then making the change of variable $w = xf_1(s)$ we have

$$\begin{aligned}\phi_{f_1}(y) &= \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_{E_+} \psi_\alpha(xf(s)y)m(ds)R(dx)\right\} \\ &= \exp\left\{k_\alpha \int_{\mathbb{R}_0} \psi_\alpha(wy)\mu(dw)\right\}, \qquad (1.3.9)\end{aligned}$$

where the measure μ is given by (1.3.8).
Following a method similar to (1.3.7) we obtain

$$\int_{\mathbb{R}_0} |w|^\alpha \mu(dw) < \infty \quad \text{when } \alpha \neq 1, \qquad (1.3.10)$$

$$\int_{\mathbb{R}_0} |w(1 + |\log|w||)|\mu(dw) < \infty \quad \text{when } \alpha = 1.$$

Because of (1.3.9) and (1.3.10) we can finally conclude that $I(f) \sim TS(\alpha, \mu; 0)$.
□

Another useful property of the stable tempered integral is that it is linear.

Property 2. *If $f_1, f_2 \in F$, then*

$$I(a_1 f_1 + a_2 f_2) = a_1 I(f_1) + a_2 I(f_2) \quad a.s. \qquad (1.3.11)$$

for any real numbers a_1 and a_2.

Proof. We have from (1.3.5) that

$$\begin{aligned}&E[\exp\{iy(I(a_1f_1 + a_2f_2) - a_1I(f_1) - a_2I(f_2))\}] \\ &\quad = E[\exp\{i(yI(a_1f_1 + a_2f_2) - (a_1y)I(f_1) - (a_2y)I(f_2))\}] \\ &\quad = \exp\left\{k_a \int_{\mathbb{R}}\int_E \psi_\alpha(x(y(a_1f_1 + a_2f_2) - (a_1y)f_1 - (a_2y)f_2))m(ds)R(dx)\right\} \\ &\quad = 1.\end{aligned}$$

The last equality follows from the fact that $\psi_\alpha(0) = 0$, and it implies (1.3.11).
□

1.4 Tempered stable random measures

We will now define the tempered stable random measure M. We will denote (Ω, F, P) the underlying probabiliy space and $L_0(\Omega)$ the set of all real random variables defined on it and let

$$\varepsilon_0 = \{A \in \varepsilon : m(A) < \infty\}.$$

With all these elements we can now define the tempered stable random measure.

Definition 2. *An independently scattered σ-additive set function*

$$M : \varepsilon_0 \to L^0(\Omega).$$

such that for each $A \in \varepsilon_0$,

$$M(A) \sim TS(\alpha, m(A)R; 0),$$

is called a tempered stable random measure of index α on (E, ε) with control measure m.

We will now prove the existence of the tempered stable random measure.

Proposition 1. *For every measure R_0 in $\mathbb{R}$ that satisfies*

$$\int_{\mathbb{R}_0} |x|^\alpha R(dx) < \infty \quad \text{when } \alpha \neq 1, \tag{1.4.12}$$

$$\int_{\mathbb{R}_0} |x(1 + |\log |x||)|R(dx) < \infty \quad \text{when } \alpha = 1, \tag{1.4.13}$$

there exists a tempered stable random measure on (E, ε) with control measure R.

Proof. We use the existence of the process $\{I(f), f \in F\}$ applied to $f = 1_A$ for $A \in \varepsilon_0$ to obtain a stochastic process $\{M(A), A \in \varepsilon_0\}$ with finite-dimensional distributions given in terms of its characteristic function by (1.3.5). Now consider $A \in \varepsilon_0$ then by (1.3.5) its easy to see using that $\psi_\alpha(0) = 0$

$$\begin{aligned}
&E[\exp\{i(M(A)y)\} \\
&\quad = \exp\left\{k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha(x(1_A(s)y))m(ds)R(dx)\right\} \\
&\quad = \exp\left\{k_\alpha \int_{\mathbb{R}_0} \psi_\alpha(xy)m(A)R(dx)\right\}.
\end{aligned}$$

Which implies that $M(A) \sim TS(\alpha, m(A)R; 0)$. We will now show that M is indeed a random measure, first we will prove that it is independently scattered. Let $A_1, \ldots, A_d$ be disjoint sets belonging to ε_0. Then using that $\psi_\alpha(0) = 0$ and that the sets $A_1, \ldots, A_d$ are disjoint

$$\begin{aligned}
&E[\exp\{i(\sum_{j=1}^d M(A_j)y_j)\}] \\
&\quad = \exp\left\{k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha(x(\sum_{j=1}^d 1_{A_j}(s)y_j))m(ds)R(dx)\right\} \\
&\quad = \prod_{i=1}^d \exp\left\{k_\alpha \int_{\mathbb{R}_0} \psi_\alpha(xy_j)m(A_j)R(dx)\right\} \\
&\quad = \prod_{i=1}^d E[\exp\{i(M(A_j)y_j)\}].
\end{aligned}$$

Which proves that $M(A_1), \dots, M(A_d)$ are independent.
The finite additivity follows from the linearity of the integral, let $A_1, \dots, A_d$ be disjoint sets belonging to ε_0 then

$$M(\bigcup_{j=1}^{d} A_j) = I(1_{\bigcup_{j=1}^{d} A_j}) = I(\sum_{j=1}^{d} 1_{A_j}) \stackrel{a.s.}{=} \sum_{j=1}^{d} I(1_{A_j}) = \sum_{j=1}^{d} M(A_j).$$

Now that we have the finite additivity for the random measure, we will prove the σ-additivity. We take $A_1, A_2, \dots \in \varepsilon_0$ disjoint sets, and we suppose that $\bigcup_{j=1}^{\infty} A_j \in \varepsilon_0$. Then using the finite additivity

$$M(\bigcup_{j=1}^{\infty} A_j) - \sum_{j=1}^{d+1} M(A_j) = M(\bigcup_{j=1}^{\infty} A_j) - M(\bigcup_{j=1}^{d+1} A_j) = M(\bigcup_{j=d+1}^{\infty} A_j).$$

Then $p - \lim_{d\to\infty} M(\bigcup_{j=d+1}^{\infty}) = 0$ implies that

$$M(\bigcup_{j=1}^{\infty} A_j) = p - \lim_{d\to\infty} \sum_{j=1}^{d} M(A_j), \tag{1.4.14}$$

and because the series $\sum_{j=1}^{d} M(A_j)$ has independent summands, we have that (1.4.14) is equivalent to

$$M(\bigcup_{j=1}^{\infty} A_j) = \lim_{d\to\infty} \sum_{j=1}^{d} M(A_j) \quad \text{a.s.}$$

Now recall that $M(\bigcup_{j=d}^{\infty} A_j) \sim TS(\alpha, m(\bigcup_{j=d}^{\infty} A_j)R; 0)$, and since m is a measure

$$m(\bigcup_{j=d}^{\infty} A_j) = \sum_{j=d}^{\infty} m(A_j), \quad \text{this sum is finite because } \bigcup_{j=d}^{\infty} A_j \in \varepsilon_0.$$

the above implies that $\lim_{d\to\infty} m(\bigcup_{j=d}^{\infty} A_j) = 0$, and so we have that

$$\begin{aligned}\lim_{d\to\infty} E[\exp\{iyM(\bigcup_{j=d}^{\infty} A_j)\}] &= \lim_{d\to\infty} \exp\left\{k_\alpha m(\bigcup_{j=d}^{\infty} A_j) \int_{\mathbb{R}_0} \psi_\alpha(xy)R(dx)\right\} \\ &= 1.\end{aligned}$$

We have then that $M(\bigcup_{j=d}^{\infty} A_j) \to 0$ as $d \to \infty$ in law, which is equivalent to

$$p - \lim_{d\to\infty} M(\bigcup_{j=d}^{\infty} A_j) = 0,$$

proving that M is σ-additive. □

1.5 Tempered stable integrals

We will define the tempered stable integral for $f \in F$ and we still call it $I(f)$. First, let us consider a simple function in F, $f = \sum_{j=1}^{d} a_j 1_{A_j}$, (where the $A_1, \ldots, A_d$ are disjoint), in this case we define the tempered stable integral as follows

$$I(f) = \int_E f(s)M(ds) = \sum_{j=1}^{d} a_j M(A_j).$$

Using (1.3.6) we have

$$E[\exp\{iy(a_j M(A_j))\}] = \exp\left\{k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha(xy(a_j 1_{A_j}(s))m(ds)R(dx)\right\}.$$

Using the independence of $M(A_1), \ldots, M(A_d)$ we have

$$\begin{aligned} E[\exp\{iy(\sum_{j=1}^{d} a_j M(A_j))\}] &= \exp\left\{\sum_{j=1}^{d} \int_{\mathbb{R}_0} \int_E \psi_\alpha(xy(a_j 1_{A_j}(s))m(ds)R(dx)\right\} \\ &= \exp\left\{\int_{\mathbb{R}_0} \int_E \psi_\alpha(xy(\sum_{j=1}^{d} a_j 1_{A_j}(s))m(ds)R(dx)\right\} \\ &= \exp\left\{k_\alpha \int_{\mathbb{R}_0} \psi_\alpha(x\omega)\mu(d\omega)\right\}, \qquad (1.5.15) \end{aligned}$$

where the measure μ has the form given in (1.3.8), and satisfies (1.3.7). The later implies that

$$\int_E f(s)M(ds) \sim TS(\alpha, \mu; 0).$$

On the other hand we notice that the integral $I(\cdot)$ is linear on the space of simple functions.
Now consider $f \in F$, then we take a sequence of simple functions $\{f^n\}_{n=1}^{\infty}$ that satisfy the following

$$\begin{aligned} f^n &\to f \quad \text{a.s. on E}, \\ |f^n| &\le P \quad \text{a.s. on E, with } P \in F. \text{ For every } n \ge 1. \end{aligned}$$

Using (1.5.15) and the linearity of the integral in the simple functions we have

$$I(f^n) - I(f^m) = I(f^n - f^m) \sim TS(\alpha, \mu; 0),$$

where for every $A \in B(\mathbb{R}_0)$,

$$\mu(A) = \int_{\mathbb{R}_0} \int_E 1_A(x(f^n(s) - f^m(s)))m(ds)R(dx).$$

Then

$$E[\exp\{iy(I(f^n - f^m)\}] = \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_E \psi_\alpha(xy(f^n(s) - f^m(s)))m(ds)R(dx)\right\}.$$

We will first find an upper bound for the function $\psi_\alpha(s)$.
When $\alpha \neq 1$ we know from (3.3) in [67] that

$$|\psi_\alpha(s)| \leq C_\alpha |s|^\alpha. \tag{1.5.16}$$

Then we note the following

$$\begin{aligned}|\psi_\alpha(x(f^n(s) - f^m(s))y)| &\leq C_\alpha |x|^\alpha |f^n(s) - f^m(s)|^\alpha y^\alpha \\ &\leq 2C_\alpha |x|^\alpha |P(s)|^\alpha y^\alpha,\end{aligned} \tag{1.5.17}$$

and using the fact that $P \in F$ and (1.4.12) we have

$$\int_{\mathbb{R}_0}\int_E |x|^\alpha |P(s)|^\alpha m(ds)R(dx) = \int_{\mathbb{R}_0} |x|^\alpha R(dx)\int_E |P(s)|^\alpha m(ds) < \infty.$$

And now for the case $\alpha = 1$, using Remark 2.5 in [67] we obtain

$$\begin{aligned}|\psi_1(s)| = |\frac{1}{2}\log(1+s^2) &- s\tan^{-1} s + i(s - s\log(1+s^2)^{\frac{1}{2}} - \tan^{-1} s| \\ &\leq |\log(1+|s|)| + \frac{\pi}{2}|s| + 2|s| + |slog(1+|s|)| \\ &\leq C_1|s|(1 + |\log|s||),\end{aligned} \tag{1.5.18}$$

the first inequality follows from the monoticity of the logarithm, and in the second one we used the bounds $\log(1+v) \leq v$ and $\log(1+v) \leq \log 2 + |\log v|$, $v > 0$.
Proceeding in the same way

$$\begin{aligned}|\psi_1(x(f^n(s) - f^m(s))y)| \leq &K_1(y)(|x(f^n(s) - f^m(s))| \\ &+ |x(f^n(s) - f^m(s))|\log|x(f^n(s) - f^m(s))|) \\ &\leq K_1(y)(|xP(s)| + |xP(s)|\log|xP(s)|),\end{aligned} \tag{1.5.19}$$

again using (1.4.12) and the fact that $P \in F$ it follows

$$\begin{aligned}&\int_{\mathbb{R}_0}\int_E K_1(y)(|xP(s)| + |xP(s)|\log|xP(s)|)m(ds)R(dx) \\ &\quad = \int_{\mathbb{R}_0}|x|R(dx)\int_E |P(s)|m(ds) + \int_{\mathbb{R}_0}|x\log|x||R(dx)\int_E |P(s)|m(ds) \\ &\qquad + \int_{\mathbb{R}_0}|x|\int_E |P(s)\log|P(s)||m(ds) < \infty.\end{aligned}$$

Then thanks to (1.5.17) and (1.5.19) we can apply the Dominanted Convergence Theorem to obtain

$$\lim_{n,m\to\infty}\int_{\mathbb{R}_0}\int_E \psi_\alpha(xy(f^n(s)-f^m(s)))m(ds)R(dx)$$
$$=\int_{\mathbb{R}_0}\int_E \lim_{n,m\to\infty}\psi_\alpha(xy(f^n(s)-f^m(s)))m(ds)R(dx)=0.$$

The last equality follows from the fact that $\psi_\alpha(s)$ is continuous in $\mathbb{R}$.
This implies that $\{I(f^n)\}_{n=1}^\infty$ is a Cauchy sequence in distribution, and so Cauchy sequence in probability. Recalling that the convergence in probability is complete, we can define $I(f)$ as the limit in probability of $\{I(f^n)\}_{n=1}^\infty$ We can proceed as in [69] to prove that the above limit does not depend on the choice of the approximating sequence. Since convergence in probability implies convergence in distribution, we have the following:

Proposition 2. *Let $f \in F$ then*

$$I(f) \sim TS(\alpha, R_f; 0).$$

Where

$$R_f(A)=\int_{\mathbb{R}_0}\int_{E_+} 1_A(xf(s))m(ds)R(dx), \quad \textit{for } A\in\mathbb{B}(\mathbb{R}_0),$$

i.e.,

$$E[\exp\{iyI(f)\}]=\exp\left\{k_\alpha\int_{\mathbb{R}_0}\int_E \psi_\alpha(xf(s)y)m(ds)R(dx)\right\}.$$

Proposition 3. *Let $f,g\in F$, $a,b\in\mathbb{R}$. Then*

$$I(af+bg)=I(af)+I(bg) \quad a.s.$$

Proof. The proof follows the ideas of Samorodnitsky and Taqqu in [69]. □

The linearity of the integral and Proposition 1, imply

Proposition 4. *For any $f_1,\dots,f_d$ in F, the characteristic function of the random vector $(I(f_1),\dots,I(f_d))$ is given by (1.3.5).*

This gives us that the vector $(I(f_1),\dots,I(f_d))$ is tempered α-stable, and that the two constructions of the tempered integral are equivalent.

1.6 Main results on tempered stable integrals

Theorem 1. *Let $X_j=\int_E f_j(x)M(dx)$, $j=1,2,\dots$ and $X=\int_E f(x)M(dx)$, where M is an α tempered stable random measure, with control measure m. If in the case $\alpha\neq 1$*

$$\lim_{j\to\infty}\int_E |f_j(s)-f(s)|^\alpha m(ds)=0, \tag{1.6.20}$$

then

$$p-\lim_{j\to\infty} X_j = X.$$

The result is still true for $\alpha = 1$ *if we assume furthermore that*

$$\lim_{j\to\infty}\int_E |f_j(s)-f(s)|\ln|f_j(s)-f(s)|m(ds) = 0. \tag{1.6.21}$$

Proof. The linearity of the tempered stable integral and Proposition 1, imply that

$$E[\exp\{iyI(f_j-f)\}] = \exp\left\{k_\alpha\int_{\mathbb{R}_0}\int_E \psi_\alpha(x(f_j(s)-f(s))y)m(ds)R(dx)\right\}.$$

Now we have for $\alpha \neq 1$ using (1.5.16)

$$\begin{aligned}
&|\int_{\mathbb{R}_0}\int_E \psi_\alpha(x(f_j(s)-f(s))y)m(ds)R(dx)| \\
&\qquad \le \int_{\mathbb{R}_0}\int_E |\psi_\alpha(x(f_j(s)-f(s))y)|m(ds)R(dx) \\
&\qquad \le \int_{\mathbb{R}_0}\int_E C_\alpha(y)|x|^\alpha|f_j(s)-f(s)|^\alpha m(ds)R(dx) \\
&\qquad = C_\alpha(y)\int_{\mathbb{R}_0}|x|^\alpha R(dx)\int_E |f_j(s)-f(s)|^\alpha m(ds).
\end{aligned}$$

And for $\alpha = 1$ using (1.5.18)

$$\begin{aligned}
&|\int_{\mathbb{R}_0}\int_E \psi_1(x(f_j(s)-f(s))y)m(ds)R(dx)| \\
&\qquad \le \int_{\mathbb{R}_0}\int_E |\psi_1(x(f_j(s)-f(s))y)|m(ds)R(dx) \\
&\qquad \le \int_{\mathbb{R}_0}\int_E C_1(y)((|x(f^n(s)-f^m(s))| \\
&\qquad + |x(f^n(s)-f^m(s))|\log|x(f^n(s)-f^m(s))|)m(ds)R(dx) \\
&\qquad = C_1(y)\int_{\mathbb{R}_0}|x|R(dx)\int_E(|(f^n(s)-f^m(s))| \\
&\qquad + |(f^n(s)-f^m(s))|\log|(f^n(s)-f^m(s))|)m(ds).
\end{aligned}$$

So using (1.6.20) in the case $\alpha \neq 1$ and (1.6.20)-(1.6.21) in the case $\alpha = 1$,

$$\lim_{j\to\infty}\left|\int_{\mathbb{R}_0}\int_E \psi_\alpha(x(f_j(s)-f(s))y)m(ds)R(dx)\right| = 0, \tag{1.6.22}$$

which implies that $\lim_{j\to\infty}(X_j - X) = 0$ in distribution, and therefore in probability. But this is equivalent to

$$p-\lim_{j\to\infty} X_j = X,$$

which proofs the proposition. □

Now we will give some conditions for the independence of tempered integrals.

Theorem 2. *Let $X_1 = \int_E f_1(s)M(ds)$ and $X_2 = \int_E f_2(s)M(ds)$ be two tempered stable integrals of index $\alpha \in (0,2)$ and control measure m. Then X_1 and X_2 are independent if*

$$f_1 f_2 = 0 \quad m\text{-a.s. on } E.$$

Proof. Since $f_1 f_2 = 0$ m-a.s. and $\psi_\alpha(0) = 0$ we have

$$\psi_\alpha\left(\sum_{j=1}^{2} y_j f_j(s)\right) = \sum_{j=1}^{2} \psi_\alpha(y_j f_j(s)) \quad m\text{-a.s.}$$

And since $\{X_j\}_{j=1}^2$ are jointly tempered α-stable, then

$$\begin{aligned} E\left[\exp\left\{i\left(\sum_{j=1}^{2} y_j X_j\right)\right\}\right] &= \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_E \psi_\alpha\left(x\sum_{j=1}^{2} y_j f_j(s)\right) m(ds)R(dx)\right\} \\ &= \prod_{j=1}^{2} \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_E \psi_\alpha\left(x y_j f_j(s)\right) m(ds)R(dx)\right\} \\ &= \prod_{j=1}^{2} E\left[\exp\left\{i\left(y_j X_j\right)\right\}\right]. \end{aligned}$$

Which proves that they are independent. □

The natural question of whether the independence of X_1 and X_2 imply $f_1 f_2 = 0$ m-a.s. on E seems difficult to answer due to the fact that ψ_α has a very complicated expression.
Now we will prove a theorem which will allow us to make a change of variable in tempered integrals.

Theorem 3. *Let M_1 and M_2 be two tempered α-stable random measures defined on (E_1, ε_1) and (E_2, ε_2) with control measures m_1 and m_2 respectively, and suppose that there exists $g : E_1 \to E_2$ such that for every $A \in \varepsilon_2$*

$$m_2(A) = \int_{E_1} 1_A(g(s)) m_1(ds). \tag{1.6.23}$$

Then if $f \in F(M_2)$ we have that

$$\int_{E_2} f(s)M(ds) \overset{d}{=} \int_{E_1} f(g(s))M_1(ds).$$

Proof. Let us consider $\alpha \neq 1$. Since $f \in L^\alpha$ using (1.6.23) it follows that

$$\int_{E_1} |f(g(s))|^\alpha m_1(ds) = \int_{E_2} |f(s)|^\alpha m_2(ds) < \infty,$$

which implies that $f(g) \in F(M_1)$, so it makes sense to define the tempered integral of $f(g)$ with respect to the tempered random measure M_1.
On the other hand let $X = \int_{E_2} f(s)M_2(ds)$, then the characteristic function of X is given by

$$E[e^{(iyX)}] = \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_{E_2} \psi_\alpha((xf(s))y)m_2(ds)R(dx)\right\}. \tag{1.6.24}$$

Now from (1.6.23) it follows that

$$\int_{E_2} \psi_\alpha((xf(s))y)m_2(ds) = \int_{E_1} \psi_\alpha(xf(g(s))y)m_1(ds).$$

So if we substitute the above relation on (1.6.24) we have that

$$\begin{aligned} E[e^{(iyX)}] &= \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_{E_2} \psi_\alpha((xf(s))y)m_2(ds)R(dx)\right\} \\ &= \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_{E_2} \psi_\alpha(xf(g(s))y)m_1(ds)R(dx)\right\} \\ &= E[e^{(iyY)}]. \end{aligned}$$

Where $Y = \int_{E_1} f(g(s))M_1(ds)$. The case $\alpha = 1$ is proved in a similar way. □

1.7 Examples

Example 1. *The α-Tempered Stable Lévy Motion.*

Let

$$X(t) = \int_0^\infty 1_{\{s\le t\}}M(ds) = \int_0^t M(ds) \quad t \ge 0,$$

where M is an αtempered stable random measure with control measure $m(ds) = ds$. Then

$$X(0) = 0 \quad \text{a.s,}$$

$$X(t) - X(r) = \int_r^t M(ds) = M([r,t]) \sim TS(\alpha, (t-r)R). \tag{1.7.25}$$

And by Proposition 2, if $0 \le t_1 < t_2 < \cdots < t_n$, then

$$\begin{aligned} &(X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1}), \\ &\qquad = \left(\int_{t_1}^{t_2} M(ds), \ldots, \int_{t_{n-1}}^{t_n} M(ds)\right). \end{aligned}$$

Observe that the entries of the above random vector are independent because their support are disjoint; which implies that the process has independent increments. The fact that the process has stationary increments follows from

(1.7.25).
Therefore the process $\{X(t), t \geq 0\}$ is a Lévy process, and by (1.7.25) the increments have a tempered stable distribution. This process has been studied by Rosiński in [67].

Example 2. *Moving Averages*

Let us take f a measurable function on $\mathbb{R}$ satisfying

$$\int_{\mathbb{R}} |f(s)|^{\alpha} ds < \infty, \quad \text{for } \alpha \neq 1,$$

$$\int_{\mathbb{R}} |f(s)(1 + |\log |f(s)||)| ds < \infty, \quad \text{for } \alpha = 1,$$

and let us take

$$X(t) = \int_{-\infty}^{\infty} f(t-s) M(ds), \quad t \in \mathbb{R},$$

where M is a tempered α-stable random measure with control measure $m(ds) = ds$. We will now prove that this process is stationary, so let us take $t_1, \ldots, t_d$, $h \in \mathbb{R}$, and real $\theta_1, \ldots, \theta_2$,

$$E[\exp\{i(\sum_{j=1}^{d} y_j X(t_j + h))\}]$$

$$= \exp\left\{ k_{\alpha} \int_{\mathbb{R}_0} \int_E \psi_{\alpha}(x(\sum_{j=1}^{d} y_j f(t_j + h - s)) ds R(dx) \right\}$$

now by the change of variable $z = s - h$ we have

$$= \exp\left\{ k_{\alpha} \int_{\mathbb{R}_0} \int_E \psi_{\alpha}(x(\sum_{j=1}^{d} y_j f(t_j - z)) dz R(dx) \right\}$$

$$= E[\exp\{i(\sum_{j=1}^{d} y_j X(t_j))\}].$$

Therefore the process is stationary.

Example 3. *Ornstein-Uhlenbeck process*

Consider $\lambda > 0$ and M a tempered α-stable random measure, with Lebesgue control measure and $\alpha \in (0, 2]$. And let us take the associated Ornstein-Uhlenbeck process

$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-x)} M(dx), \quad -\infty < t < \infty.$$

As we know taking $f(s) = e^{-\lambda x} 1_{\{0,\infty\}}(s)$ the process $X(t)$ is a moving average process, and because of the above it is stationary. And finally for fixed $s < t$,

$$X(t) - e^{-\lambda(t-s)} X(s) = \int_{s}^{t} e^{-\lambda(t-s)} M(ds).$$

Theorem 2 implies that the above random variable will be independent of any linear combination $\sum_{j=1}^{d} b_k X(u_k),\ u_k \le s,\ k = 1, \ldots, n$. Therefore $X(t) - e^{-\lambda(t-s)}X(s)$ is independent of $\sigma(X(u), u \le s)$. This implies that the tempered Ornstein-Uhlenbeck process is a Markov process. Further properties have been obtained by Rosiński in [67].

Example 4. *Tempered Stable Fractional Motion*

Let M be a tempered α-stable random measure, with Lebesgue control measure and $\alpha \in (0, 2]$ and consider

$$X(t) = \int_{-\infty}^{\infty} (|t-s|^{H-1/\alpha} - |s|^{H-1/\alpha}) M(ds), \quad -\infty < t < \infty,$$

where $0 < H < 1$, $H \ne 1/\alpha$. This integral is well defined (see [69]).
First we will prove that it has stationary increments, i.e., for any $\tau \in \mathbb{R}$

$$\{X(t) - X(0), -\infty < t < \infty\} \stackrel{d}{=} \{X(t+\tau) - X(\tau), -\infty < t < \infty\}.$$

By linearity of the tempered integral we have

$$\begin{aligned} X(t+\tau) - X(\tau) &= \int_{-\infty}^{\infty} (|t+\tau-s|^{H-1/\alpha} - |s|^{H-1/\alpha}) M(ds) \\ &\quad - \int_{-\infty}^{\infty} (|\tau-s|^{H-1/\alpha} - |s|^{H-1/\alpha}) M(ds) \\ &= \int_{-\infty}^{\infty} (|t+\tau-s|^{H-1/\alpha} - |\tau-s|^{H-1/\alpha}) M(ds). \end{aligned}$$

Then for $t_1, \ldots, t_d$ it follows

$$E[\exp\{i \sum_{j=1}^{d} y_j (X(t_j+\tau) - X(\tau))\}]$$

$$= \exp\left\{ k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha (x(\sum_{j=1}^{d} y_j (|t_j+\tau-s|^{H-1/\alpha} - |s-\tau|^{H-1/\alpha})) ds R(dx) \right\}$$

now by the change of variable $z = s - \tau$ we have

$$= \exp\left\{ k_\alpha \int_{\mathbb{R}_0} \int_E \psi_\alpha (x(\sum_{j=1}^{d} y_j (|t_j-z|^{H-1/\alpha} - |z|^{H-1/\alpha})) ds R(dx) \right\}$$

$$= E[\exp\{i \sum_{j=1}^{d} y_j (X(t_j) - X(0))\}],$$

then it has stationary increments. This process has been studied with detail by Houdré in [35].

Chapter 2

On weighted tempered moving averages processes

2.1 Introduction

Tempered stable processes have been known for quite some time as the truncated Lévy flight model used to model turbulence. The Tempered stable Lévy processes and the Tempered fractional motion have been recently studied by Rosiński [67] and Houdré and Kawai [36] respectively. The importance of these processes is that in a certain way they mix both α-stable and Gaussian trends, and so they have many applications for example in mathematical finance to model volatility or in option pricing.

We study a new class of processes which we call *Weighted tempered moving averages processes*; we use the construction of the tempered stochastic integral in [59], to define them as a stochastic integral with respect to a tempered stable random measure. The usual definition of the moving averages process is given as a stochastic integral with respect to a random measure with control measure the Lebesgue measure. Here we do it with a general control measure, we only ask that it is absolutely continuous with respect to Lebesgue measure; and its Radon-Nikodým derivative is the weight to which they owe their name. This class of processes allows more freedom in the applications to time series than the usual moving averages processes.

These processes have interesting properties; although these processes in general are no longer stationary, we prove in Section 3 that for $\alpha \in (0, 1)$, they are mixing which implies ergodicity for this class.

Finally in Section 4 we prove that these processes have a similar behavior to the one found by Rosiński [67] for the tempered Lévy motion and Houdré and Kawai [36] for the tempered fractional motion.

2.2 Preliminaries and notations

We set some notations that will be used throughout the paper.
$\mathbb{R}^d$ is the d-dimensionl euclidean space with the norm $\|\cdot\|$.
$\mathbb{R}_0^d = \mathbb{R}^d - \{0\}$, $\mathbb{B}(\mathbb{R}_0^d)$ is the Borel σ-field of $\mathbb{R}_0^d$, and $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$.
We will write $\stackrel{d}{=}$ to denote equality in law, and $\stackrel{f.d.d.}{\rightarrow}$ is used for convergence in the sense of the finite dimensional distributions.
Now we recall the definition of a tempered stable distribution as well as some of their properties, which will be used later.

Definition 3. [1] *An infinitely divisible probability measure, without Gaussian part on $\mathbb{R}_0^d$ is called tempered stable if its Lévy measure ν, in polar coordinates, has the following form*

$$\nu(B) = \int_{\mathbb{R}_0^d}\int_0^\infty 1_B(rx)r^{-\alpha-1}q(r,u)dr\sigma(du),$$

where $\alpha \in (0,2)$, σ is a finite measure on S^{d-1}, and $q : (0,\infty)\times S^{d-1} \to (0,\infty)$ is a Borel function such that $q(\cdot,u)$ is completely monotone with $q(\infty,u) = 0$ and $q(0+,u) = 1$ for each $u \in S^{d-1}$.

Now by Theorem 2.3 in Rosiński [67], the Lévy measure of a tempered stable distribution can be written in the following form

$$\nu(B) = \int_{\mathbb{R}_0^d}\int_0^\infty 1_B(sx)s^{-\alpha-1}e^{-s}dsR(dx), \quad B \in \mathbb{B}(\mathbb{R}^d),$$

where R is a unique measure defined in $\mathbb{B}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \|x\|^\alpha R(dx) < \infty.$$

In a similar way as in the stable case the two parameters α and R determine a tempered stable distribution in $\mathbb{R}^d$.
Under the assumption that in the case where $\alpha = 1$

$$\int_{\mathbb{R}_0^d} \|x\|(1+\log^+\|x\|)R(dx) < \infty,$$

the characteristic function of a tempered stable distribution in $\mathbb{R}^d$, is given by Rosiński (see [67]).

$$\widehat{\mu}(y) = \exp\left\{\int_{\mathbb{R}_0^d} \psi_\alpha(\langle y,x\rangle)R(dx) + i\langle y,b\rangle\right\} \tag{2.2.1}$$

[1] Definition 1 corresponds to the family of probability distributions named as proper tempered stable distributions by Rosiński in [67].

where

$$\psi_\alpha(s) = \begin{cases} k_\alpha[1-(1-is)^\alpha], & 0<\alpha<1 \\ (1-is)\log(1-is)+is, & \alpha=1 \\ k_\alpha[(1-is)^\alpha-1+i\alpha s], & 1<\alpha<2. \end{cases} \tag{2.2.2}$$

Here $k_\alpha = |\Gamma(1-\alpha)|, \alpha \neq 1$. And is important to note that (2.2.1) determines the pair (α, R) uniquely.
We write $\mu \sim TS(\alpha, R; b)$ if the characteristic function of μ is given by (2.2.1). Finally let (E, ε, m) be a a measure space and let us define the following

$$F = \left\{ \begin{array}{ll} L^\alpha(E,\varepsilon,m), & \text{if } \alpha \neq 1; \\ \{f \in L^1(E,\varepsilon,m) : \int_E |f(s)\log|f(s)||m(ds) < \infty\} & \text{if } \alpha = 1; \end{array} \right\}$$

Without loss of generality we will suppose that m is σ-finite, because if $f \in F$ it implies that the support of f is contained in a region of E where m is σ-finite.

2.3 Tempered stable integrals

In this section we give a quick review of the construction of tempered stable integrals given in chapter 1.

Recall the definition of a tempered stable random measure. We will denote (Ω, F, P) the underlying probabiliy space and $L_0(\Omega)$ the set of all real random variables defined on it and let

$$\varepsilon_0 = \{A \in \varepsilon : m(A) < \infty\}$$

With all these elements we can now define the tempered stable random measure.

Definition 4. *An independently scattered σ additive set function*

$$M : \varepsilon_0 \to L^0(\Omega)$$

such that for each $A \in \varepsilon_0$,

$$M(A) \sim TS(\alpha, m(A)R; 0)$$

is called a tempered stable random measure of index α on (E, ε) with control measure m.

An α-tempered stable integral is defined by

$$I(f) = \int_E f(s)M(ds),$$

where M is an independently scattered tempered stable random measure on E with control measure $m(dx)$ and $0 < \alpha < 2$, (E, m) is a σ-finite complete measure space, $f : E \to \mathbb{R}$ is measurable and such that $f \in F$.

The distribution of the process $\{I(f) : f \in F\}$ is determined by its finite dimensional distributions, given in terms of its characteristic functions by

$$\phi_{f_1,\ldots,f_d}(y_1,\ldots,y_d) = \exp\left\{k_\alpha \int_{\mathbb{R}_0}\int_E \psi_\alpha(x\langle f(s), y\rangle)m(ds)R(dx)\right\} \qquad (2.3.3)$$

where

$$f(s) = (f_1(s),\ldots,f_d(s)),$$
$$y = (y_1,\ldots,y_d).$$

Where the function ψ_α is defined in (2.2.2) and R is a measure in $\mathbb{R}_0$ satisfying

$$\int_{\mathbb{R}_0} |x|^\alpha R(dx) < \infty,$$

and when $\alpha = 1$ we ask additionally that

$$\int_{\mathbb{R}_0} |x||\log|x||R(dx) < \infty,$$

In [59] using Kolmogorov's existence theorem an alternative construction of the tempered stable integral is given as a stochastic process $\{I(f), f \in F\}$ indexed by the set of functions F.

2.4 Weighted tempered moving averages process

In this section we study the Weighted Tempered Moving Averages Process, which is a wider class of process that include the tempered moving averages process as a special case. As we have already mentioned, they have interesting behavior under different scalings similar to the tempered stable Lévy motion, and the tempered fractional motion studied by Rosiński in [67] and Houdré and Kawai in [36] respectively.

Definition 5. *Let m be a measure on $\mathbb{B}(\mathbb{R})$ absolutely continuous which respect the Lebesgue measure, a measurable function f on $\mathbb{R}$ satisfying*

$$\int_{\mathbb{R}} |f(s)|^\alpha m(ds) < \infty, \quad \textit{for } \alpha \neq 1,$$
$$\int_{\mathbb{R}} |f(s)(1 + |\log|f(s)||)|m(ds) < \infty, \quad \textit{for } \alpha = 1.$$

Define

$$X(t) = \int_{-\infty}^{\infty} f(t-s)M(ds), \qquad t \in \mathbb{R}$$

Where M is a tempered α-stable random measure with control measure m. Then we call the process $X(t)$ a weighted tempered moving averages process or WTMAP.

When the measure m is not the Lebesgue measure then the WTMAP is not always stationary, but we can prove an asymptotic result which guarantees that for the case $\alpha < 1$ the process is mixing.

2.5 Asymptotic behavior of weighted moving averages tempered stable processes

We will analyze the behavior of a weighted moving averages tempered stable process, using the function

$$\begin{aligned} I(\theta_1, \theta_2; t) := & - \ln E[\exp\{i(\theta_1 X(t) + \theta_2 X(0))\}] \\ & + \ln E[\exp\{i(\theta_1 X(t))\}] + \ln E[\exp\{i(\theta_2 X(0))\}] \end{aligned}$$

We begin by recalling a result given by Hardin Jr. in [34] which tells us that that if $\lim_{t\to\infty} I(\theta_1, \theta_2; t) = 0$ for any θ_1 and θ_2, then the process $\{X(t), t \in \mathbb{R}\}$ is mixing, that is,

$$\lim_{t\to\infty} \mid P(A \cap B) - P(A)P(B) \mid = 0,$$

for every $A \in \sigma\{X(s), s \leq t\}$ and $B \in \sigma\{X(s), s \geq t\}$. We will now prove that a weighted tempered stable moving averages process is mixing when $\alpha < 1$.

Theorem 4. *For a weighted tempered stable moving averages process with* $0 < \alpha < 1$*,*

$$\lim_{t\to\infty} I(\theta_1, \theta_2; t) = 0.$$

Proof. Suppose first that f has compact support. Then we can find $T \in \mathbb{R}$, such that if $t \geq T$ then $f(t + \cdot)$ and $f(\cdot)$ have disjoint supports. Then for $t \geq T$ and recalling that $\psi_\alpha(0) = 0$ we have

$$\begin{aligned} - \ln E[\exp\{i(\theta_1 X(t) + \theta_2 X(0))\}] & \\ &= -k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_1 f(t-x) + \theta_2 f(-x))) dx R(dz) \\ &= -k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_1 f(t-x))) dx R(dz) \\ &\quad - k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_2 f(-x))) dx R(dz) \\ &= - \ln E[\exp\{i(\theta_1 X(t))\}] - \ln E[\exp\{i(\theta_2 X(0))\}]. \end{aligned}$$

Which implies that for $t \geq T$

$$I(\theta_1, \theta_2; t) = 0,$$

and hence $\lim_{t\to\infty} I(\theta_1, \theta_2; t) = 0$.
If f does not have compact support, let $\varepsilon > 0$ be arbitrary and choose a bounded

interval K_ε such that $\int_{\mathbb{R}} |f(-x)|^\alpha 1_{K_\varepsilon^c}(-x)dx < \varepsilon$. We now note that the function $g = f1_{K_\varepsilon}$ has compact support, so we can find $T \geq 0$ such that if $t \geq T$ then $g(t+\cdot)$ and $g(\cdot)$ have disjoint supports. Then for $t \geq T$

$$\begin{aligned} I(\theta_1, \theta_2; t)_\varepsilon = \\ k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_1 f(t-x)1_{K_\varepsilon}(t-x) + \theta_2 f(-x)1_{K_\varepsilon}(-x)))dxR(dz) \\ - k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_1 f(t-x)1_{K_\varepsilon}(t-x)))dxR(dz) \\ - k_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_2 f(-x)1_{K_\varepsilon}(-x)))dxR(dz) \\ = 0. \end{aligned} \tag{2.5.4}$$

It is easy to see that if $\alpha \in (0,1)$, then for any $s, r \in \mathbb{R}$,

$$|\psi_\alpha(r) - \psi_\alpha(rs)| \leq C_\alpha |s-r|^\alpha. \tag{2.5.5}$$

So using (2.5.5) it follows that

$$\begin{aligned} J(\theta_1, \theta_2) := \Big| \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_1 f(t-x)1_{K_\varepsilon}(t-x) + \theta_2 f(-x)1_{K_\varepsilon}(-x)))dxR(dz) \\ - \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_1 f(t-x)(t-x) + \theta_2 f(-x)))dxR(dz) \Big| \\ \leq C_\alpha \int_{\mathbb{R}_0} \int_{\mathbb{R}} |(z(\theta_1 f(t-x)1_{K_\varepsilon}(t-x) + \theta_2 f(-x)1_{K_\varepsilon}(-x) \\ - \theta_1 f(t-x) + \theta_2 f(-x)))|^\alpha dxR(dz) \\ \leq C_\alpha \left(\int_{\mathbb{R}_0} |z|^\alpha R(dz) \right) \left(\int_{K_\varepsilon^c} |\theta_1 f(t-x)|^\alpha dxR(dz) \right. \\ \left. + \int_{K_\varepsilon^c} |\theta_2 f(-x)|^\alpha dxR(dz) \right) \\ < C_\alpha \left(\int_{\mathbb{R}_0} |z|^\alpha R(dz) \right) (\theta_1^\alpha + \theta_2^\alpha)\varepsilon = M_\alpha(\theta_1, \theta_2)\varepsilon. \end{aligned} \tag{2.5.6}$$

In a similar way it follows for $i = 1, 2$

$$\begin{aligned} J(\theta_i) := \Big| \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_i f(t-x)1_{K_\varepsilon}(t-x)))dxR(dz) \\ - \int_{\mathbb{R}_0} \int_{\mathbb{R}} \psi_\alpha(z(\theta_i f(t-x)(t-x)))dxR(dz) \Big| \\ \leq C_\alpha \left(\int_{\mathbb{R}_0} |z|^\alpha R(dz) \right) \left(\int_{K_\varepsilon^c} |\theta_i f(t-x)|^\alpha dxR(dz) \right) \\ < C_\alpha \theta_i^\alpha \left(\int_{\mathbb{R}_0} |z|^\alpha R(dz) \right) \varepsilon = M_\alpha(\theta_i)\varepsilon, \end{aligned} \tag{2.5.7}$$

so using (2.5.4), (2.5.6), and (2.5.7) it follows that if $t \geq T$,

$$\begin{aligned}|I(\theta_1, \theta_2; t)| &= |I(\theta_1, \theta_2; t) - I(\theta_1, \theta_2; t)_\varepsilon| \\ &\leq J(\theta_1, \theta_2) + J(\theta_1) + J(\theta_2) < (M_\alpha(\theta_1, \theta_2) + M_\alpha(\theta_1) + M_\alpha(\theta_2))\varepsilon.\end{aligned}$$

Noting that ε is arbitrary, we have for fixed θ_1, θ_2 that

$$\lim_{t\to\infty} I(\theta_1, \theta_2; t) = 0,$$

which completes the proof. □

2.6 Short and long time behavior of weighted tempered stable moving averages processes

Finally in this section we determine for a certain kind of weighted tempered moving averages process their behavior under different scalings.
We will obtain an asymptotic result for the rescaled processes in the sense of their finite dimensional distributions. As it will be seen in the next Theorem their behaviour in a short time frame is close to an α-stable weighted moving averages process, while in a long time frame its very similar to a Gaussian weighted moving averages process.
We remark that the method used to prove the next Theorem is inspired in the work of Rosiński (see [67]) for the tempered stable Lévy process.

Theorem 5. *Let us take a measure m on $\mathbb{B}(\mathbb{R})$ satisfying that m is absolutely continuous with respect the Lebesgue measure, and a measurable function f on $\mathbb{R}$ such that $f \in F$.*
Let g be the Radon-Nikodým derivate of m with respect the Lebesgue measure and define the following family of measures $\{m_h\}_{h\in\mathbb{R}}$

$$m_h(A) = \int_A g(s/h)ds \qquad \text{with } A \in \mathbb{B}(\mathbb{R}).$$

Let $\alpha \in (0, 2)$, and M_h a tempered α-stable random measure with control measure m_h, and define

$$X_h(t) = \int_{-\infty}^{\infty} f(t-s)M_h(ds).$$

(i)Short time behavior:

Suppose the following

1. *There exists a measurable function u_1 on $\mathbb{R}$ and a function $r_1 \in F$ such that $|u_1(h)f(ht)| \leq r_1(t)$ for all $t \in \mathbb{R}$.*

2. *There exists a function $L_1 \in F$, such that $\lim_{h\to 0} u_1(h)f(ht) = L_1(t)$ for all $t \in \mathbb{R}$.*

Then when $\alpha \neq 1$

$$\{h^{-1/\alpha}u_1(h)X_h(ht) : t \geq 0\} \stackrel{f.d.d.}{\to} \{cX^{(\alpha)}(t) : t \geq 0\} \quad \text{as } h \to 0+.$$

Where

$$X^{(\alpha)}(t) = c\int_{-\infty}^{\infty} L_1(t-s)M^{(\alpha)}(ds),$$

$M^{(\alpha)}$ *is an* α*-stable random measure with control measure* m*, and*

$$c = \left(k_\alpha |\cos\frac{\pi\alpha}{2}| \int_{\mathbb{R}_0} |x|^\alpha R(dx)\right)^{1/\alpha}.$$

And when $\alpha = 1$

$$\{h^{-1}u_1(h)(X_h(ht) - b(h,t)) : t \geq 0\} \stackrel{f.d.d.}{\to} \{cX^{(1)}(t) : t \geq 0\} \quad \text{as } h \to 0+.$$

Where

$$b(h,t) = \log h \int_{\mathbb{R}_0} xR(dx) \int_{-\infty}^{\infty} f(t-s)m(ds).$$

And

$$X^{(1)}(t) = c\int_{-\infty}^{\infty} L_1(t-s)M^{(1)}(ds) + \mu,$$

$$c = \frac{\pi}{2}\int_{\mathbb{R}_0} |x|R(dx),$$
$$\mu = \int_{\mathbb{R}_0}\int_{-\infty}^{\infty} x\sum_{i=1}^{k} a_i L_1(t_i - s)(1 - \log|x|)m(ds)R(dx),$$

where $M^{(1)}$ *is an 1-stable random measure with control measure* m*.*
(ii)Long time behavior:

Assume the following

1. $\int_{\mathbb{R}_0} |x|^2 R(dx) < \infty$.
2. *There exists a measurable function* u_2 *on* $\mathbb{R}$ *and a function* $r_2 \in L^2(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$*, such that* $|u_2(h)f(ht)| \leq r_2(t)$ *for all* $t \in \mathbb{R}$*.*
3. *There exists a function* $L_2 \in L^2(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$*, such that* $\lim_{h\to\infty} u_2(h)f(ht) = L_2(t)$ *for all* $t \in \mathbb{R}$*.*

Let $1 \leq \alpha < 2$*, then*

$$\{h^{-1/2}u_2(h)X(ht) : t \geq 0\} \stackrel{f.d.d.}{\to} \{cX^{(2)}(t) : t \geq 0\} \quad \text{as } h \to \infty.$$

Where

$$X^{(2)}(t) = c\int_{-\infty}^{\infty} L_2(t-s)M^{(2)}(ds),$$

$M^{(2)}$ *is an Gaussian random measure with control measure* m*, and*

$$c = \left(\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2 R(dx)\right)^{1/2}.$$

And when $0<\alpha<1$

$$\{h^{-1/2}u_2(X(ht)-k(h,t)) : t\geq 0\} \overset{f.d.d.}{\to} \{cX^{(2)}(t) : t\geq 0\} \quad \textit{as } h\to\infty.$$

Where

$$k(h,t) = \alpha\Gamma(1-\alpha)\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} xf(ht-s)m_h ds R(dx),$$

and

$$X^{(2)}(t) = c\int_{-\infty}^{\infty} L_2(t-s)M^{(2)}(ds),$$

$M^{(2)}$ *is an Gaussian random measure with control measure* m*, and*

$$c = \left(\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2 R(dx)\right)^{1/2}.$$

Proof. It suffices to show that for any reals $\{a_i\}_{i=1}^k$, and nonnegative nondecreasing reals $\{t_i\}_{i=1}^k$ $k\in\mathbb{N}$, the random variable $\sum_{i=1}^k a_i X_h(ht_i)$ converges in law to $\sum_{i=1}^k a_i X^{(\alpha)}(t_i)$, as $h\to 0$.

We note the following

$$\begin{aligned}\sum_{i=1}^k a_i(h^{-1/\alpha}u_1(h)X_h(ht_i)) &= \sum_{i=1}^k a_i h^{-1/\alpha}u_1(h)\int_{-\infty}^{\infty} f(ht_i-s)M_h(ds)\\ &= \int_{-\infty}^{\infty}\left(\sum_{i=1}^k a_i h^{-1/\alpha}u_1(h)f(ht_i-s)\right)M_h(ds).\end{aligned}$$

Now by (2.3.3) we have

$$\begin{aligned}&E\left[\exp\left\{iy\left(\sum_{i=1}^k a_i h^{-1/\alpha}u_1(h)X_h(ht_i)\right)\right\}\right]\\ &\quad= \exp\left\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty}\psi_\alpha\left(xy\left(\sum_{i=1}^k a_i h^{-1/\alpha}u_1(h)f(h(t_i-s/h))\right)\right)g(s/h)dsR(dx)\right\}\\ &\quad= \exp\left\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} h\psi_\alpha\left(xy\left(\sum_{i=1}^k a_i h^{-1/\alpha}u_1(h)f(h(t_i-s))\right)\right)m(ds)R(dx)\right\}.\end{aligned} \tag{2.6.8}$$

On the other hand it is easy to see, following the proof of Theorem 3.1 (i) in Rosiński [67], that for $\alpha \neq 1$,

$$\lim_{h\to 0} h\psi_\alpha\Big(xy\Big(\sum_{i=1}^{k} a_i h^{-1/\alpha} u_1(h) f(h(t_i - s))\Big)\Big) = \phi_\alpha\Big(xy\Big(\sum_{i=1}^{k} a_i L_1(t_i - s)\Big)\Big),$$

where $\phi_\alpha(s)$ is given by,

$$\phi_\alpha(s) = -k_\alpha \Big|\cos\frac{\pi\alpha}{2}\Big||s|^\alpha\Big(1 - \tan\frac{\pi\alpha}{2} sign(s)\Big),$$

and

$$|h\psi_\alpha(h^{-1/\alpha}s)| \leq z_\alpha |s|^\alpha,$$

where z_α is some constant depending only on ($\alpha \neq 1$).
So using the above inequality in the integrand of (2.6.8) it follows that

$$\Big|h\psi_\alpha\Big(xy\Big(\sum_{i=1}^{k} a_i h^{-(1/\alpha)} u_1(h) f(h(t_i - s))\Big)\Big)\Big| \leq z_\alpha |x|^\alpha |y|^\alpha |\sum_{i=1}^{k} a_i r_1(t_i - s)|^\alpha. \tag{2.6.9}$$

From the above calculation, and using that $r_1 \in F$ it follows that

$$\begin{aligned}\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} |x|^\alpha \Big|\sum_{i=1}^{k} a_i r_1(t_i - s)\Big|^\alpha m(ds) R(dx)& \\ \leq \int_{\mathbb{R}_0} |x|^\alpha R(dx) \int_{-\infty}^{\infty}\Big|\sum_{i=1}^{k} a_i r_1(t_i - s)\Big|^\alpha m(ds) < \infty.& \end{aligned} \tag{2.6.10}$$

Now from (2.6.9), (2.6.10), and using the bounded convergence theorem

$$\begin{aligned}&\lim_{h\to 0} E\Big[\exp\Big\{iy\Big(\sum_{i=1}^{k} a_i h^{-1/\alpha} u_1(h) X_h(ht_i)\Big)\Big\}\Big] \\ &= \exp\Big\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} \lim_{h\to 0} h\psi_\alpha\Big(xy\Big(\sum_{i=1}^{k} a_i h^{(-1/\alpha)} u_1(h) f(h(t_i - s))\Big)\Big) m(ds) R(dx)\Big\} \\ &= \exp\Big\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} \phi_\alpha\Big(xy\Big(\sum_{i=1}^{k} a_i L_1(t_i - s)\Big)\Big) m(ds) R(dx)\Big\}. \end{aligned} \tag{2.6.11}$$

Finally it is easy to show that

$$\begin{aligned}&\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} \phi_\alpha\Big(xy\Big(\sum_{i=1}^{k} a_i L_1(t_i - s)\Big)\Big) m(ds) R(dx) \\ &= -\int_{-\infty}^{\infty}\Big|cy\sum_{i=1}^{k} a_i f(t_i - s)\Big|^\alpha \Big(1 - i\tan\frac{\pi\alpha}{2}\beta\Big(sign\Big(y\sum_{i=1}^{k} a_i L_1(t_i - s)\Big)\Big) m(ds). \end{aligned} \tag{2.6.12}$$

Where

$$c = \left(\Gamma(-\alpha)\left|\cos\frac{\pi\alpha}{2}\right|\int_{\mathbb{R}_0}|x|^\alpha R(dx)\right)^{1/\alpha},$$

$$\beta = \frac{\int_{\mathbb{R}_0} x^{<\alpha>}R(dx)}{\int_{\mathbb{R}_0}|x|^\alpha R(dx)},$$

and $x^{\langle\alpha\rangle} := |x|^\alpha sign(x)$.
Then by (2.6.12) it follows that (2.6.11) is the characteristic function of a weighted α-stable moving averages process

$$X^{(\alpha)}(t) = c\int_{-\infty}^{\infty} L_1(t-s)M^{(\alpha)}(ds),$$

where $M^{(\alpha)}$ is an α-stable random measure, such that for every $A \in \mathbb{B}(\mathbb{R})$

$$M^{(\alpha)}(A) \sim S_\alpha((m(A))^{1/\alpha}, \beta, 0).$$

Now we will work the case $\alpha = 1$, then following the procedure in (2.6.8)

$$\begin{aligned} E\Big[\exp\Big\{iy\Big(\sum_{i=1}^k a_i h^{-1}u_1(h)(X_h(ht_i) - b(t_i,h))\Big)\Big\}\Big] \\ = \exp\Big\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty}\Big(h\psi_1\Big(h^{-1}u_1(h)xy\Big(\sum_{i=1}^k a_i f(h(t_i-s))\Big)\Big) \\ - ixy\Big(\sum_{i=1}^k a_i u_1(h) f(h(t_i-s))\Big)\log h\Big)m(ds)R(dx)\Big\}. \end{aligned} \tag{2.6.13}$$

It is easy to see that the function $\psi_1(s)$ satisfies the following

$$|h\psi_1(h^{-1}s) - is\log h| \le C|s|(1+|\log|s||) \quad \text{for all } h \in (0,1]. \tag{2.6.14}$$

We use the inequality in (2.6.14) to uniformly bound the integrand (2.6.13) for $h \in (0,1]$

$$\begin{aligned} \Big|h\psi_1\Big(h^{-1}u_1(h)xy\Big(\sum_{i=1}^k a_i f(h(t_i-s))\Big)\Big) - ixyu_1(h)\Big(\sum_{i=1}^k a_i f(h(t_i-s))\Big)\log h\Big| \\ \le C\Big|xy\sum_{i=1}^k a_i r_1(t_i-s)\Big|\Big(1+\Big|\log\Big|xy\sum_{i=1}^k a_i r_1(t_i-s)\Big|\Big|\Big). \end{aligned} \tag{2.6.15}$$

And using the fact that $r_1 \in F$ and the hypothesis 1, it follows that (2.6.15) is integrable.

Now it can be shown following that

$$\lim_{h\to 0} h\psi_1\Bigg(h^{-1}u_1(h)xy\Bigg(\sum_{i=1}^{k} a_i f(h(t_i-s))\Bigg)\Bigg) - ixyu_1(h)\Bigg(\sum_{i=1}^{k} a_i f(h(t_i-s))\Bigg)\log h$$
$$= -\frac{\pi}{2}\Bigg|xy\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg| + ixy\Bigg(\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg)\Bigg(1-\log\Bigg|xy\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg|\Bigg). \tag{2.6.16}$$

So using the Bounded Convergence Theorem and (2.6.16) it follows that

$$\lim_{h\to 0} E\Bigg[\exp\Bigg\{iy\Bigg(\sum_{i=1}^{k} a_i h^{-1}u_1(h)(X_h(ht_i)-b(t_i,h))\Bigg)\Bigg\}\Bigg]$$
$$= \exp\Bigg\{-\int_{-\infty}^{\infty}\Bigg|cy\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg|$$
$$\Bigg(1+i\frac{2}{\pi}\beta\Bigg(sign\Bigg(y\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg)\Bigg)\log\Bigg|y\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg|\Bigg) + i(cy)\mu\Bigg\}. \tag{2.6.17}$$

Where

$$c = \frac{\pi}{2}\int_{\mathbb{R}_0} |x|R(dx)$$

$$\beta = \frac{\int_{\mathbb{R}_0} xR(dx)}{\int_{\mathbb{R}_0} |x|R(dx)}$$
$$\mu = \int_{\mathbb{R}_0}\int_{-\infty}^{\infty}\Bigg(x\sum_{i=1}^{k} a_i L_1(t_i-s)\Bigg)(1-\log|x|)m(ds)R(dx).$$

And again (2.6.17) is the characteristic function of a 1-stable moving averages process

$$X^{(1)}(t) = c\int_{-\infty}^{\infty} L_1(t-s)M^{(1)}(ds) + \mu,$$

where $M^{(1)}$ is an 1-stable random measure, such that for every $A \in \mathbb{B}(\mathbb{R})$

$$M^{(1)}(A) \sim S_1(m(A),\beta,0).$$

Now we are going to prove ii), let us consider the case $1 \leq \alpha < 2$. We have that

$$
\begin{aligned}
&E\bigg[\exp\bigg\{iy\sum_{i=1}^{k} a_i h^{-1/2} u_2(h) X_h(ht_i)\bigg\}\bigg] \\
&= \exp\bigg\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} \psi_\alpha\bigg(xy\sum_{i=1}^{k} a_i h^{-1/2} u_2(h) f(h(t_i - s/h))\bigg) g(s/h) ds R(dx)\bigg\} \\
&= \exp\bigg\{\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} h\psi_\alpha\bigg(xyh^{-1/2} u_2(h)\sum_{i=1}^{k} a_i f(h(t_i - s))\bigg) m(ds) R(dx)\bigg\}.
\end{aligned}
\tag{2.6.18}
$$

It is easy to prove that

$$
\lim_{h\to\infty} h\psi_\alpha\bigg(xyh^{-1/2} u_2(h)\sum_{i=1}^{k} a_i f(h(t_i - s))\bigg) = -\frac{\alpha}{2}\Gamma(2-\alpha)xy\bigg(\sum_{i=1}^{k} a_i L_2(t_i - s)\bigg)^2,
\tag{2.6.19}
$$

and

$$
\bigg|h\psi_\alpha\bigg(xyh^{-1/2} u_2(h)\sum_{i=1}^{k} a_i f(h(t_i - s))\bigg)\bigg| \leq \Gamma(2-\alpha)x^2y^2\bigg(\sum_{i=1}^{k} a_i r_2(t_i - s)\bigg)^2.
\tag{2.6.20}
$$

So from the above inequality, the hypothesis on the existence of the second moment of the measure R, and recalling that $r_2 \in L^2(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$ we note the following

$$
\begin{aligned}
&\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} x^2\bigg(\sum_{i=1}^{k} a_i r_2(t_i - s)\bigg)^2 m(ds) R(dx) \\
&\qquad \leq \int_{\mathbb{R}_0} x^2 R(dx)\int_{-\infty}^{\infty}\bigg(\sum_{i=1}^{k} a_i r_2(t_i - s)\bigg)^2 m(ds) < \infty.
\end{aligned}
$$

Then from (2.6.19),(2.6.20) and using the bounded convergence theorem it follows from (2.6.18) that

$$
\begin{aligned}
&\lim_{h\to\infty} E\bigg[\exp\bigg\{iy\sum_{i=1}^{k} a_i h^{-1/2} u_2(h) X_h(ht_i)\bigg\}\bigg] \\
&\quad = \exp\bigg\{\bigg(-\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2 R(dx)\int_{-\infty}^{\infty}\bigg(\sum_{i=1}^{k} a_i L_2(t_i - s)\bigg)^2 ds\bigg) y^2\bigg\}.
\end{aligned}
\tag{2.6.21}
$$

And (2.6.21) is the characteristic function of a Gaussian moving averages processes

$$
cX^{(2)}(t) = c\int_{-\infty}^{\infty} L_2(t-s) M^{(2)}(ds).
$$

Where $M^{(2)}$ is a Gaussian random measure such that

$$M^{(2)}(A) \sim N(0, (m(A))),$$

and

$$c = \left(\frac{\alpha}{2} \Gamma(2-\alpha) \int_{\mathbb{R}_0} x^2 R(dx) \right)^{1/2}.$$

And now for the case $0 < \alpha < 1$ we have that

$$\begin{aligned} E\Bigg[\exp \Bigg\{ iy \sum_{i=1}^{k} a_i h^{-1/2} u_2(h)(X_h(ht_i) - k(h, t_i)) \Bigg\} \Bigg] &= \\ = \exp \Bigg\{ \int_{\mathbb{R}_0} \int_{-\infty}^{\infty} \Bigg(\psi_\alpha \Bigg(xy \sum_{i=1}^{k} a_i h^{-1/2} u_2(h) f(h(t_i - s/h)) \Bigg) \\ - iyh^{-1/2} u_2(h) \alpha \Gamma(1-\alpha) x \sum_{i=1}^{k} a_i f(h(t_i - s)) \Bigg) g(s/h) ds R(dx) \Bigg\} \\ = \exp \Bigg\{ \int_{\mathbb{R}_0} \int_{-\infty}^{\infty} h \Bigg(\psi_\alpha \Bigg(xyh^{-1/2} u_2(h) \sum_{i=1}^{k} a_i f(h(t_i - s)) \Bigg) \\ - iyh^{-1/2} u_2(h) \alpha \Gamma(1-\alpha) x \sum_{i=1}^{k} a_i f(h(t_i - s)) \Bigg) m(ds) R(dx) \Bigg\}. \end{aligned} \tag{2.6.22}$$

Using the same ideas it is easy to see that

$$\begin{aligned} \lim_{h \to \infty} h \Bigg(\psi_\alpha \Bigg(xyh^{-1/2} u_2(h) \sum_{i=1}^{k} a_i f(h(t_i - s)) \Bigg) \\ - \alpha \Gamma(1-\alpha) xyh^{-1/2} u_2(h) \sum_{i=1}^{k} a_i f(h(t_i - s)) \Bigg) \\ = -\frac{\alpha}{2} \Gamma(2-\alpha) \Bigg(xy \sum_{i=1}^{k} a_i L_2(t_i - s) \Bigg)^2, \end{aligned} \tag{2.6.23}$$

and

$$\begin{aligned} h \Bigg| \psi_\alpha \Bigg(xyh^{-1/2} u_2(h) \sum_{i=1}^{k} a_i f(h(t_i - s)) \Bigg) - iyh^{1/2} u_2(h) \alpha \Gamma(1-\alpha) x \sum_{i=1}^{k} a_i f(t_i - s) \Bigg| \\ \leq \frac{\alpha}{2} \Gamma(2-\alpha) x^2 y^2 \Bigg(\sum_{i=1}^{k} a_i r_2(t_i - s) \Bigg)^2. \end{aligned} \tag{2.6.24}$$

The term in (2.6.24) is integrable, so using the bounded convergence theorem and using (2.6.23) in (2.6.22) we have

$$\lim_{h\to\infty} E\Big[\exp\Big\{iy\sum_{i=1}^{k} a_i h^{-(1/2)} u_2(h)(X_h(ht_i) - k(h,t_i))\Big\}\Big]$$
$$= \exp\Big\{ -\Big(\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2 R(dx)\int_{-\infty}^{\infty}\Big(\sum_{i=1}^{k} a_i L_2(t_i - s)\Big)^2 m(ds)\Big) y^2\Big\}. \tag{2.6.25}$$

And (2.6.25) is the characteristic function of a Gaussian moving averages processes

$$cX^{(2)}(t) = c\int_{-\infty}^{\infty} L_2(t-s)M^{(2)}(ds),$$

where $M^{(2)}$ is a Gaussian random measure such that

$$M^{(2)}(A) \sim N(0,(m(A))) \quad \text{and,}$$
$$c = \Big(\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2 R(dx)\Big)^{1/2}.$$

And so the theorem is proved. □

Corollary 1. *Let us take a measure m on $\mathbb{B}(\mathbb{R})$ satisfying that m is absolutely continuous with respect the Lebesgue measure, and a measurable function f on $\mathbb{R}$ such that $f \in F \cap L^2(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$.*
Let g be the Radon-Nikodým derivate of m with respect the Lebesgue measure and define the following family of measures $\{m_h\}_{h\in\mathbb{R}}$

$$m_h(A) = \int_A g(s/h)ds \qquad \text{with } A \in \mathbb{B}(\mathbb{R}).$$

Let $\alpha \in (0,2)$, and M_h a tempered α-stable random measure with control measure m_h, and define

$$X_h(t) = \int_{-\infty}^{\infty} f(t-s)M_h(ds).$$

Suppose that

$$f(ht) = h^N f(t) \quad \text{for some } N \in \mathbb{R},$$

then when $\alpha \neq 1$
(i)Short time behavior:

$$\{h^{-(N+1/\alpha)}X_h(ht) : t \geq 0\} \overset{f.d.d.}{\to} \{cX^{(\alpha)}(t) : t \geq 0\} \quad \text{as } h \to 0,$$

where

$$X^{(\alpha)}(t) = c\int_{-\infty}^{\infty} f(t-s)M^{(\alpha)}(ds),$$

$M^{(\alpha)}$ *is an α-stable random measure with control measure m, and*

$$c = \left(\Gamma(-\alpha)\left|\cos\frac{\pi\alpha}{2}\right|\int_{\mathbb{R}_0}|x|^\alpha R(dx)\right)^{1/\alpha}.$$

And when $\alpha = 1$

$$\{h^{-(N+1)}(X_h(ht) - b(h,t)) : t \geq 0\} \overset{f.d.d.}{\rightarrow} \{cX^{(1)}(t) : t \geq 0\} \quad \text{as } h \to 0,$$

where

$$b(h,t) = \log h \int_{\mathbb{R}_0} xR(dx) \int_{-\infty}^{\infty} f(t-s)m(ds),$$

and

$$X^{(1)}(t) = c\int_{-\infty}^{\infty} f(t-s)M^{(1)}(ds) + \mu$$

$$c = \frac{\pi}{2}\int_{\mathbb{R}_0}|x|R(dx)$$
$$\mu = \int_{\mathbb{R}_0}\int_{-\infty}^{\infty} x\sum_{i=1}^{k} a_i f(t_i - s)(1-\log|x|)m(ds)R(dx),$$

where $M^{(1)}$ is an 1-stable random measure with control measure m.
(ii)Long time behavior:
Assume that

$$\int_{\mathbb{R}_0}|x|^2R(dx) < \infty,$$

and let $1 \leq \alpha < 2$, then

$$\{h^{-(N+1/2)}X(ht) : t \geq 0\} \overset{f.d.d.}{\rightarrow} \{cX^{(2)}(t) : t \geq 0\} \quad \text{as } h \to \infty,$$

where

$$X^{(2)}(t) = c\int_{-\infty}^{\infty} f(t-s)M^{(2)}(ds),$$

M^2 is an Gaussian random measure with control measure m, and

$$c = \left(\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0}x^2R(dx)\right)^{1/2}.$$

And when $0 < \alpha < 1$

$$\{h^{-(N+1/2)}(X(ht) - k(h,t)) : t \geq 0\} \overset{f.d.d.}{\rightarrow} \{cX^{(2)}(t) : t \geq 0\} \quad \text{as } h \to \infty,$$

where

$$k(h,t) = \alpha\Gamma(1-\alpha)\int_{\mathbb{R}_0}\int_{-\infty}^{\infty} xf(ht-s)m_h(ds)R(dx),$$

and

$$X^{(2)}(t) = c\int_{-\infty}^{\infty} f(t-s)M^{(2)}(ds),$$

$M^{(2)}$ *is an Gaussian random measure with control measure* m*, and*

$$c = \left(\frac{\alpha}{2}\Gamma(2-\alpha)\int_{\mathbb{R}_0} x^2R(dx)\right)^{1/2}.$$

Proof. The proof follows from Theorem 6, by taking $u_i(h) = h^{-N}$ and $r_i(t) = L_i(t) = f(t)$ for $i = 1, 2$. □

Chapter 3

Lamperti stable processes

3.1 Introduction

In recent years the interest in having more accurate models in various domains of applied probability has lead to an increasing attention paid to some special classes of Lévy processes related to the stable law, for example: the tempered stable and the layered stable processes introduced by Rosiński in [67] and Houdré and Kawai in [35], respectively. Both families of processes have nice structural and analytical properties, such as combining in short time the behavior of stable processes and in long time the behavior of a Brownian motion. They also have a series representation which may be used for sample paths simulation.

Lamperti [46] and more recently, Caballero and Chaumont [14] studied four families of Lévy processes which are related to the stable subordinator and some conditioned stable processes via the Lamperti representation of positive self-similar Markov processes. Those studies are the starting point of our work. Recall that positive self-similar Markov processes, $(X, \mathbb{P}_x)$, $x > 0$, are strong Markov process with càdlàg paths, which fulfill a scaling property, i.e. there exists a constant $\alpha > 0$ such that for any $b > 0$:

$$\textit{The law of } \; (bX_{b^{-\alpha}t},\, t \geq 0) \; \textit{under } \mathbb{P}_x \; \textit{is } \mathbb{P}_{bx}.$$

We shall refer to these processes as pssMp. According to Lamperti [46], any pssMp up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. Reciprocally, any Lévy process ξ can be expressed as the logarithm of a time changed pssMp X. In this paper we refer to this Lamperti transformation as LT_1 and the details can be seen in [46].

One of the examples treated by Lamperti in [46] is the case when $(X, \mathbf{P}_x)$ is a stable subordinator of index $\alpha \in (0, 1)$ starting from $x > 0$. Lamperti in [46], describes the characteristics of the associated Lévy process $\mathfrak{s} = (\mathfrak{s}_t, t \geq 0)$

which is again a subordinator, with no drift and with Lévy measure given by

$$\eta(\mathrm{d}x) = \frac{c_+ e^x}{(e^x - 1)^{\alpha+1}}\mathrm{d}x, \qquad x > 0.$$

The three cases of pssMp studied in [14] are related to some conditioned stable processes. The first one is the stable Lévy processes killed when it first exits from the positive half-line, here denoted by $(X^*, \mathbf{P}_x)$. The second class corresponds to that of stable processes conditioned to stay positive (see for instance [20, 26]), denoted by $(X^\uparrow, \mathbf{P}_x)$. Finally, the third class of pssMp is that of stable processes conditioned to hit 0 continuously, denoted by $(X^\downarrow, \mathbf{P}_x)$. The corresponding Lévy processes under the LT_1 transformation are denoted by $\xi^*, \xi^\uparrow$ and $\xi^\downarrow$, respectively. These three classes of Lévy processes have no gaussian component and their Lévy measure are of the type

$$\pi(\mathrm{d}x) = \left(\frac{c_+ e^{dx}}{(e^x - 1)^{\alpha+1}} \mathbb{I}_{\{x>0\}} + \frac{c_- e^{dx}}{(1 - e^x)^{\alpha+1}} \mathbb{I}_{\{x<0\}} \right) \mathrm{d}x,$$

where c_+, c_- are the constants of the Lévy measure of the original stable process and d is a positive parameter. We recall that for ξ^* the constant d equals 1 and moreover it has finite lifetime if $c_- > 0$ and, when $c_- = 0$, it has infinite lifetime and drifts to $-\infty$. For $\xi^\uparrow$ the constant d is equal to $\alpha\rho + 1$, where $\rho = \mathbf{P}_0(X_1 < 0)$. It has infinite lifetime and drifts to ∞. Finally, for the processes $\xi^\uparrow$ the constant d is $\alpha\rho$. It has infinite lifetime and drifts to $-\infty$. We remark that such processes have linear coefficients in their respective characteristic exponent that we denote by $a^*, a^\uparrow$ and $a^\downarrow$. Such constants are computed explicitly in [14] in terms of α, ρ, c_- and c_+. Actually it was proved recently in [21], that the process $\xi^\downarrow$ corresponds to $\xi^\uparrow$ conditioned to drift to $-\infty$ (or equivalently $\xi^\uparrow$ is $\xi^\downarrow$ conditioned to drift to $+\infty$).

Finally, motivated by self-similar continuous state branching processes with immigration, Patie has recently studied in [58] the family of Lévy process with no positive jumps with Lévy measure

$$\eta^*(\mathrm{d}x) = \frac{c_- e^{(\alpha+\vartheta)x}}{(1 - e^x)^{\alpha+1}}\mathrm{d}x, \qquad x < 0,$$

where $\vartheta > -\alpha$.

These Lévy processes have the advantage that the law of many functionals can be computed explicitly, for example: the first exit time from a finite interval or semi-finite interval, overshoots distributions and exponential functionals (see for instance Caballero and Chaumont [14], Chaumont et al. [21], Kyprianou and Pardo [45] and Patie [58]). We also emphasize that in some cases the Wiener-Hopf factors can be computed and scale functions in the spectrally one sided case can be obtained. Since many tractable mathematical expressions can be computed, this class seems to be an useful tool for applications and rich enough to be of particular interest.

In this work, we investigate a generalization of the Lévy processes mentioned above and we will refer to them as Lamperti stable processes. We also study these processes in higher dimensions. We will see that this class has nice structural and analytical properties close to those for tempered stable and layered stable processes.

In section 3.3 we begin by studying the Lamperti stable distributions, which are multivariate infinitely divisible distributions with no Gaussian component and whose Lévy measure is characterized by a triplet (α, f, σ), more precisely an index $\alpha \in (0,2)$, a function f, and a finite measure σ, both defined on the unit sphere in $\mathbb{R}^d$. In particular, the radial component of any of these Lévy measures is asymptotically equivalent to that of a stable distribution, with index α, near zero and has exponential decay at infinity. These distributions have a density with respect to the Lebesgue measure, have finite moments of all orders and exponential finite moments of some order. In the one dimensional case, the density is C^∞. In some particular cases, we also prove that these distributions are self-decomposable.

In section 3.4, we formally introduce the Lamperti stable processes and study their properties with emphasis in the one dimensional case, where we obtain an explicit closed form for the characteristic exponent. Motivated by the works of Rosiński [67] and Houdré and Kawai [35], we prove in section 3.5, 3.6 and 3.7 that Lamperti stable processes in a short time look like a stable process while in a large time scale they look like a Brownian motion, that they are absolute continuous with respect to its short time limiting stable process and they admit a series representation that allows simulations of their paths, respectively.

In section 3.8, we study some related processes: the Ornstein-Uhlenbeck processes whose limiting distribution is a Lamperti stable law and the Lévy processes with no positive jumps whose descending ladder height process is a Lamperti stable subordinator. Finally we illustrate with several examples the presence of Lamperti stable distributions in recent literature.

3.2 Preliminaries

Recall that positive self-similar Markov processes, $(X, \mathbb{P}_x)$, $x > 0$, are strong Markov process with càdlàg paths, which fulfill a scaling property, i.e. there exists a constant $\alpha > 0$ such that for any $b > 0$:

$$\text{The law of } (bX_{b^{-\alpha}t},\, t \geq 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{bx}.$$

We shall refer to these processes as pssMp. According to Lamperti [46], any pssMp up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. More formally, let $(X, \mathbb{P}_x)$ be a pssMp with index $\alpha > 0$, starting from $x > 0$, set

$$S = \inf\{t > 0 : X_t = 0\}$$

and write the canonical process X in the following form:

$$X_t = x \exp\left\{\xi_{\tau(tx^{-\alpha})}\right\} \qquad 0 \leq t < S\,, \tag{3.2.1}$$

where for $t < S$,

$$\tau(t) = \inf\left\{ s \geq 0 : \int_0^s \exp\{\alpha\xi_u\}\,\mathrm{d}u \geq t \right\}.$$

Then under $\mathbb{P}_x$, $\xi = (\xi_t,\ t \geq 0)$ is a Lévy process started from 0 whose law does not depend on $x > 0$ and such that:

(i) if $\mathbb{P}_x(S = +\infty) = 1$, then ξ has an infinite lifetime and drifts towards $+\infty$,

(ii) if $\mathbb{P}_x(S < +\infty,\ X(S-) = 0) = 1$, then the process ξ has an infinite lifetime and drifts towards $-\infty$.

(iii) if $\mathbb{P}_x(S < +\infty,\ X(S-) > 0) = 1$, then ξ is killed at an independent exponentially distributed random time with parameter $\lambda > 0$.

As it is mentioned in [46], the probabilities $\mathbb{P}_x(S = +\infty)$, $\mathbb{P}_x(S < +\infty,\ X(S-) = 0)$ and $\mathbb{P}_x(S < +\infty,\ X(S-) > 0)$ are 0 or 1 independently of x, so that the three classes presented above are exhaustive. Moreover, for any $t < \int_0^\infty \exp\{\alpha\xi_s\}\,\mathrm{d}s$,

$$\tau(t) = \int_0^{x^\alpha t} \frac{\mathrm{d}s}{(X_s)^\alpha}, \quad \mathbb{P}_x\text{-a.s.}$$

Therefore (3.2.1) is invertible and yields a one to one relation between the class of pssMp's killed at time S and the one of Lévy processes.

3.3 Lamperti stable distributions

In this section, we define Lamperti stable distributions on $\mathbb{R}^d$ and establish some of their basic properties. According to Theorem 14.3 in Sato [70], the Lévy measure Π of a stable distribution with index α on $\mathbb{R}^d$ in polar coordinates is of the form

$$\Pi(dr, d\xi) = r^{-(\alpha+1)}\mathrm{d}r\sigma(\mathrm{d}\xi)$$

where $\alpha \in (0, 2)$ and σ is a finite measure on S^{d-1}, the unit sphere on $\mathbb{R}^d$. The measure σ is uniquely determined by Π. Conversely, for any non-zero finite measure σ on S^{d-1} and for any $\alpha \in (0, 2)$ we can define an stable distribution with Lévy measure defined as above.

Motivated by the form of the Lévy measure of the processes mentioned in the introduction and the previous discussion, we define a new family of infinitely divisible distributions that we call Lamperti stable.

Definition 6. *Let μ be an infinitely divisible probability measure on $\mathbb{R}^d$ without Gaussian component. Then, μ is called Lamperti stable if its Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$ is given by*

$$\nu_\sigma^{\alpha,f}(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty \mathbb{I}_B(r\xi)e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)}dr, \qquad B \in \mathcal{B}(\mathbb{R}^d_0), \tag{3.3.2}$$

where $\alpha \in (0, 2)$, σ is non-zero finite measure on S^{d-1}, and $f : S^{d-1} \to \mathbb{R}$ is a measurable function such that $\gamma := \sup_{\xi\in S^{d-1}} f(\xi) < \alpha + 1$.

Note that $\nu_\sigma^{\alpha,f}$ is indeed a Lévy measure on $\mathbb{R}_0^d$. To see this, we need to verify that

$$\int_{\mathbb{R}_0^d}(1\wedge\|x\|^2)\nu_\sigma^{\alpha,f}(\mathrm{d}x)=\int_{S^{d-1}}\sigma(\mathrm{d}\xi)\int_0^\infty(1\wedge r^2)e^{rf(\xi)}(e^r-1)^{-(\alpha+1)}\mathrm{d}r<\infty.$$

On the one hand, since $e^{rf(\xi)}(e^r-1)^{-(\alpha+1)}\sim r^{-(\alpha+1)}$ as $r\to 0$, [1] we have that

$$\int_{S^{d-1}}\sigma(\mathrm{d}\xi)\int_0^1 r^2e^{rf(\xi)}(e^r-1)^{-(\alpha+1)}\mathrm{d}r<\infty.$$

One the other hand, from elementary calculations we deduce

$$\int_{S^{d-1}}\sigma(\mathrm{d}\xi)\int_1^\infty e^{rf(\xi)}(e^r-1)^{-(\alpha+1)}\mathrm{d}r\le\frac{\sigma(S^{d-1})}{(1-e^{-1})^{\alpha+1}}\int_1^\infty e^{-r(\alpha+1-\gamma)}\mathrm{d}r,$$

where $\gamma:=\sup_{\xi\in S^{d-1}}f(\xi)$. Since $\gamma<\alpha+1$, the above integral is finite and therefore $\nu_\sigma^{\alpha,f}$ is a Lévy measure.

In the one dimensional case f takes only two possible values, since $S^0=\{-1,1\}$. In the sequel, we denote these two values by $f(1):=\beta$ and $f(-1):=\delta$. From the definition of the measure σ, we have $\sigma(\{1\})=c_+$ and $\sigma(\{-1\})=c_-$. Therefore each distribution associated to the Lévy processes mentioned in the introduction belongs to the class of Lamperti stable distribution. Following the notation of the introduction, for:

- the subordinator $\mathfrak{s}$, $\beta=1$ and $c_-=0$,
- the process ξ^*, $\beta=1$ and $\delta=\alpha$,
- the process $\xi^\uparrow$, $\beta=\alpha\rho+1$ and $\delta=\alpha(1-\rho)$,
- the process $\xi^\downarrow$, $\beta=\alpha\rho$ and $\delta=\alpha(1-\rho)+1$,
- the class of Lévy processes with no positive jumps considered by Patie [58], $\delta=1-\vartheta$ and $c_+=0$.

Note that Lamperti stable distributions satisfy the divergence condition, i.e.

$$\int_0^\infty e^{rf(\xi)}(e^r-1)^{-(\alpha+1)}dr=\infty\qquad\text{for any}\qquad\xi\in S^{d-1}.$$

Thus from Theorem 27.10 in [70], we deduce that they are absolutely continuous with respect to the Lebesgue measure. Also note that the class of Lamperti stable distributions and that of layered stable distributions (see [35]) are disjoint. This follows from the following estimate

$$\frac{e^{rf(\xi)}}{(e^r-1)^{\alpha+1}}\sim e^{-(\alpha+1-f(\xi))r}\qquad\text{as}\qquad r\to\infty.$$

[1] We say that $f\sim g$ as $x\to x_0$ if for $x_0\in\mathbb{R}^d$ $\lim_{x\to x_0}f(x)/g(x)=1$.

Lamperti stable distributions do not belong in general to the class of tempered stable distributions. For instance, fix $\xi \in S^{d-1}$ and take $f(\xi) \in ((\alpha+1)/2, \alpha+1)$. It is not difficult to see that the first derivative of the function

$$q(r,\xi) = \frac{e^{rf(\xi)}}{(e^r-1)^{\alpha+1}} r^{1+\alpha},$$

is positive for $r \in (0, 2-(\alpha+1)/c)$, which implies that $q(r,\xi)$ is not completely monotone.

Proposition 5. *Let μ be a Lamperti stable distribution with Lévy measure $\nu_\sigma^{\alpha,f}$ given by (3.3.2). If $\zeta < \alpha + 1 - \gamma$, then*

$$\int_{\mathbb{R}^d} e^{\zeta\|x\|}\mu(dx) < \infty,$$

In particular, for $\kappa < \alpha + 1$ and if $f \equiv \kappa$, we have

$$\int_{\mathbb{R}^d} e^{\zeta\|x\|}\mu(dx) < \infty \qquad \text{if and only if} \qquad \zeta < \alpha + 1 - \kappa.$$

Proof. Consider

$$\int_{\{\|x\|>1\}} e^{\zeta\|x\|}\nu_\sigma^{\alpha,f}(dx) \le \sigma(S^{d-1})(1-e^{-1})^{-(\alpha+1)} \int_1^\infty e^{r(\zeta+\gamma-(\alpha+1))}dr,$$

which is finite since $\zeta < \alpha + 1 - \gamma$. Hence by Theorem 25.3 in [70], we obtain the desired result.
Next, we suppose that $f \equiv \kappa$. The former arguments imply that for $\zeta < \alpha+1-\kappa$, the Lamperti stable distribution μ has a finite exponential moment of order ζ. In a similar way, it is clear that

$$\int_{\{\|x\|>1\}} e^{\zeta\|x\|}\nu_\sigma^{\alpha,\kappa}(dx) \ge \sigma(S^{d-1}) \int_1^\infty e^{r(\zeta+\kappa-(\alpha+1))}dr.$$

This implies that $\int_{\{\|x\|>1\}} e^{\zeta\|x\|}\nu_\sigma^{\alpha,\kappa}(dx)$ is finite if and only if $\zeta < \alpha + 1 - \kappa$. □

Corollary 2. *Let μ be be a Lamperti stable distribution. Then*

$$\int_{\mathbb{R}^d} \|x\|^p \mu(dx) < \infty \quad \text{for all} \quad p > 0.$$

Our next result shows that Lamperti stable distributions belong to the Jurek class and that in some cases they are self-decomposable. We recall briefly these definitions. The class of infinitely divisible distributions for which the Lévy measure ν takes the following form

$$\nu(B) = \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty \mathbb{I}_B(r\xi)\ell(\xi,r)dr, \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}_0^d),$$

is called:

(1) Self-decomposable if $r\ell(\xi,r)$ is non negative, measurable in $\xi \in S^{d-1}$ and decreasing in $r \in (0,\infty)$.

(2) Jurek class if $\ell(\xi,r)$ is measurable in $\xi \in S^{d-1}$, and decreasing in $r \in (0,\infty)$.

Proposition 6. *Let μ be a Lamperti-stable distribution on $\mathbb{R}^d$ with Lévy measure $\nu_\sigma^{\alpha,f}$ given by (3.3.2), then μ belongs to the Jurek class. Moreover, μ is selfdecomposable if $f(\xi) \leq \alpha + 1/2$, for all $\xi \in S^{d-1}$ and $\alpha \in (0,2)$.*

Proof. In the case of a Lamperti stable distribution, we have

$$\ell(\xi,r) = \frac{e^{f(\xi)r}}{(e^r-1)^{\alpha+1}},$$

so the measurability of $r\ell(\xi,r)$ and $\ell(\xi,r)$ is clear.
In order to prove that ℓ is decreasing in $r>0$, we fix $\xi \in S^{d-1}$ and consider the derivative of $\ell_1(\cdot) = \ell(\xi,\cdot)$, i.e.

$$\ell_1'(r) = \frac{e^{f(\xi)r}}{(e^r-1)^{\alpha+2}}\Big(e^r(f(\xi)-\alpha-1) - f(\xi)\Big).$$

Hence $\ell_1'(r) < 0$ for $r > 0$, since $f(\xi) \leq \alpha + 1$. This implies that μ is in the Jurek class.

For the second part of the Proposition, we take $k(\xi,r) = r\ell(\xi,r)$. Note that the derivative of $k(\xi,r)$ with respect to r, can be written as

$$\frac{e^{f(\xi)r}}{(e^r-1)^{\alpha+2}}\Big(e^r\left[(1+f(\xi)r-(\alpha+1)r\Big)-f(\xi)\right],$$

Elementary calculations prove that k is decreasing for $r>0$, if $f(\xi) \leq \alpha + 1/2$ for all $\xi \in S^{d-1}$ and $\alpha \in (0,2)$. We leave the details to the reader. $\square$

We note that we can find $\alpha \in (0,2)$ such that if $f(\xi) > \alpha + 1/2$, the Lamperti stable distribution μ is not self-decomposable.

We finish this section with some properties of Lamperti stable distributions defined on $\mathbb{R}$. The first of which says in particular that the density of any Lamperti stable distribution belongs to C^∞.

Proposition 7. *Let μ be a Lamperti stable distribution on $\mathbb{R}$, then μ has a C^∞ density and all the derivatives of the density tend to 0 as $|x|$ tends to ∞.*

Proof: Recall that the function f takes two values, $\beta = f(1)$ and $\rho = f(-1)$ as usual. According to [56] it is enough to prove that

$$g(r) = \int_0^r x^2 \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} \mathrm{d}x, \quad \text{verifies that} \quad \liminf_{r\to 0} \frac{g(r)}{r^{2-a}} > 0, \tag{3.3.3}$$

for some $a \in (0, 2)$. But this is immediate because for r sufficiently small, we have

$$\int_0^r x^2 \frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}} \mathrm{d}x \geq K \int_0^r \frac{x^2}{x^{\alpha+1}} \mathrm{d}x = K r^{2-\alpha},$$

where $K > 0$. In particular when $a = \alpha$, the condition in (3.3.3) is satisfied and the statement follows. ■

Before we state the last result of this section, we recall the definition of a particular class of distributions which is important in risk theory (see for instance [27] and [42]).

Definition 7 (Class $\mathcal{L}^{(q)}$)**.** *Take a parameter $q \geq 0$. We shall say that a distribution function G on $[0, \infty)$ with tail $\overline{G} := 1 - G$ belongs to class $\mathcal{L}^{(q)}$ if $\overline{G}(x) > 0$ for each $x \geq 0$ and*

$$\lim_{u\to\infty} \frac{\overline{G}(u - x)}{\overline{G}(u)} = e^{qx} \quad \textit{for each } x \in \mathbb{R}.$$

The tail of any (Lévy or other) measure, finite and non-zero on (x_0, ∞) for some $x_0 > 0$, can be renormalised to be the tail of a distribution function and by extension, then is said to be in $\mathcal{L}^{(q)}$, if the associated distribution function is in $\mathcal{L}^{(q)}$.

Now, we will prove that the tail of the Lévy measure of any Lamperti stable distributions defined in $\mathbb{R}$ belongs to the class $\mathcal{L}^{(\alpha+1-\beta)}$, where $\beta = f(1)$ as usual.

Proposition 8. *Let μ be a Lamperti-stable distribution on $\mathbb{R}$, then the tail of its Lévy measure belongs to the class $\mathcal{L}^{(\alpha+1-\beta)}$. In particular when μ is defined on $\mathbb{R}_+$, we have that μ belongs to the class $\mathcal{L}^{(\alpha+1-\beta)}$.*

Proof: First, we define

$$\nu(u) = \frac{1}{K} \int_1^u \frac{e^{\beta r}}{(e^r - 1)^{\alpha+1}} \mathrm{d}r, \qquad u \geq 1,$$

where $K = \displaystyle\int_1^\infty \frac{e^{\beta r}}{(e^r - 1)^{\alpha+1}} \mathrm{d}r$. Note that ν corresponds to the distribution function associated to the tail of the Lévy measure of a Lamperti stable distribution.

From elementary calculations, we get

$$\begin{aligned}\frac{\overline{\nu}(u-x)}{\overline{\nu}(u)} &= \int_{u-x}^{\infty} \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r \left(\int_{u}^{\infty} \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r\right)^{-1} \\ &\leq (\alpha+1-\beta)e^{u(\alpha+1-\beta)} \int_{u-x}^{\infty} \frac{e^{\beta r}}{(e^r-1)^{\alpha+1}}\mathrm{d}r \\ &= e^{(\alpha+1-\beta)x}\Big(1-e^{-(u-x)}\Big)^{-\alpha-1}.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\overline{\nu}(u-x)}{\overline{\nu}(u)} &\geq (1-e^{-u})^{\alpha+1}(\alpha+1-\beta)e^{u(\alpha+1-\beta)} \int_{u-x}^{\infty} e^{(\beta-(\alpha+1))r}\mathrm{d}r \\ &= e^{(\alpha+1-\beta)x}(1-e^{-u})^{\alpha+1}.\end{aligned}$$

Therefore taking u large enough, we deduce that $\nu \in \mathcal{L}^{(\alpha+1-\beta)}$. The case when μ is defined in $\mathbb{R}_+$ follows from Proposition 3.4 in Kyprianou et al. [42]. ■

3.4 Lamperti stable Lévy processes

Here, we introduce the class of Lévy processes which is associated to Lamperti stable distributions. We also discuss, specially in the one dimensional case, a number of coarse and fine properties of their paths which are of particular interest for applications.

Definition 8. *A Lévy process without gaussian component, and linear term θ, is called Lamperti stable with characteristics $(\alpha, f, \sigma, \theta)$ if its Lévy measure is given by (3.3.2).*

In the sequel, we denote the Lamperti stable Lévy process with characteristics $(\alpha, f, \sigma, \theta)$ by $X^L = (X^L_t, t \geq 0)$. Its characteristic exponent is defined by $E[\exp(i\langle y, X^L_t\rangle)] = \exp(-t\Psi(y))$ for $t \geq 0$, $y \in \mathbb{R}^d$ where

$$\Psi(y) = i\langle y, \theta\rangle + \int_{\mathbb{R}^d_0} \Big(1 - e^{i\langle y,x\rangle} + i\langle y, x\rangle \mathbb{I}_{\{\|x\|<1\}}\Big)\nu^{\alpha,f}_{\sigma}(\mathrm{d}x), \qquad (3.4.4)$$

the measure $\nu^{\alpha,f}_{\sigma}$ has the form given in (3.3.2) and $\theta \in \mathbb{R}^d$.

The first property in study is the p-th variation of Lamperti stable processes. In particular, we prove that their p-th variation is similar to that of stable processes.

Proposition 9. *Let X^L be a Lamperti stable process with characteristics $(\alpha, f, \sigma, \theta)$.*

i) If $\alpha \in (1, 2)$, the process X^L is a.s. of finite p-th variation in every finite interval if and only if $p \in (\alpha, 2)$.

ii) The process X^L is a.s. of finite variation in every finite interval if and only if $\alpha \in (0, 1)$.

Proof: (i) From Theorem III in Bretagnolle [10], we have that for $p \in (1,2)$, the process X^L is a.s. of finite p-th variation on every finite interval if and only if

$$\int_{\{\|x\|\le 1\}} \|x\|^p \nu_\sigma^{\alpha,f}(\mathrm{d}x) < \infty.$$

Recall that $\gamma := \sup_{\xi\in S^{d-1}} f(\xi)$. From the form of the Lévy measure $\nu_\sigma^{\alpha,f}$ and some elementary calculations, we have

$$\int_{\{\|x\|\le 1\}} \|x\|^p \nu_\sigma^{\alpha,f}(\mathrm{d}x) \le \sigma(S^{d-1})e^\gamma \int_0^1 \frac{r^p}{(e^r-1)^{\alpha+1}}\mathrm{d}r \le \sigma(S^{d-1})e^\gamma \int_0^1 r^{p-(\alpha+1)}\mathrm{d}r. \tag{3.4.5}$$

On the other hand, we have

$$\begin{aligned}\int_{\{\|x\|\le 1\}} \|x\|^p \nu_\sigma^{\alpha,f}(\mathrm{d}x) &\ge \sigma\Big(\{\xi\in S^{d-1} : f(\xi)\ge 0\}\Big)\int_0^1 \frac{r^p}{(e^r-1)^{\alpha+1}}\mathrm{d}r \\ &+ \int_{S^{d-1}} \mathbb{1}_{\{f(\xi)<0\}} e^{f(\xi)}\sigma(\mathrm{d}x)\int_0^1 \frac{r^p}{(e^r-1)^{\alpha+1}}\mathrm{d}r \\ &\ge K\bigg(\sigma\Big(\{\xi\in S^{d-1} : f(\xi)\ge 0\}\Big) + \int_{S^{d-1}} \mathbb{1}_{\{f(\xi)<0\}} e^{f(\xi)}\sigma(\mathrm{d}x)\bigg)\int_0^1 r^{p-(\alpha+1)}\mathrm{d}r,\end{aligned} \tag{3.4.6}$$

for some $K > 0$. Therefore X^L is of finite p-th variation on every finite interval if and only if $p > \alpha$.
The proof of part (ii) is very similar. According to Theorem 3 of Gikhman and Skorokhod [32], it is enough to prove that

$$\int_{\{\|x\|\le 1\}} \|x\| \nu_\sigma^{\alpha,\gamma}(dx) < \infty,$$

if and only if $\alpha \in (0,1)$. But this follows from (3.4.5) and (3.4.6) taking $p = 1$, which concludes the proof. ■

Recall that the characteristic exponent of a Lévy process has a simpler expression when its sample paths have a.s. finite variation in every finite interval. In this case, we have that the characteristic exponent in (3.4.4) takes the form

$$\Psi_L(y) = -i\langle \mathbf{d}, y\rangle + \int_{\mathbb{R}_0^d} \Big(1 - e^{i\langle y,x\rangle}\Big)\nu_\sigma^{\alpha,f}(\mathrm{d}x),$$

where $\mathbf{d} = -\theta - \int_{\{\|x\|\le 1\}} x\nu_\sigma^{\alpha,f}(\mathrm{d}x)$ and, in this case, $\mathbf{d}$ is known as the drift coefficient.

In the rest of this section we work with real valued processes. We are now interested in two important properties of Lévy processes: creeping and the regularity of 0. Define for each $x \ge 0$, the first passage time

$$\tau_x^+ = \inf\Big\{t > 0 : X_t^L > x\Big\},$$

with the convention $\inf\emptyset = \infty$. We say that the Lamperti stable process X^L *creeps upwards* if for all $x \geq 0$, $\mathbf{P}_0(X^L_{\tau_x^+} = x) > 0$. If $-X^L$ creeps upwards, we say that X^L *creeps downwards*. Recall that if creeping occurs at just one x then creeping occurs at all x.

Proposition 10. *Let X^L be a Lamperti stable process with characteristics $L = (\alpha, f, \sigma, \theta)$.*

i) If $\alpha \in (0,1)$ and $\mathbf{d} > 0$, the process X^L creeps upwards.

ii) If $\alpha \in [1,2)$ and $c_+ = 0$, the process X^L creeps upwards.

iii) If $\alpha \in [1,2)$ and $c_+ > 0$, the process X^L does not creep upwards.

Proof. The first part of our statement follows directly from part (i) of Theorem 8 in [43].

From Proposition 9, for $\alpha \in [1,2)$ the process X^L is of unbounded variation. In this case, a result due to Vigon [76] says that X^L creeps upwards if and only if the following integral converges,

$$\int_0^1 \frac{x\nu_\sigma^{\alpha,f}\big([x,\infty)\big)}{H(x)}\mathrm{d}x, \quad \text{where} \quad H(x) = \int_{-x}^{0}\int_{-1}^{y} \nu_\sigma^{\alpha,f}\big((-\infty,u]\big)\mathrm{d}u\mathrm{d}y. \tag{3.4.7}$$

If $c_+ = 0$, it is clear that the above integral is equal to 0 which implies part (ii). In order to prove part (iii), we first study the case when $c_+ > 0$ and $c_- > 0$; in this case we have

$$|u|^\alpha \nu_\sigma^{\alpha,f}((-\infty,u]) = c_-|u|^\alpha \int_{-\infty}^{u} \frac{e^{-\delta x}}{(e^{-x}-1)^{\alpha+1}}\mathrm{d}x \sim \frac{c_-}{\alpha} \qquad \text{as} \qquad u \uparrow 0. \tag{3.4.8}$$

Then, it is not difficult to deduce that

$$x^{\alpha-2}H(x) \sim \frac{c_-}{(2-\alpha)(\alpha-1)\alpha} \qquad \text{as} \qquad x \downarrow 0.$$

Similar arguments as those used in (3.4.8) allows us to write

$$x^\alpha \nu_\sigma^{\alpha,f}\big([x,\infty)\big) \sim \frac{c_+}{\alpha} \qquad \text{as} \qquad x \downarrow 0.$$

Therefore,

$$\frac{x\nu_\sigma^{\alpha,f}\big([x,\infty)\big)}{H(x)} \sim \frac{(2-\alpha)(\alpha-1)c_+}{c_-}\frac{1}{x} \qquad \text{as} \qquad x \downarrow 0,$$

which implies that the integral K in (3.4.7) diverges.

Finally, if $c_+ > 0$ and $c_- = 0$ the integral (3.4.7) obviously diverges. The proof is now complete. □

We recall that for a Lévy process X a point $x \in \mathbb{R}$ is regular for $(0,\infty)$ if

$$\mathbf{P}_x(\tau^{(0,\infty)} = 0) = 1,$$

where $\tau^{(0,\infty)} = \inf\{t > 0 : X_t \in (0,\infty)\}$.

Proposition 11. *For a Lamperti stable process X^L with characteristics $(\alpha, f, \sigma, \theta)$, the point 0 is regular for $(0,\infty)$ if one of these three conditions hold:*

i) $\alpha \in [1,2)$.

ii) $\alpha \in (0,1)$ *and* $\mathbf{d} > 0$.

iii) $\alpha \in (0,1)$, $\mathbf{d} = 0$ *and* $c_+ > 0$.

Proof. (i) Recall from Proposition 9 that X^L has unbounded variation for $\alpha \in [1,2)$. Hence from Theorem 11 in [43], we deduce that 0 is regular for $(0,\infty)$.

Now we prove parts (ii) and (iii). Suppose that $\alpha \in (0,1)$. In this case X^L has bounded variation and again from Theorem 11 in [43], we know that the point 0 is regular for $(0,\infty)$ if the drift coefficient $\mathbf{d} > 0$ or if $\mathbf{d} = 0$ and the following condition holds

$$\int_0^1 \frac{x\nu_\sigma^{\alpha,f}(\mathrm{d}x)}{H_1(x)} = \infty, \quad \text{where} \quad H_1(x) = \int_0^x \nu_\sigma^{\alpha,f}(-\infty,-y)\mathrm{d}y. \tag{3.4.9}$$

It is enough to prove that (3.4.9) holds when $\mathbf{d} = 0$ and $c_+ > 0$ to conclude our proof. The case when $c_+ > 0$ and $c_- = 0$ is immediate. For the second case, i.e. when $c_+ > 0$ and $c_- > 0$, we first recall from (3.4.8) that

$$y^\alpha \nu_\sigma^{\alpha,f}((-\infty,-y]) \sim \frac{c_-}{\alpha} \qquad \text{as} \quad y \downarrow 0,$$

which implies that

$$x^{\alpha-1}H_1(x) \sim \frac{c_-}{\alpha(1-\alpha)} \qquad \text{as} \quad x \downarrow 0.$$

We observe then, that

$$\frac{x^2 e^{\beta x}(e^x-1)^{-(\alpha+1)}}{H(x)} \sim \frac{\alpha(1-\alpha)}{c_-}, \qquad \text{as} \quad x \downarrow 0,$$

which implies (3.4.9). □

Our next result deals with the computation of the characteristic exponents of Lamperti stable processes. Denote by

$$(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \qquad \text{for} \quad z \in \mathbb{C},$$

which is known as the Pochhammer symbol. And by

$$\psi(z) = \frac{d}{dz}\Gamma(z), \qquad \text{for} \quad z \in \mathbb{C}, \tag{3.4.10}$$

which is called the Digamma function.

Theorem 6. *Let X^L be a Lamperti stable process with characteristics $(\alpha, f, \sigma, \theta)$.*

i) If $\alpha \in (0,1) \cup (1,2)$, the characteristic exponent of X^L is given by

$$\Psi_L(\lambda) = i\lambda\tilde{\theta} - c_+\Gamma(-\alpha)\left((-i\lambda + 1 - \beta)_\alpha - (1-\beta)_\alpha\right) \\ - c_-\Gamma(-\alpha)\left((i\lambda + 1 - \delta)_\alpha - (1-\delta)_\alpha\right), \qquad \lambda \in \mathbb{R}.$$

ii) If $\alpha = 1$, the characteristic exponent of X^L is given by

$$\Psi_L(\lambda) = i\lambda\tilde{\theta} - c_+\Big((-i\lambda + 1 - \beta)\psi(-i\lambda + 2 - \beta) - (1-\beta)\psi(2-\beta)\Big) \\ - c_-\Big((i\lambda + 1 - \delta)\psi(i\lambda + 2 - \delta) - (1-\delta)\psi(2-\delta)\Big), \qquad \lambda \in \mathbb{R}.$$

Where ψ is the Digamma function (see 3.4.10), $\tilde{\theta}$ is given by

$$\tilde{\theta} = \begin{cases} -\mathbf{d} & \text{if } \alpha \in (0,1), \\ \theta - \Big(c_+\tilde{a}_\beta - c_-\tilde{b}_\delta + (c_+ - c_-)(1-\mathcal{C})\Big) & \text{if } \alpha = 1, \\ \theta - \Big(c_+\tilde{a}_\beta - c_-\tilde{b}_\delta + \frac{c_+ - c_-}{\alpha - 1}\Big) & \text{if } \alpha \in (1,2), \end{cases} \tag{3.4.11}$$

where $\mathcal{C}$ is the Euler constant and $\tilde{a}_\beta, \tilde{b}_\delta$ are given by:

$$\tilde{a}_\beta = \int_0^1 \frac{xe^{-x}(1 - e^{-(\alpha-\beta)x})}{(1-e^{-x})^{\alpha+1}}dx + \int_0^1 e^{-x}\frac{1 - x - e^{-x}}{(1-e^{-x})^{\alpha+1}}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha}dx,$$

$$\tilde{b}_\delta = \int_0^1 \frac{xe^{-x}(1 - e^{-(\alpha-\delta)x})}{(1-e^{-x})^{\alpha+1}}dx + \int_0^1 e^{-x}\frac{1 - x - e^{-x}}{(1-e^{-x})^{\alpha+1}}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha}dx,$$

for all $\beta, \delta < \alpha + 1$.

Proof: *i*) First we will consider the case where $\alpha \in (0,1)$. Without loss of generality, we assume that $\mathbf{d} = 0$. Since $\alpha \in (0,1)$, we know that the characteristic exponent of X^L is given by

$$\Psi_L(\lambda) = -\Bigg(c_+\int_0^\infty (e^{i\lambda x} - 1)\frac{e^{\beta x}}{(e^x - 1)^{\alpha+1}}dx \\ + c_-\int_{-\infty}^0 (e^{i\lambda x} - 1)\frac{e^{-\delta x}}{(e^{-x} - 1)^{\alpha+1}}dx\Bigg). \tag{3.4.12}$$

We compute each of these integrals which we call I_1 and I_2 respectively. Since all the computations involved are valid for all $\lambda \in \mathbb{R}$, we center our attention in the variable β. In order to compute I_1 explicitly we will define in the set $U = \{z \in \mathbb{C} : \Re(z) < \alpha + 1\}$, the following function $F : U \to \mathbb{C}$, given by

$$F(z) := \int_0^\infty (e^{i\lambda x} - 1)\frac{e^{zx}}{(e^x - 1)^{\alpha+1}}dx = \int_0^\infty (e^{i\lambda x} - 1)\frac{e^{-z_1 x}}{(1 - e^{-x})^{\alpha+1}}dx$$
$$= \int_0^1 (u^{-i\lambda} - 1)u^{z_1 - 1}(1 - u)^{-(\alpha+1)}du, \tag{3.4.13}$$

where $z_1 = \alpha + 1 - z$ and $\Re(z_1) > 0$. Then by making an integration by parts in the last integral of (3.4.13) we obtain for $\Re(z_1) > 1$

$$\int_0^1 (u^{-i\lambda} - 1)u^{z_1-1}(1 - u)^{-(\alpha+1)}du = \frac{(-i\lambda - z_1 + 1)}{\alpha}\int_0^1 u^{-i\lambda+z_1-2}(1 - u)^{-\alpha}du$$
$$+ \frac{z_1 - 1}{\alpha}\int_0^1 u^{z_1-2}(1 - u)^{-\alpha}du. \tag{3.4.14}$$

Now recalling the integral representation for the Beta function, (see [48]), we have for $\Re(a), \Re(b) > 0$

$$B(a, b) = \int_0^1 u^{a-1}(1 - u)^{b-1}du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \tag{3.4.15}$$

we can express (3.4.14), in the following form:

$$\int_0^1 (u^{-i\lambda} - 1)u^{z_1-1}(1 - u)^{-(\alpha+1)}du = \frac{-(i\lambda + z_1 - 1)}{\alpha}\frac{\Gamma(-i\lambda + z_1 - 1)\Gamma(1 - \alpha)}{\Gamma(-i\lambda + z_1 - \alpha)}$$
$$+ \frac{(z_1 - 1)}{\alpha}\frac{\Gamma(z_1 - 1)\Gamma(1 - \alpha)}{\Gamma(z_1 - \alpha)},$$

finally by the recurrence relation for the Gamma function, $\Gamma(x + 1) = x\Gamma(x)$, and the fact that $z_1 = \alpha + 1 - z$, we obtain

$$F(z) = \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda + \alpha + 1 - z)}{\Gamma(-i\lambda + 1 - z)} - \frac{\Gamma(\alpha + 1 - z)}{\Gamma(1 - z)}\right), \tag{3.4.16}$$

for $\Re(z_1) > 1$, i.e. $\Re(z) < \alpha$. So we have the desired result for $\beta < \alpha$. In order to obtain it for $\beta \in [\alpha, \alpha+1)$ we do the following: The equality (3.4.16) is valid in particular in $D_\alpha = \{z \in \mathbb{C} : \|z\| < \alpha\}$, in order to extend it to the case where $\|z\| < \alpha + 1$, we will prove first that both sides of the equality in (3.4.16) are analytic functions in the disk $D_{\alpha+1} = \{z \in \mathbb{C} : \|z\| < \alpha + 1\}$.
First we take F, and then using a series expansion we have

$$F(z) = \int_0^\infty (e^{i\lambda x} - 1)\frac{e^{zx}}{(e^x - 1)^{\alpha+1}}dx = \int_0^\infty \sum_{n=0}^\infty (e^{i\lambda x} - 1)\frac{(zx)^n}{n!}(e^x - 1)^{-(\alpha+1)}dx. \tag{3.4.17}$$

Now consider the following

$$\begin{aligned}
&\int_0^\infty \sum_{n=0}^\infty \left\| (e^{i\lambda x}-1)\frac{(zx)^n}{n!}(e^x-1)^{-(\alpha+1)} \right\| dx \\
&\quad \le \int_0^\infty \sum_{n=0}^\infty (|\lambda| x)\frac{(\|z\|x)^n}{n!}(e^x-1)^{-(\alpha+1)}dx \\
&\quad = \int_0^\infty (|\lambda| x) e^{\|z\|x}(e^x-1)^{-(\alpha+1)}dx, \qquad (3.4.18)
\end{aligned}$$

which is finite when $\|z\| < \alpha+1$, therefore we can apply Fubini's Theorem in (3.4.17) and obtain

$$F(z) = \sum_{n=0}^\infty \frac{z^n}{n!}\int_0^\infty (e^{i\lambda x}-1)\frac{x^n}{(e^x-1)^{\alpha+1}}dx = \sum_{n=0}^\infty a_n z^n, \qquad (3.4.19)$$

for $z \in W$, where

$$a_n = \frac{1}{n!}\int_0^\infty (e^{i\lambda x}-1)\frac{x^n}{(e^x-1)^{\alpha+1}}dx.$$

which implies that F is analytic in $D_{\alpha+1}$.
Since for $\|z\| < \alpha+1$ we have that $\Re(-i\lambda+\alpha+1-z) > 0$, and $\Re(\alpha+1-z) > 0$, therefore the function $G : U \to \mathbb{C}$, given by

$$G(z) = \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda+\alpha+1-z)}{\Gamma(-i\lambda+1-z)} - \frac{\Gamma(\alpha+1-z)}{\Gamma(1-z)}\right),$$

is analytic in $D_{\alpha+1}$. Since F and G are analytic in $D_{\alpha+1}$, and $F \equiv G$ in D_α, we conclude that $F \equiv G$ in $D_{\alpha+1}$, which implies that

$$I_1 = F(\beta) = \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda+\alpha+1-\beta)}{\Gamma(-i\lambda+1-\beta)} - \frac{\Gamma(\alpha+1-\beta)}{\Gamma(1-\beta)}\right),$$

for all $\beta < \alpha+1$.
Now we compute the second integral in the right-hand side of (3.4.12)

$$I_2 = \int_{-\infty}^0 (e^{i\lambda x}-1)\frac{e^{-\delta x}}{(e^{-x}-1)^{\alpha+1}}dx = \int_0^1 (u^{i\lambda}-1)u^{\delta_1-1}(1-u)^{-(\alpha+1)}du,$$

where $\delta_1 = \alpha+1-\delta$, hence following the same arguments used in the computation of I_1, we get

$$I_2 = \Gamma(-\alpha)\left(\frac{\Gamma(i\lambda+\alpha+1-\delta)}{\Gamma(i\lambda+1-\delta)} - \frac{\Gamma(\alpha+1-\delta)}{\Gamma(1-\delta)}\right),$$

for all $\delta < \alpha+1$. Therefore from the form of I_1 and I_2 we get

$$\begin{aligned}
\Psi_L(\lambda) = &-c_+\Gamma(-\alpha)\left((-i\lambda+1-\beta)_\alpha - (1-\beta)_\alpha\right) \\
&- c_-\Gamma(-\alpha)\left((i\lambda+1-\delta)_\alpha - (1-\delta)_\alpha\right).
\end{aligned}$$

for all $\beta, \delta < \alpha + 1$.
Now we consider the case where $\alpha \in (1,2)$. As in the case where $\alpha \in (0,1)$, we assume that $\theta = 0$. Since $\alpha \in (1,2)$, the characteristic exponent of X^L is given by

$$\begin{aligned}\Psi_L(\lambda) = -\Big(c_+ \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{\beta x}}{(e^x-1)^{\alpha+1}} dx \\ + c_- \int_{-\infty}^0 \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x>-1\}}\right) \frac{e^{-\delta x}}{(e^{-x}-1)^{\alpha+1}} dx\Big),\end{aligned} \tag{3.4.20}$$

We call I_1 and I_2 respectively the integrals in (3.4.20). To study I_1 to do that we define the function $G : U \to \mathbb{C}$, given by

$$\begin{aligned}G(z) :=& \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{zx}}{(e^x-1)^{\alpha+1}} dx \\ =& \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{-z_1 x}}{(1-e^{-x})^{\alpha+1}} dx \\ =& \int_0^\infty \frac{(e^{i\lambda x}-1)e^{-z_1 x} - i\lambda(1-e^{-x})e^{-x}}{(1-e^{-x})^{\alpha+1}} dx + i\lambda \int_0^1 \frac{x(e^{-x}-e^{-z_1 x})}{(1-e^{-x})^{\alpha+1}} dx \\ &+ i\lambda \int_0^1 e^{-x} \frac{1-x-e^{-x}}{(1-e^{-x})^{\alpha+1}} dx + i\lambda \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha} dx \\ =& i\lambda\tilde{a} + i\lambda I(z) + \int_0^1 \frac{(u^{-i\lambda}-1)u^{z_1-1} + i\lambda(u-1)}{(1-u)^{\alpha+1}} du,\end{aligned} \tag{3.4.21}$$

where $z_1 = \alpha + 1 - z$, $\Re(z_1) > 0$,

$$\tilde{a} = \int_0^1 e^{-x} \frac{1-x-e^{-x}}{(1-e^{-x})^{\alpha+1}} dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha} dx,$$

and $I : U \to \mathbb{C}$ is defined by

$$I(z) := \int_0^1 \frac{xe^{-x}(1-e^{-(\alpha-z)x})}{(1-e^{-x})^{\alpha+1}} dx. \tag{3.4.22}$$

We consider the last integral in (3.4.21), an integration by parts gives us for $\Re(z_1) > 2$

$$\begin{aligned}\int_0^1 \frac{(u^{-i\lambda}-1)u^{z_1-1} + i\lambda(u-1)}{(1-u)^{\alpha+1}} du =& \frac{i\lambda}{\alpha} - \frac{(z_1-1)}{\alpha} \int_0^1 (u^{-i\lambda}-1)u^{z_1-2}(1-u)^{-\alpha} du \\ &+ \frac{i\lambda}{\alpha} \int_0^1 (u^{-i\lambda+z_1-2}-1)(1-u)^{-\alpha} du.\end{aligned} \tag{3.4.23}$$

We now compute the integrals in the right-hand side of (3.4.23), using (3.4.15) and making an integration by parts, we obtain for the first integral the following

$$\begin{aligned}
&\int_0^1 (u^{-i\lambda}-1)u^{z_1-2}(1-u)^{-\alpha}du \\
&\quad = \frac{(z_1-2)}{1-\alpha}\int_0^1 (u^{-i\lambda}-1)u^{z_1-3}(1-u)^{1-\alpha}du - \frac{i\lambda}{1-\alpha}\int_0^1 u^{-i\lambda+z_1-2}(1-u)^{1-\alpha}du \\
&\quad = \frac{(-i\lambda+z_1-2)}{1-\alpha}\int_0^1 u^{-i\lambda+z_1-3}(1-u)^{1-\alpha}du - \frac{(z_1-2)}{1-\alpha}\int_0^1 u^{z_1-3}(1-u)^{1-\alpha}du \\
&\quad = \Gamma(1-\alpha)\left(\frac{\Gamma(-i\lambda+z_1)}{(-i\lambda+z_1-1)\Gamma(-i\lambda+z_1-\alpha)} - \frac{\Gamma(z_1)}{(z_1-1)\Gamma(z_1-\alpha)}\right),
\end{aligned} \tag{3.4.24}$$

and for the second integral in the right-hand side of (3.4.23)

$$\begin{aligned}
\int_0^1 (u^{-i\lambda+z_1-2}-1)(1-u)^{-\alpha}du &= \frac{1}{\alpha-1} + \frac{(-i\lambda+z_1-2)}{1-\alpha}\int_0^1 u^{-i\lambda+z_1-3}(1-u)^{1-\alpha}du \\
&= \frac{1}{\alpha-1} + \frac{(-i\lambda+z_1-2)}{1-\alpha}\frac{\Gamma(-i\lambda+z_1-2)\Gamma(2-\alpha)}{\Gamma(-i\lambda+z_1-\alpha)} \\
&= \frac{1}{\alpha-1} + \frac{\Gamma(-i\lambda+z_1)}{(-i\lambda+z_1-1)}\frac{\Gamma(1-\alpha)}{\Gamma(-i\lambda+z_1-\alpha)},
\end{aligned} \tag{3.4.25}$$

so using (3.4.24) and (3.4.25), in (3.4.23) and recalling that $z_1 = \alpha+1-z$, we get

$$\begin{aligned}
\int_0^1 \frac{(u^{-i\lambda}-1)u^{z_1-1}+i\lambda(u-1)}{(1-u)^{\alpha+1}}du &= \frac{i\lambda}{\alpha-1} \\
&\quad + \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda+\alpha+1-z)}{\Gamma(-i\lambda+1-z)} - \frac{\Gamma(\alpha+1-z)}{\Gamma(1-z)}\right),
\end{aligned}$$

if $\Re(z) < \alpha-1$.
So if we consider the function $P : U \to \mathbb{C}$ defined by

$$P(z) = \frac{i\lambda}{\alpha-1} + \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda+\alpha+1-z)}{\Gamma(-i\lambda+1-z)} - \frac{\Gamma(\alpha+1-z)}{\Gamma(1-z)}\right), \tag{3.4.26}$$

we have that

$$G(z) = i\lambda\tilde{a} + i\lambda I(z) + P(z),$$

for $\Re(z) < \alpha-1$, in particular the equality holds in the set $D_{\alpha-1} = \{z \in \mathbb{C} : \|z\| < \alpha-1\}$.
We will prove that G is analytic in $D_{\alpha+1}$, so we follow the same method as in (3.4.18) and (3.4.19) and obtain

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\int_0^\infty \left(e^{i\lambda x}-1-i\lambda x\mathbb{1}_{\{x<1\}}\right)x^n(e^x-1)^{-(\alpha+1)}dx \\
&= \sum_{n=0}^{\infty} a_n z^n,
\end{aligned}$$

for $z \in D_{\alpha+1}$, where

$$a_n = \frac{1}{n!}\int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) x^n (e^x - 1)^{-(\alpha+1)} dx.$$

In a similar way we prove that the function I defined in (3.4.22) is an entire function.
We note that if $z \in D_{\alpha+1}$, then $\Re(-i\lambda + \alpha + 1 - z) > 0$, and $\Re(\alpha + 1 - z) > 0$, which implies that the function P defined in (3.4.26) is also analytic in $D_{\alpha+1}$.
Finally since $G = i\lambda\tilde{a} + i\lambda I + P$ in $D_{\alpha-1}$, and both sides are analytic in $D_{\alpha+1}$, then $G = i\lambda\tilde{a} + i\lambda I + P$ in $D_{\alpha+1}$.
Therefore the first integral in the right-hand side of (3.4.20), I_1, is given by

$$I_1 = G(\beta) = i\lambda\left(\tilde{a}_\beta + \frac{1}{\alpha-1}\right) + \Gamma(-\alpha)\left(\frac{\Gamma(-i\lambda+\alpha+1-\beta)}{\Gamma(-i\lambda+1-\beta)} - \frac{\Gamma(\alpha+1-\beta)}{\Gamma(1-\beta)}\right),$$

where

$$\begin{aligned}\tilde{a}_\beta = \tilde{a} + I(\beta) = &\int_0^1 \frac{xe^{-x}(1-e^{-(\alpha-\beta)x})}{(1-e^{-x})^{\alpha+1}}dx + \int_0^1 e^{-x}\frac{1-x-e^{-x}}{(1-e^{-x})^{\alpha+1}}dx \\ &+ \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha}dx,\end{aligned}$$

for all $\beta < \alpha + 1$. Now we compute the second integral in the right-hand side of (3.4.20)

$$I_2 = -i\lambda\tilde{b}_\delta + \int_0^1 \frac{(u^{i\lambda}-1)u^{\delta_1-1} + i\lambda(u-1)}{(1-u)^{\alpha+1}}du,$$

where

$$\tilde{b}_\delta = \int_0^1 \frac{xe^{-x}(1-e^{-(\alpha-\delta)x})}{(1-e^{-x})^{\alpha+1}}dx + \int_0^1 e^{-x}\frac{1-x-e^{-x}}{(1-e^{-x})^{\alpha+1}}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})^\alpha}dx,$$

and $\delta_1 = \alpha + 1 - \delta$, hence following the same arguments used in the computation of I_1, we get

$$I_2 = -i\lambda\left(\tilde{b}_\delta + \frac{1}{\alpha-1}\right) + \Gamma(-\alpha)\left(\frac{\Gamma(i\lambda+\alpha+1-\delta)}{\Gamma(i\lambda+1-\delta)} - \frac{\Gamma(\alpha+1-\delta)}{\Gamma(1-\delta)}\right),$$

for all $\delta < \alpha + 1$. Therefore from the form of I_1 and I_2 we get

$$\begin{aligned}\Psi_L(\lambda) = &-i\lambda\left(c_+\tilde{a}_\beta - c_-\tilde{b}_\delta + \frac{c_+ - c_-}{\alpha-1}\right) - c_+\Gamma(-\alpha)\left((-i\lambda+1-\beta)_\alpha - (1-\beta)_\alpha\right) \\ &- c_-\Gamma(-\alpha)\left((i\lambda+1-\delta)_\alpha - (1-\delta)_\alpha\right),\end{aligned}$$

for all $\beta, \delta < \alpha + 1$.
ii) Now we will compute the characteristic exponent when $\alpha = 1$. In the following we assume that $\theta = 0$ and that $c_+ = c_- = 1$. Since $\alpha = 1$, the characteristic

exponent of X^L is given by

$$\Psi_L(\lambda) = -\Big(c_+ \int_0^\infty \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x<1\}}\right) \frac{e^{\beta x}}{(e^x-1)^2} dx + c_- \int_{-\infty}^0 \left(e^{i\lambda x} - 1 - i\lambda x \mathbb{1}_{\{x>-1\}}\right) \frac{e^{-\delta x}}{(e^{-x}-1)^2} dx\Big).$$

We will follow the same arguments used in the first part of the computation of the characteristic exponent in the case $\alpha \in (1,2)$. But to compute the two integrals in (3.4.23) we will need the following integral representation for the Digamma function (see [33])

$$\psi(z) = \int_0^1 \frac{t^{z-1}-1}{z-1} dt - \mathcal{C}, \qquad \text{for} \quad z \in \mathbb{C}, \tag{3.4.27}$$

where $\mathcal{C}$ is the Euler constant.
Now by making an integration by parts, using (3.4.27), and the recurrence relation for the Digamma function $\psi(z+1) = \psi(z) + z^{-1}$, we can express the first integral, for $\Re(z_1) > 1$, in the following form

$$\begin{aligned}\int_0^1 (u^{-i\lambda}-1)u^{z_1-2}(1-u)^{-1} du &= -\left(\int_0^1 \frac{u^{-i\lambda+z_1-2}-1}{u-1} du - \int_0^1 \frac{u^{z_1-2}-1}{u-1} du\right)\\ &= \psi(z_1-1) - \psi(-i\lambda+z_1-1)\\ &= \psi(z_1) - \frac{1}{z_1-1} - \psi(-i\lambda+z_1) + \frac{1}{-i\lambda+z_1-1},\end{aligned} \tag{3.4.28}$$

As for the second integral in the right-hand side of (3.4.23)

$$\begin{aligned}\int_0^1 (u^{-i\lambda+z_1-2}-1)(1-u)^{-1} du &= -\psi(-i\lambda+z_1-1) - \mathcal{C}\\ &= \frac{1}{-i\lambda+z_1-1} - \psi(-i\lambda+z_1) - \mathcal{C}.\end{aligned} \tag{3.4.29}$$

So using (3.4.28) and (3.4.29), in (3.4.23) and recalling that $z_1 = 2 - z$, we get

$$\begin{aligned}\int_0^1 \frac{(u^{-i\lambda}-1)u^{z_1-1} + i\lambda(u-1)}{(1-u)^2} du &= i\lambda(1-\mathcal{C}) + (-i\lambda+1-z)\psi(-i\lambda+2-z)\\ &\quad - (1-z)\psi(2-z)\end{aligned} \tag{3.4.30}$$

if $\Re(z) < 1$.
We note that (3.4.30) can be extended to the case where, $\Re(z) < 2$, by the same arguments used in the case $\alpha \in (1,2)$, we only need to remark that the function $P : U \to \mathbb{C}$ defined by

$$P(z) = i\lambda(1-\mathcal{C}) + (-i\lambda+1-z)\psi(-i\lambda+2-z) - (1-\beta)\psi(2-z),$$

is analytic in the disc $D_2 = \{z \in \mathbb{C} : \|z\| < 2\}$. This implies that (3.4.30) is true for all $z \in D_2$, so in particular

$$I_1 = G(\beta) = i\lambda(\tilde{a}_\beta + 1 - \mathcal{C}) + (-i\lambda + 1 - \beta)\psi(-i\lambda + 2 - \beta) - (1-\beta)\psi(2-\beta),$$

where

$$\tilde{a}_\beta = \int_0^1 \frac{xe^{-x}(1 - e^{-(1-\beta)x})}{(1-e^{-x})^2}dx + \int_0^1 e^{-x}\frac{1 - x - e^{-x}}{(1-e^{-x})^2}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})}dx,$$

for all $\beta < 2$.
Finally by the same arguments used in the computation of I_1, we obtain

$$I_2 = -i\lambda(\tilde{b}_\delta + 1 - \mathcal{C}) + (i\lambda + 1 - \delta)\psi(i\lambda + 2 - \delta) - (1-\delta)\psi(2-\delta),$$

where

$$\tilde{b}_\delta = \int_0^1 \frac{xe^{-x}(1 - e^{-(1-\delta)x})}{(1-e^{-x})^2}dx + \int_0^1 e^{-x}\frac{1 - x - e^{-x}}{(1-e^{-x})^2}dx + \int_1^\infty \frac{e^{-x}}{(1-e^{-x})}dx,$$

for all $\delta < 2$. Therefore from the form of I_1 and I_2 we get

$$\begin{aligned}\Psi_L(\lambda) = &-i\lambda\Big(c_+\tilde{a}_\beta - c_-\tilde{b}_\delta + (c_+ - c_-)(1-\mathcal{C})\Big)\\ &- c_+\Big((-i\lambda + 1 - \beta)\psi(-i\lambda + 2 - \beta) - (1-\beta)\psi(2-\beta)\Big)\\ &- c_-\Big((i\lambda + 1 - \delta)\psi(i\lambda + 2 - \delta) - (1-\delta)\psi(2-\delta)\Big),\end{aligned}$$

for all $\beta, \delta < 2$. ■

Using the well known relationship between the Laplace and the characteristic exponents, we obtain:

Corollary 3. *Let X^L be a Lamperti stable process with characteristics $(\alpha, f, \sigma, \theta)$.*

i) Let $\alpha \in (0,1)$ and suppose that X^L is a Lamperti stable subordinator, then its Laplace exponent is given by

$$\Phi_L(\lambda) = \mathbf{d}\lambda - c_+\Gamma(-\alpha)\Big((\lambda + 1 - \beta)_\alpha - (1-\beta)_\alpha\Big), \qquad \lambda \geq 0,$$

where $\mathbf{d} \geq 0$.

ii) Let $\alpha \in (1,2)$ and suppose that X^L has no positive jumps then its Laplace exponent is given by

$$\Phi_L(z) = -\tilde{\theta}\lambda + c_-\Gamma(-\alpha)\Big((\lambda + 1 - \delta)_\alpha - (1-\delta)_\alpha\Big), \qquad \lambda \geq 0,$$

where $\tilde{\theta}$ is given by (3.4.11).

Remark 1. *This Corollary has, as particular cases, the two recent results found in:*

i) Corollary 2, and Lemma 4, in [21], where the result is obtained by means of the Lamperti transformation.

ii) Proposition 3.1 in [58], where the Laplace exponent is obtained using special functions. The ideas in [58] as well as in [14], inspired parts of the proof if Theorem 1, specifically the decomposition (3.4.21).

Now we turn our attention to another group of properties. Let $H = (H_t, t \geq 0)$ be the increasing ladder height process of X^L (see chapter VI in [6]) and $\widehat{H} = (\widehat{H}_t, t \geq 0)$, its decreasing ladder height process. Denote by k and $\hat{k}$ for the characteristic exponents of H and $\widehat{H}$, which are subordinators, and suppose that X^L drifts to $-\infty$ and $\nu_\sigma^{\alpha,f}(0,\infty) > 0$. Under this hypothesis, the process H is a killed subordinator and we denote by Π_H for its Lévy measure. The following result give us a relation between $\nu_\sigma^{\alpha,f}$ and Π_H.

Proposition 12. *Let X^L be a Lamperti stable process with positive jumps and characteristics (α, f) such that it drifts to $-\infty$. Then, the tail of the Lévy measure of H, its increasing ladder height process, belongs to $\mathcal{L}^{(\alpha+1-\beta)}$ and*

$$\nu_\sigma^{\alpha,f}(u,\infty) \sim \hat{k}(-i(\alpha+1-\beta))\Pi(u,\infty) \qquad as \quad u \to \infty.$$

Proof: The proof follows directly from Proposition 5.3 in [42] and Proposition 4. ■

We finish this section with some properties of Lamperti stable processes with no positive jumps.

Proposition 13. *Let X^L be a Lamperti stable process with no positive jumps and characteristics $(\alpha, \delta, \sigma, \theta)$, such that $\tilde{\theta} = 0$ in (3.4.11). Then,*

i) there exist $\delta_0 \in (1,2)$ such that X_L drifts to ∞, oscillates or drifts to $-\infty$ according as $\delta \in (-\infty, \delta_0)$, $\delta = \delta_0$ or $\delta \in (\delta_0, \alpha+1)$.

ii) for $\delta \in (\delta_0, \alpha+1)$, we have that there exist $\lambda > 0$ such that

$$\mathbf{P}_0\Big(S_\infty^L > x\Big) \sim \frac{c}{\lambda k} e^{-\lambda x}, \quad as \quad x \to \infty, \tag{3.4.31}$$

where $S_\infty^L = \sup_{t\geq 0} X_t^L$, $c = -\log \mathbf{P}_0(H_1 < \infty)$, $k = \mathbf{E}_0(H_1 e^{\lambda H_1}; H_1 < \infty)$ and H is the increasing ladder height process.

iii) for $\delta \in (\delta_0, \alpha+1)$, we have that there exist $\lambda > 0$ such that

$$\mathbf{P}_0\Big(I(X^L) > x\Big) \sim K_2 x^{-\lambda}, \quad as \quad x \to \infty, \tag{3.4.32}$$

where K_2 is a positive constant and

$$I(X^L) = \int_0^\infty \exp\Big\{X_t^L\Big\} dt.$$

iv) the probability that the process X^L has increase times is 1. [2]

v) the process X^L satisfies the Spitzer's condition at ∞, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{P}(X_s^L \geq 0)ds = 1/\alpha \quad as \quad x\to\infty.$$

vi) the process X^L satisfies the following law of the iterated logarithm

$$\limsup_{x\to 0}\frac{X_t^L\Phi_L^{-1}(t^{-1}\log|\log t|)}{\log|\log t|} = \alpha^{-\alpha}(1-\alpha)^{\alpha-1} \qquad a.s., \qquad (3.4.33)$$

where Φ_L^{-1} denotes the right-continuous inverse of Φ_L^{-1}.

Proof. (i) We know that in this case $\alpha \in (1,2)$, so from Corollary VII.2 in [6], the process X^L drifts to $+\infty$, oscillates or drifts to $-\infty$ according as $\Phi_L'(0^+)$ is positive, zero or negative. Hence, from the Laplace exponent of X^L we have, using the recursion formula for the Gamma and Digamma functions, the following

$$\begin{aligned}\Phi_L'(0^+) &= c_-\Gamma(-\alpha)(1+\alpha-\delta)_\alpha(\psi(1-\delta+\alpha)-\psi(1-\delta)),\\ &= c_-\Gamma(-\alpha)\frac{\Gamma(1+\alpha-\delta)}{\Gamma(3-\delta)}\\ &\cdot\Big((2-\delta)(1-\delta)((\psi(1-\delta+\alpha)-\psi(1-\delta))+3-2\delta\Big),\\ &= g(\delta). \qquad (3.4.34)\end{aligned}$$

We have from (3.4.34) that $g(1) < 0$, and $g(2) > 0$. On the other hand, in the interval $(1,2)$, the function g is continuous and decreasing which implies that there exist $\delta_0 \in (1,2)$ such that $g(\delta_0) = 0$. Thus, we deduce that X^L drifts to ∞, oscillates or drifts to $-\infty$ according as $\delta \in (-\infty,\delta_0)$, $\delta = \delta_0$ or $\delta \in (\delta_0, \alpha+1)$.
(ii) Any Lévy process with no positive jumps which drifts to $-\infty$ has the property that its Laplace exponent has a strictly positive root. Hence for a Lamperti stable process with no positive jumps and with $\delta \in (\delta_0, \alpha+1)$, there exists $\lambda > 0$ such that

$$\mathbf{E}_0\Big(\exp\{\lambda X_1^L\}\Big) = 1,$$

i.e. that X^L satisfies the Cramér condition. Thus, the main result in [7] gives us the sharp estimate in (3.4.31).
(iii) First note that X^L is not arithmetic and that under our assumptions the Cramér condition is satisfied for some $\lambda > 0$. Hence from Lemma 4 in [63], we

[2] Recall that an instant $t > 0$ is an increase time for a path ω if for some $\epsilon > 0$,

$$\omega(t') \leq \omega(t) \leq \omega(t'') \qquad \text{for all } t \in [t-\epsilon, t] \text{ and } t'' \in [t, t+\epsilon].$$

get the sharp estimate (3.4.32) for the exponential functional $I(X^L)$.
(*iv*) Here, we need the following estimate of the Pochhammer symbol (see for instance [48]),

$$(\lambda + 1 - \delta)_\alpha \sim \lambda^\alpha \qquad \text{as} \quad \lambda \to \infty. \tag{3.4.35}$$

From Corollary VII.9 and Proposition VII.10 in [6] we know that X^L has increase times if

$$\int^\infty \lambda^{-3}\Phi_L(\lambda)\mathrm{d}\lambda < \infty,$$

which in our case is satisfied since from (3.4.35), we have

$$\Phi_L(\lambda) \sim c_-\Gamma(-\alpha)\lambda^\alpha \qquad \text{as} \quad \lambda \to \infty. \tag{3.4.36}$$

(*v*) From (3.4.36), we see that Φ_L is regularly varying at ∞ with index α. Hence, the statement follows from Proposition VII.6 in [6].
(*vi*) Since Φ_L is regularly varying at ∞ with index α, we have that its right-continuous inverse Φ_L^{-1} is regularly varying at ∞ with index $1/\alpha$ which corresponds to the Laplace exponent of the first passage time of X^L (which is a subordinator). Therefore, from Theorem III.11 in [6] we deduce the law of the iterated logarithm (3.4.33). □

3.5 Short and long time behaviour

Motivated by the works of Rosiński [67] and Houdré and Kawai [35], we study the short and long time behavior of Lamperti stable processes. In particular, we will show that this class of processes share with the tempered and layered stable processes, the peculiarity that in short time they behave like stable processes.

The convergence in distribution of processes, considered in this section, is in the functional sense, i.e. in the sense of the weak convergence of the laws of the processes on the Skorokhod space and will be denoted by "$\stackrel{d}{\to}$".

Proposition 14. *Let X^L be a Lamperti stable process with characteristics $(\alpha, f, \sigma, 0)$ and*

$$\eta_\alpha = \begin{cases} 0 & \text{if } \alpha = 1, \\ \displaystyle\int_{S^{d-1}} \xi\sigma(d\xi)\int_0^1 re^{f(\xi)r}(e^r - 1)^{-(\alpha+1)}\,dr & \text{if } \alpha \in (0,1), \\ \displaystyle\int_{S^{d-1}} \xi\sigma(d\xi)\int_1^\infty re^{f(\xi)r}(e^r - 1)^{-(\alpha+1)}\,dr & \text{if } \alpha \in (1,2). \end{cases}$$

Then,

$$\Big(h^{-1/\alpha}\big(X^L_{ht} - ht\eta_\alpha\big), t > 0\Big) \stackrel{d}{\to} (X_t, t > 0) \quad \textit{as} \quad h \to 0,$$

where $(X_t, t > 0)$ is a stable process of index α.

Proof: The proof is similar to that of the short time behaviour of layered stable process, since for each $\xi \in S^{d-1}$

$$e^{f(\xi)r}(e^r-1)^{-(\alpha+1)} \sim r^{-(\alpha+1)} \quad as \quad r \to 0.$$

Thus, we follow the proof of Theorem 3.1 in [35] with

$$q(\xi, r) = e^{f(\xi)r}(e^r-1)^{-(\alpha+1)}$$

and the desired result is obtained. ■

Theorem 7. *Let X_t^L be a Lamperti stable process with characteristics $(\alpha, f, \sigma, 0)$ and*

$$\eta_\alpha = -\int_{S^{d-1}} \xi\sigma(d\xi) \int_1^\infty r e^{f(\xi)r}(e^r-1)^{-(\alpha+1)} dr.$$

Then,

$$\Big(h^{-1/2}\big(X_{ht}^L - ht\eta_\alpha\big), t>0\Big) \xrightarrow{d} (W_t, t>0) \quad as \quad h \to \infty, \tag{3.5.37}$$

where $(W_t, t>0)$ is a centered Brownian motion with covariance matrix

$$\int_{\mathbb{R}_0^d} xx'\nu_\sigma^{\alpha,f}(dx).$$

Proof: According to a standard result on the convergence of processes with independent increments due to Skorokhod (see for instance Theorem 15.17 of Kallenberg [39]), the functional convergence (3.5.37) holds if and only if

$$h^{-1/2}\big(X_h^L - h\eta_\alpha\big) \xrightarrow{d} W_1 \quad as \quad h \to \infty.$$

Now, we introduce the following transform for positive measures, for any $r > 0$

$$(T_r\nu)(B) = \nu(r^{-1}B) \qquad \text{for} \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Note that the random variable $h^{-1/2}X_h^{LS}$ is infinitely divisible and since it has finite first moment, we may rewrite its characteristic exponent as follows;

$$ih\int_{\mathbb{R}_0^d} \langle y, x\rangle \mathbb{1}_{\{\|x\|\geq 1\}}(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) - h\int_{\mathbb{R}_0^d}\Big(e^{i\langle y,x\rangle} - 1 - i\langle y,x\rangle \mathbb{1}_{\{\|x\|\leq 1\}}\Big)(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x).$$

Hence, from Theorem 15.14 of Kallenberg [39] we only need to check the following convergences as h increases:

a) $h(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})$ converges vaguely towards 0 on $\mathbb{R}_0^d$,

b) for each $k > 0$, $\qquad h\int_{\|x\|\leq\kappa} xx'(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \to \int_{\mathbb{R}_0^d} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x),$

c) for each $k > 0$, $$h\int_{\|x\|\geq k} x(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) \to 0.$$

We first prove (a) or equivalently

$$\lim_{h\to\infty}\int_{\mathbb{R}_0^d} g(x)h(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) = 0 \tag{3.5.38}$$

for all bounded continuous functions $g : \mathbb{R}_0^d \to \mathbb{R}$ vanishing in a neighborhood of the origin. Let g be such a function satisfying that $|g| \leq C$, and that for some $\delta > 0$, $g(x) \equiv 0$ on $\{x \in \mathbb{R}_0^d : \|x\| < \delta\}$. Let $\gamma := \sup_{\xi\in S^{d-1}} f(\xi)$, then we have

$$\begin{aligned}
&\left|h\int_{\mathbb{R}_0^d} g(x)\,(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x)\right| \\
&\qquad\leq h^{1+1/2}\int_{S^{d-1}}\sigma(\mathrm{d}\xi)\int_0^\infty |g(r\xi)|e^{rf(\xi)h^{1/2}}(e^{rh^{1/2}}-1)^{-(\alpha+1)}\mathrm{d}r \\
&\qquad= \int_{S^{d-1}}\sigma(\mathrm{d}\xi)\int_\delta^\infty |g(r\xi)|\frac{(rh^{1/2})^3}{r^3}e^{rh^{1/2}\gamma}(e^{rh^{1/2}}-1)^{-(\alpha+1)}\mathrm{d}r.
\end{aligned} \tag{3.5.39}$$

On the other hand, since $\gamma < \alpha + 1$ it follows

$$\lim_{r\to\infty} r^3\frac{e^{r\gamma}}{(e^r-1)^{\alpha+1}} = 0,$$

then for $\epsilon > 0$ sufficiently small, there exist $M > 0$ such that for all $r \geq M$

$$r^3 e^{rf(\xi)}(e^r-1)^{-(\alpha+1)} < \epsilon.$$

Since $r > \delta$, we may take $h > \left(\frac{M}{\delta}\right)^2$ in (3.5.39) and obtain

$$\begin{aligned}
\left|h\int_{\mathbb{R}_0^d} g(x)\,(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x)\right| &< \epsilon\int_{S^{d-1}}\sigma(d\xi)\int_\delta^\infty |g(r\xi)|\frac{1}{r^3}\mathrm{d}r \\
&\leq \epsilon C\int_{S^{d-1}}\sigma(d\xi)\int_\delta^\infty \frac{1}{r^3}\mathrm{d}r.
\end{aligned}$$

Note that the last integral in the right-hand side of the above inequality is finite and therefore the convergence (3.5.38) follows.

Next, we prove part (b). First note that $\int_{\mathbb{R}_0^d}\|x\|^2\nu_\sigma^{\alpha,f}(\mathrm{d}x)$ is finite. This follows by similar arguments as those used in proposition 1. This implies that the integral $\int_{\mathbb{R}_0^d} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x)$ is well defined. Now take $k > 0$ fixed, and note that

$$h\int_{\{\|x\|\leq k\}} xx'(T_{h^{-1/2}}\nu_\sigma^{\alpha,f})(\mathrm{d}x) = \int_{\{\|x\|\leq h^{1/2}k\}} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x) \to \int_{\mathbb{R}_0^d} xx'\nu_\sigma^{\alpha,f}(\mathrm{d}x),$$

as h goes to ∞, which proves part (b).
Finally, we consider $k > 0$ and recall that $\gamma = \sup_{\xi \in S^{d-1}} f(\xi)$, then

$$\begin{aligned}
&\left\| h \int_{\{\|x\| \geq k\}} z(T_{h^{-1/2}} \nu_\sigma^{\alpha,f})(\mathrm{d}z) \right\| \\
&\qquad = \left\| h^{1+1/2} \int_{S^{d-1}} \xi \sigma(\mathrm{d}\xi) \int_k^\infty r e^{rf(\xi)h^{1/2}} (e^{rh^{1/2}} - 1)^{-(\alpha+1)} \mathrm{d}r \right\| \\
&\qquad \leq (1 - e^{-kh^{1/2}})^{-(\alpha+1)} \left\| h^{1+1/2} \int_{S^{d-1}} \xi \sigma(\mathrm{d}\xi) \int_k^\infty r e^{rh^{1/2}(\gamma - (\alpha+1))} \mathrm{d}r \right\| \\
&\qquad = \frac{e^{-kh^{1/2}(\alpha+1-\gamma)}}{(1 - e^{-kh^{1/2}})^{\alpha+1}} \left(\frac{hk}{\alpha + 1 - \gamma} - \frac{h^{1/2}}{(\alpha+1-\gamma)^2} \right) \left\| \int_{S^{d-1}} \xi \sigma(d\xi) \right\|,
\end{aligned}$$

which goes to 0 as $h \to \infty$ since $\gamma < \alpha + 1$. This completes the proof. ■

Let us apply the above results to the special cases treated in the introduction. In particular when we start with a stable process $(X, \mathbf{P}_x), x > 0$, of index α, applying the result in short time behavior after various transformations we return to this initial process. Recall that associated to the stable process three Lévy processes are obtained via the Lamperti representation of pssMp: $\xi^*, \xi^\uparrow, \xi^\downarrow$. Then the normalization of any of them according to proposition 10, converges weakly in the space of Skorokhod to the original stable process X, i.e.

$$X \xrightarrow{kill} X^* \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} X \qquad \text{as} \quad h \to 0$$
$$X \xrightarrow{kill} X^* \xrightarrow{DT} X^C \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} X \quad \text{as} \quad h \to 0$$

where $kill$, LT, DT and $norm$ means killing , the Lamperti representation of pssMp, Doob-transform or conditioning, and normalization of a given process, respectively. Moreover X^C is the conditioned process (to be positive or to hit 0 continuously), X^L stands for any of the Lamperti stable processes $\xi^\uparrow, \xi^\downarrow$ and ξ^*, and X_h^L is the normalization of each of them given in proposition 10. In the same spirit we could also write, using theorem 2,

$$X \xrightarrow{kill} X^* \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} W \qquad \text{as} \quad h \to \infty,$$
$$X \xrightarrow{kill} X^* \xrightarrow{DT} X^C \xrightarrow{LT} X^L \xrightarrow{norm} X_h^L \xrightarrow{d} W \quad \text{as} \quad h \to \infty,$$

where W is a centered brownian motion.

The final result of this section follows the line of reasoning of last remark but uses additional tools that we shall briefly introduce. In [15] the convergence in the Skorokhod space is studied in relation to the second Lamperti transformation (LT_2), i.e. the one that transforms Lévy processes with no negative jumps to continuous state branching processes. The problem of explosions is difficult to handle in this metric so the authors consider another metric d_∞ on the Skorohod space which given by

$$d_\infty(f,g) = 1 \wedge \inf_{\lambda\in\Lambda_\infty} \|f - g\circ\lambda\|_\infty \vee \|\lambda - I\|_\infty.$$

and where Λ_∞ is the set of increasing homeomorphisms of $[0,\infty)$ into itself. According to the authors the convergence in this metric implies it in the usual Skorohod metric. Two of the main results in [15] say that the Lamperti transform LT_2, is continuous with this new metric (proposition 4) and that a sequence $(Y^{*,n})$ of stopped Lévy processes with no negative jumps converges in this new metric towards Y, a stopped Lévy process when the sequence of the associated Laplace exponents of $(Y^{*,n})$ converges towards the associated Laplace exponent of Y (proposition 5). Therefore, a combination of the results mentioned above and proposition 8 give us the following corollary.

Corollary 4. *Let X and X^L be a stable proces of index α with no negative jumps and a Lamperti stable processes with no negative jumps with characteristics (α,β) which does not drift towards $+\infty$, respectively. Let $Y_h = LT_2(X^L_h)$ and $Y = LT_2(X)$. Then*

$$Y_h \xrightarrow{d} Y \qquad as \quad h\to 0.$$

3.6 Absolute continuity with respect to stable processes

We showed that in small times a Lamperti-stable process behaves like a stable process, now following Rosiński [67] we will relate the law of both processes. In other words, we will find a probability measure under which the law of a Lamperti stable process with characteristics (α, f, σ) is the same that the law of the short time limiting stable process with index α.

Theorem 8. *Let P and Q be two probability measures on $(\Omega,\mathcal{F})$ and such that under P the canonical process $(X_t, t\geq 0)$ is a Lamperti stable process with characteristics (α, f, σ, a), while under Q it is a stable process with index α with linear term b. Let $(\mathcal{F}_t)$ be the canonical filtration, and assume that $f\in L^2(S^{d-1},\mathbb{B}(S^{d-1}),\sigma)$. Then*

i) $P|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t>0$ if and only if

$$a-b = \begin{cases} \displaystyle\int_{S^{d-1}} \xi\sigma(d\xi)\int_0^1 re^{rf(\xi)}(e^r-1)^{-(\alpha+1)}dr, & \text{if } \alpha\in(0,1),\\ \displaystyle\int_{S^{d-1}} \xi\sigma(d\xi)\int_0^1 r(e^{rf(\xi)}(e^r-1)^{-(\alpha+1)} - r^{-(\alpha+1)})dr, & \text{if } \alpha = 1,\\ \displaystyle\int_{S^{d-1}} \xi\sigma(d\xi)\int_0^1 r(e^{rf(\xi)}(e^r-1)^{-(\alpha+1)} - r^{-(\alpha+1)})dr & \\ \displaystyle-\int_{S^{d-1}} \xi\sigma(d\xi)\int_1^\infty r^{-(\alpha+1)}dr, & \text{if } \alpha\in(1,2). \end{cases}$$

ii) For each $t > 0$,

$$\frac{dQ}{dP}\bigg|_{\mathcal{F}_t} = e^{U_t},$$

where $(U_t, t \geq 0)$ is a Lévy process defined on $(\Omega, \mathcal{F}, P)$ by

$$U_t = \lim_{\epsilon \downarrow 0} \sum_{\{s \in (0,t] : \|\Delta X_s\| > \epsilon\}} \bigg[\Big(e^{\|\Delta X_s\| f(\Delta X_s)} (e^{\|\Delta X_s\|} - 1)^{-(\alpha+1)} \|\Delta X_s\|^{\alpha+1} \Big) - t(\nu_\sigma^{\alpha,f} - \Pi) \Big(\{ z \in \mathbb{R}_0^d : \|z\| > \epsilon \} \Big) \bigg].$$

In the above right hand side, the convergence holds P-a.s. uniformly in t on every interval of positive length.

Proof: From Theorem 33.2 in Sato [70], we only need to verify that

$$\int_{\mathbb{R}_0^d} (e^{\varphi(x)/2} - 1)^2 \Pi(\mathrm{d}x) < \infty,$$

where $\varphi : \mathbb{R}_0^d \to \mathbb{R}$ is defined by

$$\frac{\mathrm{d}\nu_\sigma^{\alpha,f}}{\mathrm{d}\Pi}(x) = e^{\varphi(x)}.$$

In particular, we have $\varphi(r\xi) = \log\big(e^{rf(\xi)}(e^r - 1)^{-(\alpha+1)} r^{\alpha+1}\big)$. Thus, we need to check

$$\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty \left[\left(\frac{e^{rf(\xi)} r^{(1+\alpha)}}{(e^r - 1)^{(\alpha+1)}} \right)^{1/2} - 1 \right]^2 \frac{1}{r^{1+\alpha}} \mathrm{d}r < \infty \tag{3.6.40}$$

By Taylor expansion and the Lagrange form for the residual, we have $(e^r - 1) = re^{r\theta_r}$, where $\theta_r \in (0, 1)$. This implies

$$\frac{e^{rf(\xi)} r^{(1+\alpha)}}{(e^r - 1)^{(\alpha+1)}} = e^{r(f(\xi) - \theta_r(\alpha+1))}. \tag{3.6.41}$$

Now, noting that $f(\xi) - (\alpha + 1) \leq f(\xi) - \theta_r(\alpha + 1) \leq f(\xi)$, it follows

$$e^{r(f(\xi)-(\alpha+1))/2} - 1 \leq e^{r(f(\xi)-\theta_r(\alpha+1))/2} - 1 \leq e^{rf(\xi)/2} - 1,$$

and since $f(\xi) \leq \gamma = \sup_{\xi \in S^{d-1}} f(\xi)$, we have

$$\Big(e^{r(f(\xi)-\theta_r(\alpha+1))/2} - 1\Big)^2 \leq \Big(e^{r(f(\xi)-(\alpha+1))/2} - 1\Big)^2 \vee \Big(e^{rf(\xi)/2} - 1\Big)^2 \tag{3.6.42}$$

Using a Taylor expansion again and (3.6.42), it is clear that there exists a constant $R > 0$ such that if $r < R$, then

$$\Big(e^{r(f(\xi)-\theta_r(\alpha+1))/2} - 1\Big)^2 \leq K_3(f^2(\xi) + 1)r^2, \tag{3.6.43}$$

where K_3 is a positive constant. Hence from (3.6.41) and (3.6.43), it follows that

$$\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^R \left[\left(\frac{e^{rf(\xi)} r^{(1+\alpha)}}{(e^r-1)^{(\alpha+1)}} \right)^{1/2} - 1 \right]^2 \frac{1}{r^{1+\alpha}} \mathrm{d}r \\ \leq K_3 \left(\sigma(S^{d-1}) + \int_{S^{d-1}} f^2(\xi) d\xi \right) \int_0^R \frac{r^2}{r^{1+\alpha}} \mathrm{d}r,$$

which is finite because $\alpha \in (0,2)$ and $f \in L^2(S^{d-1}, \mathbb{B}(S^{d-1}), \sigma)$. In the case when $r > R$, we have

$$\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_R^\infty \left[\left(\frac{e^{rf(\xi)} r^{(1+\alpha)}}{(e^r-1)^{(\alpha+1)}} \right)^{1/2} - 1 \right]^2 \frac{1}{r^{1+\alpha}} \mathrm{d}r \\ \leq 4 \left((1-e^{-R})^{-(\alpha+1)} \int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_R^\infty e^{r(f(\xi)-(\alpha+1))} \mathrm{d}r + \sigma(S^{d-1}) \int_R^\infty \frac{1}{r^{1+\alpha}} \mathrm{d}r \right) \\ \leq 4\sigma(S^{d-1}) \left((1-e^{-R})^{-(\alpha+1)} \int_R^\infty e^{r(\gamma-(\alpha+1))} \mathrm{d}r + \int_R^\infty \frac{1}{r^{1+\alpha}} \mathrm{d}r \right),$$

which is also finite because $\gamma < \alpha + 1$. Therefore (3.6.40) follows.

The proof of the second statement of the Theorem follows directly from Theorem 33.2 of Sato [70]. ■

Here, we follow the same notation as in Theorem 3. Note that under the conditions of Theorem 4.1 in [35], if R is another probability measure on $(\Omega, \mathcal{F})$ under which the canonical process $X = (X_t, t \geq 0)$ is a layered stable process, we have that $R|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$. From our previous result, we obtain the corresponding result for Lamperti stable processes, i.e. that $R|_{\mathcal{F}_t}$ and $P|_{\mathcal{F}_t}$ are mutually absolutely continuous for every $t > 0$. Similar result holds for the tempered stable processes, see Theorem 4.1 in [67].

3.7 Series representations of Lamperti stable process

In this section, we establish a series representation for Lamperti stable processes which allow us to generate some of their sample paths. To this end, we will use the LePage's method found in [49]. We first introduce the following sequences of mutually independent random variables defined in $[0, T]$. Let $\{\Gamma_i\}_{i\geq 1}$ be a sequence of of partial sums of iid standard exponential random variables, $\{U_i\}_{i\geq 1}$ be a sequence of uniform random variables on $[0, T]$, and let $\{V_i\}_{i\geq 1}$ be a sequence of iid random variables in S^{d-1} with common distribution $\sigma(d\xi)/\sigma(S^{d-1})$. In order to use the LePage's method, we consider the following function $\delta^{-1} : (0,\infty) \times S^{d-1} \to \mathbb{R}_+$ given by

$$\rho^{-1}(u,\xi) := \inf\Big\{x > 0 : \rho([x,\infty),\xi) < u\Big\},$$

where

$$\rho([x,\infty),\xi) = \int_x^\infty e^{f(\xi)r}(e^r - 1)^{-(\alpha+1)}\mathrm{d}r.$$

Now, let $\{c_i\}_{i\geq 1}$ be a sequence of constants defined as follows,

$$c_i = \int_{i-1}^{i} \mathbf{E}\Big(\rho^{-1}(s/T, V_1)V_1 \mathbb{1}_{\{\rho^{-1}(s/T,V_1)\leq 1\}}\Big)ds.$$

Then from Theorem 5.1 in [68], the process

$$\left(\sum_{i=1}^{\infty}\Big(\rho^{-1}(\Gamma_i/T, V_i)V_i\mathbb{1}_{\{U_i\leq t\}} - c_i\frac{t}{T}\Big), t \in [0,T]\right),$$

converges uniformly a.s. towards a Lamperti stable process with characteristics (α, f, σ) and linear term $\theta = 0$ (in the Lévy-Khintchine formula). In particular when $f(\xi) = 1$, we have that

$$\rho^{-1}(u,\xi) = \ln(1 + (\alpha u)^{-1/\alpha}),$$

hence the series representation for a Lamperti stable Lévy process X^L with characteristics $(\alpha, 1)$, is as follows

$$X_t^L \stackrel{d}{=} \sum_{i=1}^{\infty}\left(\ln\left(1 + \left(\frac{\alpha\Gamma_i}{T}\right)^{-1/\alpha}\right)V_i\mathbb{1}_{\{U_i\leq t\}} - c_i\frac{t}{T}\right)$$

where

$$c_i = \mathbf{E}\Big(V_1\Big)\int_{i-1}^{i}\ln\left(1 + \left(\frac{\alpha s}{T}\right)^{-1/\alpha}\right)\mathbb{1}_{\{\ln(1+(\alpha s T^{-1})^{-1/\alpha})\leq 1\}}\mathrm{d}s.$$

Let us observe below some sample paths of this particular Lamperti stable process generated via the series representation.

3.8 Associated processes and examples

Here, we are interested in study some related processes to Lamperti stable distributions (or processes) and give some examples of Lamperti stable processes which appear in the literature but they are not the main objects in study. In particular, we study the Ornstein-Uhlenbeck process and the self-similar additive process related to a Lamperti stable distribution in the case when the latter is self-decomposable. We also investigate the parent process of a Lamperti stable subordinator.

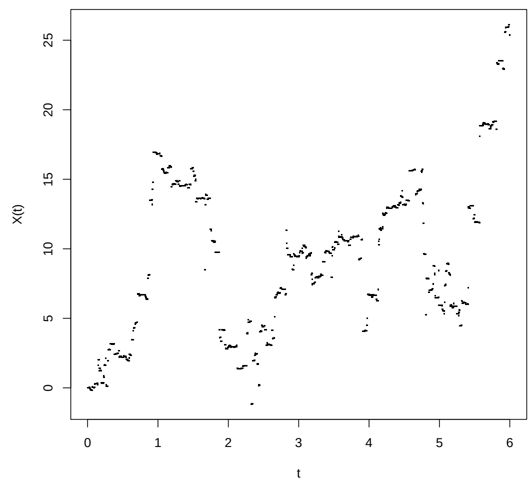


Figure 3.1: $\alpha = 0.5,\ f = 1,\ \sigma(1) = \sigma(-1) = 1.$

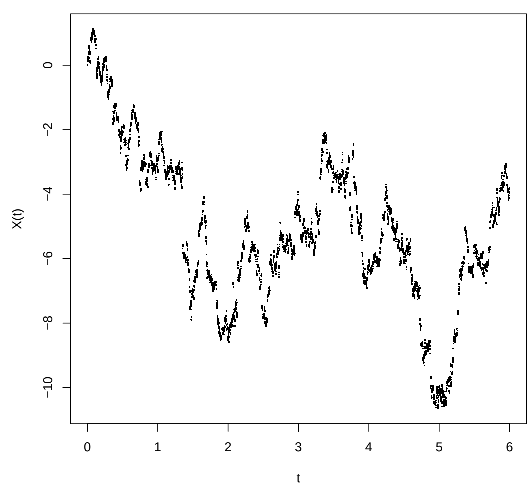


Figure 3.2: $\alpha = 1.5,\ f = 1,\ \sigma(1) = \sigma(-1) = 1.$

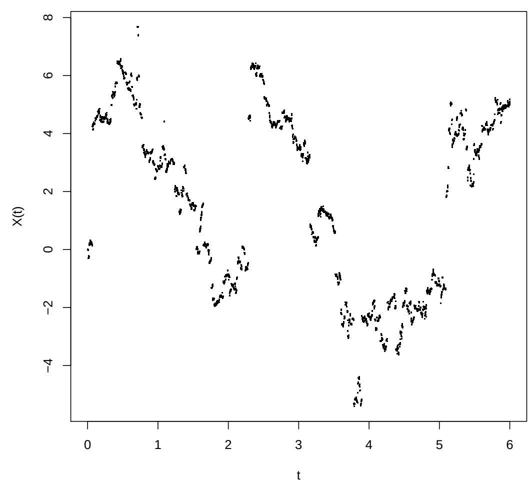


Figure 3.3: $\alpha = 1,\ f = 1,\ \sigma(1) = \sigma(-1) = 1.$

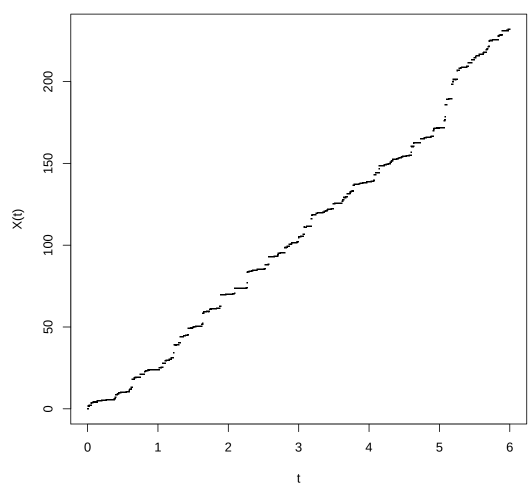


Figure 3.4: $\alpha = 0.5,\ f = 1,\ \sigma(1) = 1,\ \sigma(-1) = 0.$

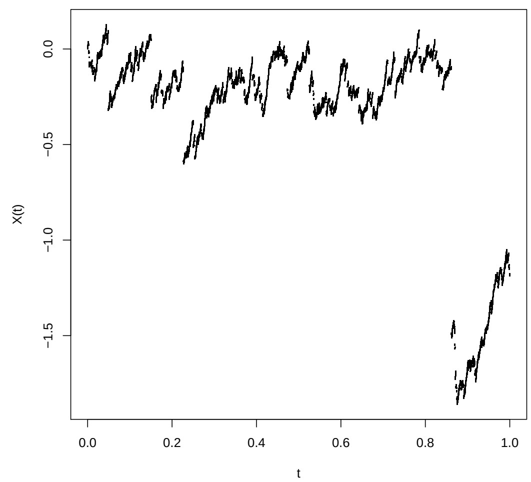


Figure 3.5: $\alpha = 1.5,\ f = 1,\ \sigma(1) = 0,\ \sigma(-1) = 1.$

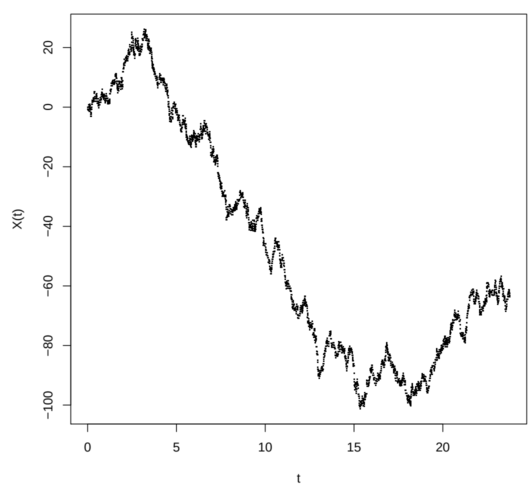


Figure 3.6: $\alpha = 1.9,\ f = 1,\ \sigma(1) = 1,\ \sigma(-1) = 1.$

3.8.1 Ornstein-Uhlenbeck type processes and self-simlar additive processes

Ornstein-Uhlenbeck type processes appear in many areas of science, for instance in physics, biology and mathematical finance. One of the particularity of these processes is that its limiting distribution is self-decomposable. Recall that a random variable Y on $\mathbb{R}^d$, distributed as a Lamperti stable law with characteristics (α, f, σ) is self-decomposable if and only if $f \leq \alpha + 1/2$. Therefore, according to Wolfe [77] and Jurek and Vervaat [38], there exists a Lévy process $Z = (Z_t, t \geq 0)$ on $\mathbb{R}^d$, with $\mathbf{E}_0(\log^+ |Z_1|) < \infty$ such that

$$Y \stackrel{law}{=} I := \int_0^\infty e^{-cs} \mathrm{d}Z_s,$$

where $c > 0$. Consequently, one can define an Ornstein-Uhlenbeck type process driven by Z with initial state U_0 and parameter $c > 0$, that is the solution of

$$U_t = U_0 + Z_t - c\int_0^t U_s \mathrm{d}s,$$

and such that the law of U_t converge towards the law of Y as t goes to ∞. From Theorem 17.5 in [70], we have that the process Z has no Gaussian component, its Lévy measure is given by

$$\Pi_Z(B) = -c\int_{S^{d-1}} \sigma(\mathrm{d}\xi) \int_0^\infty \mathbb{I}_B(r\xi) h(r,\xi) \mathrm{d}r, \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

where

$$h(r) = \frac{e^{rf(\xi)}}{(e^r - 1)^{\alpha+2}} \Big(re^r (f(\xi) - \alpha - 1) + e^r - rf(\xi) - 1 \Big);$$

and linear term

$$\upsilon = c\eta - \int_{\{\|x\| \geq 1\}} \frac{x}{\|x\|} \Pi_Z(\mathrm{d}x).$$

In the one dimensional case, the form of the Lévy measure of Z is reduced to

$$\begin{aligned} \Pi_Z(\mathrm{d}x) = c\Bigg(& c_+ \frac{e^{\beta x}}{(e^x - 1)^{\alpha+2}} \Big(x\beta + 1 - e^x + xe^x(\alpha + 1 - \beta) \Big) \mathbb{I}_{\{x>0\}} \\ & - c_- \frac{e^{-\rho x}}{(e^{-x} - 1)^{\alpha+2}} \Big(x\rho - 1 + e^{-x} + xe^{-x}(\alpha + 1 - \rho) \Big) \mathbb{I}_{\{x<0\}} \Bigg) \mathrm{d}x, \end{aligned}$$

where $\beta = f(1)$, $\rho = f(-1)$, $c_+ = \sigma(\{1\})$ and $c_- = \sigma(\{-1\})$, as usual. In this case, we have another process which is related to the Lamperti stable distribution Y, to Z and to the Ornstein-Uhlenbeck type process U. To this end, recall that in Theorem 16.1 of [70], it is showed that a distribution is self-decomposable if and only if for any fixed $H > 0$, it is the distribution of V_1 for some additive process $V = (V_t, t \geq 0)$ which is self-similar, meaning that for each $k > 0$

$$(V_{kt}, t \geq 0) \stackrel{d}{=} \Big(k^H V_t, t \geq 0 \Big).$$

We remark that self-similar additive processes can be used to model space-time scaling random phenomena that can be observed in many areas of science. In particular, they are recently used to model asset prices and the risk-neutral process (see for instance [18]) in financial mathematics.

Assume that V is the self-similar additive process associated to I, in which case V_1 has the same law I, and that $H = c$. From Theorem 1 in [37], there are two independent copies of Z denoted by $Z^{(-)} = (Z^{(-)}_t, t \geq 0)$ and $Z^{(+)} = (Z^{(+)}_t, t \geq 0)$ which are defined by

$$Z^{(-)}_t \overset{\text{(def)}}{=} \int_{e^{-t}}^{1} \frac{\mathrm{d}V_r}{r^\gamma} \qquad \text{and} \qquad Z^{(+)}_t \overset{\text{(def)}}{=} \int_{1}^{e^t} \frac{\mathrm{d}V_r}{r^\gamma}.$$

The process V can be recovered by

$$V_r = \begin{cases} \displaystyle\int_{\log(1/r)}^{\infty} e^{-ct}\mathrm{d}Z^{(-)}_t & \text{if } 0 \leq r \leq 1, \\ \displaystyle Y + \int_0^{\log(r)} e^{ct}\mathrm{d}Z^{(+)}_t & \text{if } r \geq 1, \end{cases}$$

and moreover $(U^{(+)}_t = e^{-tc}V_{e^t}, t \geq 0)$ is the Ornstein-Uhlenbeck process driven by $Z^{(+)}$ with initial state I and parameter c; and $(U^{(-)}_t = e^{tc}V_{e^{-t}}, t \geq 0)$ is the Ornstein-Uhlenbeck process driven by $-Z^{(-)}$ with initial state I and parameter $-c$.

3.8.2 Parent process

Motivated in generating new examples of scale functions, Kyprianou and Rivero [44] constructed Lévy processes with no positive jumps around a given possibly killed subordinator which plays the role of the descending ladder height process. One of our aims is to determine the characteristics of the Lévy process with no positive jumps whose descending ladder height process is a Lamperti stable subordinator.

Let X^L be a Lamperti stable subordinator with characteristics $(\alpha, \beta, \sigma, \theta)$ with zero drift and no killing rate. Since the density of its Lévy measure is decreasing, then according to Theorem 1 in [44], there is $X^{PL} = (X^{PL}_t, t \geq 0)$, a Lévy process with no positive jumps that we call *the parent process of* X^L whose Laplace exponent is given by

$$\psi_{PL}(\lambda) = \lambda \Phi_L(\lambda), \qquad \text{for} \quad \lambda \geq 0,$$

where Φ_L is the Laplace exponent of X^L. Moreover, the process X^{PL} has no Gaussian coefficient, its Lévy measure is given by

$$\Pi_{PL}(\mathrm{d}x) = c_+ \frac{e^{-\beta x}}{(e^{-x} - 1)^{\alpha+2}} \Big((\alpha + 1 - \beta)e^{-x} + \beta \Big) \mathrm{d}x \qquad \text{for} \quad x < 0,$$

and with linear term

$$b = \int_{(-\infty,1)} x \Pi_{PL}(\mathrm{d}x).$$

Note that X^{PL} oscillates or drifts to ∞ according to whether $\Phi_L(0)$ is equal zero or strictly positive. From the form of its Lévy measure, we deduce that X^{PL} is the sum of two Lamperti stable processes with no positive jumps X^1 and X^2 with characteristics $(\alpha+1,\beta+1,\sigma_1,b_1)$ and $(\alpha+1,\beta,\sigma_2,b_2)$, where $\sigma_1(\{1\})=\sigma_2(\{1\})=0$,

$$\sigma_1(\{-1\})=c_+(\alpha+1-\beta), \qquad \sigma_2(\{-1\})=c_+\beta,$$
$$b_1=\int_{(-\infty,-1)} x\Pi_1(\mathrm{d}x)-\tilde{a}_{\beta+1}, \qquad b_1=\int_{(-\infty,-1)} x\Pi_2(\mathrm{d}x)-\tilde{a}_{\beta},$$

and Π_1 and Π_2 are the respective Lévy measures of X^1 and X^2. On the other hand, the binomial expansion give us

$$\int_x^\infty \frac{e^{\beta x}}{(e^x-1)^{\alpha+1}}\mathrm{d}x=e^{-x(\alpha+1-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\alpha-\beta)_n}{n!(\alpha+1-\beta)_n}e^{-nx}, \tag{3.8.44}$$

which is clearly log-convex on $(0,\infty)$ since it is completely monotone. Hence according to Theorem in 2 [44], there is a subordinator $X^{*,L}$ with Laplace exponent Φ_L^* such that

$$\Phi_L(\lambda)=\frac{\lambda}{\Phi_L^*(\lambda)} \qquad \text{for} \quad \lambda\geq 0.$$

Moreover the subordinator $X^{*,L}$ has no drift and no killing term and the scale function of the parent process X^{PL} is determined by

$$W(x)=\int_0^x \Pi_L^*(y,\infty)\mathrm{d}y,$$

where Π_L^* is the Lévy measure of $X^{*,L}$. Note that for $\beta=1$, we have that

$$\Pi_L^*(y,\infty)=\frac{\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}(1-e^{-y})^{\alpha-1}, \qquad \text{for} \quad y>0,$$

but for $\beta\neq 1$ we do not have an explicit form for Π_L^*.

The Example 2 in [44] is related to the Lamperti stable subordinators considered above but with a given killing rate. Let us explain in detail such example in terms of Lamperti stable processes. Take $X^{L,K}$ to be a Lamperti stable subordinator with characteristics $(\alpha,\beta,\sigma,\theta)$ with zero drift and killing rate given by

$$K=\frac{c_+\Gamma(-\alpha)\Gamma(1-\beta+\alpha)}{\Gamma(1-\beta)}.$$

According to Kyprianou and Rivero [44], there is a subordinator, here now denoted by Y with no drift, no killing rate and Lévy measure given by

$$\Pi_Y(\mathrm{d}x)=\frac{1}{c_+\Gamma^2(1-\alpha)}\left((\alpha-\beta)e^{-(\alpha-\beta)x}(e^x-1)^{\alpha-1}+(2-\alpha)\frac{e^{-(\alpha-1-\beta)x}}{(e^x-1)^{2-\alpha}}\right)\mathrm{d}x,$$

which is the sum of two subordinators, one of which is a Lamperti stable with characteristics $(1-\alpha, \beta+1-\alpha, \sigma_Y)$, where

$$\sigma_Y(\{1\}) = \frac{(2-\alpha)}{c_+\Gamma^2(1-\alpha)}, \quad \text{and} \quad \sigma_Y(\{-1\}) = 0.$$

Moreover, the Laplace exponent of the subordinator Y satisfies that

$$\phi_Y(\lambda) = \frac{\lambda}{\phi_L(\lambda)}, \qquad \text{for} \quad \lambda \geq 0.$$

From the form of Π_Y, we have the restriction that $\beta < 1$. Thus, his parent process Y^P, a spectrally negative Lévy process, has Laplace exponent

$$\psi_{Y^P}(\lambda) = \frac{\lambda^2\Gamma(1-\beta+\lambda)}{\Gamma(1-\beta+\lambda+\alpha)},$$

which has no Gaussian component and its Lévy measure satisfies

$$\Pi_{Y^P}(-\infty, y) = \Pi_Y(\mathrm{d}y)/\mathrm{d}y.$$

According to Kyprianou and Rivero and by (3.8.44), its associated scale function is given by

$$W_{Y^P}(x) = -Kx + c_+\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\alpha-\beta)_n}{n!(\alpha+2-\beta)_n}\left(1-e^{-(\alpha+2-\beta+n)x}\right), \quad x \geq 0.$$

Now $Y^{*,P}$, the parent process of the Lamperti subordinator $X^{L,K}$ with killing rate K, is a spectrally Levy process which drifts to ∞, with Laplace exponent

$$\psi_{Y^{P,*}}(\lambda) = \frac{c_+\Gamma(-\alpha)\lambda\Gamma(\lambda+1-\beta+\alpha)}{\Gamma(\lambda+1-\beta)},$$

which has no Gaussian coefficient and whose Lévy measure satisfies

$$\Pi_{Y^{P,*}}(\mathrm{d}x) = \Pi_{PL}(\mathrm{d}x), \qquad x < 0,$$

with linear term

$$b = \int_{(-\infty,1)} x\Pi_{PL}(\mathrm{d}x) - K,$$

and the associated scale function is given by

$$W^*(x) = \frac{1}{c_+\Gamma^2(1-\alpha)}\int_0^x e^{-(\alpha-\beta)y}(e^y-1)^{\alpha-1}\mathrm{d}y.$$

As the process X^{PL}, $Y^{*,P}$ may be seen as the sum of two Lamperti stable process with no positive jumps Y^1 and Y^2 with characteristics $(\alpha+1, \beta+1, \sigma_1, b_1-K)$ and $(\alpha+1, \beta, \sigma_2, b_2)$, where σ_1, σ_2, b_1 and b_2 are defined as above.

It is important to note that the above example have been recently used for the risk neutral stock price model by Eberlein and Madan [28].

3.9 Examples

Examples of Lamperti stable processes appear in the literature at least in the papers mentioned in the introduction ([14, 21, 45, 58]) but they also appear (in a hidden way) in many other recent works. We will give a quick overview of some of them, not pretending to be exhaustive in this list.

In [8], we find two examples related to the factorization

$$\mathbf{e} \stackrel{law}{=} \mathbf{e}^{\alpha} \tau_{\alpha}^{-\alpha}.$$

where $\mathbf{e}$ is an exponential variable independent of the α-stable variable τ_α. The first of them is related with the exponential functional of a killed subordinator Z^1 whose Laplace exponent is given by

$$\phi_1(\lambda) = \frac{\Gamma(\alpha\lambda + 1)}{\Gamma(\alpha(\lambda - 1) + 1)}.$$

It is easy to see that it is related to the Laplace exponent Φ_L of a Lamperti stable subordinator X^L with characteristics $(\alpha, \alpha, \sigma, \theta)$, zero drift, and $\sigma(\{1\}) = \alpha/\Gamma(1-\alpha)$. The relationship between both Laplace exponents is

$$\phi_1(\lambda) = \Phi_L(\alpha\lambda) + \frac{1}{\Gamma(1-\alpha)}.$$

This subordinator is also studied in Rivero [64], where the author finds its renewal density and other related computations.

The Laplace exponent of the second subordinator, here denoted by Z^2, is given by

$$\phi_2(\lambda) = \lambda \frac{\Gamma(\alpha(\lambda - 1) + 1)}{\Gamma(\alpha\lambda + 1)},$$

and can be expressed in terms of the Laplace exponent $\Phi_{L,2}$ of a Lamperti stable subordinator $X^{L,2}$ with characteristics $(1-\alpha, 1, \sigma, \theta)$, and zero drift where $\sigma(\{1\}) = \alpha/\Gamma(1-\alpha)$. The relation between them is

$$\phi_2(\lambda) = \alpha\Phi_{L,2}(\alpha\lambda).$$

In both cases this allows us to compute the law of the exponential functional of αX^L and $\alpha X^{L,2}$ in terms of the one of Z^1 and Z^2, respectively.

There is another example in [8] which is related to the factorization

$$\mathbf{e} \stackrel{law}{=} \gamma_s^{\alpha} J_s^{(\gamma)},$$

where $s \geq \alpha$, γ_s is a Gamma r.v. with parameter s and $J_s^{(\gamma)}$ denotes a certain r.v which is independent of γ_s. In this case, the killed subordinator related to the exponential functional which has the same moments as the γ_s, can be expressed as the sum of two independent Lamperti stable procesess. In [64] further calculation are carried over concerning this subordinator.

In the paper [71] in section 5.3, the authors found the Lévy measure of the inverse of the local time at 0 of an Ornstein Uhlenbeck process driven by a standard Brownian motion and parameter $\gamma > 0$. This measure is

$$\nu(t) = \frac{\gamma^{3/2} e^{\gamma t/2}}{\sqrt{2\pi}(\sinh(\gamma t))^{3/2}} = \frac{(2\gamma)^{3/2} e^{2\gamma t}}{\sqrt{2\pi}(e^{2\gamma t} - 1)^{3/2}}$$

and the corresponding Laplace exponent is computed. It is related to a Lamperti stable distribution with characteristics $(1/2, 1, \sqrt{\gamma/\pi})$.

This computation as well as the three former examples can be carried out by recognizing that behind those measures there is a related Lamperti stable distribution and applying our Theorem 1 to calculate the corresponding Laplace exponent.

In the papers [21] , [45], [58] the main processes in study are Lamperti stable processes. All these papers share the property that many useful explicit calculations are be carried out. This is, we believe, the main advantage of this class: being at the same time a good model for many situations, allowing simulation of the paths as well as many explict calculation to be carried on.

Chapter 4

Explicit identities for Lévy processes associated to symmetric stable processes

4.1 Introduction and preliminaries

Let $Z = (Z_t = \{Z_t^{(1)}, \dots Z_t^{(d)}\}, t \geq 0)$ be a symmetric stable Lévy process of index $\alpha \in (0, 2)$ in $\mathbb{R}^d$ $(d \geq 1)$, that is, a process with stationary independent increments, its sample paths are càdlàg and

$$\mathbb{E}_0\Big(\exp\{i < \lambda, Z_t >\}\Big) = \exp\{-t\|\lambda\|^\alpha\},$$

for all $t \geq 0$ and $\lambda \in \mathbb{R}^d$. Here $\mathbb{P}_z$ denotes the law of the process Z initiated from $z \in \mathbb{R}^d$ and $< \cdot, \cdot >$ the Euclidean inner product.
The process $Z^{(k)} = (Z_t^{(k)}, t \geq 0)$ will be called the k-th coordinate process of Z. Of course, $Z^{(k)}$ is a real symmetric stable process whose characteristic exponent is given by

$$\mathbb{E}_0\Big(\exp\big\{i\theta Z_t^{(k)}\big\}\Big) = \exp\{-t|\theta|^\alpha\},$$

for all $t \geq 0$ and $\theta \in \mathbb{R}$.

Recall that Z is transient for $\alpha < d$, that is

$$\lim_{t\to\infty} \|Z_t\| = \infty \qquad \text{a.s.},$$

and it oscillates otherwise, i.e. for $\alpha \in [1, 2)$ and $d = 1$, we have

$$\limsup_{t\to\infty} Z_t = \liminf_{t\to\infty} Z_t = \infty \qquad \text{a.s.}$$

When $d \geq 2$, we have that single points are polar, i.e. for every $x, y \in \mathbb{R}^d$

$$\mathbb{P}_x(Z_t = z \quad \text{for some } t > 0) = 0.$$

In the one-dimesional case, points are polar for $\alpha \in (0,1]$ and when $\alpha \in (1,2)$ the process Z makes infinitely many jumps across a point, say x, before the first hitting time at x.

Since Z is isotropic and satisfies the scaling property with index α, i.e. for every $b > 0$

$$\textit{The law of } (bZ_{b^{-\alpha}t},\, t \geq 0) \textit{ under } \mathbb{P}_x \textit{ is } \mathbb{P}_{bx}, \tag{4.1.1}$$

the radial process $R = (R_t, t \geq 0)$ defined by $R_t = \|Z_t\|$, is a positive self-similar Markov process with index α. The fact that the radial process satisfies the scaling property follows from (4.1.1) and that it is a Markov process is explained in Millar [53]. According to Millar [53] the process R hits points if and only if the process $Z^{(1)}$ hits points. This occurs when $\alpha \in (1,2)$ as it was stated above. In what follows, we assume that the process Z is transient which implies that the radial process drifts to $+\infty$.

Recall that positive self-similar Markov processes $(X, \mathbb{Q}_x)$, $x > 0$, are strong Markov processes with càdlàg paths, which fulfill a scaling property. We shall refer to these processes as pssMp. Well-known examples of this kind of processes are: Bessel processes, stable subordinators, stable processes conditioned to stay positive, etc.

According to Lamperti [46], any pssMp up to its first hitting time of 0 may be expressed as the exponential of a Lévy process, time changed by the inverse of its exponential functional. More formally, let $(X, \mathbb{Q}_x)$ be a pssMp with index $\beta > 0$, starting from $x > 0$, set

$$S = \inf\{t > 0 : X_t = 0\}$$

and write the canonical process X in the following form:

$$X_t = x \exp\left\{\xi_{\tau(tx^{-\beta})}\right\} \qquad 0 \leq t < S\,, \tag{4.1.2}$$

where for $t < S$,

$$\tau(t) = \inf\left\{s \geq 0 : \int_0^s \exp\{\beta \xi_u\}\, \mathrm{d}u \geq t\right\}.$$

Then under $\mathbb{Q}_x$, $\xi = (\xi_t,\ t \geq 0)$ is a Lévy process started from 0 whose law does not depend on $x > 0$ and such that:

(*i*) if $\mathbb{Q}_x(S = +\infty) = 1$, then ξ has an infinite lifetime and $\limsup_{t\to+\infty} \xi_t = +\infty$, $\mathbb{P}_x$-a.s.,

(*ii*) if $\mathbb{Q}_x(S < +\infty, X(S-) = 0) = 1$, then ξ has an infinite lifetime and $\lim_{t\to\infty} \xi_t = -\infty$, $\mathbb{P}_x$-a.s.,

(*iii*) if $\mathbb{Q}_x(S < +\infty, X(S-) > 0) = 1$, then ξ is killed at an independent exponentially distributed random time with parameter $\lambda > 0$.

As mentioned in [46], the probabilities $\mathbb{Q}_x(S = +\infty)$, $\mathbb{Q}_x(S < +\infty,\, X(S-) = 0)$ and $\mathbb{Q}_x(S < +\infty,\, X(S-) > 0)$ are 0 or 1 independently of x, so that the three classes presented above are exhaustive. Moreover, for any $t < \int_0^\infty \exp\{\beta\xi_s\}\,\mathrm{d}s$,

$$\tau(t) = \int_0^{x^\beta t} \frac{\mathrm{d}s}{(X_s)^\beta}\,, \quad \mathbb{Q}_x - \text{a.s.}$$

Therefore (4.1.2) is invertible and yields a one-to-one relation between the class of pssMp's killed at time S and the one of Lévy processes.

Another important result of Lamperti [46] provides the explicit form of the generator of any PSSMP $(X, \mathbb{Q}_y)$ in terms of its underlying Lévy process. Let ξ be the underlying Lévy process associated to $(X, \mathbb{Q}_y)$ via (4.1.2) and denote by $\mathcal{L}$ and $\mathcal{M}$ for their respective infinitesimal generators. Let $\mathcal{D}_\mathcal{L}$ be the domain of the generator $\mathcal{L}$ and recall that it contains all the functions with continuous second derivatives on $[-\infty, \infty]$, and that if f is such a function then $\mathcal{L}$ acts as follows for $x \in \mathbb{R}$, where $\mu \in \mathbb{R}$ and $\sigma > 0$:

$$\mathcal{L}f(x) = \mu f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \big(f(x+y) - f(x) - f'(x)\ell(y)\big)\Pi(\mathrm{d}y) - bf(x).$$

The measure $\Pi(\mathrm{d}x)$ is the so-called Lévy measure of ξ, which satisfies

$$\Pi(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} (1 \wedge |x|^2)\Pi(\mathrm{d}x) < \infty.$$

The function $\ell(\cdot)$ is a bounded Borel function such that $\ell(y) \sim y$ as $y \to 0$. The positive constant b represents the killing rate of ξ (b=0 if ξ has infinite lifetime). Lamperti establishes the following result in [46].

Theorem 9. *If g is such that g, yg' and $y^2 g''$ are continuous on $[0, \infty]$, then they belong to the domain, $\mathcal{D}_\mathcal{M}$, of the infinitesimal generator of $(X, \mathbb{Q}_y)$, which acts as follows for $y > 0$*

$$\begin{aligned}\mathcal{M}g(y) = \mu y^{1-\beta} g'(y) + \frac{\sigma^2}{2} y^{2-\beta} g''(y) - b y^{-\beta} g(y)\\ + y^{-\beta} \int_0^\infty \big(g(yu) - g(y) - yg'(y)\ell(\log u)\big)G(\mathrm{d}u),\end{aligned}$$

where $G(\mathrm{d}u) = \Pi(\mathrm{d}u) \circ \log u$, for $u > 0$. This expression determines the law of the process $(X_t, 0 \le t \le T)$ under $\mathbb{Q}_y$.

4.2 The underlying Lévy process of R

In this section, we compute the characteristics of the underlying Lévy process in the Lamperti representation (4.1.2) of the radial process R, here denoted by ξ.

To this end, it will be useful to invoke the expression of Z as a subordinated Brownian motion. More precisely, let $B = (B_t, t \ge 0)$ be a d-dimensional

Brownian motion initiated from $x \in \mathbb{R}^d$ and let $\sigma = (\sigma_t, t \geq 0)$ be an independent stable subordinator with index $\alpha/2$ initiated from 0. Then the process $(B_{2\sigma_t}, t \geq 0)$ is a standard symmetric α-stable process.

Let us define the so-called Pochhammer symbol by

$$(z)_\alpha = \frac{\Gamma(z+\alpha)}{\Gamma(z)}, \qquad \text{for} \quad z \in \mathbb{C},$$

and the Gauss's hypergeometric function by

$${}_2\mathcal{F}_1\Big(a, b; c; z\Big) = \sum_{k=0}^{\infty} z^k \frac{(a)_k (b)_k}{(c)_k\, k!}, \qquad \text{for} \quad \|z\| < 1,$$

where $a, b, c > 0$.

Theorem 10. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is such that $f \in C_0^2(\mathbb{R}_+)$. Hence the infinitesimal generator of $R = (R_t, t \geq 0)$, denoted by M, acts as follows for $a > 0$,*

$$Mf(a) = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} a^{-\alpha} \int_0^\infty \Big(f(\rho a) - f(a) - l(\log \rho) f'(a) \Big) \times \frac{\rho^{d-1}}{(1+\rho^2)^{\alpha+d/2}} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho,$$

where

$$\bar{F}(z) = {}_2\mathcal{F}_1((\alpha+d)/4, (\alpha+d)/4 + 1/2; d/2; z) \qquad \textit{for} \quad z \in (-1, 1), \quad (4.2.3)$$

and ${}_2\mathcal{F}_1$ is the Gauss's hypergeometric function. The function l is given by

$$l(y) = \frac{y}{1+y^2} e^{(1-d)y} (1+e^{2y})^{\alpha+d/2-1} 1_{A_\epsilon}(e^y).$$

Proof. From Theorem 32.1 in [70] and the above remark, the infinitesimal generator M of $R = (R_t, t \geq 0)$ is given as follows

$$Mf = \int_0^\infty (P_s f - f) \rho(ds),$$

where

$$\rho(ds) = \frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)} s^{-(1+\alpha/2)} \mathbb{I}_{\{s>0\}} ds,$$

is the Lévy measure of 2σ, P_s is the semi-group of the d-dimensional Bessel process and f is in the domain of the infinitesimal generator of $(P_t, t \geq 0)$.

Let $x \in \mathbb{R}^d$ and f be as in the statement. Recall that according to [61] for $a = |x| > 0$, the semi-group for the d-dimensional Bessel process satisfies

$$P_s f(a) = \int_0^\infty d\rho \left(\frac{\rho}{a}\right)^{d/2-1} \frac{\rho}{s} \exp\left(-\frac{\rho^2 + a^2}{2s}\right) I_{d/2-1}\left(\frac{a\rho}{s}\right) f(\rho).$$

where $I_{d/2-1}$ is the modified Bessel function of index $d/2-1$.

Therefore putting the pieces together, it follows

$$Mf(a) = \frac{2^{\alpha/2-1}\alpha}{\Gamma(1-\alpha/2)} \int_0^\infty d\rho\, \rho^{d-1}(f(\rho)-f(a))(a\rho)^{1-d/2} \times \int_0^\infty \frac{ds}{s^{2+\alpha/2}} \exp\left(-\frac{a^2+\rho^2}{2s}\right) I_{d/2-1}\left(\frac{a\rho}{s}\right).$$

Recall that

$$I_{d/2-1}(x) = \sum_{k=0}^\infty \frac{(x/2)^{2k+d/2-1}}{\Gamma(d/2+k)k!},$$

this implies

$$\begin{aligned}
&\int_0^\infty ds \exp\left(-\frac{a^2+\rho^2}{2s}\right) I_{d/2-1}\left(\frac{a\rho}{2s}\right) s^{-2-\alpha/2} \\
&= \sum_{k=0}^\infty \int_0^\infty ds \left(\frac{a\rho}{2s}\right)^{2k+d/2-1} \frac{1}{k!} \frac{s^{-2-\alpha/2}}{\Gamma(d/2+k)} \exp\left(-\frac{a^2+\rho^2}{2s}\right) \\
&= \sum_{k=0}^\infty \left(\frac{a\rho}{2}\right)^{2k+d/2-1} \frac{1}{k!} \frac{1}{\Gamma(d/2+k)} \left(\frac{a^2+\rho^2}{2}\right)^{-2k-(\alpha+d)/2} \int_0^\infty dz\, z^{2k+(\alpha+d)/2-1} e^{-z} \\
&= 2^{\alpha/2+1} \frac{(a\rho)^{d/2-1}}{(a^2+\rho^2)^{(\alpha+d)/2}} \sum_{k=0}^\infty \left(\frac{a\rho}{(a^2+\rho^2)}\right)^{2k} \frac{\Gamma(2k+(\alpha+d)/2)}{\Gamma(k+1)\Gamma(d/2+k)}. \qquad (4.2.4)
\end{aligned}$$

Next, we consider the following property of the Gamma function

$$\Gamma(2z) = (2\pi)^{-1/2} 2^{2z-1/2} \Gamma(z)\Gamma(z+1/2),$$

and we deduce

$$\begin{aligned}
\Gamma(2k+(\alpha+d)/2) &= (2\pi)^{-1/2} 2^{2k+(\alpha+d)/2-1/2} \Gamma(k+(\alpha+d)/4)\Gamma(k+(\alpha+d)/4+1/2) \\
&= (2\pi)^{-1/2} 2^{2k+(\alpha+d)/2-1/2} \Gamma((\alpha+d)/4)\Gamma((\alpha+d)/4+1/2) \\
&\quad \times ((\alpha+d)/4)_k((\alpha+d)/4+1/2)_k,
\end{aligned}$$

where $(z)_n = \Gamma(z+n)/\Gamma(z)$. Therefore using the above identity, we see that (4.2.4) is equal to

$$\frac{2^{\alpha/2+1}(a\rho)^{d/2-1}}{(a^2+\rho^2)^{\alpha+d/2}} \frac{\Gamma((\alpha+d)/2)}{\Gamma(d/2)} \sum_{k=0}^\infty \left(\left(\frac{2a\rho}{a^2+\rho^2}\right)^2\right)^k \frac{((\alpha+d)/4)_k((\alpha+d)/4+1/2)_k}{(d/2)_k\, k!},$$

where the series from above is the Gauss's hypergeometric function

$${}_2\mathcal{F}_1\left((\alpha+d)/4, (\alpha+d)/4+1/2; d/2; \left(\frac{2a\rho}{a^2+\rho^2}\right)^2\right). \qquad (4.2.5)$$

For simplicity, we use the notation established in (4.2.3). Finally, we see that the infinitesimal generator M satisfies the following identity

$$Mf(a) = \frac{2^\alpha \alpha \Gamma((\alpha+d)/2)}{\Gamma(1-\alpha/2)\Gamma(d/2)} \int_0^\infty \rho^{d-1} \frac{(f(\rho)-f(a))}{(a^2+\rho^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2a\rho}{a^2+\rho^2}\right)^2\right) d\rho$$
$$= \frac{2^\alpha \alpha}{\Gamma(1-\alpha/2)} (d/2)_{\alpha/2} a^{-\alpha} \int_0^\infty (f(\rho a)-f(a)) \frac{\rho^{d-1}}{(1+\rho^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho. \tag{4.2.6}$$

Now let us consider the following integral

$$\int_0^\infty \frac{\log\rho}{1+\log^2\rho} \frac{1}{1+\rho^2} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) 1_{A_\epsilon}(\rho) d\rho, \tag{4.2.7}$$

where for $\epsilon > 0$, $A_\epsilon = \{\rho \geq 0 : \frac{1}{1+\epsilon} < \rho < 1+\epsilon\}$. Now we will check that the integral in (4.2.7) is equal zero, to this end consider the following

$$\int_0^\infty \frac{\log\rho}{1+\log^2\rho} \frac{1_{A_\epsilon}(\rho)}{1+\rho^2} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho = \int_{1/1+\epsilon}^1 \frac{\log\rho}{1+\log^2\rho} \frac{1_{A_\epsilon}(\rho)}{1+\rho^2} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho$$
$$= \int_1^\epsilon \frac{\log\rho}{1+\log^2\rho} \frac{1_{A_\epsilon}(\rho)}{1+\rho^2} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho. \tag{4.2.8}$$

So making $\rho' = \rho^{-1}$ in the first integral of (4.2.8) we obtain

$$\int_{1/1+\epsilon}^1 \frac{\log\rho}{1+\log^2\rho} \frac{1_{A_\epsilon}(\rho)}{1+\rho^2} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho$$
$$= -\int_1^{1+\epsilon} \frac{\log\rho'}{1+\log^2\rho'} \frac{1_{A_\epsilon}(\rho')}{1+\rho'^2} \bar{F}\left(\left(\frac{2\rho'}{1+\rho'^2}\right)^2\right) d\rho',$$

so using the above expression in (4.2.8) we have that

$$\int_0^\infty \frac{\log\rho}{1+\log^2\rho} \frac{1}{1+\rho^2} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) 1_{A_\epsilon}(\rho) d\rho = 0.$$

Now if we add the integral in (4.2.7) to (4.2.6) we obtain the following

$$Mf(a) = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} a^{-\alpha} \int_0^\infty (f(\rho a)-f(a)) \frac{\rho^{d-1}}{(1+\rho^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho$$
$$= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} a^{-\alpha} \int_0^\infty \left[(f(\rho a)-f(a)) \frac{\rho^{d-1}}{(1+\rho^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right)\right.$$

$$- \frac{\log \rho}{1+\log^2 \rho} \frac{1_{A_\epsilon}(\rho)}{(1+\rho^2)} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) f'(a)\Big] d\rho$$
$$= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} a^{-\alpha} \int_0^\infty \left(f(\rho a) - f(a) - \frac{\log \rho}{1+\log^2 \rho} \frac{(1+\rho^2)^{\alpha+d/2-1}}{\rho^{d-1}} 1_{A_\epsilon}(\rho) f'(a) \right)$$
$$\times \frac{\rho^{d-1}}{(1+\rho^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho$$
$$= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} a^{-\alpha} \int_0^\infty (f(\rho a) - f(a) - f'(a) l(\log \rho)) \frac{\rho^{d-1}}{(1+\rho^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2\rho}{1+\rho^2}\right)^2\right) d\rho, \tag{4.2.9}$$

where the function l in (4.2.9) is given by

$$l(y) = \frac{y}{1+y^2} e^{(1-d)y} (1+e^{2y})^{\alpha+d/2-1} 1_{A_\epsilon}(e^y). \tag{4.2.10}$$

□

Let us remark that the function l in (4.2.10) satisfies the following:

- It is a bounded Borel function.
- And also that $l(y) \sim y$ as $y \to 0$.

From (4.2.9) we obtain the infinitesimal generator of a pssMp in the form suggested by Lamperti's result recalled Theorem 1:

$$G(du) = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{u^{d-1}}{(1+u^2)^{(\alpha+d)/2}} \bar{F}\left(\left(\frac{2u}{1+u^2}\right)^2\right) du. \tag{4.2.11}$$

Finally to obtain the characteristics of the Lévy process $\xi = (\xi_t, t \geq 0)$ associated to $R = (R_t, t \geq 0)$ by the Lamperti transformation, from Theorem 1

$$\Pi(du) = G(du) \circ e^u,$$

so using (4.2.11) we obtain

$$\begin{aligned} \Pi(du) &= e^u \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{(d-1)u}}{(1+e^{2u})^{(\alpha+d)/2}} \bar{F}\left(\frac{4e^{2u}}{(e^{2u}+1)^2}\right) du \\ &= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{du}}{(1+e^{2u})^{(\alpha+d)/2}} \bar{F}\left(\frac{4e^{2u}}{(e^{2u}+1)^2}\right) du. \end{aligned} \tag{4.2.12}$$

And finally we note that according to [46] the lineal term μ is given by

$$\mu = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \int_{\mathbb{R}} (u 1_{\{|u|\leq 1\}} - l(u)) \frac{e^{du}}{(1+e^{2u})^{(\alpha+d)/2}} \bar{F}\left(\frac{4e^{2u}}{(e^{2u}+1)^2}\right) du.$$

So the characteristic function of ξ_t is the following:

$$E[\exp\{i\lambda\xi_t\}] = \exp\left\{i\mu t + t\int_{\mathbb{R}}\left(e^{i\lambda u} - 1 - i\lambda u 1_{\{|u|\leq 1\}}\right)\Pi(du)\right\}. \quad (4.2.13)$$

The above computations give us the following corollary:

Corollary 5. *Let ξ the Lévy process in the Lamperti representation (4.1.2) of the radial process R. The infinitesimal generator $\mathcal{A}$, of ξ, with domain $\mathcal{D}_{\mathcal{A}}$ is given*

$$\mathcal{A}f(x) = \int_{\mathbb{R}}\big(f(x+y) - f(x) - f'(x)\ell(y)\big)\Pi(\mathrm{d}y),$$

for any $f \in \mathcal{D}_{\mathcal{A}}$ and $x \in \mathbb{R}$, where

$$\Pi(\mathrm{d}y) = \frac{e^{dy}}{(1+e^{2y})^{(\alpha+d)/2}}\overline{F}\left(\frac{4e^{2y}}{(e^{2y}+1)^2}\right)\mathrm{d}y.$$

Equivalently, the characteristic exponent of ξ is given by

$$\Psi(\lambda) = i\lambda b + \int_{\mathbb{R}}\Big(1 - e^{i\lambda y} + i\lambda y \mathbb{1}_{\{|y|<1\}}\Big)\Pi(\mathrm{d}y),$$

where

$$b = \int_{\mathbb{R}}\Big(\ell(y) - y\mathbb{1}_{\{|y|\leq 1\}}\Big)\frac{e^{dy}}{(1+e^{2y})^{(\alpha+d)/2}}\overline{F}\left(\frac{4e^{2y}}{(e^{2y}+1)^2}\right)\mathrm{d}y.$$

We finish this section with a remarkable result on the decomposition of the Lévy measure of the process ξ when the dimension is $d = 1$. Such decomposition describes the structure of ξ in terms of two independent Lévy processes, each with different types of path behaviour. Recall in this case that the symmetric stable process Z is of bounded variation and so its radial part R and the Lévy process ξ. Hence, the characteristic exponent of ξ is given by

$$E[\exp\{i\lambda\xi_t\}] = \exp\left\{id\lambda t + t\int_{\mathbb{R}}\left(e^{i\lambda u} - 1\right)\Pi(du)\right\}. \quad (4.2.14)$$

Proposition 15. *Let us consider that $d = 1$, then we have the following decomposition for the process ξ:*

$$\xi \stackrel{\mathcal{L}}{=} \xi^1 + \xi^2$$

where $\xi^1 = (\xi^1_t, t \geq 0)$ is a Lamperti stable Lévy process, and $\xi^2 = (\xi^2_t, t \geq 0)$ is a compound Poisson process independent of ξ^1.

Proof. For $x \in [0,1)$ the hypergeometric function in (4.2.5) takes the following form

$$
\begin{aligned}
{}_2\mathcal{F}_1\left((\alpha+1)/4, (\alpha+1)/4+1/2; 1/2; x^2\right) &= \sum_{k=0}^{\infty} x^{2k} \frac{((\alpha+1)/4)_k((\alpha+1)/4+1/2)_k}{k!(1/2)_k} \\
&= \frac{\Gamma(1/2)}{\Gamma((\alpha+1)/4+1/2)} \frac{1}{\Gamma((\alpha+1)/4)} \sum_{k=0}^{\infty} x^{2k} \frac{\Gamma((\alpha+1)/4+k)}{\Gamma(k+1)} \frac{\Gamma((\alpha 1)/4+1/2+k)}{\Gamma(k+1/2)} \\
&= \frac{\Gamma(1/2)}{\Gamma((\alpha+1)/4+1/2)} \frac{1}{\Gamma((\alpha+1)/4)} \sum_{k=0}^{\infty} x^{2k} \frac{2^{1/2-(\alpha+1)/2-2k}\Gamma((\alpha+1)/2+2k)}{2^{1/2-2k-1}\Gamma(2k+1)} \\
&= \frac{2^{1/2-\alpha/2}\Gamma(1/2)}{(2\pi)^{1/2}2^{1/2-(\alpha+1)/2}\Gamma((\alpha+1)/2)} \cdot \frac{1}{2}\left(\sum_{k=0}^{\infty} x^k \frac{\Gamma((\alpha+1)/2+k)}{\Gamma(1+k)} + \sum_{k=0}^{\infty}(-x)^k \frac{\Gamma((\alpha+1)/2+k)}{\Gamma(1+k)}\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty} x^k \frac{((\alpha+1)/2)_k}{k!} + \sum_{k=0}^{\infty}(-x)^k \frac{((\alpha+1)/2)_k}{k!}\right) \\
&= 2^{-1}\left((1-x)^{-(\alpha+1)/2} + (1+x)^{-(\alpha+1)/2}\right). \qquad (4.2.15)
\end{aligned}
$$

Now using (4.2.15) in (4.2.12) we obtain that the Lévy measure of the process ξ when $d=1$ is given by

$$
\begin{aligned}
\Pi(dy) &= \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^y}{(1+e^{2y})^{(\alpha+1)/2}} \left(\left(1-\frac{2e^y}{e^{2y}+1}\right)^{-(\alpha+1)/2} + \left(1+\frac{2e^y}{e^{2y}+1}\right)^{-(\alpha+1)/2}\right) dy \\
&= \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} e^y \left(|e^y-1|^{\alpha+1} + (e^y+1)^{\alpha+1}\right) dy \\
&= \Pi_1(dy) + \Pi_2(dy),
\end{aligned}
$$

where Π_1 and Π_2 are given respectively by

$$
\begin{aligned}
\Pi_1(dy) &= \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \left(\frac{e^y}{(e^y-1)^{\alpha+1}} 1_{\{y\geq 0\}} + \frac{e^y}{(1-e^y)^{\alpha+1}} 1_{\{y<0\}}\right) dy \\
\Pi_2(dy) &= \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^y}{(e^y+1)^{\alpha+1}} dy.
\end{aligned}
$$

So using (4.2.14) we obtain that the characteristic function of the Lévy process ξ at time $t \geq 0$ is:

$$
\begin{aligned}
E[e^{i\lambda\xi_t}] &= \exp\left\{id\lambda t + t\int_{\mathbb{R}} \left(e^{i\lambda y}-1\right)\Pi(dy)\right\} \\
&= \exp\left\{id\lambda t + t\int_{\mathbb{R}} \left(e^{i\lambda y}-1\right)\Pi_1(dy)\right\} \\
&\times \exp\left\{\frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} ct \int_{\mathbb{R}} (e^{i\lambda y}-1)\Pi_2(dy)\right\} \\
&= E[e^{i\lambda\xi_t^1}]E[e^{i\lambda\xi_t^2}],
\end{aligned}
$$

where

$$c = \int_{\mathbb{R}} \frac{e^y}{(e^y+1)^{\alpha+1}} dy.$$

So we can conclude that for each $t \geq 0$, $\xi_t = \xi_t^1 + \xi_t^2$. Where $\xi^1 = (\xi_t^1, t \geq 0)$ is a Lamperti stable process with Lévy measure Π_1 given by

$$\Pi_1(dy) = \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \left(\frac{e^y}{(e^y-1)^{\alpha+1}} 1_{\{y\geq 0\}} + \frac{e^y}{(1-e^y)^{\alpha+1}} 1_{\{y<0\}} \right) dy.$$

On the other hand using that

$$c = \int_{\mathbb{R}} e^y (e^y+1)^{-(\alpha+1)} dy < \infty,$$

the Lévy process $\xi^2 = (\xi_t^2, t \geq 0)$ is a compound Poisson process independent of ξ^1 with rate

$$c' = \frac{2^{\alpha-1}\alpha(1/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} c,$$

and jump distribution

$$\Pi_2(dy) = c^{-1} e^y (e^y-1)^{-(\alpha+1)} dy.$$

□

4.3 Entrance laws for the process ξ: Intervals

In this section, we study the probability that the Lévy process ξ makes its first exit from an interval. In particular, we obtain some explicit identities for the one-sided exit problems.

In what follows, P will be a reference probability measure on $\mathcal{D}$ (the Skorokhod space of $\mathbb{R}$-valued càdlàg paths) under which ξ is the Lévy process described in Corollary 1 starting from 0. For any $y \in \mathbb{R}$ let

$$T_y^+ = \inf\{t \geq 0 : \xi_t > y\} \text{ and } T_y^- = \inf\{t \geq 0 : \xi_t < y\},$$

and for any $y > 0$ let

$$\sigma_y^+ = \inf\{t \geq 0 : R_t > y\} \text{ and } \sigma_y^- = \inf\{t \geq 0 : R_t < y\}.$$

Lemma 1. *Fix $-\infty < v < 0 < u < \infty$. Suppose that A is any interval in $[u, \infty)$ and B is any interval in $(-\infty, v]$. Then,*

$$P\Big(\xi_{T_u^+} \in A; T_u^+ < \infty\Big) = \mathbb{P}_x\Big(R_{\sigma_{e^u}^+} \in e^A; \sigma_{e^u}^+ < \infty\Big)$$

and

$$P\Big(\xi_{T_v^-} \in B; T_v^- < \infty\Big) = \mathbb{P}_x\Big(R_{\sigma_{e^v}^-} \in e^B; \sigma_{e^v}^- < \infty\Big),$$

where x satisfies that $\|x\| = 1$.

The proof is a straightforward consequence of the Lamperti representation and is left as an exercise. Although somewhat obvious, this lemma indicates that for the process ξ, we need to understand how the radial process R exit a positive interval around $x > 0$. Fortunately this is possible thanks to a result of Blumenthal et al. [9] who established the following for the symmetric α-stable process Z.

Define,

$$f(y, z) = \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)\left|1 - \|y\|^2\right|^{\alpha/2}\left|1 - \|z\|^2\right|^{-\alpha/2}\|y - z\|^{-d}.$$

Theorem 11 (Blumenthal et al. [9])**.** *Suppose that $\alpha < d$ and that $(Z, \mathbb{P}_x)$ is an symmetric α-stable process with values in $\mathbb{R}^d$, initiated from x. For $\|y\| < 1$ and $\|z\| \geq 1$, we have*

$$\mathbb{P}_y\left(Z_{\sigma_1^+} \in \mathrm{d}z; \sigma_1^+ < \infty\right) = f(y, z)\mathrm{d}z. \tag{4.3.16}$$

Similarly for $\|y\| > 1$ and $\|z\| \leq 1$, we have

$$\mathbb{P}_y\left(Z_{\sigma_1^-} \in \mathrm{d}z; \sigma_1^- < \infty\right) = f(y, z)\mathrm{d}z. \tag{4.3.17}$$

The one-side exit problem for ξ can be obtained from Lemma 1 and Theorem 3 as follows.

Proposition 16. *Fix $\theta \geq 0$ and $u > 0$*

$$\begin{aligned} &P\left(\xi_{T_u^+} - u \in d\theta, T_u^+ < \infty\right) \\ &= \frac{2}{\pi}\sin\frac{\pi\alpha}{2}(1 - e^{-2u})^{\alpha/2}(e^{2\theta} - 1)^{-\alpha/2}e^{d(u+\theta)}(e^{2(\theta+u)} - 1)^{-1}d\theta \end{aligned} \tag{4.3.18}$$

Proof. Let us recall that the process ξ is a Lévy process associates to the process $R = |Z|$, where Z is an α symmetric stable process in $\mathbb{R}^d$.
First we recall the following property that can be found in [21], fix $0 < u < \infty$, and suppose that A is any interval in $[u, \infty)$, then

$$P\left(\xi_{T_u^+} \in A; T_u^+ < \infty\right) = \mathbb{P}_1\left(X_{\sigma_{e^u}^+} \in e^A; \sigma_{e^u}^+ < \infty\right). \tag{4.3.19}$$

On the other hand since Z is isotropic and satisfies the scaling property it easy to see that for each $b > 0$, and any set $B \in \mathbb{B}(\mathbb{R})$, that

$$P_x(b^{-\alpha}\sigma_b^+ \in B) = P_{x/b}(\sigma_1^+ \in B).$$

This in turn implies that

$$P_{x/b}(|Z_{\sigma_1^+}| \in dy; \sigma_1^+ < \infty) = P_x(b^{-1}|Z_{\sigma_b^+}| \in dy; \sigma_b^+ < \infty). \tag{4.3.20}$$

Then using (4.3.20) we have the following

$$P_1(|Z_{\sigma^+_{e^u}}| \in [e^u, e^{u+\theta}]; \sigma^+_{e^u} < \infty) = P_1(e^{-u}|Z_{\sigma^+_{e^u}}| \in [1, e^\theta]; \sigma^+_{e^u} < \infty)$$
$$= P_{e^{-u}}(|Z_{\sigma^+_1}| \in [1, e^\theta]; \sigma^+_1 < \infty),$$

Now fix $x \in \mathbb{R}^d$ such that $|x| = e^{-u}$, and set $w_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$; then using (4.3.16) and the Poisson formula in $\mathbb{R}^d$, we have

$$P_x\left(|Z_{\sigma^+_1}| \in [1, e^\theta]; \sigma^+_1 < \infty\right)$$
$$= \int_{1<|y|<e^\theta} \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right) \sin\frac{\pi\alpha}{2}|1-|x|^2|^{\alpha/2}|1-|y|^2|^{-\alpha/2}|x-y|^{-d}dy$$
$$= \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right) \sin\frac{\pi\alpha}{2}(1-e^{-2u})^{\alpha/2} \int_{1<|y|<e^\theta} |1-|y|^2|^{-\alpha/2}|x-y|^{-d}dy$$
$$= \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right) \sin\frac{\pi\alpha}{2}(1-e^{-2u})^{\alpha/2} \int_1^{e^\theta} dr\frac{r^{d-1}}{(r^2-1)^{-\alpha/2}}$$
$$\times \int_0^\pi d\theta \frac{\sin^{d-2}\theta}{(r^2+|x|^2-2r|x|\cos\theta)^{d/2}} \tag{4.3.21}$$

On the other hand, from formula 3.665 in [33] we get for $r > 1$

$$\int_0^\pi d\theta \frac{\sin^{d-2}\theta}{(r^2+|x|^2-2r|x|\cos\theta)^{d/2}} = w_d r^{2-d}(r^2-|x|^2)^{-1} \tag{4.3.22}$$

So using (4.3.22) in (4.3.21) we obtain

$$P_x\left(|Z_{\sigma^+_1}| \in [1, e^\theta]; \sigma^+_1 < \infty\right) = \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right) \sin\frac{\pi\alpha}{2} w_d(1-e^{-2u})^{\alpha/2}$$
$$\times \int_1^{e^\theta} dr r(r^2-1)^{-\alpha/2}(r^2-|x|^2)^{-1}$$
$$= \frac{2}{\pi}\sin\frac{\pi\alpha}{2}(1-e^{-2u})^{\alpha/2}\int_1^{e^\theta} dr r(r^2-1)^{-\alpha/2}(r^2-e^{-2u})^{-1}. \tag{4.3.23}$$

So using (4.3.19), (4.3.20), and (4.3.23) we conclude that

$$P\left(\xi_{T^+_u} \le u+\theta; T^+_u < \infty\right) = P_1\left(|Z_{\sigma^+_{e^u}}| \in [e^u, e^{u+\theta}]; \sigma^+_{e^u} < \infty\right)$$
$$= P_{e^{-u}}\left(|Z_{\sigma^+_1}| \in [1, e^\theta]; \sigma^+_1 < \infty\right)$$
$$= \frac{2}{\pi}\sin\frac{\pi\alpha}{2}(1-e^{-2u})^{\alpha/2}\int_1^{e^\theta} dr r(r^2-1)^{-\alpha/2}(r^2-e^{-2u})^{-1}. \tag{4.3.24}$$

So differentiating (4.3.24) we obtain (4.3.18), which completes the proof. □

Proposition 17. *Suppose that $\alpha < d$, fix $\theta \geq 0$ and $-\infty < u < 0$, then*

$$P\left(v - \xi_{T_v^-} \in d\theta, T_v^- < \infty\right)$$
$$= \frac{2}{\pi}\sin\frac{\pi\alpha}{2}(e^{-2v}-1)^{\alpha/2}(1-e^{-2\theta})^{-\alpha/2}e^{d(v-\theta)}(1-e^{2(v-\theta)})^{-1}d\theta \quad (4.3.25)$$

Proof. Let us recall that the process ξ is a Lévy process associates to the process $R = |Z|$, where Z is an α symmetric stable process in $\mathbb{R}^d$.
First we recall the following property that can be found in [21], fix $-\infty < v < 0$, and suppose that B is any interval in $(-\infty, v]$, then

$$P\left(\xi_{T_v^-} \in B; T_v^- < \infty\right) = \mathbb{P}_1\left(X_{\sigma_{e^v}^-} \in e^B; \sigma_{e^v}^- < \infty\right). \quad (4.3.26)$$

On the other hand since Z is isotropic and satisfies the scaling property it easy to see that for each $b > 0$, and any set $B \in \mathbb{B}(\mathbb{R})$, that

$$P_x(b^{-\alpha}\sigma_b^- \in B) = P_{x/b}(\sigma_1^- \in B).$$

This in turn implies that

$$P_{x/b}(|Z_{\sigma_1^-}| \in dy; \sigma_1^- < \infty) = P_x(b^{-1}|Z_{\sigma_b^-}| \in dy; \sigma_b^- < \infty). \quad (4.3.27)$$

Then using (4.3.27) we have the following

$$\begin{aligned} P_1(|Z_{\sigma_{e^v}^-}| \in [e^{v-\theta}, e^v]; \sigma_{e^v}^- < \infty) &= P_1(e^{-v}|Z_{\sigma_{e^v}^-}| \in [e^{-\theta}, 1]; \sigma_{e^v}^- < \infty) \\ &= P_{e^{-v}}(|Z_{\sigma_1^-}| \in [e^{-\theta}, 1]; \sigma_1^- < \infty), \end{aligned}$$

Now fix $x \in \mathbb{R}^d$ such that $|x| = e^{-v}$, and set $w_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$; then using (4.3.17) and the Poisson formula in $\mathbb{R}^d$, we have

$$\begin{aligned} &P_x\left(|Z_{\sigma_1^-}| \in [e^{-\theta}, 1]; \sigma_1^- < \infty\right) \\ &= \int_{e^{-\theta}<|y|<1} \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi\alpha}{2} \times |1-|x|^2|^{\alpha/2}|1-|y|^2|^{-\alpha/2}|x-y|^{-d}dy \\ &= \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi\alpha}{2}(e^{-2v}-1)^{\alpha/2}\int_{e^{-\theta}<|y|<1}|1-|y|^2|^{-\alpha/2}|x-y|^{-d}dy \\ &= \pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi\alpha}{2}(e^{-2v}-1)^{\alpha/2}\int_{e^{-\theta}}^1 dr\frac{r^{d-1}}{(1-r^2)^{-\alpha/2}} \\ &\times \int_0^\pi d\theta\frac{\sin^{d-2}\theta}{(r^2+|x|^2-2r|x|\cos\theta)^{d/2}} \end{aligned} \quad (4.3.28)$$

On the other hand, from formula 3.665 in [33] we get for $r < 1$

$$\int_0^\pi d\theta\frac{w_d\sin^{d-2}\theta}{(r^2+|x|^2-2r|x|\cos\theta)^{d/2}} = w_d|x|^{2-d}(|x|^2-r^2)^{-1}. \quad (4.3.29)$$

So using (4.3.29) in (4.3.28) we obtain

$$P_x\left(|Z_{\sigma_1^-}|\in[e^{-\theta},1];\sigma_1^-<\infty\right)=\pi^{-(d/2+1)}\Gamma\left(\frac{d}{2}\right)\sin\frac{\pi\alpha}{2}(e^{-2v}-1)^{\alpha/2}w_d|x|^{2-d}$$
$$\times\int_{e^{-\theta}}^1 drr^{d-1}(1-r^2)^{-\alpha/2}(|x|^2-r^2)^{-1}$$
$$=\frac{2}{\pi}\sin\frac{\pi\alpha}{2}(e^{-2v}-1)^{\alpha/2}(e^{-v})^{2-d}\int_{e^{-\theta}}^1 drr^{d-1}(1-r^2)^{-\alpha/2}(e^{-2v}-r^2)^{-1}. \tag{4.3.30}$$

So using (4.3.26), (4.3.27), and (4.3.30) we conclude that

$$P\left(v-\xi_{T_v^-}\le\theta;T_v^-<\infty\right)=P_1\left(|Z_{\sigma_{e^v}^-}|\in[e^{v-\theta},e^v];\sigma_{e^v}^-<\infty\right)$$
$$=P_{e^{-v}}\left(|Z_{\sigma_1^-}|\in[e^{-\theta},1];\sigma_1^-<\infty\right)$$
$$=\frac{2}{\pi}\sin\frac{\pi\alpha}{2}(e^{-2v}-1)^{\alpha/2}(e^{-v})^{2-d}\int_{e^{-\theta}}^1 drr^{d-1}(1-r^2)^{-\alpha/2}(e^{-2v}-r^2)^{-1}. \tag{4.3.31}$$

So differentiating (4.3.31) we obtain (4.3.25), which completes the proof. □

Relatively straightforward computations yield the following proposition.

Proposition 18. *Let* $\xi_\infty=\inf_{t\ge0}\ \xi_t$. *For* $z\ge0$

$$P(-\xi_\infty\in dz)=\frac{2\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}e^{-z(d-1)}(e^{2z}-1)^{\alpha/2-1}. \tag{4.3.32}$$

Proof. Fix $0\le y<1$ and, $x\in\mathbb{R}^d$ such that $|x|=1$. Then

$$P_{x/y}\left(\sigma_1^+=\infty\right)=P_{x/y}\left(\sigma_1^+=\infty\right)=P_{a/y}\left(\inf_{t\ge0}\quad|Z_t|>1\right)$$
$$=P\left(\inf_{t\ge0}\quad a/ye^{\xi_t}>1\right)$$
$$=P\left(\inf_{t\ge0}\quad\xi_t>\log y/a\right). \tag{4.3.33}$$

On the other hand using Corollary 2 in [9] we know that

$$P_x(\sigma_1^-=\infty)=\frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\int_0^{|x|^2-1}(u+1)^{-d/2}u^{\alpha/2-1}du. \tag{4.3.34}$$

So using (4.3.33) and (4.3.34) we have that:

$$P(-\xi_\infty\le z)=\frac{\Gamma(d/2)}{\Gamma(\alpha/2)\Gamma((d-\alpha)/2)}\int_0^{e^{2z}-1}(u+1)^{-d/2}u^{\alpha/2-1}du, \tag{4.3.35}$$

so differentiating (4.3.35) we obtain (4.3.32) which completes the proof. □

4.4 Entrance laws: points

In this section we explore the two-point hitting problem for the Lévy-Lamperti process ξ. There has been little work dedicated to this theme in the past with the paper of Getoor [31] being our principle reference.

Henceforth we shall denote by $(X, \mathbf{P}_x)$ a *symmetric* α-stable process in $\mathbb{R}^d$ issued from $x > 0$ where $\alpha \in (1, 2)$. An important quantity in the forthcoming analysis is the potential kernel of the process $Z = |X|$. From [60] we know that the potential kernel has the following form for $1 < \alpha < d$

$$u(x, y) = \frac{2^{(d/2)-\alpha}\Gamma(d/2)\Gamma(d-\alpha/2)}{\Gamma(\alpha/2)}(xy)^{1-N/2}|x^2-y^2|^{\alpha/2-1}P^{1-d/2}_{-\alpha/2}\left(\frac{x^2+y^2}{|x^2-y^2|}\right),$$

for $x, y > 0$, where P^{ν}_{μ} is the usual Legendre function of the first kind.

And for $1 < \alpha < d$ and $r > 0$

$$u(r, r) = \frac{\pi^{-1/2}2^{d/2-2}\Gamma((\alpha-1)/2)}{\Gamma((\alpha+d)/2-1)}\frac{\Gamma(d/2)\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)}r^{\alpha-d}.$$

Let $B = \{r_1, r_2, \cdots, r_n\}$ where $r_1 < r_2 < \cdots < r_n$, then according to the method presented in Port [60], the matrix $U_{ij} = u(r_i, r_j)$ is invertible. Denote its inverse by $K_B(i, j)$. If $\sigma_B = \inf\{t > 0 : Z_t \in B\}$ then if

$$H_B(a, r_j) = \mathbf{P}_{|x|}(Z_{\sigma_B} = r_j;\, \sigma_B < \infty),$$

the point hitting probability in Port [60] is given by the formula

$$H_B = (|x|, r_j) = \sum_{i=1}^{n} u(|x|, r_i)K_B(r_i, r_j). \tag{4.4.36}$$

For a two point set $B = \{r_1, r_2\}$ we have that

$$K_B = \frac{1}{\Delta}\begin{pmatrix} U_{22} & -U_{12} \\ -U_{12} & U_{11} \end{pmatrix}, \tag{4.4.37}$$

where $\Delta = U_{11}U_{22} - U_{12}^2$. Then from (4.4.36)

$$\begin{aligned} H_B(|x|, r_1) &= \frac{u(|x|, r_1)u(r_2, r_2) - u(|x|, r_2)u(r_2, r_1)}{u(r_1, r_1)u(r_2, r_2) - u(r_1, r_2)^2}, \\ H_B(|x|, r_2) &= \frac{u(|x|, r_2)u(r_1, r_1) - u(|x|, r_1)u(r_1, r_2)}{u(r_1, r_1)u(r_2, r_2) - u(r_1, r_2)^2}. \end{aligned} \tag{4.4.38}$$

Recalling the definition of ξ as the Lévy-Lamperti processes associated now with the norm of a symmetric stable process in $\mathbb{R}^d$ with $1 < \alpha < d$ we have the following result.

Theorem 12. *Fix $1 < \alpha < d$ and $-\infty < v < 0 < u < \infty$. Define*

$$T_{\{v,u\}} = \inf\{t > 0 : \xi_t \in \{v, u\}\}.$$

We have

$$P\Big(\xi_{T_{\{v,u\}}} = v\Big) = f(1, e^v, e^u),$$

and

$$P\Big(\xi_{T_{\{v,u\}}} = u\Big) = f(1, e^u, e^v),$$

where

$$f(x,a,b) = \frac{\frac{u(x,a)}{u(b,a)} - \frac{u(x,b)}{u(b,b)}}{\frac{u(a,a)}{u(b,a)} - \frac{u(a,b)}{u(b,b)}}.$$

4.5 Weiner-Hopf factorization

In this section we compute explicitly the characteristic exponent of the process ξ using its Weiner-Hopf factorization. Denote by $\{(L_t^{-1}, H_t) : t \geq 0\}$ and $\{(\widehat{L}_t^{-1}, \widehat{H}_t) : t \geq 0\}$ the (possibly killed) bivariate subordinators representing the ascending and descending ladder processes of ξ (see [6] for a proper definition). Write $\kappa(\alpha,\beta)$ and $\widehat{\kappa}(\alpha,\beta)$ for their joint Laplace exponents for $\alpha,\beta \geq 0$. For convenience we will write

$$\widehat{\kappa}(0,\beta) = \widehat{q} + \widehat{\mathrm{c}}\beta + \int_{(0,\infty)} (1 - e^{-\beta x})\Pi_{\widehat{H}}(\mathrm{d}x),$$

where $\widehat{q} \geq 0$ is the killing rate of $\widehat{H}$ so that $\widehat{q} > 0$ if and only if $\lim_{t\uparrow\infty} \xi_t = \infty$, $\widehat{\mathrm{c}} \geq 0$ is the drift of $\widehat{H}$ and $\Pi_{\widehat{H}}$ is its jump measure. Similar notation will also be used for $\kappa(0,\beta)$ by replacing $\widehat{q}$, $\widehat{\xi}$, $\widehat{\mathrm{c}}$ and $\Pi_{\widehat{H}}$ by q, ξ, c and Π_H. Note that necessarily $q = 0$ since $\lim_{t\uparrow\infty} \xi_t = \infty$.

Associated with the ascending and descending ladder processes are the bivariate renewal functions V and $\widehat{V}$. The former is defined by

$$V(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty dt \cdot P(L_t^{-1} \in \mathrm{d}s, H_t \in \mathrm{d}x)$$

and taking double Laplace transforms shows that

$$\int_0^\infty \int_0^\infty e^{-\alpha s - \beta x} V(\mathrm{d}x, \mathrm{d}s) = \frac{1}{\kappa(\alpha,\beta)} \quad \text{for } \alpha,\beta \geq 0 \tag{4.5.39}$$

with a similar definition and relation holding for $\widehat{V}$. These bivariate renewal measures are essentially the Green's measures of the ascending and descending ladder processes. With an abuse of notation we shall also write $V(\mathrm{d}x)$ and $\widehat{V}(\mathrm{d}x)$ for the marginal measures $V([0,\infty), \mathrm{d}x)$ and $\widehat{V}([0,\infty), \mathrm{d}x)$ respectively. (Since we shall never use the marginals $V(\mathrm{d}s, [0,\infty))$ and $\widehat{V}(\mathrm{d}s, [0,\infty))$ there should be no confusion). Note that local time at the maximum is defined only up to a multiplicative constant. For this reason, the exponent κ can only be defined up to a multiplicative constant and hence the same is true of the measure V (and then obviously this argument applies to $\widehat{V}$).

The main result of this section is the Wiener-Hopf factorization of the characteristic exponent of the Lévy process ξ.

Lemma 2. *Let ξ be the Lévy process in the Lamperti representation (4.1.2) of the radial process R. The Laplace exponent of the descending ladder height process $\widehat{H}$ of ξ is given by*

$$\hat{\kappa}(0,\lambda) = \frac{\Gamma((d+\lambda)/2)\Gamma((d-\alpha)/2)}{\Gamma(d/2)\Gamma((d-\alpha+\lambda)/2)}. \tag{4.5.40}$$

Proof. Let us consider $T_y = \inf\{t \geq 0 : |Z_t| < y\}$, $x \in \mathbb{R}^d$, and $a = \|x\|$. Then using that $bZ^x_{b^{-\alpha}t} \overset{\mathcal{L}}{=} Z^{bx}_t$ we have

$$\begin{aligned} \mathbb{P}_x(T_y = \infty) &= \mathbb{P}_{x/y}(T_1 = \infty) = \mathbb{P}_{a/y}\left(\inf_{t\geq 0} R_t > 1\right) \\ &= \mathbb{P}\left(\inf_{t\geq 0}(a/y)e^{\xi_t} > 1\right) = \mathbb{P}\left(\inf_{t\geq 0}\xi_t > \log(y/a)\right). \end{aligned} \tag{4.5.41}$$

Now let us recall the following result in Corollary 2 in [9]

$$\mathbb{P}_x(T_1 = \infty) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{|x|^2-1} (u+1)^{-d/2} u^{\alpha/2-1} du. \tag{4.5.42}$$

Then using (4.5.42) in (4.5.41) we obtain the following

$$\mathbb{P}\left(-\inf_{t\geq 0}\xi_t < z\right) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{e^{2z}-1} (u+1)^{-d/2} u^{\alpha/2-1} du. \tag{4.5.43}$$

Also recall that $\widehat{V}$ denotes the renewal function associated with $\widehat{H}$. From Proposition VI.17 in [6], we know that

$$\begin{aligned} \widehat{V}(x) = \widehat{V}([0,x]) &= \widehat{V}([0,\infty))\mathbb{P}\left(\sup_{t\geq 0} -\xi_t \leq x\right) \\ &= \widehat{V}([0,\infty))\mathbb{P}\left(-\inf_{t\geq 0}\xi_t \leq x\right). \end{aligned}$$

As we mentioned before, it is well known that $\widehat{V}$ is unique up to a multiplicative constant which depends on the normalization of local time of ξ at its infimum. Without loss of generality we may therefore assume in the forthcoming analysis that $\widehat{V}(\infty)$, which is equal to the reciprocal of killing rate of the descending ladder height process, may be taken identically equal to 1. Hence

$$\widehat{V}(z) = \frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)} \int_0^{e^{2z}-1} (u+1)^{-d/2} u^{\alpha/2-1} \mathrm{d}u.$$

Now, let $K(\alpha, d) = \Gamma(d/2)\big(\Gamma((d-\alpha)/2)\Gamma(\alpha/2)\big)^{-1}$ and note that

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda x} U_-(x)dx &= \lambda K(\alpha, d) \int_0^\infty e^{-\lambda x} \int_0^{e^{2x}-1} (u+1)^{-d/2} u^{\alpha/2-1} du \\
&= K(\alpha, d) \int_0^\infty (u+1)^{-d/2} u^{\alpha/2-1} \int_{1/2\log(u+1)}^\infty \lambda e^{-\lambda x} dx du \\
&= K(\alpha, d) \int_0^\infty (u+1)^{-(d+\lambda)/2} u^{\alpha/2-1} du \\
&= K(\alpha, d) \int_0^1 u^{(d+\lambda-\alpha)/2-1} (1-u)^{\alpha/2-1} du \\
&= \frac{\Gamma((d+\lambda)/2)\Gamma((d-\alpha)/2)}{\Gamma(d/2)\Gamma((d+\lambda-\alpha)/2)}, \qquad (4.5.44)
\end{aligned}$$

Now recalling that

$$\hat{\kappa}(0, \lambda) = \left(\lambda \int_0^\infty e^{-\lambda x} U_-(x)dx\right)^{-1},$$

we conclude the result. □

The previous result tells gives us the explicit form of the Weiner-Hopf factor corresponding to the descending ladder height process $\widehat{H}$ of ξ. In the next proposition we will obtain the corresponding factor for the ascending ladder height process H.
First we recall the following property of the hypergeometric function ${}_2\mathcal{F}_1$ ([3] (3.1.9)).

$${}_2\mathcal{F}_1(a, b; a-b+1; x) = (1+x)^{-a}\, {}_2\mathcal{F}_1\left(a/2, (a+1)/2; a-b+1; \frac{4x}{(1+x)^2}\right). \qquad (4.5.45)$$

So the Lévy measure Π of the process ξ takes the following form

$$\begin{aligned}
\Pi(dy) &= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{dy}}{(e^{2y}+1)^{(\alpha+d)/2}} \\
&\times_2 \mathcal{F}_1\left((\alpha+d)/4, (\alpha+d/4+1/2); d/2; \frac{4e^{2y}}{(e^{2y}+1)^2}\right) dy \\
&= \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{-\alpha y}}{(1+e^{-2y})^{(\alpha+d)/2}} \\
&\times_2 \mathcal{F}_1\left((\alpha+d)/4, (\alpha+d)/4+1/2; d/2; \frac{4e^{-2y}}{(1+e^{-2y})^2}\right) 1_{\{y>0\}} dy \\
&+ \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} \frac{e^{dy}}{(1+e^{2y})^{\alpha+d/2}} \\
&\times_2 \mathcal{F}_1\left(\alpha/2+d/4, \alpha/2+d/4+1/2; d/2; \frac{4e^{2y}}{(e^{2y}+1)^2}\right) 1_{\{y<0\}} dy,
\end{aligned}$$

So using (4.5.45) we obtain

$$\Pi(dy) = A_\alpha e^{-\alpha u}\,{}_2\mathcal{F}_1\left(\alpha + d/2, \alpha + 1; d/2; e^{-2u}\right) 1_{\{u>0\}}du$$
$$+ A_\alpha e^{du}\,{}_2\mathcal{F}_1\left(\alpha + d/2, \alpha + 1; d/2; e^{2u}\right) 1_{\{u<0\}}du. \tag{4.5.46}$$

where $A_\alpha = 2^\alpha \alpha (d/2)_{\alpha/2}/\Gamma(1-\alpha/2)$. Now we can prove the following result:

Lemma 3. *Let ξ be the Lévy process in the Lamperti representation (4.1.2) of the radial process R. The Laplace exponent of the ascending ladder height process H of ξ is given by*

$$\kappa(0,\lambda) = 2^{\alpha/2}\frac{\Gamma(d/2)\Gamma(-\alpha/2)}{B((d-\alpha)/2), \alpha/2)}\frac{\Gamma((\lambda+\alpha)/2)}{\Gamma(\lambda/2)}. \tag{4.5.47}$$

Proof. In order to compute the Laplace exponent of H we will first obtain its Lévy measure μ_+. To this end we will use the following result of Vigon [76], for $x > 0$ we have

$$\bar{\mu}_+(x) = \int_0^\infty \bar{\Pi}_+(x+y)\widehat{V}(dy), \tag{4.5.48}$$

where $\bar{\mu}_+$ denotes the tail of the Lévy measures of H, $\bar{\Pi}_+$ denotes the tail of the restriction of the Lévy measure of ξ to $(0,\infty)$, and $\widehat{V}(dy)$ is the renewal measure of the subordinator $\widehat{H}$ which following (4.5.44) is given by

$$\widehat{V}(dy) = 2\frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)}(e^{2y})^{1-d/2}(e^{2y}-1)^{\alpha/2-1}dy.$$

Now let us recall from (4.5.46) that Π_+ is given by

$$\Pi_+(dx) = \frac{2^\alpha \alpha (d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)} e^{-\alpha x}\,{}_2\mathcal{F}_1((\alpha+d)/2, \alpha/2+1; d/2; e^{-2x})\mathbb{1}_{\{x>0\}}dx. \tag{4.5.49}$$

Then using (4.5.48) we have

$$\int_x^\infty \mu_+(dy) = C(\alpha,d)\int_0^\infty \left(\int_{x+y}^\infty e^{-\alpha u}F(e^{-2u})du\right)(e^{2y})^{1-d/2}(e^{2y}-1)^{\alpha/2-1}dy, \tag{4.5.50}$$

where

$$C(\alpha,d) = \frac{2^{\alpha+1}\alpha(d/2)_{\alpha/2}}{\Gamma(1-\alpha/2)}\frac{\Gamma(d/2)}{\Gamma((d-\alpha)/2)\Gamma(\alpha/2)},$$

and for notation purposes $F(x)$ denotes the hypergeometric function in (4.5.49). We will now compute the second integral in the right side of (4.5.50)

$$\int_{x+y}^\infty e^{-\alpha u}F(e^{-2u})du = \frac{1}{2}\int_0^{e^{-2(x+y)}} z^{\alpha/2-1}F(z)dz$$
$$= \alpha^{-1}\left(e^{-\alpha(x+y)}\,{}_2\mathcal{F}_1((d+\alpha)/2, \alpha/2; d/2; e^{-2(x+y)})\right). \tag{4.5.51}$$

Now let us compute the second integral, then using (4.5.51) we have

$$\begin{aligned}
&\int_0^\infty \int_{x+y}^\infty e^{-\alpha u} F(e^{-2u})(e^{2y})^{1-d/2}(e^{2y}-1)^{\alpha/2-1} du dy \\
&= \int_0^\infty \alpha^{-1}\Big(e^{-\alpha(x+y)}\, {}_2\mathcal{F}_1\big((d+\alpha)/2, \alpha/2; d/2; e^{-2(x+y)}\big)\Big)(e^{2y})^{1-d/2}(e^{2y}-1)^{\alpha/2-1} dy \\
&= \alpha^{-1} \sum_{k=0}^\infty e^{-2x(\alpha/2+k)} \frac{((d+\alpha)/2)_k(\alpha/2)_k}{(d/2)_k k!} \int_0^\infty (e^{2y})^{1-d/2-\alpha/2-k}(e^{2y}-1)^{\alpha/2-1} dy \\
&= (2\alpha)^{-1} \sum_{k=0}^\infty e^{-2x(\alpha/2+k)} \frac{((d+\alpha)/2)_k(\alpha/2)_k}{(d/2)_k k!} \int_0^1 u^{d/2+k-1}(1-u)^{\alpha/2-1} dy \\
&= (2\alpha)^{-1} \sum_{k=0}^\infty e^{-2x(\alpha/2+k)} \frac{((d+\alpha)/2)_k(\alpha/2)_k}{(d/2)_k k!} B(d/2+k, \alpha/2) \\
&= (2\alpha)^{-1} B(\alpha/2, d/2) e^{-\alpha x} \sum_{k=0}^\infty e^{-2kx} \frac{(\alpha/2)_k}{k!} \\
&= (2\alpha)^{-1} B(\alpha/2, d/2) e^{-\alpha x}(1-e^{-2x})^{-\alpha/2}.
\end{aligned}$$

Then using (4.5.50) we have that

$$\int_x^\infty \mu_+(dy) = 2^\alpha \frac{\Gamma(d/2)}{\Gamma(1-\alpha/2)} \frac{1}{B\left((d-\alpha)/2, \alpha/2\right)} (e^{2x}-1)^{-\alpha/2}. \tag{4.5.52}$$

So using (4.5.48) it is easy to see that $\mu_+(dx)$ has a density with respect to the Lebesgue measure $\mu_+(x)$, then by differentiating in (4.5.52) we obtain

$$\mu_+(x) = 2^\alpha \frac{\Gamma(d/2)}{\Gamma(1-\alpha/2)} \frac{\alpha}{B\left((d-\alpha)/2, \alpha/2\right)} \frac{e^{2x}}{(e^{2x}-1)^{\alpha/2+1}}.$$

Now let us recall that the process does not creeps, this implies that H has no drift term, and we also know that H has no killing term. Then the Laplace exponent $\kappa(0,\lambda)$ of H is given by

$$\kappa(0,\lambda) = A(\alpha, d) \int_0^\infty (e^{-\lambda x}-1) \frac{e^{2x}}{(e^{2x}-1)^{\alpha/2+1}} dx, \tag{4.5.53}$$

where

$$A(\alpha, d) = 2^\alpha \frac{\Gamma(d/2)}{\Gamma(1-\alpha/2)} \frac{\alpha}{B\left((d-\alpha)/2, \alpha/2\right)}.$$

So we have from (4.5.53) that

$$\kappa(0,\lambda) = \frac{A(\alpha, d)}{2} \int_0^\infty (e^{-\lambda/2x}-1) \frac{e^{x}}{(e^{x}-1)^{\alpha/2+1}} dx.$$

The last integral has been computed in [16], so

$$\kappa(0,\lambda) = 2^{-1}A(\alpha,d)\Gamma(-\alpha/2)\frac{\Gamma((\lambda+\alpha))/2}{\Gamma(\lambda/2)}$$
$$= \alpha 2^{\alpha-1}\frac{1}{\Gamma(1-\alpha/2)}\frac{\Gamma(d/2)\Gamma(-\alpha/2)}{B((d-\alpha)/2),\alpha/2)}\frac{\Gamma((\lambda+\alpha)/2)}{\Gamma(\lambda/2)}. \tag{4.5.54}$$

Which concludes the proof. □

The Weiner-Hopf factorization for the process ξ allows to obtain an explicit form of its characteristic exponent.

Theorem 13. *Let ξ be the Lévy process in the Lamperti representation (4.1.2) of the radial process R. Then its characteristic exponent Ψ enjoys the following Wiener-Hopf factorization*

$$\psi(\lambda) = k'\alpha 2^{\alpha-1}\frac{\Gamma(d/2)\Gamma(-\alpha/2)}{\Gamma(1-\alpha/2)\Gamma(\alpha/2)}\left(\frac{i\lambda+d}{2}\right)_{\alpha/2}\left(-\frac{i\lambda}{2}\right)_{\alpha/2}.$$

Proof. We recall the following classical result in [6] which states that there exists a constant $k' > 0$ such that

$$\psi(\lambda) = k'\kappa(0,-i\lambda)\hat{\kappa}(0,i\lambda)$$

so using (4.5.40) and (4.5.47) in (4.5.55) we obtain the result. □

Part II

Probabilistic models associated with evolutionary processes.

Chapter 5

Preliminaries

5.1 Introduction

5.1.1 What is the dynamics of a population?

The idea behind this work is to model the dynamics of a sexual population, in other words how a population changes in time. Some individuals within a population leave more offsprings than others. And as time passes the frequency of the offsprings of theses individuals will increase. This difference in the reproductive ability of individuals is known as the principle of natural selection of Darwin. Those individuals that are better adapted to their environment will have a larger number of descendants than those less adapted.

Each individual in the population has a specific genetic constitution which is called *genotype*, then during its development the *genotype* is expressed in the individual actual observable characteristics, like height, morphology, sexual efficiency, which we call *phenotype*. Then the individuals in the population characterized by their *phenotype* have to compete between themselves for resources (like food, water, etc.); the outcome of this process of competition is fundamental for the survival and reproduction ability of each individual. Those *phenotypes* with a higher reproduction ability contribute with more offsprings to the succeeding generation than the other individuals, this increases the prevalence of these *phenotypes* in the population, because these offsprings will inherit their *traits*. So the competition between *phenotypes* induces a *selection* process which is one of the driving mechanisms of evolution.

The adaptation of a phenotype depends not only on the environment but also in the constitution of the rest of the population, by means of the exploit of resources or the competition with other species in the population. This population dynamics can also affect the set of *genotypes* in the population through the main sources of variability which are *mutations* in the genome of the offsprings, or by *sexual reproduction* (meiosis, recombination), and finally the environment can also influence the way the genotype of an individual expresses to its phenotype, during the development of the individual.

In the rest of this section we will give a deeper view to the main concepts require for the rest of this work.

5.1.2 Basic genetics, haploid and diploid organisms

Mendel's (1866) primary achievement was the recognition of the particular nature of the hereditary determinants, now called *genes*. A gene may have different forms, called *alleles*. From his experiments with peas he discovered that the genes are present in pairs, one pair having inherited from the maternal parent, the other form the paternal.

The allelic composition is called the *genotype*, and the set of observable properties derived from the genotype is called the *phenotype*.

Since the 1940's it has been known that the genetic material is desoxyribonucleic acid (DNA). It consists of four bases: adenine (A), guanine (G), thymine (T), and cytosine (C). Each base is linked to a sugar and a phosphate group, yielding a nucleotide. The nucleotides are arranged along two chains to form a double-stranded helix in which the pairings A-T and G-C between the strands are formed. Therefore all genetic material is contained in each of the two strands. Three bases code for one amino acid, which are the building blocks of polypeptide chains and proteins. A gene typically represents a contiguous region of DNA coding for one polypeptide chain. Its position along the DNA is called the *locus*, sampled from a population, and a particular sequence there is called an allele.

There are two types of organisms in nature, *haploid* organisms which have only one set of chromosomes like algae or fungi, and *diploid* organisms in which chromosomes form homologous pairs, each one inherited from each parent like higher plants or animals.

5.1.3 Asexual and sexual reproduction

The phenomena of reproduction is a fundamental feature of all life, it is the biological process by which new organisms are produced in a population. In nature there are two main methods for reproduction: *asexual*, and *sexual reproduction*.

In a population with asexual reproduction an individual in the population gives birth to an individual with similar or identical genetic material, without the contribution of another organism in the population. Asexual reproduction requires more energy and therefore its faster than sexual reproduction. Its main characteristic which offers benefits and costs is that the offspring is genetically similar to the parent, this similarity will be beneficial if the parent is well adapted to a stable environment, but if the environment is changing it can mean the disappearance of the population. Asexual lineages can increase their numbers rapidly because every individual can produce viable offspring. Another advantage of asexual reproduction include the ability to reproduce without partner in situations where the density of the population is low, reducing the chance to find a partner, or in situations where a single member of the population is enough to start a population. The main source of variation in an asexual population is given by *mutations*.

Sexual reproduction is the biological process by which organisms create descendants that have a combination of genetic material contributed from two (usually) different members of the species, resulting in offsprings which are different between themselves, and from their parents. The sexual reproduction cycle supposes an alternation of generations between haploid cells (cells with a single set of chromosomes), and diploid cells (which have 2 complete sets of chromosomes). The combination of the genetic material occurs when two haploid cells fuse to produce a diploid cell called *zygote*, this process is called *fecundation*, or *mating* if it occurs in diploid, or haploid organisms respectively. Then when a descendant of the diploid cell divides by the process of *meiosis*, new haploid cells are formed. During meiosis and before the chromosomes are distributed in haploid sets, the chromosomes that belong to the diploid set exchange genetic material by the process of *genetic recombination*. In this way each cell of the new haploid generation receives a different genetic load, so each chromosome presents genes that come from one of the cells of the previous generation and genes that come from the other cell. In this way through cycles of haploidy, cellular fusion, diploidy, and meiosis the old combinations of genes disappear and new combinations are formed.

Sexual reproduction can occur in diploid and haploid organisms, the main difference is in which phase the cells proliferate (by *mitosis*) to form a new organism. In diploid organisms the cells proliferate during the diploid phase forming a multicellular organism. Meanwhile in haploid organisms the haploid cells are the ones that proliferate and the unique diploid cell is the zygote, that exists transitorily after mating (see Fig. 5.1).

5.1.4 Mitosis and meiosis

Mitosis is the process by which a cell duplicates the chromosomes in its cell nucleus, in order to generate two, identical, daughter nuclei. It is generally followed immediately by cytokinesis, which divides the nuclei, cytoplasm, organelles and cell membrane into two daughter cells containing roughly equal shares of these cellular components.

The process of mitosis is complex and highly regulated. The sequence of events is divided into phases, corresponding to the completion of one set of activities and the start of the next. These stages are prophase, prometaphase, metaphase, anaphase and telophase (see Fig. 5.2). During the process of mitosis the pairs of chromosomes condense and attach to fibers that pull the sister chromatids to opposite sides of the cell. The cell then divides to produce two identical daughter cells. The process of mitosis which produces identical cells is the basis of *asexual reproduction*. Meiosis is a process of cellular division in which a diploid cell with $2n$ chromosomes will experience two consequence cellular divisions, with the capacity to generate 4 haploid cells, i.e. with n chromosomes.

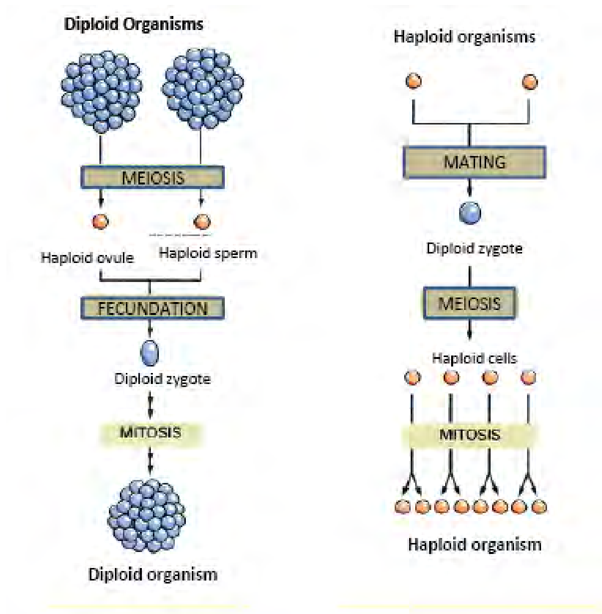


Figure 5.1: Sexual reproduction in diploids and haploids.

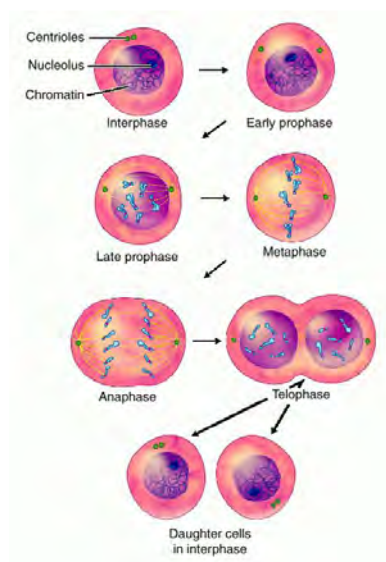


Figure 5.2: Mitosis.

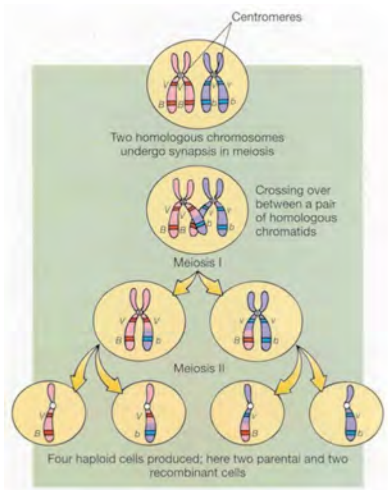


Figure 5.3: Meiosis and recombination.

During meiosis, the genome of a diploid cell, which is composed of long segments of DNA packaged into chromosomes, undergoes DNA replication followed by two rounds of division, resulting in haploid cells called gametes. Each gamete contains one complete set of chromosomes, or half of the genetic content of the original cell. Because the chromosomes of each parent undergo genetic recombination during meiosis, each gamete, and thus each zygote, will have a unique genetic blueprint encoded in its DNA. In other words, meiosis and sexual reproduction produce genetic variation.

Meiosis uses many of the same biochemical mechanisms employed during mitosis to accomplish the redistribution of chromosomes. There are several features unique to meiosis, most importantly the pairing and *genetic recombination* between chromosomes.

5.1.5 Recombination

During meiosis, different chromosomes assort independently and crossing over between two pair of homologous chromosomes may occur. Consequently, the newly formed gamete contains maternal alleles at one set of loci and paternal alleles at the complementary set. This process is called *recombination.* Since it leads to random association between alleles at different loci, recombination has the potential to combine favorable alleles of different ancestries in one gamete and to break up combinations of deleterious alleles. These properties are thought to confer a substantial evolutionary advantage to sexual species relative to asexuals (see Fig. 5.3).

5.1.6 Selection

Selection occurs when individuals of different genotypes leave different numbers of progeny because they differ in their probability to survive to reproductive age (*viability*), in their mating success, or in their average number of produced offspring (*fertility*). Darwin (1859) recognized and documented the central importance of selection as the driving force for adaptation and evolution.

Selection is measured in terms of *fitness* of individuals, i.e., by the number of progeny contributed to the next generation. There are many different measures of fitness, and it consists of several components because selection may act on ear stage of the life cycle.

5.1.7 Biological problems

How to make the transition from the genotype of an individual to its phenotype.-
In general little is known about how genes interact to produce a quantitative character. One of the most simple models to describe how we can obtain the phenotype of an individual from its genotype is the *additive genetic model* devised by Fisher and Wright.
In an haploid population, we consider l loci, and describe the state of each l-locus genotype by a vector, $\mathbf{X}$, where the components of the vector X_i describe the allelic state at locus i. The value of X_i can be taken as a real number (or belonging to a state space χ) representing the contribution of the allele to the character.

In the case of a diploid population we describe each genotype by a vector, $(\mathbf{X}, \mathbf{X'})$, where the components X_i and X_i' describe the allelic state at locus i on the paternally and maternally inherited chromosome, respectively.

In this model, the phenotypic value, P, of a single polygenic character is assumed to be determined by a component, G, attributable to the influence of the genotype, and an environmental component, E, attributable to all the non genetic effects that influence the phenotype, such that

$$P = G + E'$$

The genotypic value G, is determined additively by all allelic effects, in other words,

$$G = \sum_{i=1}^{l} X_i \quad \left(\sum_{i=1}^{l} (X_i + X_i') \quad \text{if the population is diploid} \right).$$

This rests on the assumption that the genotypic value can be approximated by the sum of the additive effects of the contributing genes. This assumption will be a good approximation if dominance and epistatic deviations are small.

Finally the the environmental component is assumed to be normally distributed with mean 0, variance σ_E^2, and independent of G.

If we neglect environmental effects then the phenotypic value of a character in an individual with genotype $\mathbf{X}$ is given by

$$P = \sum_{i=1}^{l} X_i. \tag{5.1.1}$$

How to measure selection: The fitness function.-
Natural selection acts in different ways in the phenotype of an organism. Because the phenotype is described by a large number of characters it is difficult to distinguish direct effects of selection from indirect effects. In principle, selection can usually be described in discrete time models by a *fitness function* that relates the fitness of individuals (the contribution of an organism to the offspring pool) to the traits under consideration. There are three main types of selection:

- Directional selection.- A trait is under directional selection if the fitness function is an increasing or decreasing function of the trait value.
- Stabilizing selection.- A trait is under stabilizing selection if the fitness function has a mode, or optimum, and decreases away from this mode. A typical and simple example for a fitness function to model stabilizing selection is obtained if the fitness function of individuals with genotype x is assumed to deviate quadratically as x deviates from the optimum, in other words,
$$W(x) = 1 - sx^2, \quad |x| \leq s^{-1/2},$$
which is called the quadratic optimum model.
- Disruptive selection.- It occurs if the fitness of extreme individuals is greater than that of intermediates, i.e., if the fitness function has a minimum between two peaks.

In practice, traits often experience combinations of these simple forms of selection. For more theory on the measurement of selection see Chapter VII in [11], and Lande and Arnold [47].

We know that selection acts on phenotypes, but many models with recurrence relations for gamete and genotype frequencies involve genotypic fitness. Therefore, the fitness function of an individual with genotypic value z is given by

$$W_G(z) = W_P(p(z)),$$

where W_P is the phenotypic fitness function, and the function p is the phenotype-genotype map. In case we are considering environmental effects, if we assume they are Gaussian with mean zero and variance σ_E^2 the mean fitness of individuals withe genotypic value z is given by

$$W_G(z) = E[W_P(G+E)] = \frac{1}{(2\pi\sigma_E^2)^{1/2}} \int_{\mathbb{R}} W_P(G+E)\exp[-E^2/(2\sigma_E^2)]dE.$$

As an example of how to model selection by means of the fitness function, imagine an asexually reproducing haploid population, in which we consider one gene locus at which k alleles $A_1, \dots, A_k$ occur. Individuals carrying allele A_i are assumed to have fitness W_i. Let n_i denote the number of individuals with allele A_i, and N the total size of the population. Then as we discussed before we can think the fitness W_i as the average number of offspring per individual with allele A_i. So in the next generation the number of individuals with allele A_i is $n'_i = W_i n_i$, and the total size of the population is

$$N' = \sum_{i=1}^{k} n'_i = \sum_{i=1}^{k} n_i W_i$$

So if we denote by as p_i the relative frequency of individuals with allele A_i, that is to say $p_i = n_i/N$, then in the next generation the relative frequency p'_i of individuals with allele A_i is

$$p'_i = \frac{n'_i}{N'} = \frac{n_i W_i}{\sum_{i=1}^{k} n_i W_i} = \frac{p_i W_i}{\sum_{i=1}^{k} p_i W_i} = \frac{p_i W_i}{\overline{W}}. \tag{5.1.2}$$

where $\overline{W} = \sum_{i=1}^{k} W_i p_i$ is called the mean fitness. Relation (5.1.2) is sometimes referred as *selection equation* (see [11] p.30-31).

Meanwhile in continuous time models in order to describe selection we use the *malthusian fitness parameter*. Consider we have a population at time $t > 0$ with size $N(t)$, and that an average of $b(x)\Delta t N(t)$ individuals with trait x are born in an infinitesimal time interval Δt, and that $d(x)\Delta t$ individuals with trait x die in the same time span. Then the malthusian fitness parameter for the trait x is given by $m(x) = b(x) - d(x)$, and the functions b and d are called *birth rate* and *death rate* of the trait x respectively.

Relation between the malthusian parameter and the fitness function.- For example let us consider the following simple model of the size of a population. Let $N(t)$, $t = 0, 1, 2, \dots$ be the number of individuals in generation t. If the average number of offspring per individual is W, where W is a compound measure of reproductive success and survival, the population size at generation t can be calculated from that of generation 0 according to

$$N(t) = WN(t-1) = W^2 N(t-2) = \cdots = W^t N(0). \tag{5.1.3}$$

Now lets consider a continuous time model analogous to the discrete one discussed above, assume that in a population of size $N = N(t)$ an average of $b\Delta t N(t)$ progeny are born during an infinitesimal time interval Δt and $d\Delta t N(t)$ individuals die in the same time span. The parameter b is called the birth rate and d the death rate. The total change in population number during the time interval Δt is

$$\Delta N(t) = N(t + \Delta t) - N(t) = (b(t) - d(t))\Delta t N(t). \tag{5.1.4}$$

As $\Delta t \to 0$, one obtains

$$\frac{dN(t)}{dt} = mN(t), \tag{5.1.5}$$

where $m = b - d$ is called the *Malthusian parameter* or *growth rate*. The differential equation (5.1.5) has the solution

$$N(t) = \exp(mt)N(0),$$

so that the population grows exponentially. Time can be measured in arbitrary units as long as m is measured in the reciprocal of that unit.

Now suppose we change the time scale and therefore m so that $t = 1$ is the time of the first generation, $t = 2$ is the time of the second generation, and so on. Then the relation between the Malthusian parameter m and the fitness function W is, by comparison of (5.1.3) and (5.1.6) the following,

$$W = \exp(m) \qquad \text{or} \qquad m = \log(W). \tag{5.1.6}$$

Frequency and density dependent selection.-
Density dependent selection occurs if the fitness of phenotypes depends on population size because for instance, under crowding conditions some types have reduced fitness. Frequency dependent selection means that the fitness of phenotypes depends on their frequency distribution. This typically occurs if certain types have higher fitness when rare, for example if the ability to utilize different different food resources depends on body size. An example which takes into consideration density and frequency dependent selection is the quantitative-genetic model analyzed by Slatkin [73]. He considered an approximately normally distributed phenotypic character, P, in a population of size $N(t)$. The distribution of the character is denoted by $f_P(P,t)$, its mean and variance by $\overline{P}(t)$ and $\sigma_P^2(t)$, respectively. The fitness of individuals of type P is assumed to depend on $N(t)$ and $f_P(t)$ in the following way:

$$W(P,t) = 1 + \rho - \frac{\rho N(t)}{k(P)} \int_{\mathbb{R}} \alpha(P - P')f_P(P',t)dP',$$

where $1+\rho$ is the maximum fitness in the absence of competition, $k(P)$ represents the resources that can be utilized by an individual of type P, and $\alpha(P - P')$ represents the competition between individuals of type P and P' for the limiting resource. As an example of $k(P)$, Slatkin uses a function proportional to a Gaussian density with mean P_0, which is the value of the character for which the maximum resources are available, and variance σ_k^2 whichmeasures the range of available resources. Similarly, as an example of α he uses

$$\alpha(P - P') = \exp\left[-\frac{1}{2\sigma_\alpha^2}(P - P')^2\right].$$

where σ_α^2 measures the extent of competition between individuals.

Figure 5.4: A gold-colored Midas cichlid, Cichlasoma (Amphilophus) barlowi, guarding fry.

5.1.8 Sympatric speciation

Sympatric speciation refers to the formation of two or more descendant species from a single ancestral species all occupying the same geographical location. This formation of new species without geographical barriers has often been dismissed by the argument that continuous gene flow would prevent the formation establishment of fixed genetic differences which would be necessary for the formation of species. However, a growing body of empirical data shows that closely related species often occur in sympatry (without geographical barriers), like the cichlids in East Africa (Meyer [52]), or the Midan cichlid in Nicaragua (Barluenga *et al.* [5]), which do not seem to fit the usual requirement of long periods of geographical isolation.

The most straightforward scenario for sympatric speciation requires disruptive selection favoring two substantially different phenotypes, followed by the elimination of all intermediate phenotypes. In sexual populations, the stumbling block preventing sympatric speciation is that mating between divergent ecotypes constantly scrambles gene combinations, creating organisms with intermediate phenotypes. However this mixing can be prevented if there is assortative mating instead of random mating, i.e., mating of individuals that are phenotypically similar (Doebeli [24]). The study made by Barluenga and Meyer [5] on the Midan cichlid species of fish in Nicaragua shows the prime role of assortive mating through color preference with respect to ecological speciation based on morphological traits as the jaw morph and body shape, in the Midas cichlid species. The three-spined sticklebacks, freshwater fish, that have been studied by Dolph Schluter provide an intriguing example of sympatric speciation. He found that competition favoring fishes at either extreme of body size and mouth size over those near to the mean (disruptive selection), and the fact that fish preferred mates with similar size (assortive mating), favored a a divergence into two subpopulations exploiting different food in different parts of the lake.

Chapter 6

Continuous time approximation of a model of an asexual population

6.1 Introduction

In this chapter we will work with a discrete time model for an asexual population, taking into account natural selection, and mutation. This model was introduced by Bürger [11] (p. 123-131), and has played an important role in several analyses aimed at resolving the problem of how much genetic variation can be maintained at a balance between mutation and stabilizing selection.

We find, that this model satisfies the Feyman-Kac formulae; and so is very close to the models worked by Del Moral [22]. So using some asymptotic results on the Feyman- Kac formulae we are able to obtain a stochastic particle interpretation of the model.

And finally using some hypothesis on the fitness function (weak selection) we are able to obtain an approximation in continuous time of the model.

6.2 The model

We consider an effectively infinite, haploid population, in which individuals are characterized by a trait x (phenotype), where x is an element in the trait space χ.

In mathematical terms, χ is a locally compact space, in our case a closed subset of $\mathbb{R}^d$. Trait densities and mutation distributions are taken with respect to the Lebesgue measure. In the following $B_b(\chi)$ will denote the set of bounded measurable functions on χ and $C_b(\chi)$ the set of bounded continuous functions on χ. The mutation rate of a trait x is denoted by $\mu(x)$, $0 \leq \mu(x) \leq 1$. Presently, it is not assumed that all types have the same mutation rate, though $\mu = \mu(x)$ can be constant. Conditioning on the assumption of a mutation event, the probability density of mutations from trait x to trait y is denoted by $u(x, y)$. Therefore we have $u(x, y) \geq 0$,

$$\int_{\chi} u(x, y)dy = 1 \quad \text{for all } x \in \chi,$$

and $\mu(x)u(x, y)$ is the fraction of individuals of type y that come through mutation from individuals of type x during one generation.

Let $p(n, x)$ denote the density of individuals with trait x at time n (see [11]). This means, when the trait space is discrete, that if $N(n, x)$ is the number of individuals with trait x at time n, and $N(n)$ is the population size at time n then:

$$p(n, x) = \frac{N(n, x)}{N(n)}.$$

We will assume that $p(n, x) \leq 1$ for all $n \in \mathbb{N}$ and $x \in \chi$.
The fitness for a trait x is denoted by $W(x)$, where $W(x) \geq 0$.

We define the mean fitness of the population at time n, by the following expression

$$\overline{W}(n) = \int_{\chi} W(x)p(n, x)dx.$$

Lets recall that we can think $W(x)$ for $x \in \chi$ as the average density of offspring of an individual with trait x, so $W(x)p(n, x)$ is the average number of individuals with trait x at time $n+1$. Then, provided that $0 < \overline{W}(n) < \infty$, and proceeding in a similar way as in (5.1.2) the density of types after selection $p_s(n, x)$ becomes

$$p_s(n, x) = \frac{p(n, x)W(x)}{\overline{W}(n)}.$$

So the dynamics of the population proposed by Bürger is the following, the type densities evolve according to

$$p(n+1, x) = (1 - \mu(x))p_s(n, x) + \int_{\chi} p_s(n, y)\mu(y)u(y, x)dy. \tag{6.2.1}$$

So in terms of the trait density (6.2.1) becomes

$$\begin{aligned} p(n+1, x) = &\left(\int_{\chi} W(x)p(n, x)dx\right)^{-1} \\ &\cdot \left((1 - \mu(x))p(n, x)W(x) + \int_{\chi} p(n, y)W(y)\mu(y)u(y, x)dy\right). \end{aligned} \tag{6.2.2}$$

6.3 Bürger's model and the Feynman-Kac formulae

Let $M_1(\chi)$ be the set of probability measures on χ. From now on for $\mu \in M_1(\chi)$ and $f \in B_b(\chi)$, we will denote by $\langle \mu, f \rangle$, to the integral over χ of f with respect to the probability measure μ, that is to say

$$\langle \mu, f \rangle = \int_\chi f(x)\mu(dx).$$

Consider $\eta_n \in M_1(\chi)$ with density $p(n,x)$, and a function $f \in B_b(\chi)$.

Now noting $\int_\chi u(x,y)dy = 1$, and changing the order of integration, we obtain the weak form of equation (6.2.2)

$$\begin{aligned}\int_\chi f(x)\eta_{n+1}(dx) &= \left(\int_\chi W(x)\eta_n(dx)\right)^{-1}\left(\int_\chi (1-\mu(x))W(x)f(x)\eta_n(dx)\right.\\ &\qquad \left.+\int_\chi\int_\chi f(x)\mu(y)u(y,x)W(y)\eta_n(dy)dx\right)\\ &= \left(\int_\chi W(x)\eta_n(dx)\right)^{-1}\left(\int_\chi\int_\chi f(x)\mu(y)u(y,x)W(y)\eta_n(dy)dx\right.\\ &\qquad \left.+\int_\chi\int_\chi (1-\mu(y))W(y)f(y)u(y,x)\eta_n(dy)dx\right). \qquad (6.3.3)\end{aligned}$$

Let $\eta \in M_1(\chi)$, we then define the Boltzmann-Gibbs transformation of η, $\psi : M_1(\chi) \to M_1(\chi)$ by

$$\psi(\eta) = \frac{1}{\langle \eta, W \rangle} W(x)\eta(dx). \qquad (6.3.4)$$

We also define the following probability kernel K by,

$$K(x,dy) = \mu(x)u(x,y)dy + (1-\mu(y))\delta_x(dy), \qquad (6.3.5)$$

From (6.3.3) we see that $(\eta_n)_{n\geq 0}$ satisfies the following difference equation,

$$\begin{aligned}\eta_{n+1} &= \psi(\eta_n)K\\ &\equiv \Phi(\eta_n). \qquad (6.3.6)\end{aligned}$$

Indeed let $f \in C_b(\chi)$ then using (6.3.3),

$$\begin{aligned}\langle(\psi(\eta_n)K), f\rangle &= \left\langle \psi(\eta_n), \int_\chi \mu(x)u(x,y)f(y)dy + (1-\mu(x))f(x)\right\rangle\\ &= \frac{1}{\langle \eta_n, W\rangle}\int_\chi\left(\int_\chi \mu(x)u(x,y)f(y)dy + (1-\mu(x))f(x)\right)W(x)\eta_n(dx)\\ &= \frac{1}{\langle \eta_n, W\rangle}\int_\chi\int_\chi \mu(x)u(x,y)f(y)W(x)\eta_n(dx)dy\\ &\qquad + \frac{1}{\langle \eta_n, W\rangle}\int_\chi (1-\mu(x))f(x)W(x)\eta_n(dx)\\ &= \langle \eta_{n+1}, f\rangle.\end{aligned}$$

From (6.3.6) it follows that $(\eta_n)_{n\geq 0}$ satisfies the evolution equation satisfied by the Feynman-Kac models, so following Definition 3.21 in Del Moral in [22] we find the following particle interpretation:

Definition 9. *The interacting particle model associated with the mapping* Φ *defined in (6.3.6) and with an initial measure* $\eta_0 \in M_1(\chi)$ *is an homogeneous Markov chain* $\xi^{(N)}$ *taking values at each time* $n \in \mathbb{N}$ *in the product space* χ^N*, that is*

$$\xi_n^N = (\xi_n^{(N,1)}, \ldots, \xi_n^{(N,N)}) \in \chi^N.$$

The initial configuration ξ_0^N *consists of* N *independent and identically distributed random variables with common* η_0*. Its elementary transitions on* χ^N *are given in a symbolic integral form by*

$$P_{\eta_0}^N(\xi_n^N \in dx_n | \xi_{n-1}^N) = \prod_{p=1}^N \Phi\left(\frac{1}{N}\sum_{i=1}^N \delta_{\xi_{n-1}^{(N,i)}}(dx_{n-1}^p)\right), \tag{6.3.7}$$

where $dx_n = dx_n^1 \times \cdots \times dx_n^N$ *is an infinitesimal neighborhood of a point* $x_n = (x_n^1, \ldots, x_n^N) \in \chi^N$*.*

A more explicit description of the transitions (6.3.7) in terms of a two stage genetic type process can already be done. More precisely, using the fact that the mapping under study Φ is given by (6.3.6) and that by (6.3.4) we have

$$\Psi\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}\right) = \sum_i^N \frac{W(x^i)}{\sum_{j=1}^N W(x^j)}\delta_{x^i},$$

we can consider the particles as individuals in a population of size N, and we see that the resulting evolution of these individuals can be decomposed into two separate mechanisms

$$\xi_{n-1}^N \overset{Selection}{\longrightarrow} \widehat{\xi}_{n-1}^N \overset{Mutation}{\longrightarrow} \xi_n^N.$$

These mechanisms can be modeled as follows:

Selection:

$$P_{\eta_0}^N(\widehat{\xi}_{n-1}^N \in dx | \xi_{n-1}) = \prod_{p=1}^N \sum_{i=1}^N \frac{W(\xi_{n-1}^{(N,i)})}{\sum_{j=1}^N W(\xi_{n-1}^{(N,j)})}\delta_{\xi_{n-1}^{(N,i)}}(dx^p).$$

Mutation:

$$P_{\eta_0}^N(\xi_n \in dz | \widehat{\xi}_{n-1}) = \prod_{p=1}^N K(\widehat{\xi}_{n-1}^{(N,p)}, dz^p). \tag{6.3.8}$$

Thus, we see that the individuals evolve according to the following rules. In the selection transition, at each time $n \geq 1$, each individual examines the system

ξ^N_{n-1} and chooses randomly a site $\xi^{(N,i)}_{n-1}$, $1 \leq i \leq N$, with a probability that depends on the entire configuration ξ^N_{n-1} and given by

$$\frac{W(\xi^{(N,i)}_{n-1})}{\sum_{j=1}^{N} W(\xi^{(N,j)}_{n-1})}.$$

In this mechanisms the individuals are selected for reproduction according to their fitness, the most fit individuals being more likely to be selected. In other words, this transition allows individuals to give birth to new individuals at the expense of the lesser fit individuals which die.

On the mutation mechanism we replace the individual $\widehat{\xi}^{(N,i)}_{n-1}$, $1 \leq i \leq N$, with a new individual (the offspring) according to the probability Kernel (6.3.8). According to (6.3.5) the trait of the offspring will be chosen in the following way, with probability $1-\mu(\widehat{\xi}^{(N,i)}_{n-1})$ the offspring carries the same trait of the parent $\widehat{\xi}^{(N,i)}_{n-1}$; and with probability $\mu(\widehat{\xi}^{(N,i)}_{n-1})$ the offspring is mutated, so we choose the trait of the offspring according to the probability law $U(\widehat{\xi}^{(N,i)}_{n-1}, y)dy$.

So if we define the empirical measures,

$$\eta^N_n = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi^{(N,i)}_n}, \tag{6.3.9}$$

the following asymptotic result due to Del Moral, Proposition 2.9 in [23], allows us to consider the model (6.2.2) as an approximation of the empirical measures $(\eta^N_n)_{n\geq 0}$, when the size of the population is very large (when $N \to \infty$).

Proposition 19. *For any time $n \geq 0$ and $p \geq 1$ there exists some finite constant $C^{(p)}_n < \infty$, such that for every $f \in B_b(\chi)$*

$$E(|\langle \eta^N_n, f\rangle - \langle \eta_n, f\rangle|^p)^{1/p} \leq \frac{1}{\sqrt{N}}\|f\|_\infty C^{(p)}_n.$$

The above result allows to give a microscopic interpretation of the discrete time model described by (6.2.2) as an approximation of the empirical measures $(\eta^N_t)_{t\geq 0}$ given by (6.3.9) when the size of the population is large. Given the fact that we can interpret the empirical measures $(\eta^N_t)_{t\geq 0}$ as the distribution of individuals in a population of size N, where the individuals are characterized by their traits $\xi^{(N,i)} \in \chi$ for $1 \leq i \leq N$.

6.4 Convergence to a continuous-time model

The aim of this section is to obtain a continuous time approximation of the discrete time model of Bürger [11] (p. 123-131). The main idea is to consider in a fixed time interval $[0,T]$, $\lfloor nT \rfloor$ generations of time length $1/n$. Then we make the length of each generation tend to 0, by increasing the number of generations, in other words by making n tend to infinity. We assume that between each generation of length $1/n$ the population evolves according to (6.2.2) with fitness function W^n and mutation rate μ^n.

To obtain the continuous time model we need to define the malthusian fitness parameter, its relation with the fitness function W^n is given in the following assumption:

Assumption (A1)
There exists a function $m : \chi \to \mathbb{R}$ such that $m \in B_b(\chi)$ and assume that

$$\int_\chi m(x)dx < \infty.$$

Applying (5.1.6) to the subpopulation of trait x (in an individual), we obtain

$$W^n(x) = \exp\left(\frac{1}{n}m(x)\right). \tag{6.4.10}$$

Let us explain the reason of the renormalization by $1/n$ of the malthusian parameter in (6.4.10). As the number of generations on the interval $[0, T]$ increases, we have to assume that the average number of offspring of each individual (given by the fitness function W^n) decreases; otherwise the number of individuals born in each generation will tend to infinity.

We make a second assumption related to the mutation rates μ^n:

Assumption (A2)
Suppose that there exists a function $\mu : \chi \to [0, 1]$, and that the mutation rates at each generation of length $1/n$ are given by

$$\mu^n(x) = \frac{1}{n}\mu(x).$$

And finally we make an assumption related to the mutation kernel u:

Assumption (A3)
There exist a constant $D > 0$ and a probability density function $\bar{u}$ on χ such that for all $x, y \in \chi$,

$$u(x, y) \leq D\bar{u}(x - y).$$

We now fix a certain time T, and we divide the interval $[0, T]$ in a series of intervals of length $1/n$ (the length of each generation), therefore in the interval $[0, T]$ we have a total of $\lfloor nT \rfloor$ generations. We denote the time of each generation by $t_i = \frac{m}{n}$ for $i = 0, 1, \ldots, \lfloor nT \rfloor + 1$.

Let us set $q^n(0, x) = q(0, x)$ for all $n \in \mathbb{N}$ (the initial density of individuals with genotype x), and denote by $q^n(t_i, x)$ to the density of individuals with genotype x at generation t_i (see [11]). This means, in the case of a discrete trait space, that if $N(t_i, x)$ is the number of individuals with trait x at the generation t_i, and $N(t_i)$ is the population size at the generation t_i then:

$$q^n(t_i, x) = \frac{N(t_i, x)}{N(t_i)}.$$

We will assume that $q^n(t_i, x) \leq 1$ for all $n \in \mathbb{N}$, $x \in \chi$, and $i = 0, 1, \ldots, \lfloor nT \rfloor + 1$. We also define the mean fitness at time t_i for $i = 0, 1, \ldots, \lfloor nT \rfloor + 1$, by the following expression

$$\overline{W}^n(t_i) = \int_{x \in \chi} W^n(x) q^n(t_i, x) dx.$$

We will assume that at each generation the population evolves as the discrete time model (6.2.2), in other words that for t_i with $i = 0, 1, \ldots, \lfloor nT \rfloor + 1$, $q^n(t_i, x)$ satisfies the following difference equation:

$$\begin{aligned} q^n(t_{i+1}, x) - q^n(t_i, x) &= \frac{1}{\overline{W}^n(t_i)} \Big((1 - \mu^n(x)) W^n(x) q^n(t_i, x) \\ &+ \int_\chi W^n(y) \mu^n(y) u(y, x) q^n(t_i, y) dy - \overline{W}^n(t_i) q^n(t_i, x) \Big) \\ &= \frac{1}{\overline{W}^n(t_i)} \Big((W^n(x) - \overline{W}^n(t_i)) q^n(t_i, x) \\ &+ \int_\chi W^n(y) \mu^n(y) u(y, x) q^n_{t_i}(y) dy - \mu^n(x) W^n(x) q^n(t_i, x) \Big). \end{aligned} \tag{6.4.11}$$

For each $n \in \mathbb{N}$ and $x \in \chi$ consider the function $q^n(t, x) : [0, T] \times \chi \to \mathbb{R}$, defined by:

$$q^n(t, x) = \sum_{i=0}^{\lfloor nT \rfloor} q^n(t_i, x) 1_{[t_i, t_{i+1})}(t). \tag{6.4.12}$$

The following result allows us to obtain the continuous time approximation of the discrete time model described by (6.4.11).

Theorem 14. *Admit Assumptions (A1), (A2), and (A3). Then the sequence of functions $\{q^n\}_{n \in \mathbb{N}}$ converges pointwise, as n goes to infinity, to the unique continuous function $q : [0, t] \times \chi \to \mathbb{R}$ satisfying:*

$$\begin{aligned} q(t, x) = q(0, x) &+ \int_0^t (m(x) - \overline{m}(s)) q(s, x) ds \\ &+ \int_0^t \int_\chi \mu(y) u(y, x) q(s, y) dy ds - \int_0^t \mu(x) q(s, x) ds, \end{aligned}$$

for any $x \in \chi$, and $t \in [0, T]$.

Proof. We start by taking the sum for $i = 0, 1, \ldots, \lfloor nt \rfloor$ of (6.4.11), and noting that $q^n(\cdot, x)$ is constant in $[\frac{\lfloor nt \rfloor}{n}, t]$ we obtain

$$q^n(t,x) - q^n(0,x) = \sum_{i=0}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} \Big((W^n(x) - \overline{W}^n(t_i)) q^n(t_i, x) - \mu^n(x) W^n(x) q^n(t_i, x) \Big)$$
$$+ \sum_{i=0}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} \int_\chi W^n(y) \mu^n(y) u(y,x) q^n(t_i, y) dy.$$

Using the particular form of W^n in (6.4.10) we can make a Taylor expansion around zero for fixed $x \in \chi$, and we obtain

$$W^n(x) = 1 + \frac{1}{n} m(x) + h^n(x), \tag{6.4.13}$$

the function h^n is the residual in Taylor's Theorem and has the following form

$$h^n(x) = \int_0^{\frac{m(x)}{n}} \exp(t)(\frac{m(x)}{n} - t)dt. \tag{6.4.14}$$

Using (6.4.13) we have the following expansion for $\overline{W}^n$ with $i = 0, 1, \ldots, \lfloor nt \rfloor$,

$$\overline{W}^n(t_i) = 1 + \frac{1}{n} \overline{m}^n(t_i) + \overline{h}^n(t_i),$$

where

$$\overline{m}^n(t_i) = \int_\chi m(x) q^n(t_i, x) dx,$$

and

$$\overline{h}^n(t_i) = \int_\chi h^n(x) q^n(t_i, x) dx.$$

It is easy to see using the later expansions that

$$q^n(t,x) - q^n(0,x) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n}$$
$$+ \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} (h^n(x) - \overline{h}^n(t_i)) q^n(t_i, x)$$
$$+ \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} \left(\int_\chi \mu^n(y) u(y,x) q^n(t_i, y) dy \right) - \sum_{i=1}^{\lfloor nt \rfloor} \mu^n(x) q^n(t_i, x)$$
$$+ \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} \left(\int_\chi h^n(y) \mu^n(y) u(y,x) q^n(t_i, y) dy \right) - \sum_{i=1}^{\lfloor nt \rfloor} \frac{m(x)}{n} \mu^n(x) q^n(t_i, x)$$
$$+ \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} \left(\int_\chi \frac{m(y)}{n} \mu^n(y) u(y,x) q^n(t_i, y) dy \right) - \sum_{i=1}^{\lfloor nt \rfloor} h^n(x) \mu^n(x) q^n(t_i, x). \tag{6.4.15}$$

Now we define the functions $\overline{m}^n, \overline{h}^n, \overline{W}^n : [0,T] \to \mathbb{R}$, as follows,

$$\overline{m}^n = \sum_{i=0}^{\lfloor nT \rfloor} m^n(t_i) 1_{[t_i, t_{i+1})},$$

$$\overline{h}^n = \sum_{i=0}^{\lfloor nT \rfloor} h^n(t_i) 1_{[t_i, t_{i+1})},$$

$$\overline{W}^n = \sum_{i=0}^{\lfloor nT \rfloor} W^n(t_i) 1_{[t_i, t_{i+1})},$$

for $i = 0, 1, \ldots, \lfloor nt \rfloor$.

Let us take the first sum of (6.4.15) then we have the following:

$$\begin{aligned}
&\sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} = \sum_{i=1}^{\lfloor nt \rfloor} (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} \\
&+ \sum_{i=1}^{\lfloor nt \rfloor} \left(1 - \frac{1}{\overline{W}^n(t_i)}\right) (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} \\
&= \sum_{i=1}^{\lfloor nt \rfloor} (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} + \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{n\overline{m}^n(s) - \overline{h}^n(s)}{n^2 \overline{W}^n(s)}\right) (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} \\
&= \sum_{i=1}^{\lfloor nt \rfloor} (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} + (m(x) - \overline{m}^n(\lfloor nt \rfloor)) q^n(\lfloor nt \rfloor, x)(t - \lfloor nt \rfloor) \\
&+ \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{n\overline{m}^n(s) - \overline{h}^n(s)}{n^2 \overline{W}^n(s)}\right) (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} \\
&- (m(x) - \overline{m}^n(\lfloor nt \rfloor)) q^n(\lfloor nt \rfloor, x)(t - \lfloor nt \rfloor) \\
&= \int_0^t (m(x) - \overline{m}^n(s)) q^n(s, x) ds - (m(x) - \overline{m}^n(\lfloor nt \rfloor)) q^n(\lfloor nt \rfloor, x)(t - \lfloor nt \rfloor) \\
&+ \sum_{i=1}^{\lfloor nt \rfloor} \left(\frac{n\overline{m}^n(t_i) - \overline{h}^n(t_i)}{n^2 \overline{W}^n(t_i)}\right) (m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} \\
&= \int_0^t (m(x) - \overline{m}^n(s)) q^n(s, x) ds - (m(x) - \overline{m}^n(\lfloor nt \rfloor)) q^n(\lfloor nt \rfloor, x)(t - \lfloor nt \rfloor) \\
&+ \sum_{i=1}^{\lfloor nt \rfloor} C(t_i, n)(m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i, x)}{n} \qquad (6.4.16)
\end{aligned}$$

Where for each t_i with $i = 0, 1, \ldots, \lfloor nT \rfloor + 1$ we have:

$$C(t_i, n) = \frac{n\overline{m}^n(t_i) - n^2 \overline{h}^n(t_i)}{n^2 \overline{W}^n(t_i)}.$$

Now let us consider the third sum in (6.4.15) proceeding exactly as before we obtain:

$$\sum_{i=1}^{\lfloor nt\rfloor} \frac{1}{\overline{W}^n(t_i)} \left(\int_\chi \mu^n(y)u(y,x)q^n(t_i,y)dy\right) = \sum_{i=1}^{\lfloor nt\rfloor} \left(\int_\chi \mu^n(y)u(y,x)q^n(t_i,y)dy\right)$$

$$\begin{aligned}
&+ \sum_{i=1}^{\lfloor nt\rfloor} \left(1 - \frac{1}{\overline{W}^n(t_i)}\right) \left(\int_\chi \mu^n(y)u(y,x)q^n(t_i,y)dy\right) \\
&= \int_0^t \int_\chi \mu(y)u(y,x)q^n(s,y)dyds - (t - \lfloor nt\rfloor) \int_\chi \mu(y)u(y,x)q^n(\lfloor nt\rfloor, y)dy \\
&+ \sum_{i=1}^{\lfloor nt\rfloor} C(t_i,n) \left(\int_\chi \mu^n(y)u(y,x)q^n(t_i,y)dy\right). \qquad (6.4.17)
\end{aligned}$$

Finally taking the fourth sum in (6.4.15) we obtain:

$$\sum_{i=1}^{\lfloor nt\rfloor} \mu^n(x)q^n(t_i,x) = \sum_{i=1}^{\lfloor nt\rfloor} \frac{\mu(x)}{n} q^n(t_i,x) = \int_0^t \mu(x)q^n(s,x)ds - \mu(x)q^n(t_i,x)(t - \lfloor nt\rfloor). \qquad (6.4.18)$$

Using (6.4.16), (6.4.17), and (6.4.18) in (6.4.15) allows us to express it in the following form:

$$\begin{aligned}
q^n(t,x) - q^n(0,x) = &\int_0^t (m(x) - \overline{m}^n(s))q^n(s,x)ds - \int_0^t \mu(x)q^n(s,x)ds \\
&+ \int_0^t \int_\chi \mu(y)u(y,x)q^n(s,y)dyds + H(n,x,t). \qquad (6.4.19)
\end{aligned}$$

where

$$\begin{aligned}
H(n,x,t) = &\sum_{i=1}^{\lfloor nt\rfloor} C(t_i,n)(m(x) - \overline{m}^n(t_i))\frac{q^n(t_i,x)}{n} + \sum_{i=1}^{\lfloor nt\rfloor} \frac{1}{\overline{W}^n(t_i)}(h^n(x) - \overline{h}^n(t_i))q^n(t_i,x) \\
&+ \sum_{i=1}^{\lfloor nt\rfloor} C(t_i,n) \left(\int_\chi \mu^n(y)u(y,x)q^n(t_i,y)dy\right) - \sum_{i=1}^{\lfloor nt\rfloor} \frac{m(x)}{n}\mu^n(x)q^n(t_i,x) \\
&- \sum_{i=1}^{\lfloor nt\rfloor} \frac{1}{\overline{W}^n(t_i)} \left(\int_\chi h^n(y)\mu^n(y)u(y,x)q^n(t_i,y)dy\right) \\
&+ \sum_{i=1}^{\lfloor nt\rfloor} \frac{1}{\overline{W}^n(t_i)} \left(\int_\chi \frac{m(y)}{n}\mu^n(y)u(y,x)q^n(t_i,y)dy\right) - \sum_{i=1}^{\lfloor nt\rfloor} h^n(x)\mu^n(x)q^n(t_i,x) \\
&- (t - \lfloor nt\rfloor) \int_\chi \mu(y)u(y,x)q^n(\lfloor nt\rfloor, y)dy - \mu(x)q^n(t_i,x)(t - \lfloor nt\rfloor) \\
&- \overline{m}^n(\lfloor nt\rfloor))q^n(\lfloor nt\rfloor, x)(t - \lfloor nt\rfloor). \qquad (6.4.20)
\end{aligned}$$

We will prove that $H(n,x,t) \to 0$ as $n \to \infty$ uniformly in x and $[0,T]$. We will bound each term of (6.4.20), so let us start with some inequalities that will be useful. By assumption (A1) the function m is bounded, in other words there exists $M \in \mathbb{R}_+$ such that $|m(x)| \le M$ for all $x \in \chi$. Then by (6.4.10) we have:

$$e^{-M/n} \le |W^n(x)| \le e^{M/n}. \tag{6.4.21}$$

Now using that $q^n \le 1$, we have for each t_i with $i = 0,1,\dots,\lfloor nT \rfloor + 1$ the following:

$$\begin{aligned} \bar{W}^n(t_i) &= \int_\chi W^n(x) q^n(t_i,x)dx \ge e^{-M/n} \qquad \text{and,} \\ \bar{m}^n(t_i) &= \int_\chi m^n(x) q^n(t_i,x)dx \le M. \end{aligned} \tag{6.4.22}$$

We also have using (6.4.14) that

$$|h^n(x)| \le \int_0^{\frac{m(x)}{n}} e^t \left(\frac{M}{n}\right) dt \le \left(\frac{M}{n}\right)^2 e^{M/n}, \tag{6.4.23}$$

and for each t_i with $i = 0,1,\dots,\lfloor nT \rfloor + 1$ we have

$$|\overline{h}^n(t_i)| \le \int_\chi |h^n(x)| q^n(t_i,x)dx \le \left(\frac{M}{n}\right)^2 e^{M/n}. \tag{6.4.24}$$

And finally using (6.4.21), (6.4.22), (6.4.23), and (6.4.24) we have that for each t_i with $i = 0,1,\dots,\lfloor nT \rfloor + 1$:

$$\begin{aligned} |C(t_i,n)| &= \left| \frac{n\overline{m}^n(t_i) - n^2\overline{h}^n(t_i)}{n^2\overline{W}^n(t_i)} \right| \\ &\le \frac{M}{n} e^{M/n} + \frac{M^2}{n^2} e^{2M/n}. \end{aligned}$$

We will proceed to bound each term in (6.4.20).

1. First term.-

$$\begin{aligned} \left| \sum_{i=1}^{\lfloor nt \rfloor} C(t_i,n)(m(x) - \overline{m}^n(t_i)) \frac{q^n(t_i,x)}{n} \right| &\le \sum_{i=1}^{\lfloor nt \rfloor} \frac{2M}{n} \left(\frac{M}{n} e^{M/n} + \frac{M^2}{n^2} e^{2M/n} \right) \\ &\le 2M \left(\frac{M}{n} e^{M/n} + \frac{M^2}{n^2} e^{2M/n} \right) T. \end{aligned}$$

2. Second term.-

$$\begin{aligned} \left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)} (h^n(x) - \overline{h}^n(t_i)) q^n(t_i,x) \right| &\le \sum_{i=1}^{\lfloor nt \rfloor} 2e^{M/n} \left(\frac{M}{n}\right)^2 e^{M/n} \\ &\le 2M \frac{M^2}{n} e^{2M/n} T. \end{aligned}$$

3. Third term.-

$$\begin{aligned}\left|\sum_{i=1}^{\lfloor nt\rfloor} C(t_i,n)\left(\int_{\chi}\mu^n(y)u(y,x)q^n(t_i,y)dy\right)\right| &\leq \frac{1}{n}\left(\frac{M}{n}e^{M/n}+\frac{M^2}{n^2}e^{2M/n}\right)\int_{\chi}q^n(t_i,y)dy\\ &\leq \left(\frac{M}{n^2}e^{M/n}+\frac{M^2}{n^3}e^{2M/n}\right)T.\end{aligned}$$

4. Fourth term.-

$$\left|\sum_{i=1}^{\lfloor nt\rfloor}\frac{m(x)}{n}\mu^n(x)q^n(t_i,x)\right| \leq \sum_{i=1}^{\lfloor nt\rfloor}\frac{M}{n^2} \leq \frac{M}{n}T.$$

5. Fifth term.-

$$\begin{aligned}\left|\sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{\overline{W}^n(t_i)}\left(\int_{\chi}h^n(y)\mu^n(y)u(y,x)q^n(t_i,y)dy\right)\right| &\leq \sum_{i=1}^{\lfloor nt\rfloor}\left(\frac{M}{n}\right)^2\frac{1}{n}\int_{\chi}q^n(t_i,y)dy\\ &\leq \left(\frac{M}{n}\right)^2 T.\end{aligned}$$

6. Sixth term.-

$$\begin{aligned}\left|\sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{\overline{W}^n(t_i)}\left(\int_{\chi}\frac{m(y)}{n}\mu^n(y)u(y,x)q^n(t_i,y)dy\right)\right| &\leq \sum_{i=1}^{\lfloor nt\rfloor}e^{M/n}\frac{M}{n^2}\int_{\chi}q^n(t_i,y)dy\\ &\leq e^{M/n}\frac{M}{n}T.\end{aligned}$$

7. Seventh term.-

$$\begin{aligned}\left|\sum_{i=1}^{\lfloor nt\rfloor}h^n(x)\mu^n(x)q^n(t_i,x)\right| &\leq \sum_{i=1}^{\lfloor nt\rfloor}\frac{1}{n}\left(\frac{M}{n}\right)^2 e^{M/n}\\ &\leq \left(\frac{M}{n}\right)^2 e^{M/n}T.\end{aligned}$$

8. Eighth term.-

$$\begin{aligned}\left|(t-\lfloor nt\rfloor)\int_{\chi}\mu(y)u(y,x)q^n(\lfloor nt\rfloor,y)dy\right| &\leq \frac{1}{n}\int_{\chi}q^n(\lfloor nt\rfloor,y)dy\\ &\leq \frac{1}{n}.\end{aligned}$$

9. Ninth term.-

$$|\mu(x)q^n(t_i,x)(t-\lfloor nt\rfloor)| \leq \frac{1}{n}.$$

10. Tenth term.-

$$|\overline{m}^n(\lfloor nt\rfloor))q^n(\lfloor nt\rfloor,x)(t-\lfloor nt\rfloor)|\leq\frac{M}{n}.$$

So finally using the bounds for each of the terms in (6.4.20) we just computed, we can conclude that there exists a constant $C(M,T)\geq 0$ such that:

$$|H(n,x,t)|\leq\frac{C(M,T)}{n}. \tag{6.4.25}$$

Now for any bounded function $f:\chi\times[0,T]\to\mathbb{R}$ we define:

$$\|f\|(s)=\sup_{x\in\chi}|f(x,s)|.$$

We will prove that for a fixed $t\in[0,T]$, the sequence $(q^n(\cdot,t))_{n\geq 0}$ is Cauchy uniformly in x, in other words that

$$\lim_{n,m\to\infty}\|q^n-q^m\|(s)=0.$$

So using (6.4.19) we have the following

$$\begin{aligned}
&q^n(s,x)-q^l(s,x)=\int_0^t m(x)(q^n(s,x)-q^l(s,x))ds\\
&+\int_0^t\int_\chi \mu(y)u(y,x)(q^n(s,y)-q^l(s,y))dyds\\
&-\int_0^t \mu(x)(q^n(s,x)-q^l(s,x))ds+\int_0^t \overline{m}^n(s)(q^n(s,x)-q^l(s,x))ds\\
&+\int_0^t(\overline{m}^n(s)-\overline{m}^l(s))q^l(s,x)ds+(H(n,x,t)-H(l,x,t))\\
&=\int_0^t m(x)(q^n(s,x)-q^l(s,x))ds\\
&+\int_0^t\int_\chi \mu(y)u(y,x)(q^n(s,y)-q^l(s,y))dyds\\
&-\int_0^t \mu(x)(q^n(s,x)-q^l(s,x))ds+\int_0^t m(x)q^n(s,x)(q^n(s,x)-q^l(s,x))ds\\
&+\int_0^t\int_\chi m(x)(q^n(s,x)-q^l(s,x))q^l(s,x)ds+(H(n,x,t)-H(l,x,t)).
\end{aligned} \tag{6.4.26}$$

Recalling the fact that m is bounded, then there exists a constant M, such that $m(x)\leq M$ for all $x\in\chi$. So from (6.4.25), (6.4.26), and using assumption (A3) we obtain

$$|q^n(s,x)-q^l(s,x)|\leq M\int_0^t\|q^n-q^l\|(s)ds+\int_0^t\int_\chi \mu(y)u(y,x)\|q^n-q^l\|(s)dyds$$

$$
\begin{aligned}
&- \int_0^t \mu(x)\|q^n - q^l\|(s)ds + M \int_0^t q^n(s,x)\|q^n - q^l\|(s)ds \\
&+ M \int_0^t \int_\chi \|q^n - q^l\|(s) q^l(s,x)ds + |H(n,x,t) - H(l,x,t)| \\
&\leq (3M + D + 1) \int_0^t \|q^n - q^l\|(s)ds + C(M,T) \left| \frac{1}{n} - \frac{1}{l} \right|.
\end{aligned}
$$

Then by an application of Gromwall's Lemma, it follows

$$
\|q^n - q^l\|(t) \leq C(M,T) \left| \frac{1}{n} - \frac{1}{l} \right| e^{(3M+D+1)T},
$$

So from the above inequality we have

$$
\lim_{n,l\to\infty} \|q^n - q^l\|(t) = 0.
$$

We can then conclude that, for each $t \geq 0$ and $x \in \chi$, the sequence $(q^n(t,x))_{n\geq 0}$ is Cauchy, so it exists $q(t,x) \in \mathbb{R}$ such that

$$
\lim_{n\to\infty} q^n(t,x) = q(t,x). \tag{6.4.27}
$$

Let us consider the function $q : [0,t] \times \chi \to \mathbb{R}$, obtained in (6.4.27); we will prove that it is solution to a differential equation. First we see using assumption (A1) that

$$
\int_\chi m(x) q^n(s,x)dx \leq \int_\chi m(x)dx < \infty, \tag{6.4.28}
$$

so using (6.4.28) and an application of The Dominated Convergence Theorem we obtain:

$$
\overline{m}(s) = \int_\chi m(x)q(s,x)dx = \lim_{n\to\infty} \int_\chi m(x)q^n(s,x)dx.
$$

Now we dominate each of the integrals in (6.4.19) in the following form

$$
\begin{aligned}
\int_0^t (m(x) - \overline{m}^n(s))q^n(s,x)ds &\leq t\Big(M + \int_\chi m(x)dx\Big), \\
\int_0^t \int_\chi \mu(y)u(y,x)q^n(s,y)dyds &\leq \int_0^t \int_\chi \mu(y)u(y,x)dyds, \\
\int_0^t \mu(x)q^n(s,x)ds &\leq \int_0^t \mu(x)ds.
\end{aligned} \tag{6.4.29}
$$

The fact that m is integrable with respect to the Lebesgue measure in χ, that m, q and μ are bounded, and that $\int_\chi u(y,x)dx \leq 1$, implies that each of the integrals in (6.4.29) are finite. Therefore by taking limits in (6.4.19) and applying The

Dominated Convergence Theorem we obtain in an integral form the differential equation satisfied by q:

$$
\begin{aligned}
q(t,x) = \lim_{n\to\infty} q^n(t,x) &= \lim_{n\to\infty}\left(q^n(0,x) + \int_0^t (m(x) - \overline{m}^n(s))q^n(s,x)ds\right) \\
&+ \lim_{n\to\infty}\left(\int_0^t\int_\chi \mu(y)u(y,x)q^n(s,y)dyds - \int_0^t \mu(x)q^n(s,x)ds + C(n,x,t)\right) \\
&= q(0,x) + \int_0^t (m(x) - \overline{m}(s))q(s,x)ds \\
&+ \int_0^t\int_\chi \mu(y)u(y,x)q(s,y)dyds - \int_0^t \mu(x)q(s,x)ds. \qquad (6.4.30)
\end{aligned}
$$

□

On the other hand let us consider for each $n \in \mathbb{N}$ and $t \in [0,T]$ the probability measures η_t^n, η_t with densities $q^n(t,\cdot)$, and $q(t,\cdot)$ respectively.

From (6.4.12) we know that for for fixed $n \in \mathbb{N}$, η_t^n has the following form

$$
\eta_t^n = \sum_{i=0}^{\lfloor nT\rfloor} \eta_{t_i}^n 1_{[t_i,t_{i+1})}(t), \qquad (6.4.31)
$$

so we can consider (6.4.31) as a function $\eta^n : [0,T] \to M_1(\chi)$, which is constant in each interval $[t_i, t_{i+1})$ and jumps at time t_{i+1}, for $i = 0, 1, \ldots, \lfloor nT\rfloor$, according to the probability transitions defined in (6.3.7).

Using (6.4.27) we know that for fixed $t \in [0,T]$, the sequence of densities $(q^n(t,\cdot))_{n\in\mathbb{N}}$ converges point-wise to the density $q(t,\cdot)$ as $n \to \infty$. So let us fix $t \in [0,T]$ and consider a bounded measurable function $f \in B_b(\chi)$ such that $\|f\|_\infty \leq 1$, then it is easy to see that

$$
\begin{aligned}
|\langle \eta_t^n, f\rangle - \langle \eta_t, f\rangle| &\leq \left|\int_\chi f(x)(q^n(t,x) - q(t,x))dx\right| \\
&\leq \int_\chi |q^n(t,x) - q(t,x)|dx.
\end{aligned}
$$

Therefore if we consider the variation norm defined for μ_1 and μ_2 in $M_1(\chi)$ by

$$
\|\mu_1 - \mu_2\| = \sup_{f\in B_b(\chi), \|f\|_\infty \leq 1} |\langle \mu_1, f\rangle - \langle \mu_2, f\rangle|,
$$

then by an application of Scheffé's Theorem we have the following result

$$
\lim_{n\to\infty} \|\eta_t^n - \eta_t\| \leq \lim_{n\to\infty}\int_\chi |q^n(t,x) - q(t,x)|dx = 0.
$$

Therefore the sequence of probability measures $(\eta_t^n)_{n\in\mathbb{N}}$ converges, for fixed $t \in [0,T]$, in the variation norm in $M_1(\chi)$ to the probability measure η_t as $n \to \infty$.

In the next section we will see that the flow of measures η_t for $t \in [0,T]$ generated by the densities $q(t,\cdot)$ satisfies the continuous time Feynman-Kac formulae.

6.5 The continuous time model

In the previous section we obtained the model (6.4.30) as an approximation in continuous time for the discrete time model (6.2.2). In this section we will link this model with the continuous time Feynman-Kac formulae; in fact we will prove that the flow of measures generated by the densities $q(t,x)$ satisfies the continuous time Feynman-Kac formulae.

We recall that the dynamics of the density of traits in continuous time satisfies the following differential equation

$$\frac{dq(t,x)}{dt} = \left(m(x) - \int_\chi m(y)q(t,y)dy\right) + \int_\chi \mu(y)u(y,x)q(t,y)dy - \mu(x)q(t,x), \tag{6.5.32}$$

now writing the weak form of (6.5.32), we obtain for a function $f \in C_b(\chi)$

$$\begin{aligned}
&\int_\chi f(x)q(t,x)dx - \int_\chi f(x)q(0,x)dx \\
&\quad = \int_0^t \int_\chi f(x)\left(m(x) - \int_\chi m(y)q(s,y)\right) q(x,s)dydxds \\
&\quad + \int_0^t \int_{\chi\times\chi} f(x)\mu(y)u(y,x)q(s,y)dydxds - \int_0^t \int_\chi f(x)\mu(x)q(t,x)dxds \\
&\quad = \int_0^t \int_{\chi\times\chi} f(y)m(y)q(s,x)q(s,y)dydxds - \int_0^t \int_{\chi\times\chi} f(x)m(y)q(s,y)q(s,x)dydxds \\
&\quad + \int_0^t \int_{\chi\times\chi} f(x)\mu(y)u(y,x)q(s,y)dydxds - \int_0^t \int_{\chi\times\chi} f(y)\mu(y)u(y,x)q(s,y)dydxds \\
&\quad = \int_0^t \int_{\chi\times\chi} (f(y)-f(x))m(x)q(s,x)q(s,y)dydxds \\
&\quad + \int_0^t \int_{\chi\times\chi} (f(x)-f(y))\mu(y)u(y,x)q(s,y)dydxds.
\end{aligned}$$

Now if we consider the measure $\eta_s \in M_F(\chi)$ with density $q(s,x)$ then (6.5.33) becomes

$$\begin{aligned}
\int_\chi f(x)\eta_t(dx) = \int_\chi f(x)\eta_0(dx) &+ \int_0^t \int_{\chi\times\chi} (f(x)-f(y))\mu(y)u(y,x)\eta_s(dy)dxds \\
&+ \int_0^t \int_{\chi\times\chi} (f(y)-f(x))m(x)\eta_s(dy)\eta_s(dx)ds,
\end{aligned}$$

which we can express in the following form

$$\frac{d}{dx}\langle \eta_t, f\rangle = \left\langle \eta_t, \int_\chi (f(x)-f(y))\mu(y)u(y,x)dx + \int_\chi (f(y)-f(x))m(y)u(y,x)\eta_t(dy)\right\rangle, \tag{6.5.33}$$

so by (6.5.33), the flow of measures $(\eta_t)_{t\geq 0}$ satisfies the Feynman-Kac formulae (see p.14 in Del Moral [23]).

Following Del Moral we have the following interacting particle approximation. Let

$$(\xi_t)_{t\geq 0} = (\xi_t^1, \dots, \xi_t^N)_{t\geq 0},$$

be a Markov process taking values in χ^N, where $N \geq 1$ denotes the number of particles.

At time $t \geq 0$ the generator L^N of the particle system is defined as the sum of two generators

$$L^N = L_1^N + L_2^N$$

1. The first generator L_1^N which is called the mutation generator is defined for $\phi(x_1, \dots, x_N) = \phi^1(x_1)\dots\phi^N(x_N)$, with $\phi^i \in C_b(\chi)$ for $i = 1, \dots, N$ as

$$L_1^N(\phi) = \sum_{i=1}^{N} \prod_{j\neq i} \phi^j(x_j) \int_\chi (\phi^i(x_i) - \phi^i(y))\mu(y)u(y,x)dy.$$

2. The second generator L_2^N, which is called the selection generator, is defined for any $\phi(x_1, \dots, x_N) = \phi^1(x_1)\dots\phi^N(x_N)$, with $\phi^i \in C_b(\chi)$ for $i = 1, \dots, N$, and for any $x = (x_1, \dots, x_N) \in \chi^N$ by

$$L_2^N(\phi)(x) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (\phi(x^{i,j}) - \phi(x))m(x_j),$$

where for $1 \leq i, j \leq N$ and $x = (x_1, \dots, x_N) \in \chi^N$, $x^{i,j}$ is the element of χ^N given by

$$\forall\ 1 \leq k \leq N, \qquad x_k^{i,j} = \begin{cases} x_k, & \text{if } k \neq i; \\ x_j, & \text{if } k = i. \end{cases}$$

The following result from Del Moral, Theorem 3.19 in [23], shows that for any $t \geq 0$ the empirical measures given by

$$\eta_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_t^i}(\cdot), \tag{6.5.34}$$

are good approximations of $\eta_t, t \geq 0$, for N large enough.

Theorem 15 (Del Moral [23]). *There exists a finite constant $C_T > 0$, such that for all $\phi \in B_b(\chi)$ and all $0 \leq t \leq T$,*

$$E[|\langle \eta_t^N, \phi\rangle - \langle \eta_t, \phi\rangle|] \leq C_T \frac{\|\phi\|}{\sqrt{N}}$$

where E designates the expectation relative to the process $(\xi_t)_{t\geq 0}$.

The above result allows to give a microscopic interpretation of the continuous time model described by (6.5.32) in terms of the empirical measures $(\eta_t^N)_{t\geq 0}$ given by (6.5.34). We can interpret the empirical measures $(\eta_t^N)_{t\geq 0}$ as the distribution of individuals in a population of size N, where the individuals are characterized by their traits $\xi^i \in \chi$ for $1 \leq i \leq N$. So the population evolves according to the following dynamics:

Between jumps due to the interaction between individuals, at a random time τ a given individual ξ^i_τ, with probability $\mu(\xi^i_\tau)$ (the mutation rate), will be replaced by a new individual with trait randomly chosen according to the mutation law $\mu(y)u(x, y)$. In other words at a random time τ, an individual ξ^i_τ gives birth to an offspring and dies, with probability $1 - \mu(\xi^i_\tau)$ the offspring carries the same trait ξ^i_τ; and with probability $\mu(\xi^i_\tau)$ the offspring is mutated. If a mutation occurs the trait of the offspring is chosen according to the mutation law $u(\xi^i_\tau, y)dy$.

At a different random time σ we introduce a competitive interaction between the individuals, during this stage a given individual ξ^i_σ will be replaced by a new particle ξ^j_σ, $1 \leq j \leq N$, with probability proportional to its "adaptation" (fitness) measured in terms of the malthusian parameter $m(\xi^j_\sigma)$.

So the model described (6.5.32) can be thought as an approximation when the size of the population is very large (when $N \to \infty$) of the empirical measures $(\eta_t^N)_{t\geq 0}$ given by (6.5.34).

Chapter 7

Continuous time approximation for a model of a sexual population

7.1 Introduction

In this chapter we will discuss two deterministic models to describe the dynamics of a sexual population, in discrete and continuous time.

We will also find a continuous time approximation for the discrete time model, using a weak selection hypothesis similar to the one we used in the chapter 6 where we discussed an asexual population.
We will consider a modification for haploid populations of the multilocus model developed by Nagylaki [54] and Nagylaki *et al.* [55], originally used to derive the gametic frequencies under selection and recombination for a diploid population.

The models we present in this section are under the approach of quantitative genetics, we are not interested in the size of the population that we can consider constant, and the fitness is given a priori and depends only on the genotypes.

7.2 Discrete time model

We consider an haploid randomly mating population with discrete nonoverlapping generations, in which two sexes don't need to be distinguished. Selection acts only through differential viabilities, which are constant, although the model can be formulated for frequency -and time- dependent fitness. The number of multiallelic loci, the linkage map (or recombination distribution), and epistasis (the presence of interactions between the genes at different loci), are arbitrary. We suppose that the genetic system consists of l loci and l_k alleles $A^{(k)}_{x_k}$ at locus k, for $x_k = 1, \ldots, l_k$. We use the multi-index $x = (x_1, \ldots, x_l)$ as an abbreviation for the individual $A^{(1)}_{x_1} A^{(2)}_{x_2} \ldots A^{(l)}_{x_l}$. We will call χ to the space of all possible genotypes, and we will write that $x = (x_1, \ldots, x_l) \in \chi$ to say that the individual with genotype $A^{(1)}_{x_1} A^{(2)}_{x_2} \ldots A^{(l)}_{x_l} \in \chi$. We remark that the space of genotypes χ is discrete.

Let us denote the frequency of the genotype x by $p(x)$. Once again this means that if $N(n, x)$ is the number of gametes with trait x, and N is the population size then:

$$p(x) = \frac{N(n, x)}{N(n)}.$$

There are $l_1 \ldots l_l$ different l-locus gametes. Collectively, the gamete frequencies form a vector $\mathbf{p}$, a probability vector in the corresponding simplex. The (marginal) frequency of $A^{(k)}_{x_k}$ in gametes is

$$p^{(k)}_{x_k} = \sum_{x \neq x_k} p_x$$

where the sum runs over all muti-indices x with k-th component fixed as x_k.

Let $W(x)$ denote the fitness of genotype x. We define the mean fitness of the population

$$\overline{W}(p) = \sum_{x \in \chi} W(x) p(x)$$

respectively.

Let us now derive the recursion relations for the gamete frequencies. We define $\{x, y \to z\}$ as the event that the genotype of the offspring of parents with genotypes x and y is z. Then the frequency of gamete z in the next generation is

$$p'(z) = \overline{W}^{-2} \sum_{x,y \in \chi} W(x)W(y)p(y)p(z)P(\{x, y \to z\}), \qquad (7.2.1)$$

because selection acts before recombination, and where $P(x, y \to z)$ is the probability that parents with genotypes x and y give birth to an offspring with genotype z.

Let $\mathbf{L} = \{1, \ldots, l\}$ be the set of all loci and let $\{I, J\}$ be a partition of $\mathbf{L}$, (i.e. $I \cap J = \emptyset$ and $I \cup J = \mathbf{L}$). Assume that for an individual $z = (z_1, \ldots, z_l)$ I is the subset of loci with alleles inherited from one parent, and J is the subset of loci inherited from the other. Since for the moment we are not interested in the parents, the partitions $\{I, J\}$ and $\{J, I\}$ will be identified. Without loss of generality assume that the locus 1 is included in I, and remark that since $J = L \setminus I$ the partition $\{I, J\}$ is completely determined by specifying I . Therefore from now on we will identify the partition $\{I, J\}$ with I. Consider a couple of vectors $x, y \in \chi$ and a partition I of $\mathbf{L}$. We will denote by $x_I y_J$ the vector with coordinates

$$(x_I y_J)_i = \begin{cases} x_i, & \text{if } i \in I; \\ y_i, & \text{if } i \in J. \end{cases}$$

From now on $\mathcal{I}$ will denote all the set of all possible partitions of $\mathbf{L}$, in other words $\mathcal{I} = \{I \subset \mathbf{L} : 1 \in I\}$.

To describe recombination, we first have to talk about *cross-overs* which refers

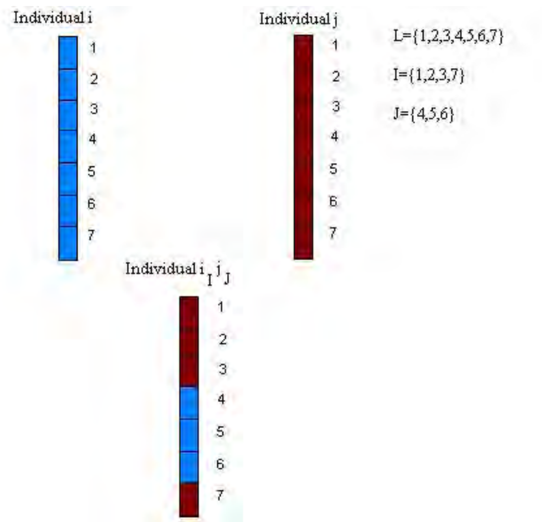


Figure 7.1: Recombination in the model.

to the special kind of exchange of chromosome parts between two chromosomes, in meiosis. Suppose we have a couple of individuals with genotypes x, y, and a partition $I \in \mathcal{I}$, and lets call R_I the event where there is reassociation (cross-overs) of the genes at the loci in I inherited by one of the parents, with the genes at the loci $J = \mathbf{L} \setminus I$ inherited from the other. We denote by r_I the probability of the event R_I, and therefore we have that $\sum_{I \in \mathcal{I}} r_I = 1$.

We remark that the event R_I is in general different from the event that parents with genotypes x and y, give birth to an offspring with genotype $x_I y_J$ or $y_I x_J$. To see this suppose that $x = y$, then the genotype of the offspring z will satisfy that $z = x_I y_J$ and that $z = y_I x_J$ even if there are no cross-overs associated with the partition I.
In order to express $P(\{x, y \to z\})$ in terms of the recombination probabilities r_I, we condition on the events R_I with $I \in \mathcal{I}$, and we obtain the following:

$$P(\{x, y \to z\}) = \sum_{I \in \mathcal{I}} P(\{x, y \to z\} \mid R_I) P(R_I) = \sum_{I \in \mathcal{I}} P(\{x, y \to z\} \mid R_I) r_I. \tag{7.2.2}$$

Now conditioned on R_I, the offspring of parents with genotypes x and y has a genotype given by $x_I y_J$ or $y_I x_j$, each with the same probability, therefore:

$$P(\{x, y \to z\} \mid R_I) = \begin{cases} 1/2, & \text{if } z = x_I y_J; \\ 1/2, & \text{if } z = y_I x_J; \\ 0, & \text{i.o.c.} \end{cases} \tag{7.2.3}$$

So using (7.2.3) in (7.2.2) it follows that:

$$P(\{x, y \to z\}) = \frac{1}{2} \sum_{I \in \mathcal{I}} r_I [1_{\{z = x_I y_J\}} + 1_{\{z = y_I y_J\}}]. \tag{7.2.4}$$

A simple calculation using (7.2.4) shows that

$$\begin{aligned} \sum_{x,y \in \chi} & W(x)W(y)p(x)p(y)P(\{x, y \to z\}) \\ &= \frac{1}{2} \sum_{x \in \chi} \sum_{I \in \mathcal{I}} r_I W(z_I x_J) W(x_I z_J) p(z_I x_J) p(x_I z_J) \\ &+ \frac{1}{2} \sum_{y \in \chi} \sum_{I \in \mathcal{I}} r_I W(z_I y_J) W(y_I z_J) p(z_I y_J) p(y_I z_J) \\ &= \sum_{x \in \chi} \sum_{I \in \mathcal{I}} r_I W(z_I x_J) W(x_I z_J) p(z_I x_J) p(x_I z_J), \end{aligned} \tag{7.2.5}$$

and finally noting that $\sum_{x \in \chi} W(x)W(z)p(x)p(z) = p(z)W(z)\overline{W}$ we can express (7.2.5) as

$$\begin{aligned} \sum_{x,y \in \chi} W(x)W(y)p(x)p(y)P(\{x, y \to z\}) &= W(z)p(z)\overline{W} - \sum_{x \in \chi} \sum_{I \in \mathcal{I}} r_I W(x)W(z)p(x)p(z) \\ &+ \sum_{x \in \chi} \sum_{I \in \mathcal{I}} r_I W(z_I x_J) W(x_I z_J) p(z_I x_J) p(x_I z_J). \end{aligned} \tag{7.2.6}$$

Therefore, substitution of (7.2.6) into (7.2.1) yields the recursion relations for the frequencies of the genotypes under the combined action of selection and recombination:

$$p'(z) = p(z)\frac{W(z)}{\overline{W}} - \theta(z).$$

Here,

$$\theta(z) = \frac{1}{\overline{W}^2} \sum_x \sum_I r_I (W(x)W(z)p(x)p(z) - W(x_I z_J)W(z_I x_J)p(x_I z_I)p(z_I x_J)),$$

represents a measure of linkage disequilibrium in genotype z. (Nagylaki 1992, Chapter 8.2; Nagylaki 1993).

Remark 2. *If in the discrete time model for haploid populations developed by Doebeli in [24], we assume random mating rather than assortive mating, then his model reduces to (7.2.1).*

7.3 A continuous deterministic model for a sexual population

A multilocus model in continuous time was developed and analyzed by Shahshahani [72] and Akin [1]; see also Pasekov [57] and Svirezhev and Passekov [74].

As in the case of the model of Nagylaki we introduced in section 7.2, we will consider a modification for haploid populations of the model of Shahshahani [72] originally used to develop in continuous time the gametic frequencies of a large randomly mating population of diploid organisms evolving solely under the influence of fitness selection and recombination.

Let $q(t,x)$ be the frequency of individuals with genotype x at time $t \geq 0$. This means that if $N(t,x)$ is the number of individuals with trait x at time t, and $N(t)$ is the population size at time t then:

$$q(t,x) = \frac{N(t,x)}{N(t)}.$$

The birth rates of all genotypes are assumed to be equal and we denote them by b, while we use $d(x)$ for the death rate of the genotype x. The model is given by the following system of ordinary differential equations,

$$\frac{dq(t,x)}{dt} = q(t,x)(m(x) - \overline{m}(t)) - \tilde{\theta}(t,x),$$

where $m(x) = b - d(x)$ is the malthusian fitness of the genotype x, $\overline{m}(t)$ is the mean fitness, and

$$\tilde{\theta}(t,x) = \sum_{y\in\chi}\sum_{I\in\mathcal{I}} br_I(q(t,x)q(t,y) - q(t,x_Iy_J)q(t,y_Ix_J)),$$

measures linkage disequilibria in the genotype x.

7.4 Convergence to a continuous time model

In this section we are interested in obtaining a continuous time approximation of the model of Nagylaki [54] we described in section 7.2. The idea is to consider on a time interval $[0,T]$, a series of $\lfloor nT \rfloor$ generations of time length $1/n$, and then accelerate time by making the time length between different generations tend to 0. Between each generation of length $1/n$ we suppose that the population evolves according to (7.2.1) with fitness function W^n.

In order to obtain a continuous time model we need first to define a malthusian fitness parameter m, the relation of this malthusian fitness parameter with the fitness function W^n is given in the following assumption:

Assumption (M)
There exists a bounded function $m : \chi \to \mathbb{R}$.
The relation with the fitness function W^n is given by the following expression, for each $x \in \chi$

$$W^n(x) = \exp\left(\frac{1}{n}m(x)\right). \tag{7.4.7}$$

This idea of performing a weak-selection approximation is similar to the one we explained in (5.1.6). As the time length between generations become smaller, the number of generations on a fixed time T increases, so we suppose that the average number of offspring of each individual (given by the fitness function W^n) decreases. This is the reason behind the renormalization by $1/n$ of the malthusian parameter m in (7.4.7).

We now fix a certain time T, and we divide the interval $[0,T]$ in a series of intervals of length $1/n$ (the length of each generation), therefore in the interval $[0,T]$, we have a total of $\lfloor nT \rfloor$ generations. We denote the time of each generation by $t_i = \frac{m}{n}$ for $i = 0, 1, \dots, \lfloor nT \rfloor + 1$.

We set $q^n(0,x) = q(0,x)$ for all $n \in \mathbb{N}$ (the initial density of individuals with genotype x), and $q^n(t_i,x)$ as the density of individuals with genotype x at generation t_i. This means that if $N(t_i,x)$ is the number of individuals with trait x at the generation t_i, and $N(t_i)$ is the population size at the generation t_i then:

$$q^n(t_i,x) = \frac{N(t_i,x)}{N(t_i)}.$$

And we define the mean fitness at time t_i for $i = 0, 1, \dots, \lfloor nT \rfloor + 1$ by the following expression

$$\overline{W}^n(t_i) = \sum_{x\in\chi} W^n(x) q^n(t_i,x).$$

In order to make the continuous time approximation of (7.2.1), we suppose that for each t_i with $i = 0, 1, \dots, \lfloor nT \rfloor + 1$, $q^n(t_i,x)$ satisfies the following difference equation:

$$q^n(t_{i+1},x) = q^n(t_i,x)\frac{W^n(x)}{\overline{W}^n(t_i)} - \theta^n(t_i), \tag{7.4.8}$$

where

$$\begin{aligned}\theta^n(t_i) = \frac{1}{(\overline{W}^n(t_i))^2} \sum_{y\in\chi}\sum_{I} r_I (&W^n(x)W^n(y)q^n(t_i,x)q^n(t_i,y) \\ &- W^n(x_I y_J)W^n(y_I x_J)q^n(t_i,x_I y_J)q^n(t_i,y_I x_J)),\end{aligned}$$

We fix $t \in [0,T]$ and for $i = 0, 1, \dots, \lfloor nt \rfloor$ we take the difference between the frequencies in each generation for a fixed genotype $x \in \chi$, so using (7.4.8) it follows that:

$$\begin{aligned}&q^n(t_{i+1},x) - q^n(t_i,x) \\ &\qquad = \frac{1}{\overline{W}^n(t_i)}(W^n(x) - \overline{W}^n(t_i))q^n(t_i,x) - \theta^n(t_i).\end{aligned} \tag{7.4.9}$$

For each $n \in \mathbb{N}$ and $x \in \chi$ consider the function $q^n(t,x) : [0,T] \times \chi \to \mathbb{R}$, defined by:

$$q^n(t,x) = \sum_{i=0}^{\lfloor nT \rfloor} q^n(t_i,x) 1_{[t_i,t_{i+1})}(t).$$

The following result allows us to obtain the continuous time approximation of the discrete time model described by (7.4.9).

Theorem 16. *Admit Assumption (M). Then the sequence of functions $\{q^n\}_{n\in\mathbb{N}}$ converges pointwise, as n goes to infinity, to the unique continuous function $q : [0,t] \times \chi \to \mathbb{R}$ satisfying for any $x \in \chi$, and $t \in [0,T]$:*

$$q(0,x) + \int_0^t (m(x) - \overline{m}(s))q(s,x)ds - \int_0^t \theta(s)ds, \tag{7.4.10}$$

where

$$\overline{m}(s) = \sum_{x\in\chi} m(x)q(s,x)dx,$$

and

$$\theta(s) = \sum_{y\in\chi}\sum_{I} r_I(q(s,x)q(s,y) - q(s,x_Iy_J)q(s,y_Ix_J)).$$

Proof. We start by taking the sum of (7.4.9) for $i = 0,1,\ldots,\lfloor nt\rfloor$, and noting that $q^n(t,x)$ is constant in $[\frac{\lfloor nt\rfloor}{n}, t]$ we obtain

$$\begin{aligned} &q^n(t,x) - q^n(0,x) \\ &= \sum_{i=1}^{\lfloor nt\rfloor} \frac{1}{\overline{W}^n(t_i)}(W^n(x) - \overline{W}^n(t_i))q^n(t_i,x) - \sum_{i=1}^{\lfloor nt\rfloor}\theta^n(t_i). \end{aligned}$$

Then making a Taylor expansion of W^n around zero for fixed $x \in \chi$, we obtain

$$W^n(x) = 1 + \frac{1}{n}m(x) + h^n(x), \tag{7.4.11}$$

the function h^n is the residual in Taylor's Theorem and is given by

$$h^n(x) = \int_0^{\frac{m(x)}{n}} \exp(t)(\frac{m(x)}{n} - t)dt. \tag{7.4.12}$$

So using (7.4.11) we have the following expansion for $\overline{W}^n$ with $i = 0,1,\ldots,\lfloor nt\rfloor$,

$$\overline{W}^n(t_i) = 1 + \frac{1}{n}\overline{m}(t_i) + \frac{1}{n^2}\overline{h}(t_i),$$

where

$$\overline{m}^n(t_i) = \sum_{x\in\chi} m(x)q^n(t_i,x)dx,$$

and

$$\overline{h}^n(t_i) = \sum_{x\in\chi} h^n(x)q^n(t_i,x)dx.$$

It is easy to see using the later expansions that

$$q^n(t,x) - q^n(0,x) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)}(m(x) - \overline{m}^n(t_i))\frac{q^n(t_i,x)}{n} + \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)}(h(x) - \overline{h}^n(t_i))q^n(t_i,x) + \sum_{i=1}^{\lfloor nt \rfloor} \widetilde{\theta}^n(t_i), \quad (7.4.13)$$

Now we define the functions $\overline{m}^n, \overline{h}^n, \overline{W}^n : [0,T] \to \mathbb{R}$, as follows,

$$\overline{m}^n(t) = \sum_{i=0}^{\lfloor nT \rfloor} \overline{m}^n(t_i) 1_{[t_i,t_{i+1})}(t),$$
$$\overline{h}^n(t) = \sum_{i=0}^{\lfloor nT \rfloor} \overline{h}^n(t_i) 1_{[t_i,t_{i+1})}(t),$$
$$\overline{W}^n(t) = \sum_{i=0}^{\lfloor nT \rfloor} \overline{W}^n(t_i) 1_{[t_i,t_{i+1})}(t).$$

Proceeding in a similar way as in (6.4.19), we can express (7.4.13) in the following form

$$q^n(t,x) - q^n(0,x) = \int_0^t (m(x) - \overline{m}^n(s))q^n(s,x)ds + \int_0^t \widetilde{\theta}^n(s)ds + H(n,x,t), \quad (7.4.14)$$

where

$$\widetilde{\theta}^n(t) = \sum_{i=0}^{\lfloor nT \rfloor} \theta^n(t_i) 1_{[t_i,t_{i+1})}(t).$$

And

$$H(n,x,t) = \sum_{i=0}^{\lfloor nT \rfloor} C(t_i,n)(m(x) - \overline{m}^n(s))\frac{q^n(t_i,x)}{n} - \widetilde{\theta}^n(\lfloor nT \rfloor)(t - \lfloor nT \rfloor) - (m(x) - \overline{m}^n(\lfloor nT \rfloor))q^n(\lfloor nT \rfloor, x)(t - \lfloor nT \rfloor) + \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)}(h(x) - \overline{h}^n(t_i))q^n(t_i,x), \quad (7.4.15)$$

where for each t_i with $i = 0, 1, \ldots, \lfloor nT \rfloor + 1$ we have:

$$C(t_i,n) = \frac{n\overline{m}^n(t_i) - \overline{h}^n(t_i)}{n^2\overline{W}^n(t_i)}.$$

We will prove that $H(n,x,t) \to 0$ as $n \to \infty$ uniformly in x and $[0,T]$. We will bound each term of (6.4.20), so let us start with some inequalities that will be useful. By assumption (M) the function m is bounded, in other words there exists $m \in \mathbb{R}_+$ such that $|m(x)| \leq m$ for all $x \in \chi$. Then by (6.4.10) we have:

$$e^{-m/n} \leq |W^n(x)| \leq e^{m/n}. \tag{7.4.16}$$

Now using that $q^n \leq 1$, we have for each t_i with $i = 0, 1, \dots, \lfloor nT \rfloor + 1$ the following:

$$\begin{aligned} \bar{W}^n(t_i) &= \int_\chi W^n(x) q^n(t_i, x) dx \geq e^{-m/n} \qquad \text{and,} \\ \bar{m}^n(t_i) &= \int_\chi m^n(x) q^n(t_i, x) dx \leq m. \end{aligned} \tag{7.4.17}$$

We also have using (7.4.12) that

$$|h^n(x)| \leq \int_0^{\frac{m(x)}{n}} e^t \left(\frac{m}{n}\right) dt \leq \left(\frac{m}{n}\right)^2 e^{m/n}, \tag{7.4.18}$$

and for each t_i with $i = 0, 1, \dots, \lfloor nT \rfloor + 1$ we have

$$|h^n(t_i)| \leq \int_\chi |h^n(x)| q^n(t_i, x) dx \leq \left(\frac{m}{n}\right)^2 e^{m/n}. \tag{7.4.19}$$

Also recalling that the genotypic space χ is finite, we denote by $|\chi|$, its cardinality, and we obtain for each t_i with $i = 0, 1, \dots, \lfloor nT \rfloor + 1$:

$$\begin{aligned} |\widetilde{\theta}^n(t_i)| &= \Bigg| \frac{1}{(\overline{W}^n(t_i))^2} \sum_{y \in \chi} \sum_I r_I (W^n(x) W^n(y) q^n(t_i, x) q^n(t_i, y) \\ &\quad - W^n(x_I y_J) W^n(y_I x_J) q^n(t_i, x_I y_J) q^n(t_i, y_I x_J)) \Bigg| \\ &\leq e^{-2m/n} \sum_{y \in \chi} \sum_I r_I 2 e^{2m/n} = 2|\chi|. \end{aligned}$$

And finally using (7.4.16), (7.4.17), (7.4.18), and (7.4.19) we have that for each t_i with $i = 0, 1, \dots, \lfloor nT \rfloor + 1$:

$$|C(t_i, n)| = \left| \frac{n \overline{m}^n(t_i) - \overline{h}^n(t_i)}{n^2 \overline{W}^n(t_i)} \right| \leq \frac{m}{n} e^{-m/n} + \frac{m^2}{n^4}.$$

We will proceed to bound each term in (7.4.15).

1. First term.-

$$\begin{aligned} \left| \sum_{i=0}^{\lfloor nT \rfloor} C(t_i, n)(m(x) - \overline{m}^n(s)) \frac{q^n(t_i, x)}{n} \right| &\leq \sum_{i=0}^{\lfloor nT \rfloor} 2 \frac{m}{n} \left(\frac{m}{n} e^{-m/n} + \frac{m^2}{n^4} \right) \\ &\leq 2mT. \end{aligned}$$

2. Second term.-

$$|(m(x) - \overline{m}^n(\lfloor nT \rfloor))q^n(\lfloor nT \rfloor, x)(t - \lfloor nT \rfloor)| \leq \frac{2m}{n}.$$

3. Third term.-

$$\left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{1}{\overline{W}^n(t_i)}(h(x) - \overline{h}^n(t_i))q^n(t_i, x) \right| \leq \sum_{i=1}^{\lfloor nt \rfloor} 2\left(\frac{m}{n}\right)^2$$
$$\leq \frac{2m^2}{n}T.$$

4. Fourth term.-

$$\left| \widetilde{\theta}^n(\lfloor nT \rfloor)(t - \lfloor nT \rfloor) \right| \leq \frac{2|\chi|}{n}.$$

So finally using the bounds for each of the terms in (7.4.19) we just computed, we can conclude that there exists a constant $C(m) \geq 0$ such that:

$$|H(n, x, t)| \leq \frac{C(m, T)}{n}. \tag{7.4.20}$$

We will prove that for a fixed $t \in [0, T]$, the sequence $(q^n)_{n \geq 0}$ is Cauchy uniformly in x, in other words that

$$\lim_{n,m \to \infty} \|q^n - q^m\|(t) \equiv \lim_{n,m \to \infty} \sup_{x \in \chi} |q^n(t, x) - q^m(t, x)| \equiv 0.$$

So using (7.4.14) we have the following

$$\begin{aligned} q^n(t, x) - q^l(t, x) &= \int_0^t m(x)(q^n(s, x) - q^l(s, x))ds \\ &+ \int_0^t (\overline{m}^n(s)q^n(s, x) - \overline{m}^l(s)q^l(s, x))ds \\ &- \int_0^t (\widetilde{\theta}^n(s) - \widetilde{\theta}^l(s))ds + (H(n, x, t) - H(l, x, t)). \end{aligned} \tag{7.4.21}$$

First we will bound the second integral in expression (7.4.21), its easy to see using the fact that we have a finite space χ, that there exists a constant K_1 not depending on n, l, such that

$$\begin{aligned} \overline{m}^n(s)q^n(s, x) - \overline{m}^l(s)q^l(s, x) &= (\overline{m}^n(s) - \overline{m}^l(s))q^n(s, x) + \overline{m}^l(s)(q^n(s, x) - q^l(s, x)) \\ &= \sum_{y \in \chi} m(y)(q^n(s, y) - q^l(s, y))q^n(s, x) \\ &+ \sum_{y \in \chi} m(y)q^l(s, y)(q^n(s, x) - q^l(s, x)) \\ &\leq K_1 \|q^n - q^l\|(s). \end{aligned} \tag{7.4.22}$$

Proceeding in a similar way, there exists a constant K_2 such that

$$\begin{aligned}\widetilde{\theta}^n(s) - \widetilde{\theta}^l(s) &= \sum_{y\in\chi}\sum_I r_I(q^n(s,x)q^n(s,y) - q^n(s,x_Iy_J)q^n(s,y_Ix_J)) \\ &- \sum_{y\in\chi}\sum_I r_I(q^l(s,x)q^l(s,y) - q^l(s,x_Iy_J)q^l(s,y_Ix_J)) \\ &= \sum_{y\in\chi}\sum_I r_I q^n(s,x)(q^n(s,y) - q^l(s,y)) \\ &+ \sum_{y\in\chi}\sum_I r_I q^l(s,y)(q^n(s,x) - q^l(s,x)) \\ &+ \sum_{y\in\chi}\sum_I r_I q^n(s,x_Iy_J)(q^n(s,y_Ix_J) - q^l(s,y_Ix_J)) \\ &+ \sum_{y\in\chi}\sum_I r_I q^l(s,y_Ix_J)(q^n(s,x_Iy_J) - q^l(s,x_Iy_J)) \\ &\le K_2\|q^n - q^l\|(s). \end{aligned} \tag{7.4.23}$$

It is easy to see using (7.4.20), (7.4.22), and (7.4.23), in (7.4.21), that

$$\begin{aligned} q^n(t,x) - q^l(t,x) \le \left(\sup_{x\in\chi}|m(x)| + K_1 + K_2\right)\int_0^t \|q^n - q^l\|(s)ds \\ + C(m,T)\left|\frac{1}{n} - \frac{1}{l}\right|. \end{aligned}$$

So from the above inequality and by an application of Gromwall's Lemma, it follows that

$$\lim_{n,l\to\infty}\|q^n - q^l\|(t) = 0.$$

We can then conclude that, for each $t \in [0,T]$ and $x \in \chi$, the sequence $(q^n(t,x))_{n\ge 0}$ is Cauchy, so it exists $q(t,x) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} q^n(t,x) = q(t,x). \tag{7.4.24}$$

Lets consider the function $q : [0,T]\times\chi \to \mathbb{R}$, obtained in (7.4.24); we will prove that its solution to a differential equation.

It is no difficult to see that all the terms in the integrals of (7.4.14) are bounded. Indeed this follows from the fact that the space χ is finite, and that $q^n(x,t) \le 1$, for all $x \in \chi$, $t \in [0,T]$ and $n \in \mathbb{N}$.

Moreover by an application of the Dominated Convergence Theorem we can take limits in (7.4.14) and obtain

$$\begin{aligned} q(t,x) = \lim_{n\to\infty} q^n(t,x) &= q(0,x) + \lim_{n\to\infty}\int_0^t (m(x) - \overline{m}^n(s))q^n(s,x)ds \\ &+ \lim_{n\to\infty}\left(\int_0^t \widetilde{\theta}^n(s)ds + C(n,x,t)\right) \\ &= q(0,x) + \int_0^t (m(x) - \overline{m}(s))q(s,x)ds - \int_0^t \theta(s)ds, \end{aligned} \tag{7.4.25}$$

where

$$\overline{m}(s) = \sum_{x \in \chi} m(x) q(s,x) dx = \lim_{n \to \infty} \sum_{x \in \chi} m(x) q^n(s,x) dx,$$

and

$$\theta(s) = \sum_{y \in \chi} \sum_{I} r_I (q(s,x)q(s,y) - q(s,x_I y_J) q(s, y_I x_J)).$$

□

Remark 3. *Equation (7.4.25) is a particular case of the model of Shashahani [72] described in section 7.3, when the birth rates satisfy that $b = 1$. This allows to consider the model of Shashahani (7.4.25) as a weak selection continuous time approximation when $b = 1$, of the model of Nagylaki (7.2.1) for haploid populations.*

Chapter 8

Individual-based probabilistic model for sexual populations

8.1 Introduction

In this paper we are interested in modeling the evolution of a sexual population, taking into account genotypic variation given by recombination and natural selection through the competition between the individuals. To this end we use a microscopic approach, i.e. we focus on the stochastic evolution of each individual, we use a multilocus genetics model to describe recombination which is a fundamental factor in the variability of sexual populations.
The second objective of this work is to understand the phenomena of sympatric speciation, which remains an important problem in biology. By sympatric speciation we mean the apparition of new species without any geographical isolation between them. The study of sympatric speciation has gained much attention in the last few years and several discrete time macroscopic models have been proposed (Doebeli [24], [25]), (Bürguer [12], [13]), but without any real justification from the microscopic point of view. The advantage of our approach is that we can understand better the biological assumptions under which our model is based.
This work begins with the microscopic description of a population with a finite number of individuals, in which each individual is characterized by its genotype, which is described by a vector of allelic values for a given number of loci. Following [19] the dynamics of the population is modeled by a Markov point process, in which the stochastic dynamics of the population is described in continuous time by its generator taking into account recombination in the birth of a new individual, and death as influenced by the traits of an individual and its interaction with the rest of the population. The nature of recombination relies in the

fact that when an individual is born its genotype will be a random combination of the allelic values of the genotypes of the parents. We refer to the state space formed by all the possible vector of allelic values as the space of genotypes, and we will call genotype to the vector of allelic values. We also incorporate assortive mating in our model by assuming that probability of mating between a couple of individuals increases by the similarity of their phenotypes.
The process will be a solution to a stochastic differential equation driven by Poisson random measures (section 8.2). We will give an algorithmic construction of the process in section 8.3, that we will help us to realize certain simulations, for different parameters. Next in section 8.4 we will prove that the point population process is a measure-valued semimartingale and we will compute some of its characteristics. Then we will work on a particular renormalization of the point process based on a large population limit, to obtain a deterministic integro-differential equation (section 8.5), which in the case that the total number of possible phenotypes is finite or infinite, reduces to the models obtained in [72] and [25] respectively (section 8.6).

8.2 Population point process

In the following we will consider an haploid population in which a given trait value (skin color, eye color, sex, etc.) is determined additively by l diallelic loci.

We model the evolution of the population as a stochastic interacting individual system, where each individual is characterized by a vector of allelic values or genotype, $x = (x_1, \dots, x_l)$, where the coordinate x_i, gives the value of the allele at the ith loci for $i = 1, \dots, l$. For the particular choice of the space of all possible genotypes χ we will consider two cases:

- We will consider $\chi = \{0,1\}^l$, in the case that the number of total possible phenotypes is finite (see [24]), like eye color or the number of abdominal bristles on a fruit fly.
- In the case of quantitative traits, such as body weight, brain volume, i.e. traits which exhibit almost continuous variation, we will consider $\chi = [0,1]^l$.

The trait value of an individual will be determined additively by its vector of allelic values that is to say we have a model without resampling; in other words, for a given individual with genotype, $x = (x_1, \dots, x_l)$, its phenotype or trait, is given by

$$p(x) = \sum_{i=1}^{l} x_i.$$

This allows us to consider the trait space as

- $\chi_p = \{0, 1, \dots, l\}$ if the genotypic space χ is $\{0,1\}^l$,

- or $\chi_p = [0, l]$ if the genotypic space χ is $[0, 1]^l$.

We will consider a model with sexual recombination, and the approach to describe it is the following. The genotype $z \in \chi$ of the offspring of a couple of individuals $x, y \in \chi$ will be randomly chosen according to the probability law $D(x, y; z)\mu(dz)$. For the particular form D and μ we consider depending on χ, two cases:

- Suppose that we have a finite space of genotypes, i.e. we consider that $\chi = \{0, 1\}^l$ (see Section 2). To describe sexual recombination in this example we will need the following.
  Let $\mathbf{L} = \{1, \ldots, l\}$ be the set of all loci and let $\{I, J\}$ be a partition of $\mathbf{L}$, (i.e. $I \cap J = \emptyset$ and $I \cup J = \mathbf{L}$). Assume that for an individual $z = (z_1, \ldots, z_l)$ I is the subset of loci with alleles inherited from one parent, and J is the subset of loci inherited from the other one. Since for the moment we are not interested in the parents, the partitions $\{I, J\}$ and $\{J, I\}$ will be identified. Without loss of generality assume that the locus 1 is included in I, and remark that since $J = L \setminus I$ the partition $\{I, J\}$ is completely determined by specifying I . Therefore from now on we will identify the partition $\{I, J\}$ with I.

  Consider a couple of vectors $x, y \in \chi$ and a partition I of $\mathbf{L}$. We will denote by $x_I y_J$ the vector with coordinates

  $$(x_I y_J)_i = \begin{cases} x_i, & \text{if } i \in I; \\ y_i, & \text{if } i \in J. \end{cases}$$

  From now on $\mathcal{I}$ will denote the set of all possible partitions of $\mathbf{L}$, in other words $\mathcal{I} = \{I \subset \mathbf{L} : 1 \in I\}$.
  For each partition $I \in \mathcal{I}$ we denote by r_I the probability of reassociation of the genes at the loci in I, inherited from one parent, with the genes at the loci J, inherited from the other one (see [11] p. 54-56). In other words r_I is the probability that the genotype of the offspring of a couple of individuals $x, y \in \chi$ is $x_I y_J$, which implies that $\sum_{I \in \mathcal{I}} r_I = 1$.
  We take the measure μ defined on χ as the uniform measure on the space of genotypes χ, in other words

  $$\mu(dz) = \sum_{x \in \chi} \delta_x(dz),$$

  And define
  $$D(x, y; z) = \sum_{I \in \mathcal{I}} r_I 1_{\{x_I y_J\}}(z).$$

- Consider a continuous space of genotypes χ, i.e. $\chi = [0, 1]^l$. In this case, we take as the measure μ the Lebesgue measure in $\mathbb{R}^l$, and $D(x, y; z)$ as the density of a probability measure on χ, for example a Gaussian law with mean equal to $(x + y)/2$ and conditioned to stay on χ (see [25]).

We will denote by $M_F(\chi)$ the set of all finite non negative measures on χ. Let also $\mathcal{M}$ denote the subset of $M_F(\chi)$ consisting in all the finite point measures:

$$\mathcal{M} = \left\{ \sum_{i=1}^{n} \delta_{x^i}, n \geq 0, x_1, \ldots, x_n \in \chi \right\}.$$

Here and below, δ_x denotes the Dirac mass at x. For any $m \in M_F(\chi)$, any measurable function $f \in \chi$, we set $\langle m, f \rangle = \int_\chi f dm$.

Our objective is to study the stochastic process ν_t, taking its values in $\mathcal{M}$, and describing the distribution of individuals and genotypes at time t. We define

$$\nu_t = \sum_{j=1}^{N(t)} \delta_{X_t^i},$$

$N(t) \in \mathbb{N}$ denotes the number of individuals alive at time t, and $X_t^1, \ldots, X_t^{N(t)}$ describing the individuals' genotypes (in χ).

In our model we will introduce assortive mating, assuming that the individuals in the population with similar traits mate more frequently. Several examples of assortive mating can be found in models of sexual populations: Doebeli [24] [25], Bürger [13], Matessi et al. [50], Gavrilets and Boake [30]. There also exists strong evidence that assortive mating occurs in several species in nature, like with the cichlid fishes [5], or the funnel-web spiders [62].

Following [24] we define a mating function $\alpha(p(x), p(y))$, which expresses the probability of mating between a couple of individuals with genotypes $x, y \in \chi$, we remark that the mating function is symmetric and depends only on their phenotypes $p(x), p(y) \in \chi_p$.

Following Doebeli (see [25]), in our model we consider that all the individuals have the same per capita birth rate b. So for a population $\nu = \sum_{i=1}^{I} \delta_{x^i}$ and a couple of genotypes in the population $x^i, x^j \in \chi$, we define a birth rate by

$$b(p(x^i), p(x^j)) = \frac{b\alpha(p(x^i), p(x^j))1_{\{i \neq j\}}}{\sum_{j=1, j\neq i}^{N} \alpha(p(x^i), p(x^j))}, \tag{8.2.1}$$

the indicator function only means that we do not allow an individual to mate with itself, and as we can see from (8.2.1) we have to normalize the birth rate by the total amount of mating the individual x^i participates in, so that if we take the total contribution of the individual x^i to the offspring pool, this is equal to b, in other words

$$\sum_{j=1}^{N} b(p(x^i), p(x^j)) = \sum_{j=1}^{N} \frac{b\alpha(p(x^i), p(x^j))1_{\{i \neq j\}}}{\sum_{j=1, j\neq i}^{N} \alpha(p(x^i), p(x^j))} = b.$$

We also define for a population $\nu = \sum_{i=1}^{I} \delta_{x^i}$ and a genotype $x \in \chi$, a death rate $d(p(x), (U \circ p) * \nu(x))$=$d(p(x), \sum_{i=1}^{I} U(p(x) - p(x^i)))$ of individuals with genotype x. U denotes the interaction kernel affecting mortality. Note that the birth and death rates, and the interaction kernel U depend uniquely on the phenotype of the individuals. And finally let $D(x, y; z)\mu(dz)$ the probability that the offspring of two individuals with genotypes x, y respectively, give birth to an individual with genotype z.

Thus the population has the following dynamics. The initial population is characterized by a (possibly random) counting measure $\nu_0 \in \mathcal{M}$ at time 0. Each couple of individuals with genotypes $x, y \in \chi$ at time t has an exponentially distributed random birth "clock" with parameter $b(p(x), p(y))$. And each individual with genotype $x \in \chi$ has an additional exponentially distributed random death "clock", independent of its associated birth "clocks", with parameter $d(p(x), (U \circ p) * \nu(x))$.

If the birth clock of a couple of individuals with genotypes $x, y \in \chi$ rings, they produce an offspring with a genotype given by z according to the probability law $D(x, y; z)\mu(dz)$. If the death clock of an individual rings, this individual dies and disappears.
When one of this events occurs all clocks are reset to 0.

8.2.1 Construction of the process

Let us justify the construction of a Markov process which follows the dynamics of the population we described in the previous section. We will make the assumption that the trait dependency is at most linear in the death rate. Specifically we will work under the following hypotheses:

Assumptions ($C1$)

There exist constants $\bar{d}, \bar{U}$ such that for each $\nu = \sum_{i=1}^{N} \delta_{x^i}$ and for $x, y \in \chi$,

$$d(p(x),(U \circ p) * \nu(x)) \leq \bar{d}(1 + N),$$

$$U(x) \leq \bar{U}.$$

We assume that there exists a constant $C > 0$ and a function $\bar{D} : \chi \to \mathbb{R}$, such that for each $x, y \in \chi$ these two conditions hold:

$$D(x, y; z) \leq C\bar{D}(z) \qquad \text{and} \qquad \int_{\chi} \bar{D}(z)\mu(dz) = 1. \tag{8.2.2}$$

Let us see that the two conditions in (8.2.2) are satisfied for the particular two possible choices for D and μ at the beginning of section 8.2.

- In the first case we assume that the space of genotypes is finite and given by $\chi = \{0, 1\}^l$. In this case we take $\mu(dz) = \sum_{x \in \chi} \delta_x(dz)$ and the recombination kernel as $D(x, y; z) = \sum_{I \in \mathcal{I}} r_I 1_{\{x_I y_J\}}(z)$. Now let us see that if we take $\overline{D}(z) = 1/2^l$, and $C = 2^l$, this particular choice satisfies (8.2.2).

So let us take $x, y \in \chi$, and consider the following:

$$D(x,y;z) = \sum_{I \in \mathcal{I}} r_I 1_{\{x_I y_J\}}(z) \leq \sum_{I \in \mathcal{I}} r_I = 1 = C\overline{D}(z),$$

and

$$\int_\chi \overline{D}(z)\mu(dz) = \sum_{x \in \chi} \int_\chi \overline{D}(z)\delta_x(dz) = \sum_{x \in \chi} \overline{D}(x) = \sum_{x \in \chi} \sum_{I \in \mathcal{I}} r_I 1_{\{x_I y_J\}}(x)$$
$$= \sum_{I \in \mathcal{I}} r_I = 1.$$

- In the second case the space of genotypes is given by $\chi = [0,1]^l$. In this case we take the measure μ as the Lebesgue measure on χ and the recombination kernel $D(x,y;z)$ as the density of a probability measure on χ, which we assume is continuous for each $x, y \in \chi$. So if we consider $\overline{D}(z) = 1$ and $C = 1$, it is straightforward using the fact that $D(x,y;z)$ is a density, to see that $D(x,y;z) \leq C\overline{D}(z)$ for any $x, y \in \chi$. Now let us check the second condition in (8.2.2):

$$\int_\chi D(x,y;z)\mu(dz) \leq \int_\chi dz = 1.$$

Finally we assume that there exists a constant $\underline{\alpha} \geq 0$ such that for $x, y \in \chi$,

$$\underline{\alpha} \leq \alpha(p(x), p(y)) \leq 1.$$

These assumptions ensure that there exists a constant $\bar{C}$, such that the total event rate, for a population counting measure $\nu = \sum_{i=1}^N \delta_{x^i}$ is bounded by $\bar{C}N(1+N)$.
Let us give a pathwise description of the population process $(\nu_t)_{t\geq 0}$, the following is inspired in the work of Fournier and Méléard [29]. We introduce the following notation.

Definition 10. *Let $\mathbb{N}^* = \mathbb{N} \setminus 0$. Let $H = (H^1, H^2, \dots, H^k, \dots) : \mathcal{M} :\to (\chi)^{\mathbb{N}*}$ be defined by $H(\sum_{i=1}^n \delta_{x^i}) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}, 0, \dots, 0, \dots)$, where $x_{\sigma(1)} \preceq \cdots \preceq x_{\sigma(n)}$, for some arbitrary order on χ. (for example the lexicographic order).*

This function H allows us to overcome the following (purely notational) problem. Choosing a genotype uniformly among all genotypes in a population $\nu \in \mathcal{M}$ consists in choosing i uniformly in $(1, \dots, \langle \nu, 1 \rangle)$, and then in choosing the individual number i (from the arbitrary order point of view). The genotype of such an individual is thus $H^i(\nu)$.
We now introduce the probabilistic elements we will need.

Definition 11. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a (sufficiently large) probability space. On this space, we consider the following three independent random elements:*

- *a $\mathcal{M}$-valued random variable ν_0 (the initial distribution),*
- *a Poisson random measure $M_1(ds, di, dj, dz, d\theta)$ on $[0,\infty)\times\mathbb{N}^*\times\mathbb{N}^*\times\chi\times\mathbb{R}^+$, with intensity measure $ds(\sum_{k\geq 1}\delta_k(di))(\sum_{k\geq 1}\delta_k(dj))$ $\mu(dz)d\theta$ (the birth Poisson measure).*
- *a Poisson random measure $M_2(ds, di, d\theta)$ on $[0,\infty)\times\mathbb{N}^*\times\mathbb{R}^+$, with intensity measure $ds(\sum_{k\geq 1}\delta_k(di))d\theta$ (the death Poisson measure).*

Definition 12. *Assume $(C1)$. A $(\mathcal{F})_{t\geq 0}$-adapted stochastic process $\nu = (\nu)_{t\geq 0}$ is called a population process if a.s., for all $t \geq 0$,*

$$\nu_t = \nu_0 + \int_{[0,t)\times\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{R}^+\times\chi} \delta_z 1_{\{i\leq\langle\nu_{s-},1\rangle\}} 1_{\{j\leq\langle\nu_{s-},1\rangle\}} 1_{\{i\neq j\}}$$

$$1_{\{\theta\leq b\alpha(p(H^i(\nu_{s-})),p(H^j(\nu_{s-})))D(H^i(\nu_{s-}),H^j(\nu_{s-});z)\bar{\alpha}(p(H^i(\nu_{s-})))^{-1}\}} M_1(ds, di, dj, d\theta, dz)$$

$$- \int_{[0,t)\times\mathbb{N}^*\times\mathbb{R}^+} \delta_{H^i(\nu_{s-})} 1_{\{i\leq\langle\nu_{s-},1\rangle\}} 1_{\{\theta\leq d(p(H^i(\nu_{s-})),(U\circ p)*\nu_{s-}(H^i(\nu_{s-}))\}} M_2(ds, di, d\theta), \tag{8.2.3}$$

where

$$\bar{\alpha}(p(H^i(\nu_{s-}))) = \sum_{j=1, j\neq i}^{\langle\nu_{s-},1\rangle} \alpha(p(H^i(\nu_{s-})), p(H^j(\nu_{s-}))). \tag{8.2.4}$$

The indicator functions in θ are related to the rates. As mentioned in section 8.2, in the first indicator function that involves θ we have to normalize the rate of mating between a couple of individuals by the function $\bar{\alpha}$ so that the birth rate of each individual remains constant and equal to b. In other words we normalize by the total amount of mating each individual participates in (see [25]) .
We will obtain now the infinitesimal generator for the population process. The following work is adapted from Fournier and Méléard [29]

Proposition 20. *Assume $(C1)$ and consider a solution $(\nu_t)_{t\geq 0}$ to equation (8.2.3) such that $E[\sup_{t\leq T}\langle\nu_t,1\rangle^2] < +\infty$, for all $T \geq 0$. Then $(\nu)_{t\geq 0}$ is a Markov process. Its infinitesimal generator L is defined for all measurable and bounded mappings $\phi : \mathcal{M} \to \mathbb{R}$, all ν in $\mathcal{M}$, by (8.4.14) below. In particular the law of $(\nu)_{t\geq 0}$ does not depend on the chosen order $\preceq$.*

Proof. The fact that $(\nu)_{t\geq 0}$ is a Markov process is classical. Let us now consider a function ϕ as in the statement. With our notation $\nu_0 = \sum_{i=1}^{\langle\nu_0,1\rangle}\delta_{H^i(\nu_0)}$.
Now using that a.s., $\phi(\nu_t) = \phi(\nu_0) + \sum_{s\leq t}(\phi(\nu_{s-} + (\nu_s - \nu_{s-})) - \phi(\nu s))$, we

obtain

$$\phi(\nu_t) = \phi(\nu_0) + \int_{[0,t)\times\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{R}^+\times\chi} (\phi(\nu_{s-} + \delta_{z)}) - \phi(\nu_{s-}))$$

$$1_{\{\theta \le b\alpha(p(H^i(\nu_{s-})),p(H^j(\nu_{s-})))D(H^i(\nu_{s-}),H^j(\nu_{s-});z)\bar{\alpha}(p(H^i(\nu_{s-})))^{-1}\}}$$

$$1_{\{i\le\langle\nu_{s-},1\rangle\}}1_{\{j\le\langle\nu_{s-},1\rangle\}}1_{\{i\neq j\}}M_1(ds,di,dj,d\theta,dz)$$

$$+ \int_{[0,t)\times\mathbb{N}^*\times\mathbb{R}^+} (\phi(\nu_{s-} - \delta_{H^i(\nu_{s-})}) - \phi(\nu_{s-}))1_{\{i\le\langle\nu_{s-},1\rangle\}}$$

$$1_{\{\theta\le d(p(H^i(\nu_{s-})),(U\circ p)*\nu_{s-}(H^i(\nu_{s-})))\}}M_2(ds,di,d\theta).$$

Taking expectations, we obtain

$$\begin{aligned} E[\phi(\nu_t)] &= E[\phi(\nu_0)] \\ &+ E\Big[\int_0^t \sum_{i=1}^{\langle\nu_s,1\rangle}\sum_{j=1}^{\langle\nu_s,1\rangle}\int_\chi (\phi(\nu_s+\delta_z) - \phi(\nu_s)) \\ &\qquad \frac{b\alpha(p(H^i(\nu_s)),p(H^j(\nu_s)))1_{\{i\neq j\}}}{\sum_{j=1,j\neq i}^{\langle\nu_s,1\rangle}\alpha(p(H^i(\nu_s)),p(H^j(\nu_s)))} D(H^i(\nu_{s-}),H^j(\nu_{s-});z)\mu(dz)ds\Big] \\ &+ E\Big[\int_0^t \sum_{i=1}^{\langle\nu_s,1\rangle}(\phi(\nu_s - \delta_{(H^i(\nu_s))}) - \phi(\nu_s))d(p(H^i(\nu_s)),(U\circ p)*\nu_s(H^i(\nu_s))ds\Big]. \end{aligned} \tag{8.2.5}$$

Now using the conditions in Assumption (C1) and the fact that ϕ is bounded, we have the following:

$$\begin{aligned} &E\Big[\int_0^t \sum_{i=1}^{\langle\nu_s,1\rangle}\sum_{j=1}^{\langle\nu_s,1\rangle}\int_\chi |\phi(\nu_s+\delta_z) - \phi(\nu_s)| \\ &\qquad \frac{b\alpha(p(H^i(\nu_s)),p(H^j(\nu_s)))1_{\{i\neq j\}}}{\sum_{j=1,j\neq i}^{\langle\nu_s,1\rangle}\alpha(p(H^i(\nu_s)),p(H^j(\nu_s)))} D(H^i(\nu_{s-}),H^j(\nu_{s-});z)\mu(dz)ds\Big] \\ &\le 2b\|\phi\|_\infty E\Big[\int_0^t\langle\nu_s,1\rangle ds\Big] \le 2b\|\phi\|_\infty tE\Big[\sup_{s\le t}\langle\nu_s,1\rangle\Big] \\ &\le 2b\|\phi\|_\infty tE\Big[\Big(\sup_{s\le t}\langle\nu_s,1\rangle^2 + 1\Big)\Big] < +\infty, \end{aligned} \tag{8.2.6}$$

where the last term is finite by hypothesis. Now we will bound the death term in (8.2.5), once again using assumptions (C1) we obtain:

$$\begin{aligned}
&E\Big[\int_0^t \sum_{i=1}^{\langle\nu_s,1\rangle} |\phi(\nu_s-\delta_{(H^i(\nu_s))})-\phi(\nu_s)|d(p(H^i(\nu_s)),(U\circ p)*\nu_s(H^i(\nu_s))ds\Big] \\
&\qquad \le 2\|\phi\|_\infty E\Big[\int_0^t \sum_{i=1}^{\langle\nu_s,1\rangle} \bar{d}(1+\langle\nu_s,1\rangle)ds\Big] \\
&\qquad \le 2\bar{d}\|\phi\|_\infty E\Big[\int_0^t (1+\langle\nu_s,1\rangle^2)ds\Big] \\
&\qquad \le 2\bar{d}\|\phi\|_\infty tE\Big[\Big(1+\sup_{s\le t}\langle\nu_s,1\rangle^2\Big)\Big] < +\infty. \qquad (8.2.7)
\end{aligned}$$

So using (8.2.6), (8.2.7) and applying Fubini's Theorem in (8.2.5) we have:

$$\begin{aligned}
&E[\phi(\nu_t)] = E[\phi(\nu_0)] \\
&\quad + \int_0^t E\Big[\sum_{i=1}^{\langle\nu_s,1\rangle}\sum_{j=1}^{\langle\nu_s,1\rangle}\int_\chi (\phi(\nu_s+\delta_z)-\phi(\nu_s)) \\
&\qquad\qquad \frac{b\alpha(p(H^i(\nu_s)),p(H^j(\nu_s)))1_{\{i\neq j\}}}{\sum_{j=1,j\neq i}^{\langle\nu_s,1\rangle}\alpha(p(H^i(\nu_s)),p(H^j(\nu_s)))} D(H^i(\nu_{s-}),H^j(\nu_{s-});z)\mu(dz)\Big]ds \\
&\quad + \int_0^t E\Big[\sum_{i=1}^{\langle\nu_s,1\rangle}(\phi(\nu_s-\delta_{(H^i(\nu_s))})-\phi(\nu_s))d(p(H^i(\nu_s)),(U\circ p)*\nu_s(H^i(\nu_s))\Big]ds.
\end{aligned}$$

Differentiating this expression, evaluating at $t=0$, leads to

$$\begin{aligned}
L\phi(\nu) = &\sum_{i=1}^{\langle\nu,1\rangle}\sum_{j=1}^{\langle\nu,1\rangle}\int_\chi \frac{b\alpha(p(x^i),p(x^j))1_{\{i\neq j\}}}{\sum_{j=1,j\neq i}^{\langle\nu,1\rangle}\alpha(p(x^i),p(x^j))} D(x^i,x^j;z)(\phi(\nu+\delta_z)-\phi(\nu))\mu(dz) \\
&+ \sum_{i=1}^{\langle\nu,1\rangle}(\phi(\nu-\delta_{x^i})-\phi(\nu))d(p(x^i),(U\circ p)*\nu(x^i)). \qquad (8.2.8)
\end{aligned}$$

□

Now we will show the existence and moment properties for the population process.

Theorem 17. *(i)Assume* (1) *and that* $E[\langle\nu_0,1\rangle]<\infty$. *Then the population process defined by (8.2.3) is well defined on* $\mathbb{R}_+$.
(ii)If furthermore for some $p\ge 1$, $E[\langle\nu_0,1\rangle^p]<\infty$, *then for any* $T<\infty$,

$$E[\sup_{s\in[0,T]}\langle\nu_s,1\rangle^p]<\infty. \qquad (8.2.9)$$

Proof. We will first prove (ii). We consider the process $(\nu)_{t\geq 0}$. We introduce for each n the following stopping time $\tau_n = \{t \geq 0, \langle \nu_t, 1\rangle \geq n\}$. Then using Assumption $(C1)$ and dropping the non-positive terms,

$$\sup_{s\in[0,t\wedge\tau_n]} \langle \nu_s, 1\rangle^p \leq \langle \nu_0, 1\rangle^p + \int_{[0,t)\times\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{R}^+\times\chi} ((\langle \nu_s, 1\rangle + 1)^p - \langle \nu_s, 1\rangle^p)$$
$$1_{\{\theta\leq b\alpha(p(H^i(\nu_{s-})),p(H^j(\nu_{s-})))D(H^i(\nu_{s-}),H^j(\nu_{s-});z)\bar{\alpha}(p(H^i(\nu_{s-})))^{-1}\}}$$
$$1_{\{i\leq\langle\nu_{s-},1\rangle\}}1_{\{j\leq\langle\nu_{s-},1\rangle\}}1_{\{i\neq j\}}M_1(ds, di, dj, d\theta, dz),$$

Using the inequality $(1+x)^p - x^p \leq C_p(1 + x^{p-1})$, taking expectations, and noting that $\int_\chi D(x, y; z)\mu(dz) = 1$ for each $x, y \in \chi$, we thus obtain, the value C_p changing from line to line:

$$\begin{aligned} E[\sup_{s\in[0,t\wedge\tau_n]} \langle \nu_s, 1\rangle^p] &\leq E[\langle \nu_0, 1\rangle^p] \\ &+ E\left(\int_0^{t\wedge\tau_n} \sum_{i=1}^{\langle\nu_{s-},1\rangle} \sum_{j=1}^{\langle\nu_{s-},1\rangle} C_p b(1 + \langle \nu_{s-}, 1\rangle^{p-1}) \frac{\alpha(p(x^i), p(x^j))1_{\{i\neq j\}}}{\sum_{j=1,j\neq i}^{\langle\nu,1\rangle} \alpha(p(x^i), p(x^j))} ds\right) \\ &\leq C_p\left(1 + E\left(\int_0^t (1 + \langle \nu_{s\wedge\tau_n}, 1\rangle^p)ds\right)\right). \end{aligned}$$

So we conclude using the Gronwall's Lemma that for any $T < \infty$, there exists a constant $C_{p,T}$, not depending on n, such that

$$E[\sup_{t\in[0,T\wedge\tau_n]} \langle \nu_t, 1\rangle^p] \leq C_{p,T}. \tag{8.2.10}$$

As in Fournier and Méléard [29] we deduce that the sequence τ_n converges a.s. to infinity. Indeed, if this is not the case, we can find $T_0 < \infty$ such that $\epsilon_{T_0} = \mathbb{P}(\sup_n \tau_n < T_0) > 0$, this implies that $E[\sup_{t\in[0,T\wedge\tau_n]}\langle \nu_t, 1\rangle^p] \geq \epsilon_{T_0} n^p$, which leads to a contradiction with (8.2.10). Now using Fatou's Lemma and letting n tend to infinity in (8.2.10) we obtain (8.2.9).

Point (i) is a consequence of Point (ii). Indeed, one builds the solution $(\nu)_{t\leq 0}$ step by step. One only has to check that the sequence of jump instants τ_n goes a.s. to ∞ as n tends to ∞. But this follows from (8.2.9) with $p = 1$. □

8.3 Simulation

8.3.1 Simulation algorithm

In this section we give an algorithmic construction of the population process introduced in (8.2.3), using a variation of the algorithms constructed by Fournier and Méléard in [29] and Champagnat et al. in [19].

Step 0. We start by simulating the initial state ν_0 and set $T_0 = 0$.

Step 1. Then we compute the total event rate, which is given by $m(0) = m_1(0) + m_2(0)$, where

$$m_1(0) = b\langle \nu_0, 1\rangle \qquad m_2(0) = \bar{d}\langle \nu_0, 1\rangle(1 + \langle \nu_0, 1\rangle)$$

Simulate τ_1 as an exponentially distributed random variable, with rate $m(0)$, and we set $T_1 = T_0 + \tau_1$. Then we set $\nu_t = \nu_0$ for $t < T_1$. Then with probability m_1/m_0, and m_2/m_0 choose to go to Step 1.1, or Step 1.2 respectively.

Step 1.1. Choose randomly i and j from $\{1, \ldots, \langle \nu_0, 1\rangle\}^2$. Choose the genotype of the offspring according to the law $D(H^i(\nu_0), H^j(\nu_0); z)\mu(dz)$. Do nothing with probability $1-b\alpha(p(H^i(\nu_0)), p(H^j(\nu_0)))D(H^i(\nu_0), H^j(\nu_0); z)1_{\{i\neq j\}}/C\bar{D}(z)\bar{\alpha}(p(H^i(\nu_0))$ where $\bar{\alpha}$ is given in (8.2.4), otherwise add a new individual to the population with genotype z.

Step 1.2. Choose randomly i from $\{1, \ldots, \langle \nu_0, 1\rangle\}$. Then with probability $1 - d(H^i(\nu_0), (U \circ p) * \nu(H^i(\nu_0)))/\bar{d}(1 + \langle \nu_0, 1\rangle$ do nothing, otherwise remove the ith individual from the population.

Step 2. Compute the total event rate given by $m(T_1) = m_1(T_1) + m_2(T_1)$, where

$$m_1(T_1) = b\langle \nu_{T_1}, 1\rangle \qquad m_2(T_1) = \bar{d}\langle \nu_{T_1}, 1\rangle(1 + \langle \nu_{T_1}, 1\rangle)$$

Simulate τ_2 as an exponentially distributed random variable, with rate $m(T_1)$, and we set $T_2 = T_1 + \tau_2$. Then we set $\nu_t = \nu_{T_1}$ for $t \in [T_1, T_2)$ and so on.

8.3.2 Simulation examples and sympatric speciation

In this section we will show simulations of some particular examples. As mentioned in Section 5.8 the most straightforward scenario for sympatric speciation is characterized by two conditions: assortive mating, and disruptive selection favoring two phenotypes; so we will check in the following simulations that under these two conditions sympatric speciation holds. We will begin by describing the different parameters we will be using in our simulations.

In the following simulations we consider a discrete genotypic space, specifically $\chi = \{0, 1\}^l$. We consider that the population has a constant intrinsic grow rate, which we will denote by b, that does not depend on the particular phenotypic value of the individuals. Following the above discussion we introduce assortive mating, assuming that individuals in the population with similar traits mate more frequently. To be explicit, we choose this mating function as:

$$\alpha(x, y) = \exp(-\theta(x - y)^2), \tag{8.3.11}$$

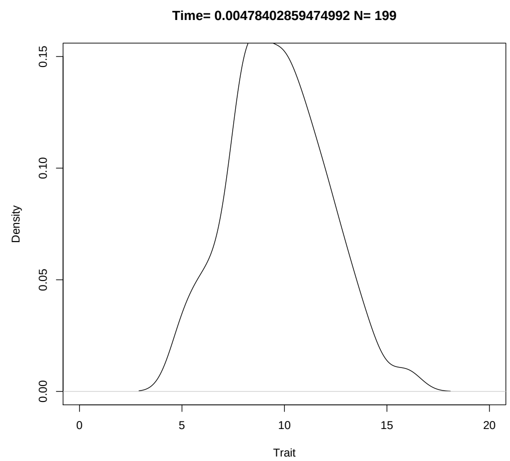


Figure 8.1: Initial density of individuals.

(see for example [13]).
The degree of assortive mating is given by the parameter θ, if the value of θ is high we have a high degree of assortativeness, while taking $\theta = 0$ we obtain random mating, which means that the probability of mating for any couple of individuals in the population is always 1.
To include disruptive selection in our simulations, then following Doebeli [24], we consider a symmetric bimodal resource distribution. For this we use the function

$$R_1(x) = R_0 \cdot \exp\left(\frac{(x - c/4)^2}{2\mu^2}\right)$$

and

$$R_2(x) = R_0 \cdot \exp\left(\frac{(x - 3c/4)^2}{2\mu^2}\right)$$

then we set,

$$R(x) = \min\{R_1(x), R_2(x)\}. \tag{8.3.12}$$

We remark that the two minima of this function are $c/4$ and $3c/4$, this implies that individuals with traits near these two minima will have an ecological advantage over the others by a better exploit of the resources. The parameter μ

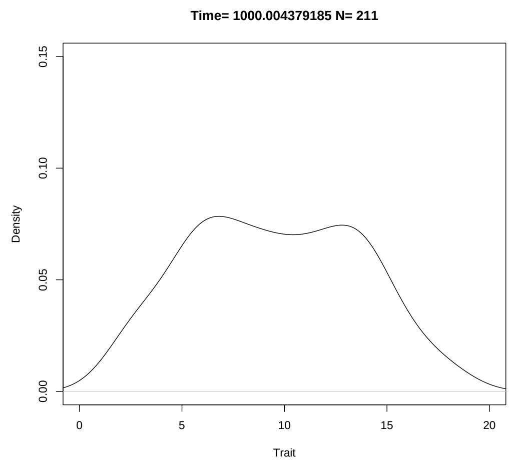


Figure 8.2: $\theta = 0$, Random mating.

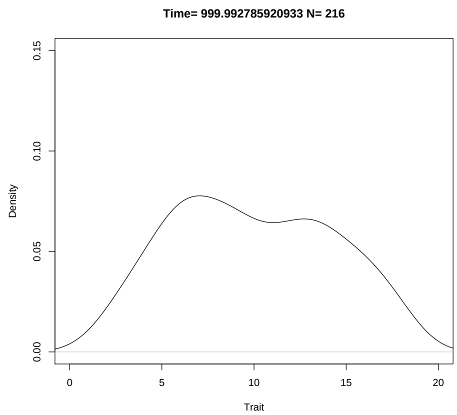
determines the strength of the competition between the individuals. A small value of μ indicates a high level of competition, meanwhile for large values of μ we obtain that the competition is less intense.
Another aim of these simulations is to show that sympatric speciation is a very common phenomenon in nature. To favor the appearance of speciation we will consider symmetric competition instead of asymmetric, in other words we consider the following interaction kernel which is modification of the logistic competition model of Kisdi in [40]:

$$U(x-y) = \frac{2}{K}\left(1 - \frac{1}{1+1.2\exp(-4(x-y)^2)}\right). \tag{8.3.13}$$

The parameter K scales the strength of the competition as well as the population size. Finally the death parameter takes the following form:

$$d(x, U * \nu(x)) = R(x) \cdot \int_{\chi} U(x-y)\nu(dy).$$

- We simulated the population for a lapse of 1000 units of time.



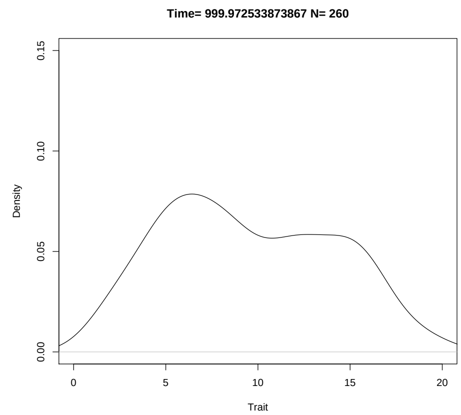


Figure 8.3: $\theta = 0.0001$.

Figure 8.4: $\theta = 0.01$.

- We took the number of loci in the genotypes of the individuals as $l = 20$, so the genotypic space is $\chi = \{0, 1\}^{20}$.
- The birth rate is $b = 5$, and the parameter K in the interaction kernel (8.3.13) is $K = 1$.
- For the bimodal resource distribution we took $K_0 = 1$, $\mu = 0.05$, and $c = 20$ in other words we favor the phenotypes with values 5 and 15.
- For the genotype of the offspring we used the recombination law we described in section 8.2 for a discrete space of genotypes, with $r_I = 2^{19}$ for all $I \in \mathcal{I}$.
- Finally in other to study how assortive mating favors the appearance of speciation, we will vary the parameter θ in (8.3.11), we will show the results of the simulations for $\theta = 0.0001, 0.01, 0.1, 100$ and random mating which is the case when $\theta = 0$.

We simulated an initial distribution of $N = 1000$ individuals. To this end we simulated the value at each loci according to a Bernoulli distribution with parameter 0.5. In Fig. 8.1 we see the plot of the initial density of individuals, which is unimodal, with individuals concentrated around the phenotype with value 10.

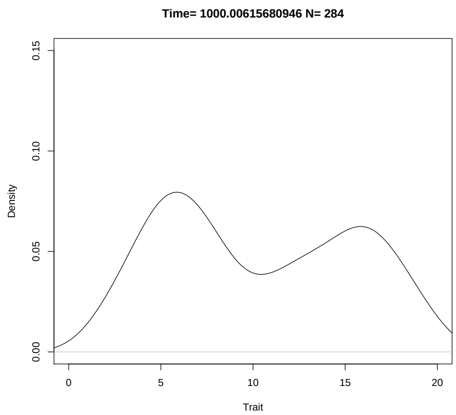


Figure 8.5: $\theta = 1$.

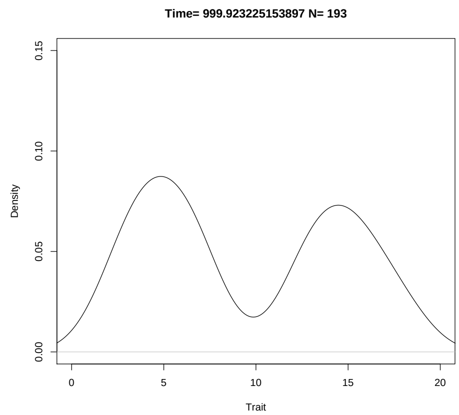


Figure 8.6: $\theta = 100$.

Fig. 8.2 shows the case when $\theta = 0$ which corresponds to random mating, in this case the mating function in (8.3.11) reduces to $\alpha = 1$. As we can see from the picture the population is still concentrated around the phenotype 10, in other words we see no evidence of speciation in this case. In this case even with the bimodal resource distribution the continuous gene flow allowed by random mating does not allow the formation of two different modes.
In the next couple of pictures (Fig. 8.2 and Fig. 8.4) we introduce assortive mating with small values of θ. And we see that even with small amounts of assortment a certain degree of speciation starts to form, but it is not enough to induce the formation of two modes in the density of the population.

Now in Fig. 8.5 we see that the level of bimodality increases with respect to the previous simulations, and finally in Fig. 8.6 we observe that the density of individuals is bimodal and we have two maxima in the density of the population at the phenotypes 5 and 15 with a minimum at 10. We can give the following interpretation, with $\theta = 100$ the degree of assortment is high enough two induce the formation of two modes reproductively isolated of each other, corresponding to the phenotypes 5 and 15. This formation of a bimodal distribution appears typically after 1000 units of time, which implies that it forms relatively fast.
Also we can conclude from the previous pictures that the degree of bimodality increases with the level of assortment, and that in the extreme case of random mating the distribution remains essentially unimodal. So these simulations sup-

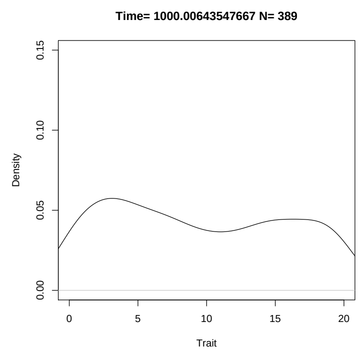


Figure 8.7: No bimodal resource distribution $R = 1$, $\theta = 0.01$.

port the assumption we detailed in Section 5.8 that assortive mating is a key ingredient in the phenomena of sympatric speciation.

Finally Fig. 8.7 shows a simulation with strong assortive mating ($\theta = 100$) but with no bimodal resource distribution, in other words we took the function R in (8.3.12), as $R \equiv 1$. So we have no ecological advantage for any phenotype over the others, unlike the previous cases. Then from the picture we can see that even with strong assortive mating there is no evidence of bimodality in the density of the population. Without disruptive selection favoring two phenotypes, assortive mating is not enough to induce sympatric speciation.

Therefore from the results of these simulations we can conclude that under the presence of strong assortive mating and disruptive selection favoring a couple of phenotypes in the population, sympatric speciation can be a common phenomena in nature. Another important fact is that in our model sympatric speciation occurs rather fast (around 1000 units of time) unlike speciation induced by geographical barriers which requires large periods of time. This is supported by the empirical evidence found by Barluenga et al. [5] for the cichlid fishes.

8.4 Martingale properties

In this section we follow Fournier and Méléard [29] and give the martingale properties of the process $(\nu_t)_{t\geq 0}$, which are fundamental for our approach.
In order to do this, assume that $\nu \in \mathcal{M}$ then for each $x \in \text{supp}(\nu)$ we consider the following function $T^x : \mathcal{M} \to \mathcal{M}$ defined by $T^x(\nu) = \nu - \delta_x$. Then using this function we can write the infinitesimal generator of the process, defined by (8.2.8), in the following form:

$$
\begin{aligned}
L\phi(\nu) = &\sum_{i=1}^{\langle \nu,1\rangle}\sum_{j=1}^{\langle \nu,1\rangle}\int_\chi \frac{b\alpha(p(x^i),p(x^j))D(x^i,x^j;z)}{\sum_{j=1}^{\langle \nu,1\rangle}\alpha(p(x^i),p(x^j)) - \alpha(p(x^i),p(x^i))}(\phi(\nu+\delta_z)-\phi(\nu))\mu(dz)\\
&-\sum_{i=1}^{\langle \nu,1\rangle}\frac{b\alpha(p(x^i),p(x^i))D(x^i,x^j;z)}{\sum_{j=1}^{\langle \nu,1\rangle}\alpha(p(x^i),p(x^i)) - \alpha(p(x^i),p(x^i))}(\phi(\nu+\delta_{x^i})-\phi(\nu))\\
&+\sum_{i=1}^{\langle \nu,1\rangle}(\phi(\nu-\delta_{x^i})-\phi(\nu))d(p(x^i),(U\circ p)*\nu(x^i))\\
=&\int_\chi\int_\chi\int_\chi \frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(\nu_s)(dy)}(\phi(\nu+\delta_z)-\phi(\nu))\mu(dz)T^x(\nu_s)(dy)\nu_s(dx)\\
&+\int_\chi(\phi(\nu-\delta_x)-\phi(\nu))d(p(x),(U\circ p)*\nu_s(x))\nu_s(dx). \qquad (8.4.14)
\end{aligned}
$$

Theorem 18. *Assume (C1), and that for some* $p \geq 2$, $E[\langle \nu_0, 1\rangle] < \infty$.
(i)For all measurable functions ϕ *from* $\mathcal{M}$ *into* $\mathbb{R}$ *such that for some constant* C, *for all* $\nu \in \mathcal{M}$, $|\phi(\nu))| + |L\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^p)$, *the process*

$$\phi(\nu_t) - \phi(\nu_0) - \int_0^t L\phi(\nu_s)ds,$$

is a càdlàg $(\mathcal{F}_t)_{t\geq 0}$*-martingale starting from 0.*
(ii)Point (i) applies to any function $\phi(\nu) = \langle \nu, f\rangle^q$, *with* $0 \leq q \leq p-1$ *and with* f *bounded and measurable on* χ.
(iii) For such a function f, *the process*

$$
\begin{aligned}
M_t^f =&\langle \nu_t, f\rangle - \langle \nu_0, f\rangle\\
&-\int_0^t\int_{\chi\times\chi\times\chi}\frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(\nu_s)(dy)}f(z)\mu(dz)T^x(\nu_s)(dy)\nu_s(dx)ds\\
&+\int_0^t\int_\chi f(x)d(p(x),(U\circ p)*\nu_s(x))\nu_s(dx)ds, \qquad (8.4.15)
\end{aligned}
$$

is a càdlàg square integrable martingale starting from 0 *with quadratic variation* *given by*

$$\langle M^f\rangle_t = \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(\nu_s)(dy)} f^2(z)\mu(dz)T^x(\nu_s)(dy)\nu_s(dx)ds$$
$$+\int_0^t\int_\chi f^2(x)d(x,(U\circ p)*\nu_s(x))\nu_s(dx)ds. \quad (8.4.16)$$

Proof. First of all, note that point (i) is immediate thanks to Proposition 2 and (8.2.9). Point (ii) follows from a straightforward computation using (8.4.14). To prove (iii), we first assume that $E[\langle\nu_0,1\rangle^3]<\infty$. We apply (i) with $\phi(\nu)=\langle\nu,f\rangle$. This yields that M^f is a martingale. To compute its bracket, we first apply (i) with $\phi(\nu)=\langle\nu,f\rangle^2$, and obtain that

$$\langle\nu_t,f\rangle^2-\langle\nu_0,f\rangle^2$$
$$-\int_0^t\int_{\chi\times\chi\times\chi}\frac{b\alpha(p(x),p(y))(2f(z)\langle\nu_s,f\rangle+f^2(z))}{\int_\chi\alpha(p(x),p(y))T^x(\nu_s)(dy)}D(x,y;z)\mu(dz)T^x(\nu_s)(dy)\nu_s(dx)ds$$
$$-\int_0^t\int_\chi(-2f(x)\langle\nu_s,f\rangle+f^2(x))d(x,(U\circ p)*\nu_s(x))\nu_s(dx)ds, \quad (8.4.17)$$

is a martingale. On the other hand we use Itô's formula to compute $\langle\nu_t,f\rangle^2$ from (8.4.15). We obtain

$$\langle\nu_t,f\rangle^2-\langle\nu_0,f\rangle^2$$
$$-\int_0^t 2\langle\nu_s,f\rangle\int_{\chi\times\chi\times\chi}f(z)\frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi\alpha(p(x),p(y))T^x(\nu_s)(dy)}\mu(dz)T^x(\nu_s)(dy)\nu_s(dx)ds$$
$$+\int_0^t 2\langle\nu_s,f\rangle\int_\chi f(x)d(x,(U\circ p)*\nu_s(x))\nu_s(dx)ds-\langle M^f\rangle_t, \quad (8.4.18)$$

is a martingale. Comparing (8.4.17) and (8.4.18) leads to (8.4.16). The extension to the case where only $E[\langle\nu_0,1\rangle^2]<\infty$ is straightforward, since even in this case, $E[\langle M^f\rangle_t]<\infty$ thanks to (8.2.9) with $p=2$. □

8.5 Large-population renormalizations of the individual-based process

In this section we are interested in a particular sequence of renormalizations $(X_t^K)_{K\in\mathbb{N}}$ of the microscopic model for sexual populations introduced in (8.2.3). These series of renormalizations are done in the following manner, we increase the initial size of the population (which is of order K), while decreasing the size of the individuals and the intensity of their interaction (which we assume of order $1/K$). The result of this section is that for each $T>0$ the sequence of renormalized process $(X_t^K)_{K\in\mathbb{N}}$ converges in law in the space $\mathbb{D}([0,T],\mathcal{M}_F(\chi)$

as K goes to infinity to the unique solution $\xi \in \mathcal{C}([0,T], \mathcal{M}_F(\chi))$ of the following equation: for any bounded function $f : \chi \to \mathbb{R}$

$$\langle \xi_t, f\rangle = \langle \xi_0, f\rangle + \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s(dy)} D(x,y;z)\mu(dz)\xi_s(dy)\xi_s(dx)ds$$
$$- \int_0^t \int_\chi f(x)d(x, U * \xi_s(x))\xi_s(dx)ds. \quad (8.5.19)$$

This result is similar to the results due to Fournier and Méléard [29], and to Champagnat and Méléard [19] in the case of asexual populations.
To this end assume that for any K, the set of parameters U_K, V_K, b_K, and d_K, satisfy the Assumption $(C1)$. Let ν_t^K be the counting measure of the population at time t. We define the measure-valued Markov process $(X_t^K)_{t\geq 0}$ by

$$X_t^K = \frac{1}{K}\nu_t^K.$$

As the system goes to infinity we need to assume the following
Assumption $(C2)$:
The parameters U_K, b_K, d_K are all continuous, $\zeta \to d(x,\zeta)$ is Lipschitz for any $x \in \chi$, and

$$U_K(x) = \frac{U(x)}{K}.$$

The biological interpretation of this renormalization is simple, we assume that if the initial number of individuals in the population increases, by a scale factor of K, then the biomass of each individual must decrease by a scale factor of $1/K$. This follows from the fact that the amount of resources in the population is fixed and that they are partitioned among the individuals, so larger populations must be made up of smaller individuals. The renormalization of U_K is related to the decrease of competition for resources, so the parameter K can be thought as scaling the resources available.
The generator L^K of $(\nu_t^K)_{t\geq 0}$ is given by (8.4.14), with parameters U_K, b_K, and d_K. The generator $\overline{L}^K$ of $(X_t^K)_{t\geq 0}$ is obtained by writing, for any measurable function ϕ from $\mathcal{M}(\chi)$ into $\mathbb{R}$ and any $\nu \in \mathcal{M}(\chi)$,

$$\overline{L}^K\phi(\nu) = \partial_t E_\nu[\phi(X_t^K)]_{t=0} = \partial_t E_{K\nu}[\phi(\nu_t^K/K)]_{t=0} = L^K\phi^K(K\nu),$$

where $\phi^K(\mu) = \phi(\mu/K)$. Then we get

$$\overline{L}^K\phi(\nu) = K\int_\chi\int_\chi\int_\chi \frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(K\nu_s)(dy)}(\phi(\nu + \frac{\delta_z}{K}) - \phi(\nu))\mu(dz)T^x(K\nu_s)(dy)\nu_s(dx)$$
$$+ K\int_\chi (\phi(\nu - \frac{\delta_x}{K}) - \phi(\nu))d(p(x), (U\circ p) * \nu_s(x))\nu_s(dx).$$

By a similar proof as the one of section 8.4, we may summarize the moment and martingale properties of X^K.

Proposition 21. *Assume that for some* $p \geq 2$, $E[\langle X_0^K, 1\rangle] < +\infty$.

(i)For any $T > 0$, $E(\sup_{t\in[0,T]}\langle X_t^K, 1\rangle) < +\infty$.

(ii)For all measurable functions ϕ *from* M_F *into* $\mathbb{R}$ *such that for some constant* C, *for all* $\nu \in M_F$, $|\phi(\nu))| + |L^K\phi(\nu)| \leq C(1 + \langle \nu, 1\rangle^p)$, *the process*

$$\phi(X_t^K) - \phi(X_0^K) - \int_0^t L^K\phi(X_s^K)ds,$$

is a càdlàg martingale.

(iii)For each bounded measurable function f, *the process*

$$\begin{aligned} M_t^{K,f} =& \langle X_t^K, f\rangle - \langle X_0^K, f\rangle \\ & - \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x), p(y))D(x, y; z)}{\int_\chi \alpha(p(x), p(y))T^x(K\nu_s)(dy)} f(z)\mu(dz)T^x(K\nu_s)(dy)\nu_s(dx)ds \\ & + \int_0^t \int_\chi f(x)d(p(x), (U\circ p) * \nu_s(x))\nu_s(dx)ds \end{aligned} \tag{8.5.20}$$

is a càdlàg square integrable martingale starting from 0 *with quadratic variation given by*

$$\begin{aligned} \langle M^{K,f}\rangle_t =& \frac{1}{K}\int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x), p(y))D(x, y; z)}{\int_\chi \alpha(p(x), p(y))T^x(K\nu_s)(dy)} f^2(z)\mu(dz)T^x(\nu_s)(dy)\nu_s(dx)ds \\ & + \frac{1}{K}\int_0^t \int_\chi f^2(x)d(x, (U\circ p) * \nu_s(x))\nu_s(dx)ds. \end{aligned} \tag{8.5.21}$$

We obtain the deterministic nature of the approximation by studying the quadratic variation of the martingale term, given in (8.5.21).

8.5.1 Uniqueness of the solution

In the next theorem we will prove the uniqueness of the solution to (8.5.19).

Theorem 19. *Assume Assumptions* $(C1)$ *and* $(C2)$. *Then for an initial condition* $\xi_0 \in \mathcal{M}(\chi)$, *the solution to (8.5.19) is unique in* $C([0, T], M_F(\chi))$. *In other words let* $(\xi_t^1)_{t\in[0,T]}$ *and* $(\xi_t^2)_{t\in[0,T]}$ *be two solutions to (8.5.19) then*

$$\xi_t^1 = \xi_t^2 \qquad \textit{for all } t \in [0, T].$$

Proof. First we consider the variation norm in $\mathcal{M}(\chi)$, defined for μ_1 and μ_2 in $\mathcal{M}(\chi)$ by

$$\|\mu_1 - \mu_2\|_{TV} = \sup_{f\in L^\infty(\overline{\chi}), \|f\|_\infty\leq 1} |\langle \mu_1 - \mu_2, f\rangle|.$$

Note that the solutions to (8.5.19) are continuous and have finite total mass in finite time, to see the later consider $t \in [0,T]$ and $(\xi_t)_{t\in[0,T]}$ a solution to (8.5.19) then

$$\begin{aligned}\langle \xi_t, 1\rangle &= \langle \xi_0, 1\rangle + \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s(dy)} D(x,y;z)\mu(dz)\xi_s(dy)\xi_s(dx)ds \\ &- \int_0^t \int_\chi f(x)d(x, U * \xi_s(x))\xi_s(dx)ds \\ &\leq \langle \xi_0, 1\rangle + b\int_0^t \langle \xi_s, 1\rangle ds,\end{aligned}$$

and using Gronwall's Lemma we conclude that

$$\langle \xi_t, 1\rangle \leq \langle \xi_0, 1\rangle \exp(bt) < \infty. \tag{8.5.22}$$

Now returning to the problem of uniqueness, consider $(\xi_t^1)_{t\in[0,T]}$, and $(\xi_t^2)_{t\in[0,T]}$ two solutions of (8.5.19) with the same initial condition $\xi_0 \in \mathcal{M}(\chi)$.
By (8.5.22) these solutions have a finite total mass for $t \in [0,T]$, so we can assume that $A_T = \sup_{t\in[0,T]}\langle \xi_t^1 + \xi_t^2, 1\rangle < \infty$.
Let f be a bounded measurable function defined in $\overline{\chi}$ such that $\|f\|_\infty \leq 1$, then we obtain

$$\begin{aligned}\langle \xi_s^1 - \xi_s^2, f\rangle &= \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)} D(x,y;z)\mu(dz)\xi_s^1(dy)\xi_s^1(dx)ds \\ &- \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)} D(x,y;z)\mu(dz)\xi_s^2(dy)\xi_s^2(dx)ds \\ &- \int_0^t \int_\chi f(x)d(x, U*\xi_s^1(x))\xi_s^1(dx)ds \\ &+ \int_0^t \int_\chi f(x)d(x, U*\xi_s^2(x))\xi_s^2(dx)ds.\end{aligned} \tag{8.5.23}$$

Now we can express (8.5.23) in the following form

$$\begin{aligned}\langle \xi_s^1 - \xi_s^2, f\rangle &= \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)} D(x,y;z)\mu(dz)(\xi_s^1 - \xi_s^2)(dy)\xi_s^1(dx)ds \\ &+ \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)} D(x,y;z)\mu(dz)\xi_s^2(dy)(\xi_s^1 - \xi_s^2)(dx)ds \\ &- \int_0^t \int_{\chi\times\chi\times\chi} \left(\frac{b\alpha(p(x),p(y))}{\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)} - \frac{b\alpha(p(x),p(y))}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)} \right) \\ &\qquad\qquad f(z)D(x,y;z)\mu(dz)\xi_s^2(dy)\xi_s^1(dx)ds \\ &- \int_0^t \int_\chi f(x)d(x, U*\xi_s^1(x))(\xi_s^1 - \xi_s^2)(dx)ds\end{aligned}$$

$$-\int_0^t \int_\chi f(x)(d(p(x),(U\circ p)*\xi_s^1(x)) - d(p(x),(U\circ p)*\xi_s^2(x)))\xi_s^2(dx)ds. \tag{8.5.24}$$

Let us consider the first integral in (8.5.24), so using Assumption ($C1$) we have the following

$$\left|\int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)} D(x,y;z)\mu(dz)(\xi_s^1-\xi_s^2)(dy)\xi_s^1(dx)ds\right|$$

$$\begin{aligned}
&\leq \int_0^t \int_\chi \left|\int \chi\times\chi \frac{b\alpha(p(x),p(y))|f(z)|}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)} D(x,y;z)\mu(dz)(\xi_s^1-\xi_s^2)(dy)\right|\xi_s^1(dx)ds\\
&\leq b\int_0^t \int_\chi \frac{\|\xi_s^1-\xi_s^2\|_{TV}}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)}\xi_s^1(dx)ds\\
&\leq b\underline{\alpha}^{-1}\int_0^t \|\xi_s^1-\xi_s^2\|_{TV}ds. \qquad (8.5.25)
\end{aligned}$$

Now let us turn our attention to the second integral in (8.5.24), in this case we have

$$\begin{aligned}
&\left|\int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)} D(x,y;z)\mu(dz)\xi_s^2(dy)(\xi_s^1-\xi_s^2)(dx)ds\right|\\
&\qquad\leq b\int_0^t \left|\int_\chi (\xi_s^1-\xi_s^2)(dx)\right| ds\\
&\qquad\leq b\int_0^t \int_\chi \|\xi_s^1-\xi_s^2\|_{TV}ds. \qquad (8.5.26)
\end{aligned}$$

Now proceeding as before we can bound the third integral in (8.5.24) in the following way

$$\left|\int_0^t \int_{\chi\times\chi\times\chi}\left(\frac{b\alpha(p(x),p(y))f(z)D(x,y;z)}{\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)} - \frac{b\alpha(p(x),p(y))f(z)D(x,y;z)}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)}\right)\mu(dz)\xi_s^2(dy)\xi_s^1(dx)ds\right|$$

$$\begin{aligned}
&\leq \int_0^t \int_{\chi\times\chi\times\chi} \frac{b|f(z)|D(x,y;z)\left|\int_\chi \alpha(p(x),p(y))(\xi_s^1-\xi_s^2)(dy)\right|}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)}\mu(dz)\xi_s^2(dy)\xi_s^1(dx)ds\\
&\leq \int_0^t \int_{\chi\times\chi} \frac{b\|\xi_s^1-\xi_s^2\|_{TV}}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)\int_\chi \alpha(p(x),p(y))\xi_s^2(dy)}\xi_s^2(dy)\xi_s^1(dx)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t \int_\chi \frac{b\|\xi_s^1-\xi_s^2\|_{TV}}{\int_\chi \alpha(p(x),p(y))\xi_s^1(dy)}\xi_s^1(dx)ds\\
&\leq b\underline{\alpha}^{-1}\int_0^t \|\xi_s^1-\xi_s^2\|_{TV}ds. \qquad (8.5.27)
\end{aligned}$$

On the other hand by using the hypothesis related to the death term in Assumption ($C1$) on the fourth integral of (8.5.24) we have

$$\begin{aligned}\left|\int_0^t \int_\chi f(x)d(x, U * \xi_s^1(x))(\xi_s^1 - \xi_s^2)(dx)ds\right| &\leq \overline{d}\int_0^t (1 + \langle \xi_s^1, 1\rangle)\|\xi_s^1 - \xi_s^2\|_{Tv}ds \\ &= \overline{d}(1 + A_T)\int_0^t \|\xi_s^1 - \xi_s^2\|_{TV}ds.\end{aligned} \tag{8.5.28}$$

Finally by Assumption ($C2$) we know that the function d is Lipschitz continuous in the second variable, we denote the constant L_d. Therefore we obtain for the last integral in (8.5.24) the following

$$\begin{aligned}&\left|\int_0^t \int_\chi f(x)(d(p(x), (U \circ p) * \xi_s^1(x)) - d(p(x), (U \circ p) * \xi_s^2(x)))\xi_s^1(dx)ds\right| \\ &\quad\leq L_d \int_0^t \int_\chi \left|\int_\chi (U \circ p)(x - y)(\xi_s^1 - \xi_s^2)(dy)\right|\xi_s^1(dx)ds \\ &\quad\leq L_d\overline{U}\int_0^t \int_\chi \|\xi_s^1 - \xi_s^2\|_{TV}\xi_s^1(dx)ds \\ &\quad\leq L_d\overline{U}A_T\int_0^t \|\xi_s^1 - \xi_s^2\|_{TV}ds.\end{aligned} \tag{8.5.29}$$

Then using (8.5.25), (8.5.26), (8.5.27), (8.5.28), and (8.5.29) in (8.5.24) we obtain

$$|\langle \xi_t^1 - \xi_t^2, f\rangle| = (2b\underline{\alpha}^{-1} + b + \overline{d}(1 + A_T) + L_dA_T\overline{U})\int_0^t \|\xi_s^1 - \xi_s^2\|_{TV}ds. \tag{8.5.30}$$

So taking the supremum over all functions f such that $\|f\|_\infty \leq 1$ and applying Gronwall's Lemma we conclude that for all $t \in [0, T]$

$$\|\xi_s^1 - \xi_s^2\|_{TV} = 0. \tag{8.5.31}$$

And therefore uniqueness holds for (8.5.19). □

In the next section we will prove that the sequence of laws $Q^K = \mathcal{L}(X^K)$ is tight in the space $\mathcal{P}(\mathbb{D}([0, T]), \mathcal{M}_F(\chi))$.

8.5.2 Tightness of the sequence of laws $Q^K = \mathcal{L}(X^K)$

In the following we will denote by $(\mathcal{M}_F(\chi), w)$ and by $(\mathcal{M}_F(\chi), v))$ the space of finite measures over χ endowed with the topology of weak and vague convergence respectively.

And in this section we will prove the following:

Theorem 20. *Assume Hypothesis (C1), (C2), and also that*

$$\sup_{K\in\mathbb{N}} E\left(\langle X_0^K, 1\rangle^3\right) < \infty. \tag{8.5.32}$$

Then the sequence of laws $Q^K = \mathcal{L}(X^K)$ *is tight in the space* $\mathcal{P}(\mathbb{D}([0,T]),(\mathcal{M}_F(\chi),w))$.

Proof. First we will start by establishing the tension of the laws of $(X^K)_{K\in\mathbb{N}}$ as probability measures over the space $\mathbb{D}([0,T]),(\mathcal{M}_F(\chi),v)$, in other words when $\mathcal{M}_F(\chi)$ is endowed with the topology of the vague convergence. To achieve this we will use a result established by Roelly ([65], Theorem 2.1). It suffices to prove that for any continuous bounded function over χ the sequence of processes $(\langle X^K, f\rangle)_{K\in\mathbb{N}}$ is tight over $\mathbb{D}([0,T],\mathbb{R})$.

We recall that the martingale part $M^{K,f}$ of $\langle X^K, f\rangle$ is defined in (8.5.20) and we define the finite variation part of $\langle X^K, f\rangle$ by:

$$V_t^{K,f} = -\frac{1}{K}\int_0^t\int_\chi f(x)d(x,(U\circ p)*X_s^K(x))X_s^K(dx)ds$$
$$+\frac{1}{K}\int_0^t\int_{\chi\times\chi\times\chi}\frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(X_s^K)(dy)}f(z)\mu(dz)T^x(X_s^K)(dy)X_s^K(dx)ds.$$

Using Proposition 3 we know that $\langle X^K, f\rangle$ is a semimartingale, and therefore we can use the Aldous-Rebolledo Criterion:

Suppose that Y^n is a square integrable semimartingale, then if we write V_t^n for the corresponding predictable finite variation process and $\langle M^n\rangle_t$ for the quadratic variation of the martingale part, we have the following:

Theorem 21. *(The Aldous-Rebolledo Criterion [2]).- Let* $\{Y^n\}_{n\geq 1}$ *be a sequence of real valued semimartingales with càdlàgs paths. Suppose that the following conditions are satisfied:*

i) For each fixed T, $\{\sup_{t\in[0,T]}|Y_t^n|\}_{n\geq 1}$ *is tight.*

ii) Given a sequence of stopping times τ_n, *bounded by* T, *for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *and* n_0 *such that*

$$\sup_{n\geq n_0}\sup_{\theta\in[0,\delta]}\mathbb{P}\big[|V^n(\tau_n+\theta)-V^n(\tau_n)|>\varepsilon\big]\leq\varepsilon, \tag{8.5.33}$$

and

$$\sup_{n\geq n_0}\sup_{\theta\in[0,\delta]}\mathbb{P}\big[|\langle M^n\rangle_{\tau_n+\theta}-\langle M^n\rangle_{\tau_n}|>\varepsilon\big]\leq\varepsilon. \tag{8.5.34}$$

Then the sequence $\{Y^n\}_{n\geq 1}$ *is tight.*

So proving the conditions in the Aldous-Rebolledo Criterion for the semimartingale $\langle X^K, f\rangle$, will imply the tension of the laws of X^K in $\mathcal{P}(\mathbb{D}([0,T]), \mathcal{M}_F(\chi))$, with $\mathcal{M}_F(\chi)$ endowed with the vague topology.

We will begin by proving the first point, it suffices to show that

$$\sup_K E\Big[\sup_{t\in[0,T]}\langle X_t^K,1\rangle^3\Big]<\infty. \tag{8.5.35}$$

Let us begin by recalling that $X^K=\frac{1}{K}\nu_t^k$ so following step by step the proof of Theorem 2 (ii), with $p=3$ it is easy to see that there exists a constant $C_T>0$, that does not depend on K such that

$$E\Big[\sup_{t\in[0,T]}\langle\nu_t^K,1\rangle^3\Big]\le C_T E[\sup_{t\in[0,T]}\langle\nu_0^K,1\rangle^3]. \tag{8.5.36}$$

So if we divide both terms in (8.5.36) by K^3, and use the fact that by hypothesis $\sup_K E[\sup_{t\in[0,T]}\langle X_0^K,1\rangle^3]<+\infty$, we easily conclude (8.5.35).
The next step is to prove the tightness of the laws of the martingale part and of the drift part of the semimartingale $\langle X^K,f\rangle$.
To this end consider $\delta\ge 0$ and a sequence of stopping times $(\tau_n)_{n\ge1}$ such that $0\le\tau_n\le T$ for all $n\ge1$, then taking $\theta\in[0,\delta]$ and using Proposition 3 we obtain

$$\begin{aligned}
&E[\langle M^{K,f}\rangle_{\tau_n+\theta}-\langle M^{K,f}\rangle_{\tau_n}]\\
&\quad=E\Big[\frac{1}{K}\int_{\tau_n}^{\tau_n+\theta}\int_{\chi\times\chi\times\chi}\frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi\alpha(p(x),p(y))T^x(X_s^K)(dy)}f^2(z)\mu(dz)T^x(X_s^K)(dy)X_s^K(dx)ds\\
&\quad+\frac{1}{K}\int_{\tau_n}^{\tau_n+\theta}\int_\chi f^2(x)d(x,(U\circ p)*X_s^K(x))X_s^K(dx)ds\Big]\\
&\quad\le\frac{1}{K}\|f\|_\infty^2E\Big[b\int_{\tau_n}^{\tau_n+\theta}\langle X_s^K,1\rangle ds+\bar d\int_{\tau_n}^{\tau_n+\theta}\int_\chi(1+\langle X_s^K,1\rangle)X_s^K(dx)ds\Big]\\
&\quad\le\frac{1}{K}\|f\|_\infty^2E\Big[b\int_{\tau_n}^{\tau_n+\theta}\langle X_s^K,1\rangle ds+\bar d\int_{\tau_n}^{\tau_n+\theta}(\langle X_s^K,1\rangle+\langle X_s^K,1\rangle^2)ds\Big]\\
&\quad\le\frac{2}{K}\|f\|_\infty^2(b+\bar d)E\Big[\theta(1+\sup_{s\in[0,T]}\langle X_s^K,1\rangle^2)\Big]\\
&\quad\le 2\|f\|_\infty^2(b+\bar d)\theta\Big(1+\sup_K E\big[\sup_{s\in[0,T]}\langle X_s^K,1\rangle^2\big]\Big)\\
&\quad\le C_f\delta
\end{aligned}\tag{8.5.37}$$

where in the last inequality we used (8.5.35). Therefore using (8.5.37) we have

$$\begin{aligned}
\sup_n\sup_{\theta\in[0,\delta]}\mathbb{P}\big[|\langle M^{K,f}\rangle_{\tau_n+\theta}-\langle M^{K,f}\rangle_{\tau_n}\big|>\varepsilon]&\le\frac{1}{\varepsilon}E[\langle M^{K,f}\rangle_{\tau_n+\theta}-\langle M^{K,f}\rangle_{\tau_n}]\\
&\le\frac{C_f\delta}{\varepsilon},
\end{aligned}\tag{8.5.38}$$

which with an adequate choice of $\delta>0$ proves (8.5.34). Now we will prove the tightness of the finite variation part $V^{K,f}$ of $\langle X^K,f\rangle$ by making the following computation

$$
\begin{aligned}
&E[\langle V^{K,f}\rangle_{\tau_n+\theta} - \langle V^{K,f}\rangle_{\tau_n}]\\
&\quad = E\Big[\frac{1}{K}\int_{\tau_n}^{\tau_n+\theta}\int_{\chi\times\chi\times\chi}\frac{b\alpha(p(x),p(y))D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(X_s^K)(dy)} f(z)\mu(dz)T^x(X_s^K)(dy)X_s^K(dx)ds\\
&\quad + \frac{1}{K}\int_0^t\int_\chi f(x)d(x,(U\circ p)*X_s^K(x))X_s^K(dx)ds\Big]\\
&\quad \leq 2\|f\|_\infty(b+\bar{d})\theta\Big(1+\sup_K E\big[\sup_{s\in[0,T]}\langle X_s^K,1\rangle^3\big]\Big)\\
&\quad \leq \bar{C}_f\delta.
\end{aligned}
$$

So proceeding as in (8.5.38), we obtain that (8.5.33) holds.
The previous steps imply the tightness of the laws of X^K in $\mathcal{P}(\mathbb{D}([0,T]),\mathcal{M}_F(\chi))$ when $\mathcal{M}_F(\chi)$ is endowed with the vague topology.

By Prohorov's Theorem, it is possible to extract from $(X^K)_{K\in\mathbb{N}}$ a convergent subsequence in law in $\mathbb{D}([0,T],(\mathcal{M}_F(\chi),v))$. Let us denote this subsequence by $(X^{\phi(K)})_{K\in\mathbb{N}}$ and by X a process with the law of the limit law of the previous subsequence. Noting that since we also proved the tension of the sequence $(\langle X^K,1\rangle)_{K\in\mathbb{N}}$, it is possible to choose the subsequence in such a way that $(\langle X^{\phi(K)},1\rangle)_{K\in\mathbb{N}}$ converges in law to $\langle X,1\rangle$ in $\mathbb{D}([0,T].\mathbb{R})$. Also by construction we know that:

$$\sup_{t\in[0,T]}\sup_{f\in L^\infty,\|f\|_\infty}|\langle X_t^K,f\rangle - \langle X_{t-}^K,f\rangle| \leq \frac{1}{K}.$$

this implies that the limit process X is a.s. strongly continuous. Using Theorem 3 of Méléard and Roelly [51], the subsequence $(X^{\phi(K)})_{K\in\mathbb{N}}$ converges also in law in $\mathbb{D}([0,T],(\mathcal{M}_F(\chi),w))$, where $\mathcal{M}_F(\chi)$ is endowed with the topology of the weak convergence. Applying Prohorov's Theorem again we can deduce that the sequence $(X^K)_{K\in\mathbb{N}}$ is tight in $\mathbb{D}([0,T],(\mathcal{M}_F(\chi),w))$. □

8.5.3 Characterization of the limit

In the previous section we showed that the sequence of laws of X^K is tight in $\mathcal{P}(\mathbb{D}([0,T]),\mathcal{M}_F(\chi))$ in the case when $\mathcal{M}_F(\chi)$ is endowed with the weak topology. In this section we will prove that the sequence X^K converges in law in $\mathcal{P}(\mathbb{D}([0,T]),(\mathcal{M}_F(\chi),w))$ to the unique solution of (8.5.19).

Theorem 22. *Admit Assumptions* $(C1)$ *and* $(C2)$. *Assume moreover that the initial conditions* X_0^K *converge in law and for the weak topology on* $M_F(\chi)$ *as* K *increases, to a finite deterministic measure* ξ_0, *and that*
$\sup_K E(\langle X_0^K,1\rangle^3) < +\infty$.
Then for any $T>0$, *the process* $(X_t^K)_{t\geq 0}$ *converges in law, in the Skorohod space* $D([0,T],M_F(\chi))$, *as* K *goes to infinity, to the unique deterministic continuous*

function $\xi \in C([0,T], M_F(\chi))$ *satisfying for any bounded* $f : \chi \to \mathbb{R}$

$$\langle \xi_t, f\rangle = \langle \xi_0, f\rangle + \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\xi_s(dy)} D(x,y;z)\mu(dz)\xi_s(dy)\xi_s(dx)ds$$
$$- \int_0^t \int_\chi f(x)d(x, U * \xi_s(x))\xi_s(dx)ds.$$

Proof. We will check that a.s. the process $(X)_{t\geq 0}$ is solution to (8.5.19). Using (8.5.35) we have that

$$E[\sup_{t\in[0,T]} \langle X_t, 1\rangle] \leq (1 + \sup_K E[\sup_{t\in[0,T]} \langle X_t^K, 1\rangle^3]) < +\infty,$$

which implies that $\sup_{t\in[0,T]}\langle \xi_t, 1\rangle < +\infty$ a.s. for each $T > 0$.

Now following standard density arguments it suffices to show that ξ is solution to (8.5.19) for any $f \in \mathcal{C}_b(\chi)$ and all $t > 0$. So let us take $f \in \mathcal{C}_b(\chi)$ and $t > 0$ fixed.

Let $\nu \in \mathcal{C}([0,\infty), \mathcal{M}_F(\chi))$ and consider the following

$$\Psi_t^1(\nu) = \langle \nu_t, f\rangle - \langle \nu_0, f\rangle + \int_0^t \int_\chi f(x)d(x, (U\circ p) * \nu_s(x))\nu_s(dx)ds$$
$$- \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\nu_s(dy)} D(x,y;z)\mu(dz)\nu_s(dy)\nu_s(dx)ds, \tag{8.5.39}$$

and

$$\Psi_t^2(\nu) = \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\nu_s(dy)} D(x,y;z)\mu(dz)\nu_s(dy)\nu_s(dx)ds$$

$$- \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(K\nu_s)(dy)} \mu(dz)T^x(K\nu_s)(dy)\nu_s(dx)ds.$$

If we show that

$$E_Q[|\Psi_t^1(X)|] = 0, \tag{8.5.40}$$

then a.s. $\Psi_t^1(X) = 0$, which would imply that X solves (8.5.19).
Let us recall that by (8.5.20) it follows that

$$M_t^{K,f} = \Psi_t^1(X^K) + \Psi_t^2(X^K). \tag{8.5.41}$$

By a simple computation using Proposition 3, Assumption $(C2)$, and (8.5.35) we have that

$$E[|M_t^{K,f}|^2] = E[\langle M_t^{K,f}\rangle^2] \leq \frac{1}{K} C_f E\left[\int_0^t (1 + \langle X_s^K, 1\rangle^2)ds\right], \tag{8.5.42}$$

which goes to 0 as K goes to infinity.
Next we have to deal with $\Psi_t^2(X^K)$, the convergence of this term to 0 follows from the fact that $T^x(KX^k) = KX^K - \delta_x$ for $x \in \text{supp } X^K$, and the following computation

$$
\begin{aligned}
|\Psi_t^2(X^K)| &= \Big| \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)}{\int_\chi \alpha(p(x),p(y))\nu_s(dy)} D(x,y;z)\mu(dz)X_s^K(dy)X_s^K(dx)ds \\
&- \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))f(z)D(x,y;z)}{\int_\chi \alpha(p(x),p(y))T^x(K\nu_s)(dy)} \mu(dz)T^x(K\nu_s)(dy)X_s^K(dx)ds \Big| \\
&= \Big| \int_0^t \int_\chi \frac{1}{A(x)} \Big(\int_\chi b\alpha(p(x),p(y)) \int_\chi f(z)D(x,y;z)\mu(dz)X_s^K(dy) \int_\chi \alpha(p(x)p(y))T^x(KX_s^K)(dy) \\
&- \int_\chi b\alpha(p(x),p(y)) \int_\chi f(z)D(x,y;z)\mu(dz)T^x(KX_s^K)(dy) \int_\chi \alpha(p(x),p(y))X_s^K(dy) \Big) X_s^K(dx) \Big| \\
&= \Big| \int_0^t \int_\chi \frac{b\alpha(p(x),p(x))}{A(x)} \Big(\int_\chi \alpha(p(x),p(y)) \int_\chi f(z)D(x,x;z)\mu(dz)X_s^K(dy) \\
&- \int_\chi \alpha(p(x),p(y)) \int_\chi f(z)D(x,y;z)\mu(dz)X_s^K(dy)) \Big) X_s^K(dx)ds \Big| \\
&= \Big| \int_0^t \int_\chi \frac{b\alpha(p(x),p(x))}{KA(x)} \Big(\int_\chi \alpha(p(x),p(y)) \int_\chi f(z)D(x,x;z)\mu(dz)T^x(KX_s^K)(dy) \\
&- \int_\chi \alpha(p(x),p(y)) \int_\chi f(z)D(x,y;z)\mu(dz)T^x(KX_s^K)(dy)) \Big) X_s^K(dx)ds \Big| \\
&\leq \frac{2b\|f\|_\infty}{K} \int_0^t \int_\chi \frac{\alpha(p(x),p(x))}{\int_\chi \alpha(p(x),p(y))X_s^K(dy)} X_s^K(dx)ds \leq \frac{C_{f,t}}{K},
\end{aligned}
\tag{8.5.43}
$$

where $A(x) = \int_\chi \alpha(p(x),p(y))X_s^K(dy) \int_\chi \alpha(p(x),p(y))T^x(KX_s^K)(dy)$.

So using (8.5.41),(8.5.42) and (8.5.43) we have the following

$$\lim_{K\to\infty} E[|\Psi_t^1(X^K)] = 0. \tag{8.5.44}$$

Now for fixed $t \in [0,T]$ we will show the continuity of Ψ_t^1.
First we will need the following lemma:

Lemma 4. *Let $(\nu^K)_{K\geq 0}$ be a sequence of probability measures on χ, such that $\nu^K \xrightarrow{\mathcal{L}} \nu$. If for each probability measure ν, on χ, we define*

$$\psi(\nu) = \left(\int_\chi \alpha(p(x),p(y))\nu(dy) \right)^{-1} \nu(dx),$$

where the function α is bounded and continuous. Then $\psi(\nu^K) \xrightarrow{\mathcal{L}} \psi(\nu)$.

Proof. By hypothesis $\nu^k \xrightarrow{\mathcal{L}} \nu$, so we use Skorohod's Representation Theorem (see for instance Theorem 4.30 in [39]). We know there exists a probability space which we denote $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ with some random variables $(\eta^K)_{K\geq 0}$ with $\mathcal{L}(\eta^K) = \nu^K$ and a random variable η with $\mathcal{L}(\eta) = \nu$, such that $\eta^K \to \eta$ $\bar{P}$ a.s.

So let us take $f \in \mathcal{C}_b(\chi)$, then using that $\underline{\alpha} \leq \alpha(p(x), p(y))$ for any $x, y \in \chi$, and The Dominated Convergence Theorem we have:

$$\begin{aligned}
\lim_{K\to\infty} \langle \psi(\nu^K), f\rangle &= \lim_{K\to\infty} \int_\chi \frac{f(x)}{\int_\chi \alpha(p(x), p(y)\nu^K(dy)} \nu^K(dx) \\
&= \lim_{K\to\infty} \int_{\bar{\Omega}} \frac{f(\eta^K(\omega))}{\int_{\bar{\Omega}} \alpha(p(\eta^K(\omega)), p(\eta^K(\omega'))\bar{P}(d\omega')} \bar{P}(d\omega) \\
&= \int_{\bar{\Omega}} \frac{f(\eta(\omega))}{\int_{\bar{\Omega}} \alpha(p(\eta(\omega)), p(\eta(\omega'))\bar{P}(d\omega')} \bar{P}(d\omega) \\
&= \int_\chi \frac{f(x)}{\int_\chi \alpha(p(x), p(y)\nu(dy)} \nu(dx) = \langle \psi(\nu), f\rangle. \qquad (8.5.45)
\end{aligned}$$

Noting that (8.5.45) holds for any $f \in \mathcal{C}_b(\chi)$, we have that $\psi(\nu^K) \xrightarrow{\mathcal{L}} \psi(\nu)$. □

So since X is a.s. strongly continuous, since f is continuous, thanks to the continuity of the parameters (Assumption (C1) and (C2)), and using Lemma 4 we have that the function Ψ_t is a.s. continuous at X. Furthermore using (8.5.39) we have for any $\nu \in \mathbb{D}([0,\infty), M_F(\chi))$

$$\begin{aligned}
|\Psi^1_t(\nu)| &\leq |\langle \nu_t, f\rangle| + |\langle \nu_0, f\rangle| + \left| \int_0^t \int_\chi f(x) d(x, (U\circ p) * \nu_s(x))\nu_s(dx)ds \right| \\
&\left| \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x), p(y))f(z)}{\int_\chi \alpha(p(x), p(y))\nu_s(dy)} D(x, y; z)\mu(dz)\nu_s(dy)\nu_s(dx)ds \right| \\
&\leq \|f\|_\infty \Big(\langle \nu_t, 1\rangle + \langle \nu_0, 1\rangle + \bar{d} \int_0^t \int_\chi (1 + \langle \nu_s, 1\rangle)\nu_s(dx)ds \\
&+ b \int_0^t \int_{\chi\times\chi} \frac{\alpha(p(x), p(y))}{\int_\chi \alpha(p(x), p(y))\nu_s(dy)} \nu_s(dy)\nu_s(dx)ds \Big) \\
&\leq \|f\|_\infty \Big(2 \sup_{s\in[0,T]} \langle \nu_s, 1\rangle + \bar{d} \int_0^t (\langle \nu_s, 1\rangle + \langle \nu_s, 1\rangle^2)ds + b \int_0^t \langle \nu_s, 1\rangle ds \Big) \\
&\leq (2 + \bar{d}T + bT)\|f\|_\infty \sup_{s\in[0,T]} (1 + \langle \nu_s, 1\rangle^2) \\
&= \overline{C}_{f,T} \sup_{s\in[0,T]} (1 + \langle \nu_s, 1\rangle^2). \qquad (8.5.46)
\end{aligned}$$

So using (8.5.35) and (8.5.46) we see that the sequence $(\Psi_t(X^K))_K$ is uniformly integrable and thus

$$\lim_{K\to\infty} E[|\Psi^1_t(X^K)|] = E[\lim_{K\to\infty} |\Psi^1_t(X^K)|] = E[|\Psi^1_t(X)|]. \qquad (8.5.47)$$

Associating (8.5.44) and (8.5.47) we conclude that (8.5.40) holds.

□

8.6 Examples

In this section we work with two particular forms of equation (8.5.19). For the two different choices of the space of genotypes and the parameters, equation (8.5.19) re-establishes the models of Shashahani [72] and Doebeli [25] from microscopic individual processes, showing that a large population is the only biological assumption needed to scale up to macroscopic evolutionary dynamics. In other words the renormalization procedure in section 8.5 gives the good scales between the coefficients to approach the microscopic individual-process by the deterministic equations of these two particular models.

8.6.1 Finite number of alleles

Suppose that we have a finite space of genotypes, i.e. we consider that $\chi = \{0,1\}^l$. Following the discussion in section 8.2 we take the measure μ defined on χ as the uniform measure on the space of genotypes χ, in other words

$$\mu(dz) = \sum_{x\in\chi} \delta_x(dz).$$

And define

$$D(x,y;z) = \sum_{I\in\mathcal{I}} r_I 1_{\{x_I y_J\}}(z),$$

then in this particular example (8.5.19) becomes

$$\begin{aligned}\langle \xi_t, f\rangle = \langle \xi_0, f\rangle + \int_0^t \int_{\chi\times\chi} \sum_{I\in\mathbf{I}} \frac{r_I b\alpha(p(x),p(y))}{\int_\chi \alpha(p(x),p(y))\xi_s(dy)} f(x_I y_J)\xi_s(dy)\xi_s(dx)ds \\ - \int_0^t \int_\chi f(x) d(x, U * \xi_s(x))\xi_s(dx)ds. \qquad (8.6.48)\end{aligned}$$

In the following we will not consider competition in the model, only natural death of the individuals, and that the mating between the individuals is random, in other words that $\alpha(x,y) = 1$ for $x, y \in \chi$.

Then by noting that the solution to (8.6.48) is a discrete measure, we have for fixed $z \in \chi$ the following differential equation for the number of individuals $\xi_t(z)$, with the genotype $z \in \chi$

$$\partial_t \xi_t(z) = \sum_{x\in\chi}\sum_I \frac{1}{\langle \xi_t, 1\rangle} b r_I \xi_t(z_I x_J)\xi_t(x_I z_J) - d(p(z))\xi_t(z). \qquad (8.6.49)$$

We remark that all the genotypes have the same birth rate, this follows from the fact that

$$b = \sum_{y\in\chi} b\frac{\xi_t(y)}{\langle \xi_t, 1\rangle}.$$

So noting that $\sum_I r_I = 1$, we can express relation (8.6.49), in the following form

$$\partial_t \xi_t = (b - d(p(z)))\xi_t(z) - \sum_{y\in\chi}\sum_I \frac{br_I}{\langle \xi_t, 1\rangle}(\xi_t(z)\xi_t(y) - \xi_t(z_I y_J)\xi_t(y_I z_J)). \tag{8.6.50}$$

Now lets consider the actual frequency of individual with a given trait, we denote this frequency by $q_t(x)$ and its given by,

$$q_t(x) = \frac{\xi_t(x)}{\langle \xi_t, 1\rangle}.$$

To make a relation with previous deterministic models, we will look at the dynamics of the frequency of a given trait $z \in \chi$, so we consider the following:

$$\partial_t q_t(x) = \frac{\partial_t \xi_t(x)}{\langle \xi_t, 1\rangle} - \frac{\partial_t \langle \xi_t, 1\rangle}{\langle \xi_t, 1\rangle^2}\xi_t(x). \tag{8.6.51}$$

We now compute the differential equation satisfied by the total population size, so setting $f \equiv 1$ in equation (8.6.48) we obtain,

$$\begin{aligned}\partial_t \langle \xi_t, 1\rangle &= \sum_{x\in\chi}\sum_{y\in\chi}\sum_I \frac{br_I}{\langle \xi_t, 1\rangle}\xi_t(x)\xi_t(y) - \sum_{x\in\chi} d(p(x))\xi_t(x)\\ &= b\langle \xi_t, 1\rangle - \sum_{x\in\chi} d(p(x))\xi_t(x).\end{aligned} \tag{8.6.52}$$

So using (8.6.51), and (8.6.52) in (8.6.50) we have

$$\begin{aligned}\partial_t q_t(z) = (b - d(p(z)))q_t(z) &- \sum_{y\in\chi}\sum_I br_I(q_t(z)q_t(y) - q_t(z_I y_J)q_t(y_I z_J))\\ - bq_t(z) &- \left(\sum_{x\in\chi} d(p(x))q_t(x)\right)q_t(z).\end{aligned} \tag{8.6.53}$$

We will denote by $m(p(z))$ the malthusian parameter associated to the genotype $z \in \chi$, defined as

$$m(p(z)) = b - d(p(z)),$$

and the mean malthusian parameter defined as $\overline{m}(t) = \sum_{x\in\chi} m(p(x))q_t(x)$, so (8.6.53) becomes

$$\begin{aligned}\partial_t q_t(z) = m(p(z))q_t(z) &- \sum_{y\in\chi}\sum_I br_I q_t(z)q_t(y) - q_t(z_I y_J)q_t(y_I z_J))\\ &- \left(\sum_{x\in\chi}(b - d(p(x)))q_t(x))\right)q_t(z)\\ &= (m(p(z)) - \overline{m}(t))q_t(z) - \sum_{y\in\chi}\sum_I br_I(q_t(z)q_t(y) - q_t(z_I y_J)q_t(y_I z_J)).\end{aligned} \tag{8.6.54}$$

Equation (8.6.54) describes the dynamics of the continuous time multilocus model developed and analyzed in [72] (p. 8), and [1] (p. 5-11), introduced in section 7.3.

8.6.2 Model with continuous genotypic space

We consider a continuous space of genotypes χ, i.e. $\chi = [0,1]^l$. In this case as mentioned in section 8.2, μ is the Lebesgue measure in $\mathbb{R}^l$, and $D(x,y;z)$ is the density of a Gaussian law with mean equal to $(x+y)/2$ and conditioned to stay on χ.
Then (8.5.19) takes the following form

$$\langle \xi_t, f\rangle = \langle \xi_0, f\rangle + \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))}{\int_\chi \alpha(p(x),p(y))\xi_s(dy)} f(z)D(x,y;z)dz\xi_s(dy)\xi_s(dx)ds$$
$$- \int_0^t \int_\chi f(x)d(x, U * \xi_s(x))\xi_s(dx)ds. \qquad (8.6.55)$$

In this case it is easy to see by taking $f = 1_A$ in (8.6.55) for any Borel set A with Lebesgue measure zero, that if ξ_0 has density with respect to the Lebesgue measure then ξ_t has density with respect to the Lebsegue measure for every $t \geq 0$.
So if we denote this density by $\xi_t(x)$, we have that

$$\langle \xi_t, f\rangle = \int_\chi f(x)\xi_t(x)dx.$$

So we can write (8.6.55) in the following form

$$\partial_t \xi_t(x) = \int_{\chi\times\chi} \frac{b\alpha(p(z),p(y))}{\int_\chi \alpha(p(z),p(y))\xi_t(y)dy} D(z,y;x)\xi_t(y)\xi_t(z)dydz$$
$$- d(x, (U\circ p) * \xi_t(x))\xi_t(x),$$

which has the same form as the model recently developed by Doebeli in [25].

8.7 Appendix

We will present the code of the program for the simulations we presented in section 4.32 for the variation of the model of Kisdi [40] to verify the appearance of sympatric speciation under the conditions of disruptive selection and random mating. The code is given for the program R, which is a free software environment for statistical computing and graphics, it can be downloaded in this web site **http://www.r-project.org/**.
Now we introduce the code for the model we presented in section 4.32, the lines preceded by the # symbol ar just explanatory notes and not part of the code.

```
###############################################################################
#                                                                             #
# Code for the simulation of the variation of the model of Kisdi presented in #
#                                                                             #
# section 4.32 following the simulation algorithm presented in 4.31.          #
```

8.6.2 Model with continuous genotypic space

We consider a continuous space of genotypes χ, i.e. $\chi = [0,1]^l$. In this case as mentioned in section 8.2, μ is the Lebesgue measure in $\mathbb{R}^l$, and $D(x,y;z)$ is the density of a Gaussian law with mean equal to $(x+y)/2$ and conditioned to stay on χ.
Then (8.5.19) takes the following form

$$\langle \xi_t, f\rangle = \langle \xi_0, f\rangle + \int_0^t \int_{\chi\times\chi\times\chi} \frac{b\alpha(p(x),p(y))}{\int_\chi \alpha(p(x),p(y))\xi_s(dy)} f(z)D(x,y;z)dz\xi_s(dy)\xi_s(dx)ds - \int_0^t\int_\chi f(x)d(x,U*\xi_s(x))\xi_s(dx)ds. \quad (8.6.55)$$

In this case it is easy to see by taking $f = 1_A$ in (8.6.55) for any Borel set A with Lebesgue measure zero, that if ξ_0 has density with respect to the Lebesgue measure then ξ_t has density with respect to the Lebsegue measure for every $t \geq 0$.
So if we denote this density by $\xi_t(x)$, we have that

$$\langle \xi_t, f\rangle = \int_\chi f(x)\xi_t(x)dx.$$

So we can write (8.6.55) in the following form

$$\partial_t \xi_t(x) = \int_{\chi\times\chi} \frac{b\alpha(p(z),p(y))}{\int_\chi \alpha(p(z),p(y))\xi_t(y)dy} D(z,y;x)\xi_t(y)\xi_t(z)dydz - d(x,(U\circ p)*\xi_t(x))\xi_t(x),$$

which has the same form as the model recently developed by Doebeli in [25].

8.7 Appendix

We will present the code of the program for the simulations we presented in section 4.32 for the variation of the model of Kisdi [40] to verify the appearance of sympatric speciation under the conditions of disruptive selection and random mating. The code is given for the program *R*, which is a free software environment for statistical computing and graphics, it can be downloaded in this web site **http://www.r-project.org/**.
Now we introduce the code for the model we presented in section 4.32, the lines preceded by the # symbol ar just explanatory notes and not part of the code.

```
################################################################################
#                                                                              #
# Code for the simulation of the variation of the model of Kisdi presented in  #
#                                                                              #
# section 4.32 following the simulation algorithm presented in 4.31.           #
```

```
    #                                                                              #
    ################################################################################
    rm(list=ls(all=TRUE))
###########  Declaration of the variables.
    datelim <- 1000          # time we will simulate
    K <- 200
    l <- 20          # number of loci
    b <- 5           # intrinsic birth rate of the simulation
    ###########################
    # Parameters of the model.
    ###########################
    sigma<- 0.01            #degree of assortment
    set.seed(1)
    genotypes<- array(data=rbinom(K*l,1,0.5), dim=c(K,l))
    # simulation of the initial distribution of the individuals
        traits<- array(rowSums(genotypes),dim=c(K,1))
        # trait distribution of the initial population
    ind<- function(x,y)
    {
        if( abs(x-y)==0 )
        {
            return(0)
        }
        else
        {
            return(1)
        }
    }
     # condition which will not allow an individual to mate with himself.
    alpha<- function(x,y)
            dnorm(x-y,sd=sigma)*sqrt(2*pi)*sigma
    }
    # mating function
    {
    alpha2<- function(x,traits)
    {
            dnorm(x-traits,sd=sigma)*sqrt(2*pi)*sigma
    }
     # total mating individual with genotype x is involved.
    R<- function(x)
    {
        return(min(exp((x-5)^2/20),exp((x-15)^2/20)))
    }
    # bimodal resource distribution
    U<- function(x,traits)
    {
```

```
        return(sum(2*(1-(1/(1+1.2*exp(-4*(x-traits)^2))))))
    }
     # competition Kernel
    d<- function(x,traits)
    {
        return(U(x,traits)*R(x))
    }
    # death rate for individual with genotype x
    barx <- 20
    underx <- 0
    pasfichier<-50
    set.seed(1)
############################################################################
    chemin <- "C:/Documents and Settings"
    dates <- array(data=0,dim=1)
    psize<-K         # Initial size of the population
    set.seed(1)
    ##########################
    # Indicators
    ##########################
    nbdeaths <- 0
    nbirths <- 0
    ##########################
    # Saved data.
    ##########################
    write(t(c(0, traits)),
        file = paste(chemin,"savedtraits2effective.txt",sep=""),
            ncolumns =2,append = TRUE)
    ##########################
    # Simulations
    ##########################
    set.seed(2)
    t<- 2
    while(dates[t-1]<datelim && psize>0)
    {
        dates2<-dates[t-1]
        testpassage<-0
        bound <- b*psize+ (4/K)*psize^(2)   # total event rate
        while(testpassage==0 && psize>0)
        {
            # Proposition for the time of the next event.
            interval<-rexp(1,bound)
            dates2<-dates2+interval
            # Choice of the individuals.
            indivi <- ceiling(runif(1,min=0,max=psize))
            indivj <- ceiling(runif(1,min=0,max=psize))
```

```
# Calculation of the bounds of each type of event.
m1 <- b*psize/(bound)
m2 <- m1+((K/4)*psize^(2)/(bound))
# Birth of an individual.
theta <- runif(1, min=0, max=1)
if(0 <= theta && theta<m1)  #choice of the event of birth
{
    n1 <- (b*alpha(traits[indivi],traits[indivj])*ind(indivi,indivj))
    /(sum(alpha2(traits[indivj],traits)))  #acceptance or rejection
    theta2<- runif(1,min=0,max=1)
        if(0 <= theta2 && theta2<n1) #recombination
    {
        dates<-c(dates,dates2)
        fix<- array(data=rbinom(l-1,1,0.5),dim=c(l-1,1))
        I<- c(rbind(1,fix))
        T<- c(array(data=c(1), dim=c(l,1)))
        J<- (T-I)
        x2<- (genotypes[indivi,]*I)+(genotypes[indivj,]*J)
        genotypes <- rbind(genotypes,x2)
        psize<- psize+1
        traits<- array(rowSums(genotypes),dim=c(psize,1))
        nbirths <- nbirths+1
        testpassage=1
        t <- t+1
    }
}
else if(m1 <= theta && theta<m2) # choice of event of death
{
    n2<- d(traits[indivi],traits)/((4/K)*psize^(2))
    theta3<- runif(1, min=0,max=1)
    #acceptance or rejection of the death event
    if(0 <= theta3 && theta3<n2)
    {
        dates<-c(dates,dates2)  #death of an individual
        if(psize>1)
        {
            if(indivi==1)
            {
                genotypes<-genotypes[2:psize,]
            }
            else if(indivi==psize)
            {
                genotypes<-genotypes[1:(psize-1),]
            }
            else
            {
```

```
                        genotypes<-rbind(array(data=genotypes
                        [1:(indivi-1),],dim=c(indivi-1,l)),
                        array(data=genotypes[(indivi+1):psize,],
                        dim=c(psize-indivi,l)))
                        }
                    }
                psize<- psize-1
                traits<- array(rowSums(genotypes),dim=c(psize,1))
                nbdeaths <- nbdeaths+1
                title<- paste("Time=",dates[t],"N=",psize)
                plot(density(traits),xlab="Trait", ylab="Density",
                xlim=c(0,barx),ylim=c(0, 0.15), main=title )
                 #density plot
                testpassage=1
                t <- t+1
                }
            }
        }
        if(identical(all.equal(t/pasfichier,as.integer(t/pasfichier)),TRUE) && psize>0)
        {
            # 4) Savedtraits
            tempdensity<-
            write(t(density(traits, give.Rkern=FALSE)$y),
                    file = paste(chemin,"savededensity.txt",sep=""),
                        ncolumns = 512,append = TRUE)
        }
    }
save.image(paste(chemin,"savedtraits.RData",sep=""))
```

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