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IDEALS AND FILTERS ON COUNTABLE SETS

T E S I S

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DAVID MEZA ALCÁNTARA

DIRECTOR DE TESIS: DR. MICHAEL HRUSAK

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Contents

Resumen

El propósito del presente trabajo es estudiar diversas propiedades acerca de los ideales y filtros sobre conjuntos numerables, a los que también llamaremos *ideales sobre* ω o *filtros sobre* ω; y clasificar estos ideales y filtros mediante relaciones de orden o propiedades de definibilidad. Esencialmente, estu-diamos propiedades combinatorias, invariantes cardinales, relaciones de orden y buscamos objetos críticos para tales propiedades. Usualmente, tales objetos críticos son ideales definibles, la mayoría de los casos borelianos y ocasionalmente coanalíticos. Nuestra principal herramienta es el orden de Katětov, el cual provee de un lenguaje útil que nos permite traducir propiedades combinatorias en funciones e ideales definibles.

En el primer capítulo introducimos los conceptos estudiados a lo largo de este trabajo, tales como las propiedades combinatorias (P-ideales, ideales altos), los invariantes cardinales, las relaciones de orden, y agregamos ademas una larga lista de ejemplos de ideales, para los cuales calculamos sus invariantes cardinales y su complejidad analítica. Frecuentemente establecemos la ubicación de estos ideales en las diferentes relaciones de orden.

El segundo capítulo está dedicado a estudiar las propiedades combinatorias de los ideales sobre ω entre los ideales en general. En este capítulo se exploran los límites de los resultados que pueden ser obtenidos sin hacer uso de la definibilidad de los ideales. Específicamente, se discuten aspectos como la destructibilidad de los ideales en extensiones genéricas, las propiedades de tipo Ramsey y los ideales críticos para las familias más estudiadas de ultrafiltros.

En el tercer capítulo hacemos uso del poder que provee la definibilidad en los ideales. Aquí aparecen argumentos de determinación de juegos, invariantes cardinales, y absolutez y genericidad. Este capítulo es protagonizado por los ideales F_σ y los P-ideales analíticos, debido a las importantes caracterizaciones de éstos en términos de submedidas inferiormente semicontinuas,

dadas por Mazur y Solecki, respectivamente.

Finalmente, el cuarto capítulo está dedicado al estudio de un juego muy natural que involucra pares de ideales Borel y que fué llamado *el juego de comparación*. Este juego permite clasificar en una estructura (casi?) bien ordenada a los ideales Borel, donde las anticadenas podrían tener cardinalidad a lo más 2. Esta clasificación es muy cercana a la jerarquía de Borel, al orden de Wadge y al orden monótono de Tukey.

Las principales contribuciones de este trabajo son:

(1) La *identificación de algunos ideales críticos para propiedades combinatorias* tales como los ideales Ramsey (ver teorema 2.2.2), los ideales ω-separadores borelianos, (ver teorema 3.2.1) y muchos otros, en cuyo caso se encuentran algunas clases de ultrafiltros como los P-puntos, los Q-puntos, los ultrafiltros selectivos y los ultrafiltros rápidos (ver sección 2.8).

(2) El *cálculo de los invariantes cardinales* de algunos ideales particulares, como el ideal $\mathcal{S}$ de Solecki (ver teorema 1.6.2), el ideal $\mathcal{ED}_{fin}$ cuyos invariantes de hecho complementan en cierto sentido al diagrama de Cichón (ver teorema 1.6.6) y el ideal $\mathcal{G}_{fc}$, los cuales fueron estudiados de manera independiente por Minami (ver teorema 1.6.21).

(3) El *cálculo de las complejidades de algunos ideales* como los de Cantor-Bendixson y los productos de Fubini.

(4) El *análisis de la propiedad de Ramsey* y propiedades relacionadas, como P^+, Q^+ y algunas versiones débiles de la propiedad de Ramsey. Con respecto a la propiedad local de Ramsey en ideales borelianos, no hemos conseguido un ejemplo de tales ideales, de modo que conjeturamos que de hecho no los hay, de modo que el ideal $\mathcal{R}$ generado por los conjuntos homogéneos de la gráfica random sería localmente mínimo entre los ideales Borel en el orden de Katětov. Al menos conocemos una clase grande de ideales borelianos, la clase de aquellos ideales $\mathscr{I}$ cuyo cociente $\mathcal{P}(\omega)/\mathscr{I}$ es propio o la extensión genérica de éste no agrega nuevos números reales, que está incluida en la clase de los ideales que están localmente por encima del ideal $\mathcal{R}$ en el orden de Katětov (ver teoremas 2.4.5 y 3.3.1).

Hemos encontrado fuertes relaciones entre la propiedad y P^+ y la propiedad F_σ. El teorema 2.8.3 provee un criterio para P-puntos en términos de la extendibilidad a ideales F_σ en ideales borelianos. El teorema 3.2.7 muestra la equivalencia entre (a) ser extendible a un ideal P^+, (b) ser extendible a un ideal F_σ y (c) ser extendible a un P-ideal maximal. Hemos aislado un ideal $\mathscr{I}_0$ que es P^+ y no es F_σ, pero tal ideal no es alto (ver teorema 4.3.1). En este momento no tenemos un ejemplo de ideal P^+ alto que no sea F_σ.

Respecto a la propiedad Q^+, nuestro resultado es el teorema 3.2.1 que muestra la equivalencia entre (a) ser Q-ideal, (b) no estar debajo de $\mathcal{ED}_{fin}$ en el orden de Katětov, (c) no ser ω-separador, (d) no ser ω-intersectante y (e) tener uniformidad no-numerable, para todo ideal Borel.

Finalmente, hemos encontrado ideales críticos y ejemplos de ideales que satisfacen algunas versiones débiles de la propiedad de Ramsey.

(5) Hemos estudiado profusamente las *propiedades estructurales del orden de Katětov* y mostramos que el orden de Katětov, incluso restringido al segmento de los ideales F_σ, es muy complicado. El teorema 3.1.1 muestra que hay un encaje de ordenes parciales del álgebra $\mathcal{P}(\omega)/fin$ ordenada por $\subseteq^*$ en la familia de los ideales sumables ordenada por el orden de Katětov. El teorema 3.4.1 de la Dicotomía de Categotía de Hrušák parte la clase de los ideales Borel en dos subclases: los que son destructibles por Cohen forcing y los que o bien no son P^+ o no son Q^+.

(6) El *análisis de la patología de las submedidas inferiormente semicontinuas* tiene su piedra angular en el teorema 3.6.5 de la Dicotomía de Medida de Hrušák, el cual divide a la familia de los P-ideales analíticos en dos clases, una de ellas la de los ideales que no son demasiado patológicos, que tiene como ideal crítico a $\mathcal{Z}$, el ideal de conjuntos con densidad asintótica cero; y la otra, que tiene a los ideales que localmente son muy patológicos y que tiene al ideal $\mathcal{S}$ de Solecki como ideal crítico. El teorema 3.7.5 muestra la relación entre satisfacer el lema de Fatou, satisfacer la propiedad de Fubini y ser patológico.

Finalmente, mencionaremos que hemos resuelto las preguntas publicadas que a continuación se describen:

(1) El teorema 2.1.17 responde una pregunta de Brendle y Yatabe [8], quienes probaron que un ideal $\mathscr{I}$ es Random-destructible si y sólo si existe un conjunto $tr(\mathcal{N})$-positivo X tal que $\mathscr{I} \leq_K tr(\mathcal{N}) \restriction X$. Brendle y Yatabe preguntaron si el conjunto X puede ser omitido, y por nuestro teorema, la respuesta es afirmativa.

(2) En Filipow et al [17] los autores preguntaron si es verdad que la relación $\omega \longrightarrow (\mathscr{I}^+)^2_2$ implica $\omega \longrightarrow (\mathscr{I}^+)^2_3$; y también preguntaron si $\omega \longrightarrow (\mathscr{I}^+)^2_2$ implica que $\omega \longrightarrow (\mathscr{I}^+)^2_n$ para toda $n > 2$. El teorema 2.7.6 responde ambas preguntas en negativo.

(3) En [44], Solecki demostró que el ideal $\mathcal{S}$ es crítico con respecto a satisfacer el lema de Fatou, en el orden de Katětov, y preguntó si el ideal $\mathcal{S}$ puede ser reemplazado por el ideal $\mathcal{G}_{fc}$ de gráficas con número cromático

finito en tal caracterización. El teorema 3.5.4 responde la pregunta de Solecki en negativo.

En el último capítulo estudiamos un problema de Farah, en el que preguntó si todos los ideales $F_{\sigma\delta}$ tienen una forma simple, cercana en espíritu a las caracterizaciones que Mazur y Solecki hicieron de los ideales F_σ y los P-ideales analíticos, respectivamente.

Preface

The aim of this work is to study several properties about ideals and filters on countable sets, which be called *ideals on* ω or *filters on* ω, and to classify those ideals and filters with order relations or definability properties. Essentially, we study combinatorial properties, cardinal characteristics, order relations and look for critical ideals for combinatorial properties. Usually that critical objects are definable ideals, most of them Borel and exceptionally co-analytic ideals. Our principal tool is the Katětov order which provides a useful language making us able to translate combinatorial properties into functions and definable ideals.

In the first chapter we introduce the concepts studied along this work, such that the combinatorial properties (P-ideals, tall ideals), cardinal characteristics, order relations, and we add a long list of examples of ideals with a calculation of their cardinal invariants and their analytic complexity. Sometimes we establish which position they occupy in some order relations.

The second chapter is dedicated to study combinatorial properties of ideals on ω among general ideals. Destructibility of ideals by forcing extensions, Ramsey and related properties and critical ideals for classical families of ultrafilters are studied in this chapter. We explore the limits of results that can be proved without using definability of the ideals.

In the third chapter we make use of the power of definability. Game theoretic proofs, cardinal invariants, forcing and absoluteness arguments appear in this chapter. The F_σ and analytic P-ideals are the protagonists, since two powerful theorems, one of them due to Mazur and the other due to Solecki relate lower semicontinuous submeasures with both classes of ideals.

In forth chapter we study a very natural game involving pairs of Borel ideals, that we have called the *comparison game*. It allows to classify ideals in an almost(?) well ordered structure, where the antichains just can be of cardinality at most 2. Such classification is very near to Borel hierarchy, Wadge order and monotone Tukey order.

The main contributions of this work are:

(1) The *identification of some critical ideals for combinatorial properties* like Ramsey ideals (see theorem 2.2.2), Borel ω-splitting ideals (see theorem 3.2.1) and many others, in which case are the classical properties about ultrafilters, like P-point, Q-point, selectivity and rapid ultrafilters (see section 2.8).

(2) The *calculation of cardinal characteristics* of some particular ideals like Solecki's ideal $\mathcal{S}$ (see theorem 1.6.2), the ideal $\mathcal{ED}_{fin}$ whose cardinal characteristics actually complete in some sense the Cichoń diagram (see theorem 1.6.6), and the ideal $\mathcal{G}_{fc}$ which were studied independently by Minami (see theorem 1.6.21).

(3) The *calculation of complexity of ideals* like Cantor Bendixson ideals and Fubini products (see proposition 1.6.16).

(4) The *analysis of the Ramsey property* and related combinatorial properties like P^+, Q^+ and some weak versions of Ramsey property. Concerning to the local Ramsey property among Borel ideals we do not have any example of such ideal, and then we conjecture that the *random graph ideal* $\mathcal{R}$ is locally minimal among Borel ideals in the Katětov order. At least we know that a very large class of Borel ideals, the class of ideals such that the quotient forcing $\mathcal{P}(\omega)/\mathscr{I}$ is proper or does not add new real numbers, is included in the class of ideals which are locally Katětov-above $\mathcal{R}$ (see theorems 2.4.5 and 3.3.1).

About P^+ property we have found some very strong relations with the property F_σ. Theorem 2.8.3 gives a criterion for P-point ultrafilters in terms of the extendability to F_σ-ideals of Borel ideals. Theorem 3.2.7 shows the equivalence between (a) to be extendable to a P^+ ideal, (b) to be extendable to an F_σ ideals and (c) to be extendable to a maximal P-ideal. We have isolated a P^+ ideal $\mathscr{I}_0$ which is not F_σ, but such ideal is not tall (see theorem 4.3.1). At this moment we do not have any example of a P^+ tall ideal which is not F_σ.

Concerning with the Q^+ property, the result is theorem 3.2.1 which shows the equivalence between to be Q-ideal, not to be Katětov above $\mathcal{ED}_{fin}$, not be ω-splitting ideal, not be an ω-hitting ideal and to have uncountable uniformity number for all Borel ideals.

Finally we have found critical ideals and examples of ideals satisfying some weak versions of Ramsey property.

(5) The *structural properties of Katětov order* have been extensively stud-

ied and we have show that Katětov order, even restricted to a segment of F_σ-ideals is very complicated. Theorem 3.1.1 shows that there is an order embedding from the algebra $\mathcal{P}(\omega)/fin$ ordered by $\subseteq^*$ into the family of summable ideals with the Katětov order. Hrušák's Category Dichotomy Theorem 3.4.1 splits the class of Borel ideals in two subclasses: the Cohen-destructible ideals and the ideals such that either are not P^+ or are not Q^+-ideals.

(6) The *analysis of pathology of lower semicontinuous submeasures* has its cornerstone in the Hrušák's Measure Dichotomy Theorem 3.6.5, which splits the family of analytic P-ideals in two classes, one of them the non-too-pathological ideals, which have $\mathcal{Z}$ as a critical ideal; and the ideals with a very pathological restriction, whose critical ideal is the Solecki's ideal $\mathcal{S}$. Theorem 3.7.5 shows the relationship between fulfil the Fatou's lemma, satisfy the Fubini property and be pathological.

Finally, we have answered some published questions like the following:

(1) Theorem 2.1.17 answers a question of Brendle and Yatabe [8] who proved that an ideal $\mathscr{I}$ is Random-destructible if and only if there is a $tr(\mathcal{N})$ positive set X such that $\mathscr{I} \leq_K tr(\mathcal{N}) \restriction X$. They asked if the set X could be omitted in their result, and our proposition answers in the positive.

(2) In Filipow et al [17] the authors ask if is it true that $\omega \longrightarrow (\mathscr{I}^+)^2_2$ implies $\omega \longrightarrow (\mathscr{I}^+)^2_3$ and $\omega \longrightarrow (\mathscr{I}^+)^2_2$ implies $\omega \longrightarrow (\mathscr{I}^+)^2_n$ for all $n > 2$. Theorem 2.7.6 answers this question in the negative.

(3) In [44], Sławomir Solecki proved that the ideal $\mathcal{S}$ is critical with respect to fulfil Fatou's lemma in Katětov order, and he asked if the ideal $\mathcal{S}$ could be replaced by the ideal $\mathcal{G}_{fc}$ of graphs with finite chromatic number in such characterization. Theorem 3.5.4 answers Solecki's question in the negative.

In the last chapter we also study a problem of Farah, asking whether all $F_{\sigma\delta}$ ideals are of a simple form, close in spirit to the powerful characterization by Mazur and Solecki of F_σ-ideals and analytic P-ideals.

Preliminaries

This pre-chapter is a compilation of set theoretical and topological concepts and results that we use in this work. The familiarized reader can skip this preliminary chapter.

0.1 Set theoretic and topological conventions

This work is framed by the standard set theory Zermelo-Fraenkel with axiom of Choice. About set theoretical matters we follow Kunen's book [33], and for general topology matters we follow Willard's book [49]. Any ordinal number α is the set of all those ordinal numbers which are smaller than α. Natural numbers are the finite ordinal numbers. A cardinal number is an ordinal number which is no equipotent with any ordinal smaller than it. The expressions X^Y, $X^{<\kappa}$, $[X]^\kappa$ and $[X]^{<\kappa}$ respectively mean, the set of all functions from Y into X, the set of all functions from some $\alpha < \kappa$ into X, the set of all subsets of X with cardinality κ and the set of all subsets of X with cardinality smaller than κ, where X and Y are sets and κ is a cardinal number. Given a function f and a set A, $f''A$ denotes the set $\{f(x) : x \in A\}$ and $ran(f) = \{y : (\exists x)(f(x) = y)\}$. For projections of a subset A of a cartesian product $X \times Y$ we agree the following notation: $(A)_x = \{y \in Y : (x, y) \in A\}$ and $(A)^y = \{x \in X : (x, y) \in A\}$, where $x \in X$ and $y \in Y$. We say that a family $\mathcal{X} \subseteq \mathcal{P}(\omega)$ is *ω-hitting* (see [11]) if for any countable family $\langle X_n : n < \omega \rangle$ of infinite subsets of ω, there is an element X of $\mathcal{X}$ such that $X \cap X_n$ is infinite, for all $n < \omega$; and we will say that $\mathcal{X}$ is *ω-splitting* if for any countable family $\{X_n : n < \omega\}$ of infinite subsets of ω there exists an element X of $\mathcal{X}$ such that $|X \cap X_n| = |X_n \setminus X| = \aleph_0$, for all $n < \omega$.

About descriptive set theory we use results and notations appearing in Kechris' book [30]. For any topological space X the *Borel σ-algebra* of X is the minimal σ-algebra on X containing all open sets of X, and it is denoted by $Borel(X)$. When X is a metrizable space, such σ algebra is stratified as follows. Δ^0_1 is the family of clopen sets, Σ^0_1 is the family of all open sets, Π^0_α is the family of complements of Σ^0_α sets, and for $\alpha > 1$, Σ_α is the family of all unions of countable families of sets taken from the classes Π_β with $\beta < \alpha$; and $\Delta^0_\alpha = \Sigma^0_\alpha \cap \Pi^0_\alpha$, for all $\alpha < \omega_1$. Hence, Σ^0_2 is the class of F_σ sets, Π^0_2 is the class of G_δ sets, Σ^0_3 is the family of $G_{\delta\sigma}$ sets, Π^0_3 is the class of $F_{\sigma\delta}$ sets and so on. Finally, $Borel(X) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$. If X i s a Polish space and $A \subseteq X$, then A is an analytic sets if there is a Borel subset (equivalently, closed subset) B of ω^ω and a continuous mapping $f : \omega^\omega \to X$ such that $f''B = A$. A subset C of X is co-analytic if its complement is an analytic subset of X.

We denote cones by $\langle s\rangle = \{f \in X^\omega : s \subseteq f\}$ where $s \in X^{<\omega}$. By $s^\frown t$ we denote the *concatenation* of the sequences s and t of X and if $t \in X$ then $s^\frown t$ denotes $s^\frown(t)$. For any tree $T \subseteq X^{<\omega}$ we will denote the family of *branches* of T by $[T]$ and for any $t \in T$, $succ_T(t) = \{n : t^\frown n \in T\}$. If $\mathcal{A}$ is a family of subsets of ω and $T \subseteq \omega^{<\omega}$ is a tree, then T is an $\mathcal{A}$-branching tree if for all $t \in T$, $succ_T(t) \in \mathcal{A}$.

0.2 Absoluteness

Two classical results about absoluteness of formulae are used in this work. Proofs of these theorems can be seen in Miller's book [39].

Theorem 0.2.1 (Mostowski's absoluteness). *Suppose $M \subseteq N$ are two transitive models of ZFC^* and θ is a Σ^1_1 sentence with parameters in M. Then*

$$M \models \theta \quad \textit{if and only if} \quad N \models \theta.$$

$\square$

Theorem 0.2.2 (Schoenfield absoluteness). *If $M \subseteq N$ are transitive models of ZFC^* and $\omega_1^N \subseteq M$, then for any $\Sigma^1_2(x)$ sentence θ with parameter $x \in M$ (and then, for $\Pi^1_2(x)$ sentences too)*

$$M \models \theta \quad \textit{if and only if} \quad N \models \theta.$$

$\square$

0.3 Determinacy

One of the most powerful tools used in this work is *determinacy.* Let A be a non-empty set and $X \subseteq A^\omega$. The *game* $G(A, X)$ consists of two *players*, called Player I and Player II, who take turns choosing elements of A, constructing a sequence $\langle a_i : i < \omega\rangle \in A^\omega$. Player I wins if $\langle a_i : i < \omega\rangle \in X$ and Player II wins otherwise. A *winning strategy* for Player I is a function $\varphi : \bigcup_n A^{2n} \to A$ such that if $x \in A^\omega$ satisfies $x(2n+1) = \varphi(x \restriction 2n)$ for all $n < \omega$, then $x \in X$. A winning strategy for Player II is also a function $\psi : \bigcup_n A^{2n+1} \to A$ such that if $x \in A^\omega$ satisfies $x(2n+2) = \psi(x \restriction 2n+1)$ for all $n < \omega$, then $x \notin X$.

In most cases, games are defined by rules, that is, not every sequence of choices of elements of A is allowed by the game. In this case we think that the game is not played in the whole of the tree $A^{<\omega}$ but in a pruned subtree T of $A^{<\omega}$, whose nodes are called the *legal positions* of the game. We usually denote this kind of games by $G(T, Y)$, where $Y \subseteq [T]$. A game with rules $G(T, Y)$ is always equivalent with a game $G(A, X)$ since rules can be coded in such way that any winning strategy for $G(A, X)$ induces a winning strategy for $G(T, Y)$ and any winning strategy for $G(T, Y)$ induces a winning strategy for $G(A, X)$.

A game $G(T, Y)$ is *determined* if there is a winning strategy for one of the two players. Under Axiom of Choice there are sets $X \subseteq \omega^\omega$ such that the game $G(\omega, X)$ is not determined. Bernstein sets are such examples. However, the complexity of Y as a subspace of the product A^ω of the discrete space A can guarantee the determinacy of the game $G(T, Y)$.

Theorem 0.3.1 (Gale and Stewart)**.** *If Y is a closed or open subset of $[T]$ then $G(T, Y)$ is determined.* □

And moreover

Theorem 0.3.2 (D. A. Martin)**.** *If Y is a Borel subset of $[T]$ then $G(T, Y)$ is determined.* □

We will use determinacy of games several times in this work. A typical game used is defined as follows: Let $\mathcal{U}$ be an ultrafilter on ω, and $\mathscr{I}$ a Borel ideal on ω. In step i Player I chooses an element $U_i \in \mathcal{U}$ and then Player II chooses an element $n_i \in U_i$. Player I wins if $\{n_i : i < \omega\} \in \mathscr{I}$.

We use Martin's theorem in order to prove the determinacy of this game by taking $A = \mathcal{P}(\omega)$, $T \subseteq A^{<\omega}$ defined by $t = \langle B_0, B_1, \ldots, B_n\rangle \in T$ if for any $i \leq n$, $B_i \in \mathcal{U}$ if i is even, $|B_i| = 1$ (and then, $B_i = \{n_i\}$ for a unique $n_i < \omega$) if i is odd and $B_{i+2} \cap [0, n_i] = \emptyset$ if i is even; and $X \subseteq A^\omega$ defined by $x = \langle B_i : i < \omega \rangle \in X$ if $\bigcup_{i<\omega} B_{2i} \in \mathscr{I}$. Note that T is not the pruned tree of legal positions of $G(\mathcal{U}, \mathscr{I})$, however the tree T is naturally isomorphic to the tree of legal positions of the game. On the other hand, note that the function $f : T \to \mathcal{P}(\omega) = 2^\omega$ given by $f(x) = \bigcup_{i<\omega} B_{2i}$ is continuous, since initial segments of that unions are determined by finite sequences. Then, since $X = f^{-1}[\mathscr{I}]$ we have that X is Borel and by Martin's theorem the game is determined.

0.4 Cardinal invariants of the continuum

Some cardinal invariants of the continuum will be used in this work. We use some facts about the *Baire space* ω^ω ordered by $\leq^*$. For $f, g \in \omega^\omega$, we say $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n < \omega$. A family $F \subseteq \omega^\omega$ is *bounded* if there is $h \in \omega^\omega$ such that $f \leq^* h$ for all $f \in F$; and we say F is *dominating* if for any $g \in \omega^\omega$ there is $f \in F$ such that $g \leq^* f$. Two very usual cardinal invariants are the minimal cardinality of an unbounded family $\mathfrak{b}$, and the minimal cardinality of a dominating family $\mathfrak{d}$.

For any ideal $\mathscr{I}$ on a set X, the following cardinal numbers are defined:

- $\mathsf{add}(\mathscr{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge \bigcup \mathcal{A} \notin \mathscr{I}\}$,
- $\mathsf{cov}(\mathscr{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge \bigcup \mathcal{A} = X\}$,
- $\mathsf{non}(\mathscr{I}) = \min\{|Y| : Y \subseteq X \wedge Y \notin \mathscr{I}\}$ and
- $\mathsf{cof}(\mathscr{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge (\forall I \in \mathscr{I})(\exists A \in \mathcal{A})(I \subseteq A)\}$.

Cardinal invariants of any ideal $\mathscr{I}$ satisfy

$$\begin{tikzcd} & \mathsf{cov}(\mathscr{I}) \arrow[dr] & \\ \mathsf{add}(\mathscr{I}) \arrow[ur] \arrow[dr] & & \mathsf{cof}(\mathscr{I}) \\ & \mathsf{non}(\mathscr{I}) \arrow[ur] & \end{tikzcd} \tag{1}$$

where every arrow points at the bigger or equal cardinal number. Cardinal invariants of two important σ-ideals on the real line $\mathbb{R}$ (or equivalently the Cantor space 2^ω or the Baire space ω^ω) are considered: $\mathcal{M}$ denotes de σ-ideal of meager sets and $\mathcal{N}$ denotes de σ-algebra of null sets (with respect to the Lebesgue measure). The following picture shows some inequalities holding and is known as the *Cichoń's diagram.*

$$\begin{tikzcd}
\mathsf{cov}(\mathcal{N}) \arrow[r] & \mathsf{non}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathsf{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathsf{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathsf{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathsf{non}(\mathcal{N}) \arrow[uu]
\end{tikzcd}$$

For more about Cichoń's diagram see [2].

Other usual cardinal invariants come from $[\omega]^\omega$ ordered by *almost inclusion.* We say $A \subseteq^* B$ if and only if $A \setminus B$ is finite; and we say $A =^* B$ if $A \triangle B$ is finite. Let $\mathcal{A}$ be a subfamily of $[\omega]^\omega$. We say $\mathcal{A}$ is a *splitting family* if for any set $B \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $|B \cap A| = |B \setminus A| = \aleph_0$. We say $\mathcal{A}$ is a *reaping family* if for any $B \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $A \subseteq^* B$ or $A \cap B =^* \emptyset$. $\mathcal{A}$ is *open* if for any $A \in \mathcal{A}$ and any $B \subseteq^* A$, $B \in \mathcal{A}$; and is dense if for any $B \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $A \subseteq^* B$. We say that $\mathcal{A}$ has the *strong finite intersection property* if any finite subfamily $\mathcal{B}$ of $\mathcal{A}$ has an infinite intersection. A set $B \subseteq \omega$ is a *pseudointersection* of $\mathcal{A}$ if $B \subseteq^* A$ for all $A \in \mathcal{A}$. A *tower* is an ordinal-indexed family $\langle A_\alpha : \alpha < \lambda \rangle$ of infinite subsets of ω such that $A_\alpha \subseteq^* A_\beta$ if $\beta < \alpha < \lambda$ and there are no infinite pseudointersections for it. Then the cardinal invariants are defined by (1) $\mathfrak{s}$ is the minimal cardinality of a splitting family, (2) $\mathfrak{r}$ is the minimal cardinality of a reaping family, (3) $\mathfrak{h}$ is the minimal cardinality of a set of open dense families with empty intersection, (4) $\mathfrak{t}$ is the minimal cardinality of a tower and (5) $\mathfrak{p}$ is the minimal cardinality of a family with the strong finite intersection property without infinite pseudointersections. The inequalities holding for these cardinal invariants are displayed in the following picture, which is known as the *van Douwen's diagram.*

$$\begin{tikzcd}
\mathfrak{r} & & \mathfrak{d} & \\
& \mathfrak{b} \arrow[ul] \arrow[ur] & & \mathfrak{s} \arrow[ul] \\
& & \mathfrak{h} \arrow[ul] \arrow[ur] & \\
& & \mathfrak{t} \arrow[u] & \\
& & \mathfrak{p} \arrow[u] & \\
& & \aleph_1 \arrow[u] &
\end{tikzcd}$$

For more about van Dowen's diagram and cardinal characteristics of the continuum see Blass' article [5].

0.5 Forcing

We refer the reader to Kunen's book [33] for the elementary theory of forcing. We identify the non-separative preorder $\langle [\omega]^\omega, \subseteq \rangle$ with its separative quotient $\mathcal{P}(\omega)/fin$ and in general, we will identify the non-separative forcing $\langle \mathscr{I}^+, \subseteq \rangle$ with its separative quotient $\mathcal{P}(\omega)/\mathscr{I}$, for any ideal $\mathscr{I}$. By σ-closedness $\mathcal{P}(\omega)/fin$ does not add new real numbers, ω-sequences of real numbers nor Borel sets. Moreover, in the extension $V[G]$, the generic filter G is a selective ultrafilter.

Proofs which use selective ultrafilters are supported by the following forcing and absoluteness argument. If we want to prove φ, we take a model V of ZFC^* and then we go to the forcing extension $V[G]$ where G is a $\mathcal{P}(\omega)/fin$-generic ultrafilter over V. We note that in $V[G]$, G is a selective ultrafilter and then the argument is correct in $V[G]$ and so $V[G] \models \varphi$. Finally, if φ is a sufficiently absolute sentence (see Schoenfield's theorem 0.2.2) as for example, a Katětov relation between two Borel ideals (see 1.5.3), then $V \models \varphi$.

Finally, we will say that a forcing $\mathbb{P}$ is *ω^ω-bounding* (or adds a dominating real) if it adds a real $\dot{r}$ such that $\mathbb{P} \Vdash$ "$(\forall f \in \omega^\omega \cap V)(f \leq^* \dot{r})$". We say that $\mathbb{P}$ *adds an unbounded real* if there is a $\mathbb{P}$-name $\dot{r}$ for a real number such that $\mathbb{P} \Vdash$ "$(\forall f \in \omega^\omega \cap V)(\dot{r} \nleq^* f)$".

Chapter 1

Ideals and their cardinal invariants

Let X be a non-empty set. An *ideal* on X is a family $\mathscr{I}$ of subsets of X satisfying:

- $\emptyset \in \mathscr{I}$ and $X \notin \mathscr{I}$,
- if $A, B \in \mathscr{I}$ then $A \cup B \in \mathscr{I}$ and
- if $A \subseteq B$ and $B \in \mathscr{I}$ then $A \in \mathscr{I}$.

Unless we specify the contrary, if $\mathscr{I}$ is an ideal on X then the family of finite subsets of X is included in $\mathscr{I}$. If φ is a bijective function from X onto Y, and $\mathscr{I}$ is an ideal on X we can think of $\mathscr{I}$ as an ideal on Y, since $\{\varphi'' I : I \in \mathscr{I}\}$ is an ideal on Y isomorphic to $\mathscr{I}$. We are interested essentially in ideals on countable sets. Sometimes, we are interested in σ-ideals on the real line $\mathbb{R}$, the Cantor space 2^ω or the Baire space ω^ω. A *σ-ideal* on X is an ideal which is closed under unions of countable subfamilies of the ideal.

The notion of a filter is dual to the notion of an ideal. A *filter* on X is a family $\mathscr{F}$ of subsets of X satisfying:

- $\emptyset \notin \mathscr{F}$ and $X \in \mathscr{F}$,
- if $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$ and
- if $A \subseteq B$ and $A \in \mathscr{F}$ then $B \in \mathscr{F}$.

As in the case of ideals, unless we specify the contrary, if $\mathscr{F}$ is a filter on X then $\mathscr{F}$ contains every cofinite subset of X. Maximal filters are called *ultrafilters*. Given an ideal $\mathscr{I}$ on X, we denote by $\mathscr{I}^*$ the family (filter) $\{A \subseteq X : X \setminus A \in \mathscr{I}\}$, which is called the *dual filter* of $\mathscr{I}$; and we denote by $\mathscr{I}^+ = \{A \subseteq X : A \notin \mathscr{I}\}$ the set of $\mathscr{I}$-positive sets of X. Dually, given a filter $\mathcal{F}$ on X, we denote by $\mathcal{F}^* = \{\omega \setminus A : A \in \mathcal{F}\}$. $\mathcal{F}^*$ is called the *dual ideal* of $\mathcal{F}$.

Given an $\mathscr{I}$-positive set Y, the *restriction of* $\mathscr{I}$ *in* Y is defined by

$$\mathscr{I} \restriction Y = \{I \cap Y : I \in \mathscr{I}\}.$$

If $A, B \subseteq X$ then we say that A and B are $\mathscr{I}$*-almost disjoint* if $A \cap B \in \mathscr{I}$; and we say that A is $\mathscr{I}$*-almost contained in* B (in symbols, $A \subseteq^{\mathscr{I}} B$) if $A \setminus B \in \mathscr{I}$.

A family $\mathcal{A} \subseteq \mathscr{I}$ is a *base* for $\mathscr{I}$ if for any $I \in \mathscr{I}$ there is $A \in \mathcal{A}$ such that $I \subseteq A$. Hence, $\mathsf{cof}(\mathscr{I})$ is the minimal cardinality of a base for $\mathscr{I}$. A family $\mathcal{B} \subseteq \mathscr{I}$ is a *subbase* for $\mathscr{I}$ if the family of all finite unions of members of $\mathcal{B}$ is a base for $\mathscr{I}$. Given a family $\mathcal{C}$ of subsets of X, if there is not a finite subfamily $\mathcal{D}$ of $\mathcal{C}$ such that $\bigcup \mathcal{D} = X$ then the *ideal generated by* $\mathcal{C}$ is the minimal ideal containing $\mathcal{C}$. Note that $\mathcal{C}$ is a subbase of the ideal generated by $\mathcal{C}$.

1.1 Ideals on ω

Let $\mathscr{I}$ be an ideal on a set X. We say $\mathscr{I}$ is an *ideal on* ω if X is a countable set and $\mathscr{I}$ contains all finite subsets of X. In general we will assume that such countable set X is ω, the first infinite ordinal number.

Fin denotes the ideal of all finite subsets of ω.

We say that an ideal $\mathscr{I}$ on ω is:

- *tall* if for every infinite subset A of ω there is I in $\mathscr{I}$ such that $I \cap A$ is infinite. Equivalently, $\mathscr{I}$ is tall if for all $\mathscr{I}$-positive set X the restriction $\mathscr{I} \restriction X$ is not the family of finite subsets of X.

- a *P-ideal* if for every sequence $\langle I_n : n < \omega\rangle$ of elements of $\mathscr{I}$ there exists an element I of $\mathscr{I}$ such that $I_n \subseteq^* I$, for all $n < \omega$.

1.2 Measure and category of ideals on ω

Ideals on ω, as subsets of the power set $\mathcal{P}(\omega)$, can be seen as subspaces of Cantor's space 2^ω, and so, they can be studied through their analytic complexity and their topological and measure-theoretic properties.

Proposition 1.2.1 (Folklore). *The minimal possible complexity of ideals on ω is F_σ.*

Proof. Since **Fin** is countable, it is an F_σ ideal. Let $\mathscr{I}$ be an ideal on ω. Since every ideal on ω is a dense subset of 2^ω, $\mathscr{I}$ can not be a closed set. Since $\mathscr{I}^*$ is a subspace of 2^ω isomorphic to and disjoint from $\mathscr{I}$, by Baire category $\mathscr{I}$ can not be G_δ. □

Proposition 1.2.2 (0-1 Law, Sierpiński). *Let $\mathscr{I}$ be an ideal on ω. Then*

(1) If $\mathscr{I}$ has the Baire property then $\mathscr{I}$ is meagre.

(2) If $\mathscr{I}$ is Lebesgue measurable then $\mathscr{I}$ is a null set.

Proof. Note that the *switch* function $sw : 2^\omega \to 2^\omega$ given by $sw(x)(n) = 1 - x(n)$ for all $n < \omega$ is a homeomorphism. Even more, for any $t \in 2^{<\omega}$, the function sw_t given by $sw_t(x)(n) = t(n)$ if $n < |t|$ and $sw_t(x)(n) = 1 - x(n)$ if $n \geq |t|$ is an autohomeomorphism of the clopen set $\langle t \rangle$.

(1) Assume that $\mathscr{I}$ has the Baire Property and is not meagre. Let $t \in 2^{<\omega}$ be such that $U = \langle t \rangle$ is a basic clopen set of 2^ω such that $\mathscr{I} \cap U$ is comeagre. Since sw_t witnesses that $\mathscr{I} \cap U$ is homeomorphic to $\mathscr{I}^* \cap U$, we have two disjoint comeagre subsets of U, a contradiction with Baire Category theorem.

(2) Let λ denote the Lebesgue measure on 2^ω. Assume that $\mathscr{I}$ is Lebesgue measurable and $\lambda(\mathscr{I}) > 0$. Note that for any basic clopen set $U = \langle t \rangle$,

$$\lambda(\mathscr{I} \cap U) = \lambda(\mathscr{I})\lambda(U) \tag{1.1}$$

since for any $s \in 2^{<\omega}$ with $|s| = |t|$, $\lambda(\mathscr{I} \cap U) = \lambda(\mathscr{I} \cap \langle s \rangle)$. Then equation 1.1 also holds for any open set U. We will prove that $\lambda(\mathscr{I}) = 1$, by contradiction. Let suppose that there is a closed set C with positive measure and disjoint with $\mathscr{I}$. Let U be an open set such that $C \subseteq U$, and $\lambda(C) < \lambda(U) < \frac{\lambda(C)}{1-\lambda(\mathscr{I})}$. Then $1 - \lambda(\mathscr{I}) < \frac{\lambda(C)}{\lambda(U)}$ and then $\lambda(\mathscr{I}) > \frac{\lambda(U)-\lambda(C)}{\lambda(U)}$. Hence, $\lambda(\mathscr{I} \cap U) > \lambda(U \setminus C)$ and then, $C \cap \mathscr{I} \neq \emptyset$, a contradiction. Now, since sw is a function that preserves measures of sets, we have that $\mathscr{I}$ and $\mathscr{I}^*$ are disjoint full measure sets, a contradiction. □

A very important result about category of filters (and ideals) is the following theorem due to Jalali-Naini and Talagrand. We will say that a filter $\mathcal{F}$ is *bounded* if the family of increasing enumeration functions e_A of the elements A of $\mathcal{F}$ is a $\leq^*$-bounded family.

Theorem 1.2.3 (Jalali-Naini [25], Talagrand [46] see [2] theorem 4.1.2)**.** *Let $\mathcal{F}$ be a filter on ω. The following conditions are equivalent.*

1. *$\mathcal{F}$ has the Baire property,*

2. *$\mathcal{F}$ is meager,*

3. *$\mathcal{F}$ is bounded,*

4. *there is a partition of ω in intervals $\{I_n : n < \omega\}$ such that for any $F \in \mathcal{F}$, $F \cap I_n \neq \emptyset$ for all but finitely many $n < \omega$ and*

5. *there is a function $f \in \omega^\omega$ fin-to-one such that $\{f^{-1}[F] : F \in \mathcal{F}\}$ is the cofinite filter.* □

A well-known corollary of Jalali-Naini - Talagrand theorem is the following.

Corollary 1.2.4 (Folklore)**.** *For any meager ideal $\mathscr{I}$ and any $\mathscr{I}$-positive set X there is an almost-disjoint family $\mathcal{A}$ of $\mathscr{I}$-positive subsets of X of cardinality $\mathfrak{c}$.*

Proof. If $\mathscr{I}$ is meager then $\mathscr{I}^*$ is meager and by Jalali-Naini - Talagrand theorem there is a partition of ω in intervals $\{I_n : n < \omega\}$ such that for any $F \in \mathscr{I}^*$, $F \cap I_n \neq \emptyset$ for all but finitely many $n < \omega$. Then, if $B \subseteq \omega$ is such that $I_n \subseteq B$ for infinitely many n then $X \in \mathscr{I}^+$. Define $S = \{n : I_n \cap X \neq \emptyset\}$ and $J_n = I_n \cap X$ for all $n \in S$. It is clear that S is infinite, and then there is a bijective function $\psi : 2^{<\omega} \to S$. For each $x \in 2^\omega$, define $A_x = \bigcup_{n<\omega} J_{\psi(x \restriction n)}$. Hence the family $\mathcal{A} = \{A_x : x \in 2^\omega\}$ is the required one. □

An ideal $\mathscr{I}$ is *hereditarily meager* if for any $\mathscr{I}$-positive set X the restriction $\mathscr{I} \restriction X$ is a meager ideal.

Note that by the 0-1 law, any Borel, analytic or co-analytic ideal is hereditarily meager, since such ideals are hereditarily Borel, analytic or co-analytic, respectively. Recall that all analytic and co-analytic sets have the Baire property (see [30], theorem 29.5).

The following lemma shows that partial orders of the form $\mathcal{P}(\omega)/\mathscr{I}$ with $\mathscr{I}$ hereditarily meager ideal are never c.c.c.

Lemma 1.2.5 (Disjoint Refinement Lemma for Definable Ideals). *If $\mathscr{I}$ is a hereditarily meager ideal and $\{X_n : n < \omega\}$ is a family of $\mathscr{I}$-positive sets then there is a pairwise disjoint family $\{Y_n : n < \omega\}$ of $\mathscr{I}$-positive sets such that $Y_n \subseteq X_n$ for all $n < \omega$.*

Proof. For any $n < \omega$ we construct an $\mathscr{I}$-almost disjoint family $\mathcal{A}_n$ as follows. Let $\mathcal{A}_0$ be an $\mathscr{I}$-almost disjoint family of $\mathscr{I}$-positive subsets of X_0 of cardinality $\mathfrak{c}$. For any n, we have two cases: (1) if $X_{n+1} \setminus \bigcup_{k \leq n} X_k$ is $\mathscr{I}$-positive then we choose an $\mathscr{I}$-almost disjoint family $\mathcal{B}_n$ of $\mathscr{I}$-positive subsets of $X_{n+1} \setminus \bigcup_{k \leq n} X_k$ of cardinality $\mathfrak{c}$ and then define $\mathcal{A}_{n+1} = \mathcal{A}_n \cup \mathcal{B}_n$, and (2) if X_{n+1} is $\mathscr{I}$-almost contained in $\bigcup_{k \leq n} X_k$ then we have two new cases: (a) if there is an element A in $\mathcal{A}_n$ $\mathscr{I}$-almost contained in X_{n+1} then we take an $\mathscr{I}$-almost disjoint family $\mathcal{C}_n$ of positive subsets of A of cardinality $\mathfrak{c}$ and then we define $\mathcal{A}_{n+1} = (\mathcal{A}_n \setminus \{A\}) \cup \mathcal{C}_n$, and (b) if every $A \in \mathcal{A}_n$ is $\mathscr{I}$-almost disjoint with X_{n+1} then we take an $\mathscr{I}$-almost disjoint family $\mathcal{D}_n$ of $\mathscr{I}$-positive subsets of X_{n+1} of cardinality $\mathfrak{c}$, and then we define $\mathcal{A}_{n+1} = \mathcal{A}_n \cup \mathcal{D}_n$. Note that in every case, $\mathcal{A}_{n+1}$ is an $\mathscr{I}$-almost disjoint family of cardinality $\mathfrak{c}$. Define $\mathcal{A} = \bigcup_{n<\omega} \bigcap_{k \geq n} \mathcal{A}_k$. Note that $\mathcal{P}(X_n) \cap \mathcal{A}$ is uncountable for all n, since only countably many of the elements of $\mathcal{A}_n$ could have been modified. Finally, we can recursively define Y_n as follows: pick $Y_0 \in \mathcal{A}$ contained in X_0 and $Y_{n+1} = Z_{n+1} \setminus \bigcup_{k \leq n} Y_k$, where Z_{n+1} is an element of $\mathcal{A}$ contained in X_{n+1}. □

1.3 Ideals and submeasures

A natural way to define ideals on ω is by using submeasures. They provide ideals with low complexity and good combinatorial properties.

Definition 1.3.1. A *submeasure on* a set X is a function $\varphi : \mathcal{P}(X) \to [0, \infty]$ satisfying:

- $\varphi(\emptyset) = 0$,
- (Monotonicity) If $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$ and
- (Subadditivity) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$.

If φ is a submeasure on ω and satisfies:

- (Lower semicontinuity) $\varphi(A) = \lim_{n \to \infty} \varphi(A \cap n)$

then φ is called a *lower semicontinuous submeasure*, which is abbreviated by lscsm. For any lscsm φ, the finite and exhaustive ideals are defined as follows:

- $Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$ and
- $Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}$.

It is immediate that $Fin(\varphi)$ is an F_σ-ideal and $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal. The following theorem claims that the converses are valid too.

Theorem 1.3.2. *Let $\mathscr{I}$ be an ideal on ω. Then*

- *(Mazur [38]) If $\mathscr{I}$ is an F_σ-ideal then there is a lscsm φ such that $\mathscr{I} = Fin(\varphi)$ and*
- *(Solecki [43]) If $\mathscr{I}$ is an analytic P-ideal then there is a lscsm φ such that $\mathscr{I} = Exh(\varphi)$.*

  If $\mathscr{I}$ is an F_σ P-ideal then there is a lscsm φ such that $\mathscr{I} = Exh(\varphi) = Fin(\varphi)$. □

It follows that many ideals given in the examples section 1.6 are F_σ-ideals or analytic P-ideals.

1.4 Cardinal invariants of ideals on ω

Let $\mathscr{I}$ be a tall ideal on ω. The cardinal invariants of $\mathscr{I}$ are defined as follows (see [20]).

- $\mathsf{add}^*(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \wedge (\forall I \in \mathscr{I})(\exists J \in \mathscr{A})(|J \setminus I| = \aleph_0)\}$.
- $\mathsf{cov}^*(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \wedge (\forall X \in [\omega]^{\aleph_0} \cap \mathscr{I})(\exists J \in \mathscr{A})(|X \cap J| = \aleph_0)\}$.
- $\mathsf{non}^*(\mathscr{I}) = \min\{|\mathscr{X}| : \mathscr{X} \subseteq [\omega]^{\aleph_0} \wedge (\forall I \in \mathscr{I})(\exists X \in \mathscr{X})(I \cap X =^* \emptyset)\}$.

Cardinal invariants of ideals on ω satisfy the following relations:

$$\begin{tikzcd} & \mathsf{cov}^*(\mathscr{I}) \arrow[dr] & \\ \mathsf{add}^*(\mathscr{I}) \arrow[ur] \arrow[dr] & & \mathsf{cof}(\mathscr{I}) \\ & \mathsf{non}^*(\mathscr{I}) \arrow[ur] & \end{tikzcd} \tag{1.2}$$

where every arrow points at the bigger or equal cardinal number.

Remark 1.4.1. Let $\mathscr{I}$ be a tall ideal on ω. Then

- $\mathsf{non}^*(\mathscr{I}) = \min\{|\mathscr{X}| : \mathscr{X} \subseteq [\omega]^{\aleph_0} \wedge (\forall I \in \mathscr{I})(\exists X \in \mathscr{X})(I \cap X = \emptyset)\}$ and
- $\mathsf{cov}^*(\mathscr{I}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \wedge (\forall X \in [\omega]^{\aleph_0})(\exists J \in \mathscr{A})(|X \cap J| = \aleph_0)\}$.

Proof. Note that if $\mathscr{X}$ witnesses $\mathsf{non}^*(\mathscr{I})$ then $\mathscr{X}' = \{X \setminus n : X \in \mathscr{X} \wedge n < \omega\}$ is a witness for $\mathsf{non}^*(\mathscr{I})$ too; and since tall ideals have elements intersecting infinitely each infinite set, we are done. □

The following remark is an immediate consequence of definitions.

Remark 1.4.2. Let $\mathscr{I}$ be a tall ideal on ω. Then,

(1) $\mathscr{I}$ is a P-ideal iff $\aleph_0 < \mathsf{add}^*(\mathscr{I})$ and

(2) $\mathscr{I}$ is ω-splitting iff $\aleph_0 < \mathsf{non}^*(\mathscr{I})$.

□

1.5 Orders on ideals on ω

We now introduce four (pre)orders between ideals on ω and then we will discuss how they impact on cardinal invariants of ideals. Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω.

- (Katětov order) $\mathscr{I} \leq_K \mathscr{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathscr{J}$, for all $I \in \mathscr{I}$.

- (Katětov-Blass order) $\mathscr{I} \leq_{KB} \mathscr{J}$ if there is a finite-to-one function $f : \omega \to \omega$ such that $f^{-1}[I] \in \mathscr{J}$, for all $I \in \mathscr{I}$.
- (Rudin-Keisler order) $\mathscr{I} \leq_{RK} \mathscr{J}$ if there is a function $f : \omega \to \omega$ such that $A \in \mathscr{I}$ if and only if $f^{-1}[I] \in \mathscr{J}$.
- (Tukey order) $\mathscr{I} \leq_T \mathscr{J}$ if there is a function $f : \mathscr{I} \to \mathscr{J}$ such that for every $\subseteq$-bounded set $X \subseteq \mathscr{J}$, $f^{-1}[X]$ is $\subseteq$-bounded in $\mathscr{I}$.

We will say $\mathscr{I}$ and $\mathscr{J}$ are *Katětov-equivalent* if $\mathscr{I} \leq_K \mathscr{J}$ and $\mathscr{J} \leq_K \mathscr{I}$. Analogously *Katětov-Blass*, *Rudin-Keisler* and *Tukey-equivalent* are defined.

Remark 1.5.1. Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω.

(1) If $\mathscr{I} \subseteq \mathscr{J}$ then $\mathscr{I} \leq_K \mathscr{J}$.

(2) If $X \in \mathscr{I}^+$ then $\mathscr{I} \leq_K \mathscr{I} \restriction X$ and $\mathscr{I} \restriction X \leq_T \mathscr{I}$.

Proof. Id_ω is a witness of $\mathscr{I} \leq_K \mathscr{J}$. Inclusion of X into ω is a witness of $\mathscr{I} \leq_K \mathscr{I} \restriction X$, and inclusion of $\mathscr{I} \restriction X$ into $\mathscr{I}$ is a witness of $\mathscr{I} \restriction X \leq_T \mathscr{I}$. $\square$

Theorem 1.5.2. *Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω. Then*

(1) If $\mathscr{I} \leq_K \mathscr{J}$ then $\mathsf{non}^*(\mathscr{I}) \leq \mathsf{non}^*(\mathscr{J})$ and $\mathsf{cov}^*(\mathscr{J}) \leq \mathsf{cov}^*(\mathscr{I})$.

(2) If $\mathscr{I} \leq_T \mathscr{J}$ then $\mathsf{cof}(\mathscr{I}) \leq \mathsf{cof}(\mathscr{J})$ and $\mathsf{add}^*(\mathscr{J}) \leq \mathsf{add}^*(\mathscr{I})$.

Proof. (1) Let $\mathscr{X}$ witness the definition of $\mathsf{non}^*(\mathscr{J})$ and let $f : \omega \to \omega$ be a witness to $\mathscr{I} \leq_K \mathscr{J}$. Define $\mathscr{Y} = \{f[X] : X \in \mathscr{X}\}$. Given $I \in \mathscr{I}$, $f^{-1}[I] \in \mathscr{J}$ and so, there exists a $X \in \mathscr{X}$ such that $X \cap f^{-1}[I]$ is finite. Hence, $f[X] \cap I$ is finite, proving that $\mathsf{non}^*(\mathscr{I}) \leq |\mathscr{Y}| \leq \mathsf{non}^*(\mathscr{J})$.

Now, let $\mathscr{A}$ witness the definition of $\mathsf{cov}^*(\mathscr{I})$. We claim that

$$\mathscr{B} = \{f^{-1}[A] : A \in \mathscr{A}\} \cup \{f^{-1}[F] : F \in \mathbf{Fin}\}$$

is a witness to the definition of $\mathsf{cov}^*(\mathscr{J})$. Let X be an infinite subset of ω. If $f[X]$ is infinite then there exists $A \in \mathscr{A}$ such that $A \cap f[X]$ is infinite and hence $X \cap f^{-1}[A]$ is infinite. If $f[X]$ is finite then $X \subseteq f^{-1}[f[X]] \in \mathscr{B}$. In both cases, X has an infinite intersection with a member of $\mathscr{B}$.

(2) Let $f : \mathscr{I} \to \mathscr{J}$ be a Tukey reduction and $\mathscr{X}$ a base for $\mathscr{J}$. For every $X \in \mathscr{X}$ there exists an $I_X \in \mathscr{I}$ such that $I \subseteq I_X$ if $f(I) \subseteq X$. Then, $\{I_X : X \in \mathscr{X}\}$ is cofinal in $\mathscr{I}$.

Now, let $\mathscr{A}$ be a witness for $\mathsf{add}^*(\mathscr{I})$. Note that direct image of a $\subseteq$-unbounded subset under a Tukey reduction is $\subseteq$-unbounded. Hence, $\{f(A) : A \in \mathscr{A}\}$ is unbounded. $\square$

By Schoenfield absoluteness theorem we show that the Katětov order among Borel ideals is absolute for models $M \subseteq N$ such that $\omega_1^N \subseteq M$.

Proposition 1.5.3. *If $\mathscr{I}$ and $\mathscr{J}$ are Borel ideals then the relation $\mathscr{I} \leq_K \mathscr{J}$ is absolute for models $M \subseteq N$ such that $\omega_1^N \subseteq M$.*

Proof. If $\mathscr{I}$ and $\mathscr{J}$ are Borel ideals then there are Σ_0-formulae $\varphi_{\mathscr{I}}$ and $\varphi_{\mathscr{J}}$ such that for any $A \subseteq \omega$,

$$A \in \mathscr{I} \quad \text{if and only if} \quad \varphi_{\mathscr{I}}(A) \text{ holds}$$

and

$$A \in \mathscr{J} \quad \text{if and only if} \quad \varphi_{\mathscr{J}}(A) \text{ holds.}$$

Then we can express Katětov order between $\mathscr{I}$ and $\mathscr{J}$ by the formula:

$$(\exists f \in \omega^\omega)\,(\forall I \in \mathcal{P}(\omega))\,(\forall J \in \mathcal{P}(\omega))$$
$$\big[\varphi_{\mathscr{I}}(I) \to \big((\forall n \in \omega)\,(f(n) \in I \leftrightarrow n \in J) \to \varphi_{\mathscr{J}}(J)\big)\big]$$

which is a Σ^1_2 formula. $\square$

1.6 Examples

In this section we define most of the ideals and families of ideals on ω which we have isolated as critical ideals for combinatorial or measure theoretic properties. In most cases, we calculate their cardinal invariants and analytic complexity.

Fin and $\emptyset$

Fin is the ideal of finite subsets of ω. **Fin** is the unique countable ideal on ω so, its complexity is F_σ and its cardinal invariants are trivial. **Fin** is not a tall ideal. By $\emptyset$ we denote the ideal whose unique element is the empty set. Strictly speaking, $\emptyset$ is not an ideal on ω, as we stipulated that all ideals on ω contain all the finite subsets of ω, but it will be useful to have notation for it.

nwd

nwd is the ideal on the set of rational numbers $\mathbb{Q}$ whose elements are the nowhere dense subsets of $\mathbb{Q}$. nwd is a tall non P-ideal, and its analytic complexity is $F_{\sigma\delta}$ (see corollary 4.3.5). A result of Keremedis, reformulated and proved by Balcar, Hernández-Hernández and Hrušák ([1] Theorem 1.4) shows that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$. In Theorem 1.6 of [1] was proved that $\mathsf{cof}(\mathsf{nwd}) = \mathsf{cof}(\mathcal{M})$. Any countable base for open sets of $\mathbb{Q}$ is a witness for $\mathsf{non}^*(\mathsf{nwd}) = \aleph_0$.

Solecki's ideal $\mathcal{S}$

Solecki's ideal $\mathcal{S}$ is the ideal on the countable set

$$\Omega = \{A \in Clop(2^\omega) : \lambda(A) = \frac{1}{2}\},$$

which is generated by the subsets of Ω with non-empty intersection. Equivalently, a subbase for $\mathcal{S}$ is the family of all subsets of Ω of the form:

$$I_x = \{A \in \Omega : x \in A\}$$

where x is an element of 2^ω.

Proposition 1.6.1. *$\mathcal{S}$ is a tall F_σ- ideal.*

Proof. Let $\{A_n : n < \omega\}$ be an infinite subset of Ω, and define $Y = \{x \in 2^\omega : (\exists^\infty n)(x \in A_n)\}$. We will see that $\delta = \lambda(Y) \geq \frac{1}{2}$. Suppose $\delta < \frac{1}{2}$. Define $B_n = A_n \setminus Y$. So, $\lambda(B_n) \geq \frac{1}{2} - \delta = \varepsilon > 0$. Define for every $a \in [\omega]^{<\omega}$

$$C_a = \{x \in 2^\omega : x \in B_n \leftrightarrow n \in a\}.$$

Then, $\{Y\} \cup \{C_a : a \in [\omega]^{<\omega}\}$ is a countable partition of 2^ω in measurable sets. Let G be a finite subset of $[\omega]^{<\omega}$ such that $\lambda(Y \cup \bigcup_{a\in G} C_a) \geq 1 - \frac{\varepsilon}{2}$. If $n \notin \bigcup G$ then $B_n \cap \bigcup_{a\in G} C_a = \emptyset$. But $\lambda(B_n) \geq \varepsilon$ for all n. This is a contradiction.

The function $\varphi(A) = \min\{|X| : X \subseteq 2^\omega \wedge (\forall a \in A)(\exists x \in X)(x \in a)\}$ is a lscsm such that $\mathcal{S} = Fin(\varphi)$, proving that $\mathcal{S}$ is an F_σ-ideal. □

On the cardinal invariants of $\mathcal{S}$ we have the following result.

Theorem 1.6.2. *The following holds.*

(1) $\mathsf{add}^*(\mathcal{S}) = \mathsf{non}^*(\mathcal{S}) = \aleph_0$,

(2) $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ *and*

(3) $\mathsf{cof}(\mathcal{S}) = \mathfrak{c}$.

Proof. In order to prove (1), it will be enough to find a countable family $\mathcal{A}$ of subsets of Ω such that for every $I \in \mathcal{S}$ exists $A \in \mathcal{A}$ such that $A \cap I =^* \emptyset$. For every $s \in \bigcup_{2 \leq n < \omega} 2^n$ define $A_s = \{A \in \Omega : A \cap \langle s \rangle = \emptyset\}$. Note that A_s is infinite for all s, since $\lambda(\langle s \rangle) \leq \frac{1}{4}$. Moreover, if F is a finite subset of $\bigcup_{2 \leq n < \omega} 2^n$ and $\lambda(\bigcup_{s \in F} \langle s \rangle) \leq \frac{1}{4}$ then $B_F = \{A \in \Omega : (\forall s \in F)(\langle s \rangle \cap A = \emptyset)\}$ is infinite. Define $\mathcal{A}$ as the family of all B_F with F a finite subset of $\bigcup_{2 \leq n < \omega} 2^n$. Now, given $A \in \mathcal{S}$, there exist $x_0, \ldots, x_k \in 2^\omega$ such that $A \subseteq \bigcup_{i \leq n} I_{x_i}$. Pick s_i an initial segment of x_i such that $\sum_{i=1}^{k} \lambda(\langle s_i \rangle) \leq \frac{1}{4}$. Hence, for $F = \{s_i : i \leq k\}$ we have $A \cap B_F = \emptyset$.

In order to prove (2), we need the following result. Here, λ^* denotes the outer Lebesgue measure on 2^ω.

Lemma 1.6.3. *Let X be a subset of 2^ω.*

(a) If $\lambda^*(X) < \frac{1}{2}$ then there exists an infinite subset $\mathcal{A}$ of Ω such that $I_x \cap \mathcal{A} =^* \emptyset$, for all $x \in X$.

(b) If $\lambda^*(X) > \frac{1}{2}$ then for every infinite subset $\mathcal{A}$ of Ω there exists $x \in X$ such that $|I_x \cap \mathcal{A}| = \aleph_0$.

Proof. (a) Let U be an open subset of 2^ω such that $\lambda(U) < \frac{1}{2}$ and $X \subseteq U$. Let $\langle U_n : n < \omega \rangle$ be an increasing family of clopen sets such that $U = \bigcup_n U_n$. For every $n < \omega$, choose $A_n \in \Omega$ such that $A_n \cap U_n = \emptyset$. Then, if $x \in X$ then there is $n < \omega$ such that $x \in U_n$ and then $x \in A_k$ implies $k < n$.

(b) Let $\{A_n : n < \omega\}$ be a countable subset of Ω, and define $Y = \{x \in 2^\omega : \exists^\infty n (x \in A_n)\}$. In the proof of proposition 1.6.1 was proved that $\lambda(Y) = \delta \geq \frac{1}{2}$. Given $\lambda^*(X) > \frac{1}{2}$, we have $X \cap Y \neq \emptyset$. □

Let $\mathcal{T}$ be a witness for $\mathsf{cov}^*(\mathcal{S})$. For every $T \in \mathcal{T}$ pick a finite subset a_T of 2^ω such that $T \subseteq \bigcup_{x \in a_T} I_x$. Define $X = \bigcup_{T \in \mathcal{T}} a_T$ and $\mathcal{T}' = \{I_x : x \in X\}$. Note that $\mathcal{T}'$ is a witness for $\mathsf{cov}^*(\mathcal{S})$. By (a) of previous lemma, we have $\lambda^*(X) \geq \frac{1}{2}$, and so, $X \notin \mathcal{N}$. Obviously, $|X| = \mathsf{cov}^*(S)$, proving $\mathsf{non}(\mathcal{N}) \leq \mathsf{cov}^*(\mathcal{S})$.

On the other hand, let $X \subseteq 2^\omega$ be a witness for $\mathsf{non}(\mathcal{N})$. By defining $X + Fin = \{y \in 2^\omega : (\exists x \in X)(y =^* x)\}$ and proving $\lambda^*(X + Fin) \geq \frac{1}{2}$ we will be done (by (b) in previous lemma). Actually, arguments given in the proof of measure 0-1 law (proposition 1.2.2(2)) prove that $\lambda^*(X + Fin) = 1$.

Now, we will prove (3). Let $\mathcal{A}$ be a subset of $\mathcal{S}$ with $|A| < \mathfrak{c}$. For every $B \in \mathcal{A}$, pick a finite subset a_B of 2^ω such that $B \subseteq \bigcup_{x \in a_B} I_x$. Note that $|\bigcup_{B \in \mathcal{A}} a_B| < \mathfrak{c}$. Let $y \in 2^\omega \setminus \bigcup_{B \in \mathcal{A}} a_B$. Note that, for every $B \in \mathcal{A}$, $I_y \setminus B \supseteq I_y \setminus \bigcup_{x \in a_B} I_x$ and $|I_y \setminus \bigcup_{x \in a_B} I_x| = \aleph_0$. □

$\mathcal{ED}$

The *eventually different ideal* is defined by

$$\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists m, n \in \omega)(\forall k \geq n)(|(A)_k| \leq m)\}.^1$$

$\mathcal{ED}$ is the ideal on the countable set $\omega \times \omega$ generated by vertical lines and graphs of functions in ω^ω. It can be thought as the ideal on ω generated by the elements of an infinite partition of ω in infinite sets, and selectors of this partition. $\mathcal{ED}$ is an F_σ-ideal, since $\mathcal{ED} = Fin(\varphi)$ where φ is the lscsm defined by $\varphi(A) = \min\{n < \omega : (\forall m \geq n)|A_m| \leq n\}$, for all $A \subseteq \omega$.

On the cardinal invariants of $\mathcal{ED}$ we have the following result.

Theorem 1.6.4. *The following conditions hold.*

(1) $\mathsf{add}^*(\mathcal{ED}) = \mathsf{non}^*(\mathcal{ED}) = \aleph_0$,

(2) $\mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$ *and*

(3) $\mathsf{cof}(\mathcal{ED}) = \mathfrak{c}$.

Proof. (1) Let us proof that the set $\{\{n\} \times \omega : n < \omega\}$ is a witness for $\mathsf{non}^*(\mathcal{ED})$. If A is an infinite subset of $\omega \times \omega$ then there are two cases: (1) There exists $n < \omega$ such that $|(A)_n| = \aleph_0$, in such case $\{n\} \times \omega$ has an infinite intersection with A; and (2) A intersects an infinite number of columns, and in such case a selector of that intersections is in $\mathcal{ED}$.

In order to prove (2) we need de following result due to Bartoszyński and Miller.

Lemma 1.6.5 ([2], Lemma 2.4.8)**.** *For any cardinal κ the following are equivalent:*

(a) $\kappa < \mathsf{non}(\mathcal{M})$,

[1] See the notation for columns in products given in the preliminaries.

(b) $(\forall F \in [\omega^\omega]^\kappa)(\exists g \in \omega^\omega)(\exists X \in [\omega]^\omega)(\forall f \in F)(\forall^\infty n \in X)(f(n) \neq g(n))$ *and*

(c) [2] $(\forall F \in [\mathcal{C}]^\kappa)(\exists g \in \omega^\omega)(\forall S \in F)(\forall^\infty n)(g(n) \notin S(n))$. □

Let $\mathcal{F}$ be a subset of ω^ω with minimal cardinality such that

$$(\forall g \in \omega^\omega)(\forall X \in [\omega]^\omega)(\exists f \in \mathcal{F})(\exists^\infty n \in X)(f(n) = g(n)).$$

Define $\mathcal{A} = \mathcal{F} \cup \{\{n\} \times \omega : n < \omega\}$ (we are identifying every function $f \in \omega^\omega$ with its graph $\{(n, f(n)) : n < \omega\}$). Obviously $\mathcal{A} \subseteq \mathcal{ED}$. We claim that $\mathcal{A}$ is a covering family. Let A be an infinite subset of $\omega \times \omega$. If there exists $n < \omega$ such that $(A)_n$ is infinite, then $(A)_n$ is an infinite subset of an element of $\mathcal{A}$. If the set $X = \{n < \omega : (A)_n \neq \emptyset\}$ is infinite then there exists $f \in \mathcal{F}$ such that $f(n) = \min((A)_n)$ for an infinite number of elements n of X. Hence, $f \cap A$ is infinite. On the other hand, let $\mathcal{A}$ be a subset of $\mathcal{ED}$ with $|\mathcal{A}| < \mathsf{non}(\mathcal{M})$. For every $A \in \mathcal{A}$, let $n_A < \omega$ such that $|(A)_k| \leq n_A$ for all $k \geq n_A$, and define a function $S_A : \omega \to [\omega]^{\leq n_A}$ by:

$$S_A(n) = \begin{cases} \emptyset & \text{if } n < n_A \\ (A)_n & \text{if } n \geq n_A. \end{cases}$$

Note that $|\{S_A : A \in \mathcal{A}\}| = |\mathcal{A}|$, and by previous lemma, there is $g \in \omega^\omega$ such that for every $A \in \mathcal{A}$, $g(n) \notin S_A(n)$, for almost all $n < \omega$. Hence, $g \cap A$ is finite for all $A \in \mathcal{A}$, and so, $\mathcal{A}$ is not a covering family.

Finally we prove (3). We define a perfect subset C of $\mathcal{ED}$ as follows. Let $\psi : 2^{<\omega} \to \omega$ be defined by $\psi(t) = n$ if and only if t is the n-th element of $2^{<\omega}$ in the lexicographical order. For each $f \in 2^\omega$, let us define

$$A_f = \{(2^n, \psi(f \upharpoonright n)) : n < \omega\}.$$

Then, let $C = \{A_f : f \in 2^\omega\}$. Note that for any $I \in \mathcal{ED}$, the set

$$\{f \in 2^\omega : A_f \subseteq I\}$$

is just a finite set. Hence, for any family $\mathcal{A} \subseteq \mathscr{I}$ with less than $\mathfrak{c}$ elements, there is $f \in 2^\omega$ such that $A_f \not\subseteq I$ for all $I \in \mathcal{A}$. □

[2] $\mathcal{C} = \{S \in ([\omega]^{<\omega})^\omega : \sum_{n=1}^{\infty} \frac{|S(n)|}{n^2} < \infty\}$, definition 2.3.2 in [2].

$\mathcal{ED}_{fin}$

The ideal $\mathcal{ED}_{fin}$ is defined as the restriction of $\mathcal{ED}$ to the set

$$\Delta = \{(n, m) : m \leq n < \omega\}.$$

Also $\mathcal{ED}_{fin}$ can be thought as the ideal on ω generated by the selectors of a partition $\{I_n : n < \omega\}$ of ω in finite sets and such that $|I_n| \xrightarrow[n\to\infty]{} \infty$. $\mathcal{ED}_{fin}$ is a tall ideal (every infinite subset of $\omega \times \omega$ intersects an infinite number of columns, and a selector from that intersections is an infinite element of $\mathcal{ED}_{fin}$); and is F_σ since it is a restriction of an F_σ-ideal. $\mathcal{ED}_{fin}$ is an ω-splitting ideal, and moreover, is critical with respect to this property in Katětov-Blass order among Borel ideals, as we prove in theorem 3.2.1.

About the cardinal invariants of $\mathcal{ED}_{fin}$ we have the following result.

Theorem 1.6.6. *The following conditions hold.*

(1) $\mathsf{add}^*(\mathcal{ED}_{fin}) = \aleph_0$*,*

(2) $\mathfrak{s} \leq \mathsf{cov}^*(\mathcal{ED}_{fin})$ *and* $\mathsf{non}^*(\mathcal{ED}_{fin}) \leq \mathfrak{r}$*,*

(3) $\mathsf{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$ *,*

(4) $\mathsf{non}(\mathcal{M}) = \max\{\mathsf{cov}^*(\mathcal{ED}_{fin}), \mathfrak{b}\}$*, and*

(5) $\mathsf{cof}(\mathcal{ED}_{fin}) = \mathfrak{c}$*.*

Proof. There is not a set A in $\mathcal{ED}_{fin}$ such that every row $(\omega \setminus n) \times \{n\}$ is almost contained in A.

The first part of (2) will be shown in the proof of theorem 1.6.23.

Let us prove the second part of (2). We will say that a family $\mathcal{R}$ is *hereditarily reaping* if for every $X \in \mathcal{R}$ and every infinite subset Y of X there is R in $\mathcal{R}$ such that $R \subseteq Y$ or $R \subseteq X \setminus Y$.

Lemma 1.6.7. $\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R}$ *is hereditarily reaping*$\}$.

Proof. It will be enough to prove that there is a hereditarily reaping family of cardinality $\mathfrak{r}$. Let $\mathcal{Q}$ be a reaping family of cardinality $\mathfrak{r}$. Define $\mathcal{Q}_0 = \mathcal{Q}$ and by recursion, for any $n < \omega$, let $\mathcal{Q}_{n+1}$ be such that for any $A \in \mathcal{Q}_n$, $\mathcal{Q}_{n+1} \cap \mathcal{P}(A)$ is a reaping family of cardinality $\mathfrak{r}$. So, $\mathcal{R} = \bigcup_{n<\omega} \mathcal{Q}_n$ is a hereditarily reaping family. $\square$

Let $\mathcal{R}$ be a hereditarily reaping family, and for every $R \in \mathcal{R}$ and $n < \omega$ define

$$X_{R,n} = \{(m,n) : m \geq n \wedge m \in R\}.$$

We will see that $\mathcal{A} = \{X_{R,n} : R \in \mathcal{R} \wedge n < \omega\}$ witnesses $\mathsf{non}^*(\mathcal{ED}_{fin})$. Let I be in $\mathcal{ED}_{fin}$, and choose a family $\{f_i : i \leq N\}$ of functions such that $I \subseteq \bigcup_{i \leq N} f_i$. Define $A_j = \{k : (\exists i \leq N)(f_i(k) = j)\}$, for $j \leq N$. Let R_0 be in $\mathcal{R}$ such that $R_0 \cap A_0 = \emptyset$ or $R_0 \subseteq A_0$. In general, for $1 \leq j \leq N$ we can find $R_j \in \mathcal{R}$ such that $R_j \cap (R_{j-1} \cap A_j) = \emptyset$ or $R_j \subseteq R_{j-1} \cap A_j$. If the first case holds for some $j \leq N$ we are done, because for such j minimal, we have that $X_{R_j,j} \cap I = \emptyset$. Suppose that $R_j \subseteq R_{j-1} \cap A_j$ for all $1 \leq j \leq N$. Then, for any $k \in R_N$, $(I)_k = N+1$, and then, $X_{R_N,N+1} \cap I = \emptyset$.

In order to prove (3) we will need the following lemma, due to Bartoszyński and Miller.

Lemma 1.6.8 ([2], lemma 2.4.2)**.** *For any cardinal κ the following conditions are equivalent:*

1. $\kappa < \mathsf{cov}(\mathcal{M})$, *and*

2. $(\forall F \in [\omega^\omega]^\kappa)(\forall G \in [[\omega]^\omega]^\kappa)(\exists g \in \omega^\omega)(\forall f \in F)(\forall X \in G)$ $(\exists^\infty n \in X)(f(n) = g(n))$. $\square$

Let $\mathscr{X}$ be a subset of $[\Delta]^{\aleph_0}$ with $|\mathscr{X}| < \mathsf{cov}(\mathcal{M})$. For every $X \in \mathscr{X}$ define $G_X = \{n < \omega : (X)_n \neq \emptyset\}$ and let $f_X \in \omega^\omega$ be a function such that $f_X(n) \in (X)_n$. By lemma 1.6.8, there is a function $g \in \omega^\omega$ such that $f_X(n) = g(n)$ for infinitely many elements n of G_X, for all $X \in \mathscr{X}$. Then, $\Delta \cap g$ is an element of $\mathcal{ED}_{fin}$ intersecting infinitely to every element of $\mathscr{X}$, proving $|\mathscr{X}| < \mathsf{non}^*(\mathcal{ED}_{fin})$. It is a well known fact that $\mathsf{cov}(\mathcal{M}) \leq \mathfrak{d}$. Let κ be a cardinal lower than $\mathfrak{d}$ and $\mathsf{non}^*(\mathcal{ED}_{fin})$. We will use and prove the following lemma.

Lemma 1.6.9. *Let κ be an infinite cardinal. The following are equivalent.*

(a) $\kappa < \mathsf{non}^*(\mathcal{ED}_{fin})$ *and*

(b) for every bounded family $\mathcal{F}$ of κ functions in ω^ω and every family $\mathcal{A}$ of κ infinite subsets of ω there exists a function $g \in \omega^\omega$ such that for all $f \in \mathcal{F}$ and $A \in \mathcal{A}$, $f(n) = g(n)$ for infinitely many $n \in A$.

Proof. Assume that κ satisfies (b) and let $\mathcal{B}$ be a family of κ infinite subsets of Δ. For every $B \in \mathcal{B}$, let $X_B = \{n : (B)_n \neq \emptyset\}$ and $f_B : \omega \to \omega$ such that $f_B(n) \in (B)_n$ if $n \in X_B$, and $f_B(n) = 0$ if not. The families $\mathcal{F} = \{f_B : B \in \mathcal{B}\}$ and $\mathcal{A} = \{X_B : B \in \mathcal{B}\}$ have cardinality κ, and so, there exists a function $g \in \omega^\omega$ such that for all $B \in \mathcal{B}$ there are infinitely many $n \in X_B$ such that $g(n) = f_B(n)$, showing that g has an infinite intersection with B.

Now, assume that $\kappa < \mathsf{non}^*(\mathcal{ED}_{fin})$, $\mathcal{F} \subseteq \omega^\omega$ and $\mathcal{A} \subseteq [\omega]^\omega$ have cardinality κ, and $\mathcal{F}$ is bounded by an increasing function $h \in \omega^\omega$. We will identify every $f \in \mathcal{F}$ with a subset of an $\mathcal{ED}_{fin}$-positive subset Δ' of Δ, as follows: Define $X = h''\omega$, $\Delta' = \prod_{n\in X} n$, $A' = h''A$ if $A \in \mathcal{A}$, and for each $f \in \mathcal{F}$, define $f' : X \to \omega$ by $f'(n) = f(h^{-1}(n))$. So, $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is a family of infinite subsets of Δ'. Let $\mathcal{B} = \{f' \restriction A' : f \in \mathcal{F} \wedge A \in \mathcal{A}\}$. Since $|\mathcal{B}| = \kappa$, there exists $I \in \mathcal{ED}_{fin}$ such that $I \cap B$ is infinite for all $B \in \mathcal{B}$. Let $\{g_i : i \leq N\}$ be a set of functions in ω^ω such that $I \subseteq \bigcup_{i\leq N} g_i$. Choose $B_{f,A} = \{n \in A' : f'(n) = g_i(n)\}$, for some $i \leq N$ such that $|(f' \restriction A') \cap g_i| = \aleph_0$, and define $\mathcal{C} = \{B_{f,A} : f \in \mathcal{F} \wedge A \in \mathcal{A}\}$. By (1) $|\mathcal{C}| \leq \kappa < \mathfrak{r}$, and so, there exists $Y \in [\omega]^\omega$ such that $|Y \cap B_{f,A}| = \omega = |B_{f,A} \setminus Y|$ for all f and A. Inside Y, by considering $\{B_{f,A} \cap Y : f \in \mathcal{F} \wedge A \in \mathcal{A}\}$, there is a partition Y_0', Y_1' of Y such that $|Y_0' \cap B_{f,A}| = \aleph_0 = |Y_1' \cap B_{f,A}|$ for all f and A. Inductively, by considering such kind of partitions we can find a partition $\{Y_i : i \leq N\}$ of Y such that for all $i \leq N$, $|B_{f,A} \cap Y_i| = \aleph_0$. Now, define $g(n) = g_i(n)$ if $n \in Y_i$ and $g(n) = 0$ if $n \notin Y$. Given f and A, if $i \leq N$ is such that $B_{f,A} = \{n \in A' : f'(n) = g_i(n)\}$ then $f'(n) = g(n)$ for infinitely many $n \in Y_i \cap A'$, and so, $f(n) = g(h(n))$ for infinitely many $n \in h^{-1}[Y_i] \cap A$. □

Let us prove that $\kappa < \mathsf{cov}(\mathcal{M})$, by using lemma 1.6.8. Let F and G be families as in 1.6.8(2).

Claim. *There exists $h \in \omega^\omega$ such that for all $X \in G$, $f(n) < h(n)$ for infinitely many $n \in X$.*

Proof of the claim. For every f, A, let e_A be the enumeration function of A and let $h_{f,A} \in \omega^\omega$ be such that $h_{f,A}(n) \geq f(e_A(i))$ for all $i \leq n$. A function h which is not dominated by the family $\{h_{f,A} : A \in \mathcal{A} \wedge f \in \mathcal{F}\}$ does the work. □

Now, for every $f \in F$ define $f' \in \omega^\omega$ such that $f'(n) = f(n)$ if $f(n) < h(n)$ and $f'(n) = 0$ otherwise; define $C_{f,A} = \{n \in A : f(n) < h(n)\}$, $\mathcal{A}' = \{C_{f,A} :$

$f \in F \wedge A \in G\}$ and $\mathcal{F}' = \{f' : f \in F\}$. $\mathcal{F}'$ is bounded and so, by lemma 1.6.9, there is $g \in \omega^\omega$ such that for all $f \in \mathcal{F}$ and for all $A \in \mathcal{A}$, $g(n) = f'(n)$ for infinitely many $n \in C_{f,A}$ and in consequence, $g(n) = f(n)$ for infinitely many $n \in A$.

In order to prove (4), note that $\mathfrak{b} \leq \mathsf{non}(\mathcal{M})$ and note that $\mathcal{ED} \leq_K \mathcal{ED}_{fin}$ and so, $\mathsf{cov}^*(\mathcal{ED}_{fin}) \leq \mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$. We are going to use the following lemma.

Lemma 1.6.10 ([2], theorem 2.4.7)**.** $\mathsf{non}(\mathcal{M})$ *is the size of the smallest family* $\mathcal{F} \subseteq \omega^\omega$ *such that for every* $g \in \omega^\omega$ *there is an element* f *of* $\mathcal{F}$ *such that* $f(n) = g(n)$ *for infinitely many* $n \in \omega$. □

Let κ be a cardinal bigger than $\mathsf{cov}^*(\mathcal{ED}_{fin})$ and bigger than $\mathfrak{b}$. Let $\mathcal{G} = \{f_\alpha : \alpha < \kappa\}$ be an unbounded family of functions in ω^ω, and let G_α a witness of $\mathsf{cov}^*(\mathcal{ED}_{fin})$ in $\Delta_\alpha = \{\langle n, m\rangle : m \leq f_\alpha(n)\}$, for all $\alpha < \kappa$. Without loose of generality we can assume that every element I of G_α is the graph of a function $f \in \omega^\omega$. We will prove that $\mathcal{F} = \bigcup_{\alpha<\kappa} G_\alpha$ is such that for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ such that $f(n) = g(n)$ for infinitely many $n \in \omega$. Given $g \in \omega^\omega$, let $\alpha < \kappa$ be such that $f_\alpha \not\leq^* g$. Then, $g \cap \Delta_\alpha$ is infinite and so, there is $I \in G_\alpha$ such that $I \cap (g \cap \Delta_\alpha)$ is infinite. Since I is the graph of a function in $\mathcal{F}$, we are done. The proof of 1.6.4(3) is also a proof for (5). □

Remark 1.6.11. Parts (3) and (4) of previous theorem are particularly relevant since they show that $\mathsf{non}^*(\mathcal{ED}_{fin})$ and $\mathsf{cov}^*(\mathcal{ED}_{fin})$ complete in a sense the Cichoń's diagram, because they make $\mathsf{cov}(\mathcal{M})$ be the minimum of a pair of cardinal invariants, one of them $\mathfrak{d}$; and make $\mathsf{non}(\mathcal{M})$ be the maximum of a pair of cardinal invariants, one of them $\mathfrak{b}$.

The inequalities $\mathfrak{b} < \mathsf{non}(\mathcal{M})$ and $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$ are both known to be relatively consistent with ZFC, $\mathfrak{b} < \mathsf{non}(\mathcal{M})$ holds in the Random real model and $\mathsf{cov}(\mathcal{M}) < \mathfrak{d}$ holds in the Laver model. We will prove that the inequalities $\mathsf{cov}^*(\mathcal{ED}_{fin}) < \mathfrak{b}$ and $\mathsf{non}^*(\mathcal{ED}_{fin}) > \mathfrak{d}$ are both consistent.

Theorem 1.6.12. *$Con(ZFC)$ implies $Con\,(ZFC + \mathsf{cov}^*(\mathcal{ED}_{fin}) < \mathsf{add}(\mathcal{M}))$ and $Con\,(ZFC + \mathsf{cof}(\mathcal{M}) < \mathsf{non}^*(\mathcal{ED}_{fin}))$.*

Before proving this theorem, it is necessary to introduce some concepts and a result of preservation under finite support iteration of Laver type forcing with the Fréchet filter as a parameter.

Laver forcing $\mathbb{L}_{Fr}$ is defined as the family of all perfect trees $T \subseteq \omega^{<\omega}$ such that there is a node $stem(T) \in T$ such that for all $t \in T$ either $t \subseteq stem(T)$

or $stem(T) \subseteq t$ and for all node $t \supseteq stem(T)$, $succ_T(t)$ is a cofinite subset of ω. The order is the inclusion.

Let us introduce the following notation:

$$Fn(\Delta) = \{A \subseteq \Delta : \forall k, m_1, m_2\, ((k, m_1), (k, m_2) \in A \to m_1 = m_2)\}.$$

We will say that a forcing $\mathbb{P}$ *strongly preserves* $\mathsf{cov}^*(\mathcal{ED}_{fin})$ if for any $\mathbb{P}$-name $\dot{A}$ such that $\mathbb{P} \Vdash$ "$\dot{A} \in Fn(\Delta)$", there is a countable family $\{B_n : n < \omega\} \subseteq Fn(\Delta)$ such that for all $B \in Fn(\Delta)$ for which $|B \cap B_n| = \aleph_0$ for all n, it happens that $\mathbb{P} \Vdash$ "$|B \cap \dot{A}| = \aleph_0$". Note that if $\mathbb{P}$ strongly preserves $\mathsf{cov}^*(\mathcal{ED}_{fin})$ then $\mathbb{P} \Vdash$ "$2^\omega \cap V$ is a $\mathsf{cov}^*(\mathcal{ED}_{fin})$ family". We will prove that the finite support iteration of Laver forcing with Fréchet filter of arbitrary length strongly preserves $\mathsf{cov}^*(\mathcal{ED}_{fin})$.

Lemma 1.6.13. *Let $\dot{h}$ be a $\mathbb{L}_{Fr}$-name for an element of $Fn(\Delta)$. Then there is a countable subfamily $\{h_n : n < \omega\}$ of $Fn(\Delta)$ such that for any $g \in Fn(\Delta)$, if $|g \cap h_n| = \aleph_0$ for all n then $\mathbb{L}_{Fr} \Vdash$ "$|\dot{h} \cap g| = \aleph_0$".*

Proof. By shrinking $\dot{h}$ we can assume that the increasing enumeration $\{\dot{d}_n : n < \omega\}$ of $dom(\dot{h})$ is such that $d_n > \dot{l}_{gen}$, where $\dot{l}_{gen}$ is the $\mathbb{L}_{Fr}$-generic real. For any $n < \omega$ and $s \in \omega^{<\omega}$ define $rk_n(s)$ as follows:

- $rk_n(s) = 0$ iff there is $T \in \mathbb{L}_{Fr}$ with $stem(T) = s$ and there is $m < \omega$ such that $T \Vdash$ "$\dot{d}_n = m$",
- $rk_n(s) \leq \alpha$ iff there are infinitely many $k < \omega$ such that $rk_n(s^\frown k) < \alpha$, and
- $rk_n(s) = \min\{\alpha : rk_n(s) \leq \alpha\}$.

First, we claim that every $s \in \omega^\omega$ has a rk_n for all $n < \omega$. Suppose not. Let $s \in \omega^{<\omega}$ be without a rk_n. Then for any $m < \omega$ there are only finitely many $k < \omega$ such that $rk_n(s^\frown k)$ is defined. Hence, we can construct a tree $T \in \mathbb{L}_{Fr}$ such that for all $t \in T$, $rk_n(t)$ is not defined; but for an extension $T' \leq T$ deciding $\dot{d}_n$ we have a contradiction by considering $stem(T')$. Now, for $n < \omega$ and $s \in \omega^{<\omega}$ with $rk_n(s) = 1$, let us define $h_{n,s} : dom(h_{n,s}) \to \omega$ such that $m \in dom(h_{n,s})$ iff there is $k < \omega$ and there is a tree $T \in \mathbb{L}_{Fr}$ with $stem(T) = s^\frown k$ such that $T \Vdash$ "$\dot{d}_n = m$"; and
$h_{n,s}(m) =$
$\min\{l : \exists k(\exists T \in \mathbb{L}_{Fr}) s^\frown k \text{ prefers } \dot{h}(m) = l \wedge stem(T) = s^\frown k \wedge T \Vdash \text{“}m = \dot{d}_n\text{”}\}$.

Let us note that if $rk_n(s) = 1$ then there exist infinitely many $m < \omega$ for which there is $k < \omega$ such that $s^\frown k$ prefers $\dot{d}_n = m$. If not, there were m such that for infinitely many k, $s^\frown k$ prefers $\dot{d} = m$, and then, s itself prefers $\dot{d} = m$, showing that $rk_n(s) = 0$. Let g be an element of $Fn(\Delta)$ such that $|g \cap h_{n,s}| = \aleph_0$ for all n and s possible. By contradiction, suppose that there is a tree $T \in \mathbb{L}_{Fr}$ and there is $m < \omega$ such that $T \Vdash$ "$(\forall n \geq m)(\dot{h}(\dot{d}_n) \neq g(\dot{d}_n))$". Let s be the stem of T and $n \geq m$ such that $rk_n(s) > 0$. By following inside T we can find $t \in T$ with $s \subseteq t$ and $rk_n(t) = 1$. By cofiniteness of $succ_T(t)$, there is a $k \in succ_T(t)$ and $M < \omega$ such that $t^\frown k$ prefers $\dot{h}(M) = g(M)$, and hence there is an extension T' of T such that $T' \Vdash$ "$\dot{d}_n = M \ \wedge \ \dot{h}(M) = g(M)$", a contradiction. $\square$

Note that easily we can strength the previous lemma by adding the same condition for any sequence $\langle \dot{h}_n : n < \omega \rangle$ of $\mathbb{L}_{Fr}$-names for elements of $Fn(\Delta)$.

Strong preservation of $\mathsf{cov}^*(\mathcal{ED}_{fin})$ property is even satisfied by finite support iteration of $\mathbb{L}_{Fr}$.

Lemma 1.6.14. *Let α be an ordinal number and let $\mathbb{L}^\alpha_{Fr}$ be the finite support iteration of $\mathbb{L}_{Fr}$ with length α. Then, if $\dot{h}$ is an $\mathbb{L}^\alpha_{Fr}$-name for an element of $Fn(\Delta)$ then there is a countable subfamily $\{k_n : n < \omega\}$ of $Fn(\Delta)$ such that for any $g \in Fn(\Delta)$, if $|g \cap k_n| = \aleph_0$ for all n then $\mathbb{L}^\alpha_{Fr} \Vdash$ "$|\dot{h} \cap g| = \aleph_0$".*

Proof. By induction on α. For successor step $\alpha+1$ we can consider a sequence $\langle \dot{h}_n : n < \omega \rangle$ of $\mathbb{L}^\alpha_{Fr}$-names for elements of $Fn(\Delta) \cap V[G]$ where G is an $\mathbb{L}^\alpha_{Fr}$-generic filter on V, such that

$$V[G] \models (\forall g \in Fn(\Delta)) \Big((\forall n < \omega |g \cap h_n| = \aleph_0) \to \mathbb{L}_{Fr} \Vdash \text{“}|\dot{h} \cap g| = \aleph_0\text{”} \Big)$$

and by inductive hypothesis and the note on previous lemma, in V there is a countable family $\{\dot{k}_n : n < \omega\}$ such that for any $g \in Fn(\Delta)$ such that $|g \cap \dot{k}_n| = \aleph_0$ for all $n < \omega$ it happens that $\mathbb{L}^\alpha_{Fr} \Vdash$ "$\forall n < \omega |g \cap \dot{h}_n| = \aleph_0$", and then, if G' is a $\mathbb{L}_{Fr}$-generic filter on $V[G]$ then $V[G][G'] \models |g \cap \dot{h}| = \aleph_0$.

For $cf(\alpha) = \omega$, let $\langle \alpha_m : m < \omega \rangle$ an increasing sequence with $\sup_{k<\omega} \alpha_k = \alpha$ and inductively assume that for each k, the forcing $\mathbb{L}^{\alpha_{k+1}-\alpha_k}_{Fr}$ strongly preserves $\mathsf{cov}^*(\mathcal{ED}_{fin})$-families, and let $\dot{h}$ be a $\mathbb{L}^\alpha_{Fr}$-name such that $\mathbb{L}^\alpha_{Fr} \Vdash$ "$\dot{h} \in Fn(\Delta)$". Let G be a $\mathbb{L}^\alpha_{Fr}$ generic ultrafilter on V. For any $m < \omega$, let $p^m \in G$ be such that p^m decides $\dot{h} \restriction m$. Let $n_m < \omega$ be such that $p^m \in \mathbb{L}^{\alpha_{n_m}}_{Fr}$, and choose a countable family of $\mathbb{L}^{\alpha-\alpha_{n_m}}_{Fr}$-names $\{\dot{k}^m_i : i < \omega\}$ such that if $V[G] = V[G \restriction \mathbb{L}^{\alpha_{n_m}}_{Fr}][G^m]$ then

$$V[G \restriction \mathbb{L}^{\alpha_{n_m}}_{Fr}] \models (\forall B)((\forall i)(|B \cap \dot{k}^m_i| = \aleph_0) \to \mathbb{L}^{\alpha-\alpha_{n_m}}_{Fr} \Vdash \text{“}|B \cap \dot{h}| = \aleph_0\text{”}).$$

By inductive hypothesis for any $m, i < \omega$ there is a family $\{k^j_{m,i} : j < \omega\}$ such that for any g, if $|g \cap k^j_{m,i}| = \aleph_0$ for all $j < \omega$ then $\mathbb{L}^{\alpha_{n_m}}_{Fr} \Vdash$ "$|g \cap \dot{k}_{m,i} = \aleph_0|$", and hence if $|g \cap k^j_{m,i}| = \aleph_0$ for all $m, i, j < \omega$ then $V[G] \models |g \cap \dot{h}| = \aleph_0$.

For $cf(\alpha) > \omega$ we only note that $\dot{h}$ is essentially a $\mathbb{L}^{\beta}_{Fr}$-name for some $\beta < \alpha$ and then by inductive hypothesis we have that $\mathbb{L}^{\beta+1}_{Fr} \Vdash$ "$|g \cap \dot{h}| = \aleph_0$". $\square$

Now we can prove the theorem 1.6.12.

Proof of theorem 1.6.12. Let V be a model of CH and let $\dot{A}$ be a $\mathbb{L}^{\omega_2}_{Fr}$-name for an infinite subset of Δ and let G be a $\mathbb{L}^{\omega_2}_{Fr}$-generic ultrafilter on V. Without lose of generality, we can assume that $V[G] \models \dot{A} \in Fn(\Delta)$. Since $\mathbb{L}^{\omega_2}_{Fr}$ strongly preserves $\mathsf{cov}^*(\mathcal{ED}_{fin})$, we have that there is $\langle B_n : n < \omega \rangle \in Fn(\Delta) \cap V$ such that if $B \in Fn(\Delta) \cap V$ is such that $|B \cap B_n| = \aleph_0$ for all n then $\mathbb{L}^{\omega_2}_{Fr} \Vdash$ "$|B \cap \dot{A}| = \aleph_0$". Then we can find such set B intersecting every B_n in an infinite set as follows: Let $\{D_n : n < \omega\}$ be a disjoint refinement of the family $\{dom(B_n) : n < \omega\}$ (use lemma 1.2.5 for $\mathscr{I} = \mathbf{Fin}$) and define $B \in Fn(\Delta)$ by $B = \bigcup_n B_n \restriction D_n$. Hence, we have proved that $Fn(\Delta) \cap V$ is a $\mathsf{cov}^*(\mathcal{ED}_{fin})$-family in $V[G]$ and since $V \models CH$, we conclude that it is a $\mathsf{cov}^*(\mathcal{ED}_{fin})$-family of cardinality $\aleph_1$. It is well known, $\mathbb{L}^{\omega_2}_{Fr}$ adds a Cohen real and adds an unbounded real, hence $V[G] \models \mathsf{cov}^*(\mathcal{M}) = \mathfrak{b} = \omega_2$, and since $\mathsf{add}(\mathcal{M}) = \min\{\mathsf{cov}(\mathcal{M}), \mathfrak{b}\}$ we have proved the first consistency result. For the second one we refer the reader to an informal communication of Jörg Brendle [7] where it is proved that if $V \models MA + \neg CH$ and G is a $\mathbb{L}^{\omega_1}_{Fr}$ generic ultrafilter then $V[G] \models \mathsf{cof}(\mathcal{M}) = \omega_1 \wedge \mathsf{non}^*(\mathcal{ED}_{fin}) > \omega_1$. $\square$

The ideal $\mathcal{ED}_{fin}$ is critical with respect to ω-splitting property in Katětov order among Borel ideals, i.e. $\mathscr{I}$ is a Borel ω-splitting ideal if and only if $\mathscr{I} \geq_{KB} \mathcal{ED}_{fin}$. Consequently $\mathsf{non}^*(\mathscr{I}) \geq \mathsf{non}^*(\mathcal{ED}_{fin})$ for all the ω-splitting Borel ideals, and then, uniformities of Borel ideals $\mathscr{I}$ satisfy either $\mathsf{non}^*(\mathscr{I}) = \aleph_0$ or $\mathsf{non}^*(\mathscr{I}) \geq \mathsf{non}^*(\mathcal{ED}_{fin})$. This highlights how important the cardinal $\mathsf{non}^*(\mathcal{ED}_{fin})$ is.

conv

The ideal conv is defined as the ideal on $\mathbb{Q} \cap [0, 1]$ generated by sequences in $\mathbb{Q} \cap [0, 1]$ convergent in $[0, 1]$. In other words, $A \in \mathsf{conv}$ if and only if there is $N \in \omega$ such that for every $\varepsilon > 0$ there is $a \in [A]^N$ such that for all but finitely many $n \in A$ there is $m \in a$ such that $|n - m| < \varepsilon$. Clearly, conv is a

Borel ideal, moreover, its complexity is $F_{\sigma\delta\sigma}$. It is clear that $\mathsf{conv} \subseteq \mathsf{nwd}$. In subsection 3.2 we will see that conv is a critical ideal for a property closely related to the extendability of ideals to F_σ ideals. About cardinal invariants of conv we have the following theorem.

Theorem 1.6.15. *The following holds.*

(1) $\mathsf{add}^*(\mathsf{conv}) = \mathsf{non}^*(\mathsf{conv}) = \aleph_0$ *and*

(2) $\mathsf{cov}^*(\mathsf{conv}) = \mathsf{cof}(\mathsf{conv}) = \mathfrak{c}$.

Proof. (1) Let $\langle x_n : n < \omega\rangle$ a strictly increasing sequence of real numbers in $(0, 1]$ and for each $n < \omega$, define $A_n = (x_{n-1}, x_n)$ (put $x_{-1} = 0$). Hence, if $I \in \mathsf{conv}$ then $|I \cap A_n| < \aleph_0$ for all but finitely many $n < \omega$.

(2) Let κ be a cardinal number smaller than $\mathfrak{c}$, $\{I_\alpha : \alpha < \kappa\} \subseteq \mathsf{conv}$ and $X = \{x \in [0, 1] : (\exists\alpha < \kappa)(x \in \overline{I_\alpha})\}$. Then $|X| = \kappa$. Take $y \in [0, 1] \setminus X$ and choose a sequence $\langle q_n : n < \omega\rangle$ in $\mathbb{Q} \cap [0, 1]$ which converges to y. Then $I_\alpha \cap \{y_n : n < \omega\}$ is finite for all $\alpha < \kappa$. □

Cantor-Bendixson ideals

Let α be a countable ordinal and $A \subseteq \mathbb{Q} \cap [0, 1]$. We shall denote by $A^{(\alpha)}$ the α-th Cantor-Bendixson derivative of the closure in $[0, 1]$ of A. We shall denote by $CB_\alpha = \{A \subseteq \mathbb{Q} \cap [0, 1] : A^{(\alpha)} = \emptyset\}$. CB_α will be called the α-th *Cantor-Bendixson ideal*. Note that every CB_α is an ideal over $\mathbb{Q} \cap [0, 1]$ and $CB_\alpha \subset CB_\beta$ if $\alpha < \beta$. In fact, $CB = \bigcup_{\alpha<\omega_1} CB_\alpha$ is an ideal on $\mathbb{Q}$, and it is equal to $\overline{\mathsf{ctbl}}$, the ideal of subsets of $\mathbb{Q} \cap [0, 1]$ with countable closure in $\mathbb{R}$, which is contained in nwd. Moreover, CB_2 is equal to conv. So, we have the following chain of ideals:

$$\{\emptyset\} = CB_0 \subset \mathbf{Fin} = CB_1 \subset \mathsf{conv} = CB_2 \subset CB_3 \subset \cdots \subset CB = \overline{\mathsf{ctbl}} \subset \mathsf{nwd}$$

In this moment we do not know if $CB_{\alpha+1} \leq_K CB_\alpha$ for some $\alpha < \omega_1$. Finally, we note that for any limit ordinal $\alpha < \omega_1$, the complexity of the ideal CB_α is at most Σ_α and the complexity of $CB_{\alpha+n}$ is at most $\Sigma_{\alpha+2n}$. In order to proof that, we will need to fix a family of clopen subsets of $\mathbb{Q} \cap [0, 1]$ as follows. Let $\mathcal{C}_n = \{C^n_k : k < 2^n\}$ be a partition of $\mathbb{Q} \cap [0, 1]$ in clopen intervals with length smaller than $\frac{1}{n}$ and refining $\mathcal{C}_{n-1}$ (if $n > 0$). Then $\mathcal{C} = \bigcup_{n<\omega} \mathcal{C}_n$ is a countable base for the usual topology of $\mathbb{Q} \cap [0, 1]$.

Now we will proceed by induction. $CB_1 = \mathbf{Fin}$ is an F_σ ideal. If α is a limit ordinal then $CB_\alpha = \bigcup_{\gamma<\alpha} CB_\gamma$ and then the result follows immediately. Finally, we note that $A \in CB_{\alpha+1}$ if and only if there is $m < \omega$ such that for all n, $|\{C^n_k : A \cap C^n_k \notin CB_\alpha\}| \leq m$, proving that complexity of $CB_{\alpha+1}$ is at most the complexity of CB_α plus 2.

Fubini Products

Let $\mathscr{I}$, $\mathscr{J}$ be ideals on ω. *Fubini Product* $\mathscr{I} \times \mathscr{J}$ is defined by

$$\mathscr{I} \times \mathscr{J} = \{A \subseteq \omega \times \omega : \{n : (A)_n \notin \mathscr{J}\} \in \mathscr{I}\}.$$

We have noted that Fubini product of Borel ideals is also a Borel ideal. Actually we have the following result about complexity of Fubini products of Borel ideals.

Proposition 1.6.16. *Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω. Then*

- *If $\mathscr{I}$ is a Σ_α ideal and $\mathscr{J}$ is a Σ_β ideal then $\mathscr{I} \times \mathscr{J}$ is a $\Sigma_{\beta+\alpha}$ ideal.*
- *If $\mathscr{I}$ is a Σ_α ideal and $\mathscr{J}$ is a Π_β ideal then $\mathscr{I} \times \mathscr{J}$ is a $\Sigma_{\beta+\alpha}$ ideal.*
- *If $\mathscr{I}$ is a Π_α ideal and $\mathscr{J}$ is a Σ_β ideal then $\mathscr{I} \times \mathscr{J}$ is a $\Pi_{\beta+\alpha}$ ideal.*
- *If $\mathscr{I}$ is a Π_α ideal and $\mathscr{J}$ is a Π_β ideal then $\mathscr{I} \times \mathscr{J}$ is a $\Pi_{\beta+\alpha}$ ideal.*

Proof. If $\mathscr{I}$ is a Σ_α ideal (respectively Π_α ideal) then there is a sequence $\langle F_n : n < \omega \rangle$ of sets such that every F_n belongs Π_{α_n} (resp. F_n belongs Σ_{α_n}) with $\alpha_n < \alpha$ and $\mathscr{I} = \bigcup_{n<\omega} F_n$ (resp. $\mathscr{I} = \bigcap_{n<\omega} F_n$). Let us define $\psi : 2^{\omega\times\omega} \to 2^\omega$ by

$$\psi(A) = \{k : (A)_k \notin \mathscr{J}\}.$$

Then, $\mathscr{I} = \bigcup_{n<\omega} \psi^{-1}[F_n]$ (resp. $\mathscr{I} = \bigcap_{n<\omega} \psi^{-1}[F_n]$). About such function ψ we have the following result.

Lemma 1.6.17. *If $\mathscr{J}$ is a Σ_β or Π_β ideal and $s \in 2^{<\omega}$ then $\psi^{-1}[\langle s \rangle]$ is a $\Sigma_{\beta+1}$ and $\Pi_{\beta+1}$ set.*

Proof of lemma. $\psi^{-1}[\langle s \rangle] = \{A \subseteq \omega \times \omega : (s_i = 0 \text{ implies } (A)_i \in \mathscr{I})\} \cap \{A \subseteq \omega \times \omega : (s_i = 1 \text{ implies } (A)_i \notin \mathscr{I})\}$. Then $\psi^{-1}[\langle s \rangle]$ is an intersection of a Σ_β set with a Π_β set. $\square$

Inductively, is easy to prove that if F is a Σ_γ subset of 2^ω then $\psi^{-1}[F]$ is a $\Sigma_{\beta+\gamma}$ subset of $2^{\omega\times\omega}$ and if F is a Π_γ subset of 2^ω then $\psi^{-1}[F]$ is a $\Pi_{\beta+\gamma}$ subset of $2^{\omega\times\omega}$, for all $\gamma < \omega_1$. By using such result for the F_n sets, we are done. $\square$

$\mathbf{Fin} \times \emptyset$

$\mathbf{Fin} \times \emptyset$ can be thought as the ideal generated by an infinite partition of ω whose elements are infinite sets. This is not a tall ideal, and therefore is Katětov equivalent to **Fin**. This ideal is not a P-ideal.

$\emptyset \times \mathbf{Fin}$

$\emptyset \times \mathbf{Fin}$ can be thought as an ideal $\mathscr{I}$ for which there is a partition in infinitely many infinite sets $\{P_n : n < \omega\}$ of ω, such that $I \in \mathscr{I}$ if and only if $I \cap P_n$ is finite for all $n < \omega$. This is not a tall ideal and consequently is Katětov equivalent with **Fin**, but $\emptyset \times \mathbf{Fin}$ is minimal with respect to Tukey order among analytic P-ideals, by a theorem of Todorčević [48]. $\emptyset \times \mathbf{Fin}$ also can be thought as the ideal on $\omega\times\omega$ generated by the sets $\Delta_f = \{(n,m) : m \leq f(n)\}$ with $f \in \mathcal{F}$, where $\mathcal{F}$ is a dominant family of functions in ω^ω. That also prove that $\mathsf{cof}(\emptyset \times \mathbf{Fin}) = \mathfrak{d}$. $\emptyset \times \mathbf{Fin}$ is a P-ideal, because given a family $\{I_n : n < \omega\} \subseteq \emptyset \times \mathbf{Fin}$ there is a family $\{f_n : n < \omega\} \subseteq \omega^\omega$ such that $(I_n)_k \subseteq f_n(k)$ for all n and k; and by taking a function g which dominates f_n for all n; we have that $\Delta_g = \{(i,j) : j \leq g(i)\} \supseteq^* I_n$ for all n.

$\mathbf{Fin} \times \mathbf{Fin}$

$\mathbf{Fin} \times \mathbf{Fin}$ is the ideal on $\omega \times \omega$ generated by columns and areas below the graphs of functions in ω^ω, that is:

$$\mathbf{Fin} \times \mathbf{Fin} = \{A \subseteq \omega \times \omega : \exists (f \in \omega^\omega)(\forall^\infty n)(\forall m \in (A)_n)(m \leq f(n))\}.$$

Moreover, $\mathbf{Fin} \times \mathbf{Fin}$ can be seen as an ideal on ω generated by an infinite partition $\{P_n : n < \omega\}$ of ω in infinite sets, and the sets $A \subseteq \omega$ such that $|A \cap P_n| < \aleph_0$ for all $n < \omega$. Actually, it is very easy to see that

Proposition 1.6.18. *For any ideal $\mathscr{I}$ on ω, $\mathscr{I} \geq_K$ $\mathbf{Fin} \times \mathbf{Fin}$ if and only if there is a partition $\{Q_n : n < \omega\}$ of ω in infinite sets such that every Q_n is in $\mathscr{I}$ and every $A \subseteq \omega$ satisfying $|A \cap Q_n| < \aleph_0$ is in $\mathscr{I}$.* $\square$

$\mathbf{Fin} \times \mathbf{Fin}$ is a tall ideal, but is not a P-ideal, essentially by the same reasons as $\mathcal{ED}$. Additionally, $\mathbf{Fin} \times \mathbf{Fin}$ is not an ω-splitting ideal since there are no elements of $\mathbf{Fin} \times \mathbf{Fin}$ infinitely intersecting all columns in $\omega \times \omega$. By proposition 1.6.16, the complexity of $\mathbf{Fin} \times \mathbf{Fin}$ is $F_{\sigma\delta\sigma}$. About cardinal invariants of $\mathbf{Fin} \times \mathbf{Fin}$ we have the following results.

Theorem 1.6.19 (Folklore). *The following conditions hold.*

1. $\mathsf{add}^*(\mathbf{Fin} \times \mathbf{Fin}) = \mathsf{non}^*(\mathbf{Fin} \times \mathbf{Fin}) = \aleph_0$,
2. $\mathsf{cov}^*(\mathbf{Fin} \times \mathbf{Fin}) = \mathfrak{b}$ *and*
3. $\mathsf{cof}(\mathbf{Fin} \times \mathbf{Fin}) = \mathfrak{d}$.

Proof. (1) $\mathbf{Fin} \times \mathbf{Fin}$ does not split columns $\{n\} \times \omega$.

Let us see (2). Let $\mathcal{A}$ be a witness $\mathsf{cov}^*(\mathbf{Fin} \times \mathbf{Fin})$. For every $A \in \mathcal{A}$ pick $f_A \in \omega^\omega$ such that $(\forall^\infty n < \omega)(\forall m \in (A)_n)(m \leq f_A(n))$. The family $\{f_A : A \in \mathcal{A}\}$ is an unbounded family, proving $\mathfrak{b} \leq \mathsf{cov}^*(\mathbf{Fin} \times \mathbf{Fin})$. In order to prove the other inequality, we use the following result.

Lemma 1.6.20 (Roitman [40], see [2] lemma 1.3.3). $\mathfrak{b}$ *is the minimal cardinality of a family of increasing functions in ω^ω such that for every function $g \in \omega^\omega$ and for all infinite subset X of ω there exists $f \in \mathcal{F}$ such that $g(n) < f(n)$ for infinitely many $n \in X$.* □

Let $\mathcal{F}$ be a family satisfying previous lemma, and for every $f \in \mathcal{F}$ let Δ_f be the area under f. $\mathcal{A} = \{\Delta_f : f \in \mathcal{F}\} \cup \{\{n\} \times \omega : n < \omega\}$ is a subset of $\mathbf{Fin} \times \mathbf{Fin}$ of cardinality $\mathfrak{b}$ and if Y is an infinite subset of $\omega \times \omega$, there are two cases: (a) $(\exists n)(|(Y)_n| = \aleph_0)$, and in such case we are done, and (b) $(\exists^\infty n)((Y)_n \neq \emptyset)$, and in such case, doing $X = \{n : (\exists m)((n, m) \in Y)\}$ and defining $g \in \omega^\omega$ be such that $g(n) = \min(Y)_n$ if $n \in X$, there exists $f \in \mathcal{F}$ such that $g(n) < f(n)$ for infinitely many $n \in X$, and so, Δ_f has infinitely many elements of Y.

Now we will prove (3). Let $\mathcal{F}$ a dominating family of functions in ω^ω of cardinality $\mathfrak{d}$. For every $f \in \mathcal{F}$ and $n < \omega$, define

$$A_{f,n} = \{(i, j) \in \omega \times \omega : n \leq i \to j \leq f(i)\}.$$

The family $\mathcal{A} = \{A_{f,n} : f \in \mathcal{F} \wedge n < \omega\}$ is a cofinal subset of $\mathbf{Fin} \times \mathbf{Fin}$ of cardinality $\mathfrak{d}$. On the other hand, let $\mathcal{A}$ be a cofinal subset of $\mathbf{Fin} \times \mathbf{Fin}$ and pick $f_A \in \omega^\omega$ like in proof of (2), for $A \in \mathcal{A}$. Given $f \in \omega^\omega$, there is $A \in \mathcal{A}$ such that $\Delta_f \subseteq^* A$. So, $f \leq^* f_A$, proving $\{f_A : A \in \mathcal{A}\}$ is a dominating family in ω^ω. □

Ideals based on graphs

A *graph* G is a pair $\langle V, E\rangle$, where V is a set and $E \subseteq [V]^2$. Elements of V are the *vertices* and elements of G are the *edges* of the graph. $G = \langle V, E\rangle$ is a *complete* graph if $E = [V]^2$. A *coloring* for a graph $G = \langle V, E\rangle$ is a function φ from V into a set X such that for all $v, w \in V$ if $\{v, w\} \in E$ then $\varphi(v) \neq \varphi(w)$. The *chromatic number* $\chi(G)$ of a graph G, is defined as the minimal $|X|$ such that there is a coloring φ from V into X.

Along this work, we will say $G \subseteq [\omega]^2$ is a graph on ω if $\langle \omega, G\rangle$ is a graph.

$\mathcal{G}_{fc}$

$\mathcal{G}_{fc}$ is the ideal of *graphs with finite chromatic number*, i.e.

$$\mathcal{G}_{fc} = \{G \subseteq [\omega]^2 : \chi(G) < \omega\}.$$

$\mathcal{G}_{fc}$ is a tall ideal since every infinite graph has an infinite subgraph with chromatic number 2. The ideal $\mathcal{G}_{fc}$ is an F_σ ideal since the function φ defined such that for all $A \subseteq \omega$,

$$\varphi(A) = \min\{|\mathcal{B}| : A \subseteq \bigcup \mathcal{B} \wedge (\forall G \in \mathcal{B})\chi(G) = 2\}$$

is a lscsm and $\mathcal{G}_{fc} = Fin(\varphi)$.

We have isolated the cardinal invariant of the continuum $\mathfrak{s}_2$ defined as follows. A *pair-splitting family* $\mathcal{P}$ is a family of infinite subsets of ω such that for any infinite set $A \subseteq [\omega]^2$, there is $P \in \mathcal{P}$ such that $|P \cap a| = 1$ for infinitely many elements $a \in A$. $\mathfrak{s}_2$ is the minimal cardinality of a pair-splitting family. This cardinal invariant was introduced in an independent context by Hiroaki Minami, and we have presented it in [23]. Minami has defined the cardinal invariant $\mathfrak{r}_2$ dual of $\mathfrak{s}_2$ as the minimal cardinality of a pair-reaping family. A *pair-reaping family* is a family $\mathcal{R}$ of infinite sets of pairs of natural numbers such that for any infinite $A \subseteq \omega$ there is $R \in \mathcal{R}$ that R is not pair-split by A.

About cardinal invariants of $\mathcal{G}_{fc}$ we have the following results:

Theorem 1.6.21. *The following holds.*

1. $\mathsf{add}^*(\mathcal{G}_{fc}) = \aleph_0$,

2. $\mathsf{cov}^*(\mathcal{G}_{fc}) = \mathfrak{s}_2$,

3. $\mathsf{non}^*(\mathcal{G}_{fc})$ *is the minimal cardinality of a family* $\mathcal{A} \subseteq [[\omega]^2]^\omega$ *such that for any finite partition* $\mathcal{P}$ *of* ω *there is an element* A *of* $\mathcal{A}$ *such that for every set* $a \in A$ *there is* $P \in \mathcal{P}$ *such that* $a \subseteq P$*, and*

4. $\mathfrak{r}_2 \leq \mathsf{non}^*(\mathcal{G}_{fc}) \leq \mathfrak{r}$.

Proof. (1) For every $n < \omega$ define the set $A_n = \{\{k, m\} : k \leq n \wedge m \neq k\}$. Note that A_n has finite chromatic number and every set which almost-contains all A_n has an infinite complete subgraph. (2) Fix $\mathcal{T} \subseteq [\omega]^\omega$ with $|\mathcal{T}| < \mathsf{cov}^*(\mathcal{G}_{fc})$, and define $\mathcal{T}' = \mathcal{T} \cup \{\omega \setminus n : n \in \omega\}$. For every $A \subseteq \omega$ we will denote by I_A the set$\{\{n, m\} : n \in A \wedge m \in \omega \setminus A\}$. Since $|\mathcal{T}'| < \mathsf{cov}^*(\mathcal{G}_{fc})$, there exists an infinite $X \subseteq [\omega]^2$ such that $X \cap I_T$ is finite, for all $T \in \mathcal{T}'$. Moreover, $\{m : \{m, n\} \in X\}$ is finite, for all $n \in \omega$. Therefore, X has an infinite subset Y whose elements are pairwise disjoint. That Y is not split by elements of $\mathcal{T}$ follows from the fact that I_T is finite for all $T \in \mathcal{T}$. We conclude that $\mathsf{cov}^*(\mathcal{G}_{fc}) \leq \mathfrak{s}_2$.

On the other hand, define $J_A = \{\{n, m\} : n \in A \wedge m \in \omega \setminus A\}$, for $A \subseteq \omega$. Note that $\{J_A : A \subseteq \omega\}$ is a subbase of $\mathcal{G}_{fc}$. It shall be enough to prove that if $\mathcal{T} \subseteq [\omega]^\omega$ is not a pair splitting family then $\{J_T : T \in \mathcal{T}\}$ is not covering. Let $\mathcal{P}$ be an infinite set of pairwise disjoint pairs of natural numbers such that no $T \in \mathcal{T}$ splits $\mathcal{P}$. Then, there are not $T \in \mathcal{T}$ splitting $\bigcup \mathcal{P}$.

(3) Note that if P is a finite partition of ω then $G_P = \{\{n, m\} : (\exists a \neq b \in P)(n \in a \wedge m \in b)\} \in \mathcal{G}_{fc}$, and moreover, $\{G_P : P \text{ is a finite partition of } \omega\}$ is a base of $\mathcal{G}_{fc}$. Then, if $\mathcal{A}$ is a family as in (3) then $\mathcal{A}$ itself witnesses $\mathsf{non}^*(\mathcal{G}_{fc})$; and if $\mathcal{B}$ is a witness of $\mathcal{G}_{fc}$ then defining $\mathcal{A}$ as the family of finite changes of elements of $\mathcal{B}$ we are done.

(4) follows directly from (3) since given a hereditarily reaping family $\mathcal{R}$ we construct a family $\mathcal{A}$ as in (3) as follows: For any $R \in \mathcal{R}$, let $\{n_k^R : k \in \omega\}$ be an enumeration of R and define $I_R = \{\{n_k^R, n_{k+1}^R\} : k \in \omega\}$. Define $\mathcal{A} = \{I_R : r \in \mathcal{R}\}$. Then, if $\mathcal{P} = \{P_0, \ldots, P_n\}$ is a finite partition of ω then either there is $R \in \mathcal{R}$ such that $R \subseteq P_0$ or $R \subseteq \bigcup_{0<i\leq n} P_i$. In first case we are done. In second case we can find $R_1 \in \mathcal{R}$ such that either $R_1 \subseteq R \cap P_1$ or $R_1 \subseteq R \cap \bigcup_{1<i\leq n} P_i$. We can repeat this procedure while second case stills holding, and in step $n-1$, we will have $R_{n-1} \subseteq P_n$ for some $R_{n-1} \in \mathcal{R}$ and then we will be done. $\square$

About the position of $\mathcal{G}_{fc}$ in Katětov order we have the following results:

Theorem 1.6.22. *The following relations hold.*

(1) $\mathcal{S} \leq_{KB} \mathcal{G}_{fc}$ *and*

(2) $\mathcal{ED}_{fin} \leq_{KB} \mathcal{G}_{fc}$.

Proof. (1) Remember that for $x \in 2^\omega$, the set $I_x = \{C \in \Omega : x \in C\} \in \mathcal{S}$. Let $f : [\omega]^2 \to \Omega$ be given by $f(\{n, m\}) = \{x \in 2^\omega : x(n) \neq x(m)\}$. It will be enough to prove that $J_x = f^{-1}[I_x]$ is finitely chromatic for all $x \in 2^\omega$. In fact, $\chi(J_x) = 2$ since x itself is a coloring for J_x.

(2) Define $f : [\omega]^2 \to \Delta$ by $f(\{n, m\}) = \langle \max\{n, m\}, \min\{n, m\}\rangle$. We will prove that f is a Katětov function. It will be enough to prove that for every $g \in \prod_{0<n<\omega} n$, $J_g = f^{-1}[\{\langle n, g(n)\rangle : n < \omega\}] \in \mathcal{G}_{fc}$. In fact, $\chi(J_n) = 2$, since we can define a coloring ψ as follows. $\psi(0) = 0$ and for all $n > 0$, define $\psi(n) = 1 - \psi(g(n))$. ψ is well defined since g is a regressive function, and immediately we have that if $\{m, n\} \in J_g$ then $\psi(m) \neq \psi(n)$. □

About $\mathfrak{s}_2$ and $\mathfrak{r}_2$ we have the following results:

Theorem 1.6.23. $\mathfrak{s} \leq \mathfrak{s}_2 \leq \min\{\mathsf{non}(\mathcal{N}), \mathsf{cov}^*(\mathcal{ED}_{fin})\}$ *and* $\max\{\mathsf{cov}(\mathcal{N}), \mathsf{cov}(\mathcal{M})\} \leq \mathfrak{r}_2 \leq \mathfrak{r}$.

Proof. By theorems 1.6.22 and 1.6.21(2) we have that $\mathfrak{s}_2 = \mathsf{cov}^*(\mathcal{G}_{fc}) \leq \mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ and $\mathfrak{s}_2 = \mathsf{cov}^*(\mathcal{G}_{fc}) \leq \mathsf{cov}^*(\mathcal{ED}_{fin}) \leq \mathsf{non}(\mathcal{M})$. Let $\kappa < \mathfrak{s}$ be a cardinal number and let $\mathcal{A}$ be a family of κ infinite subsets of ω. Since $\mathcal{A}$ is not a splitting family, there is an infinite subset B of ω such that $B \subseteq^* A$ or $B \cap A =^* \emptyset$ for all $A \in \mathcal{A}$. Let $\{n_k : k < \omega\}$ be an enumeration of B and define $G = \{\{n_k, n_{k+1}\} : k < \omega\}$. $\mathcal{A}$ is not a pair-splitting family since for any $A \in \mathcal{A}$, for all but finitely many $p \in G$, either $p \subseteq A$ or $p \cap A = \emptyset$. Hence $\kappa < \mathfrak{s}_2$. The inequality $\mathfrak{r}_2 \leq \mathfrak{r}$ was proved in 1.6.21(4). Let κ be a cardinal and let $\mathcal{A}$ be a family of infinite sets of pairs of natural numbers with $|\mathcal{A}| = \kappa$. In order to prove that $\mathcal{A}$ is not a pair-reaping family we can suppose without loss of generality that every $A \in \mathcal{A}$ is pairwise disjoint. For any infinite pairwise disjoint $A \subseteq [\omega]^2$ we define $D_A = \{X \in [\omega]^{\aleph_0} : X \text{ splits } A\}$. Note that for any A, D_A is a comeager subset of 2^ω, since the family of sets $X \in [\omega]^{\aleph_0}$ such that $|a \cap X| = 1$ for some $a \in A$ is open and dense. Moreover the measure of D_A is 1 since, given $a \in X$ and $\varepsilon > 0$, we can find an open set U in 2^ω with measure less than ε such that every set which does not split A belongs to U. Then, if $\kappa < \mathsf{cov}(\mathcal{M})$ or $\kappa < \mathsf{cov}(\mathcal{N})$ then there exists an infinite $X \in \bigcap_{A \in \mathcal{A}} D_A$, witnessing $\mathcal{A}$ is not a pair-reaping family. □

$\mathcal{G}_c$

The ideal $\mathcal{G}_c$ of *graphs without infinite complete subgraphs* is defined as the family of subsets I of $[\omega]^2$ such that for every infinite $X \subseteq \omega$ there exists $n \neq m \in X$ such that $\{n, m\} \notin I$. In other words, $\mathcal{G}_c$ consists of all graphs I on ω that do not have infinite complete subgraphs. That such family is an ideal is a direct consequence of Ramsey's theorem.

Lemma 1.6.24. *$\mathcal{G}_c$ is a co-analytic ideal.*

Proof. Define $F = \{(B, A) \in [\omega]^{\aleph_0} \times \mathcal{P}([\omega]^2) : [B]^2 \subseteq A\}$. Clearly, $\mathcal{G}_c$ is the complement of the projection of F. Let us prove that F is closed. Let φ be a bijection between $[\omega]^2$ and $\omega \setminus \{0\}$ such that $\varphi\{i, j\} \geq \max\{i, j\}$, and let $\{(X_n, Y_n) : n \in \omega\} \subseteq F$ be a sequence convergent to (B, A). Then, for all $N < \omega$, there exists $k(N) > N$ such that $(\forall n \geq k(N))(X_n \cap k(N) = B \cap k(N))$. If $i \neq j \in B$, then $\{i, j\} \subseteq B \cap k(\varphi\{i, j\}) = X_{\varphi\{i,j\}} \cap k(\varphi\{i, j\}) = X_m \cap k(\varphi\{i, j\})$, for all $m \geq k(\varphi\{i, j\})$. So, $\{i, j\} \in Y_m$ for all $m \geq k(\varphi\{i, j\})$, and therefore, $\{i, j\} \in A$, proving that $(B, A) \in F$. □

Additionally, $\mathcal{G}_c$ is tall (every infinite subset of $[\omega]^2$ has an infinite subgraph without infinite complete subgraphs), but it is not a P-ideal (every infinite pseudointersection of the family $\{A_n : n \in \omega\}$, where $A_n = \{\{k, m\} : k \leq n \vee m \leq n\}$, has an infinite complete subgraph). Since $\mathcal{G}$ is not a P-ideal we know that $\mathsf{add}^*(\mathcal{G}_c) = \aleph_0$. About other cardinal invariants of $\mathcal{G}_c$ we need to establish some preliminary results.

Lemma 1.6.25. **Fin** $\times$ **Fin** $\leq_K \mathcal{G}_c$.

Proof. Define $f : [\omega]^2 \to \omega \times \omega$ by $f(\{n, m\}) = (\min\{n, m\}, \max\{n, m\})$. If $X \in \mathbf{Fin} \times \mathbf{Fin}$ then there exists a $N \in \omega$ such that for every $n \geq N$, X_n is finite. Hence for almost every $n \in \omega$, $\{m \in \omega : \{n, m\} \in f^{-1}[X]\}$ is finite, and therefore, $f^{-1}[X]$ can not contain an infinite complete subgraph. □

A set $A \subseteq \omega$ is *homogeneous for* a given graph G if either $[A]^2 \subseteq G$ or $[A]^2 \cap G = \emptyset$. Andreas Blass in [5] defined the following. If $G \subseteq [\omega]^2$ and $H \subseteq \omega$ then H is *almost homogeneous for* G if there is a finite set $F \subseteq \omega$ such that $H \setminus F$ is homogeneous for G. $\mathfrak{par}_2$ is defined as the smallest cardinality of a family $\mathcal{F}$ of subsets of $[\omega]^2$ such that no single infinite subset of ω is almost homogeneous for all $G \in \mathcal{F}$. Theorem 3.5 of [5], proves that $\mathfrak{par}_2 = \min\{\mathfrak{b}, \mathfrak{s}\}$, and in particular, $\mathfrak{h} \leq \mathfrak{par}_2$. We will use this cardinal relation in subsection 2.6 in order to prove that the ideal nwd satisfies a Ramsey type property.

Theorem 1.6.26. $\min\{\mathfrak{b}, \mathfrak{s}\} = \mathfrak{par}_2 \leq \mathsf{cov}^*(\mathcal{G}_c) \leq \min\{\mathfrak{b}, \mathfrak{s}_2\}$.

Proof. Let $\mathcal{A}$ be a subset of $\mathcal{G}_c$ with $|\mathcal{A}| < \mathfrak{par}_2$. Note that every $A \in \mathcal{A}$ defines a partition of $[\omega]^2$. Hence, there exists $X \in [\omega]^\omega$ such that $[X \setminus F]^2 \subseteq A$ or $[X \setminus F]^2 \subseteq [\omega]^2 \setminus A$, for all $A \in \mathcal{A}$. First case is not possible since A has no infinite complete subgraphs. Hence $[X \setminus F]^2 \subseteq [\omega]^2 \setminus A$. Let Y be an infinite pseudointersection of $\{[X \setminus F]^2 : F \in [\omega]^{<\omega}\}$. Then, $A \cap Y$ is finite, for all $A \in \mathcal{A}$. The other inequalities are consequence of previous lemma, $\mathcal{G}_{fc} \subseteq \mathcal{G}_c$ and 1.7.1(1). $\square$

Question 1.6.27. *Are the inequalities in 1.6.26 consistently strict?*
Is $\mathsf{cov}^*(\mathcal{G}_c) = \min\{\mathfrak{b}, \mathfrak{s}_2\}$*?*

The random graph ideal $\mathcal{R}$

Given a graph G on ω, we can define a possibly improper ideal $\mathscr{I}_G$ as the ideal generated by the set of all the subsets of ω which are homogeneous for G. We investigate the ideal $\mathcal{R}$ on ω generated by the homogeneous sets in the *random graph* E, which is going to be defined below (see [9]).

Let $\{X_n : n < \omega\}$ be an independent family of subsets of ω. We can suppose that the family satisfies $n \in X_m$ if and only if $m \in X_n$, for all $n < \omega$ (if not, we can do finite changes and get that family). The set $E = \{\{n, m\} : m \in X_n\}$ is the set of edges of the random graph. The following property is the crucial one for random graph.

Lemma 1.6.28. *Let E be the random graph defined above. Given a and b disjoint finite subsets of ω there is $k < \omega$ such that $\{\{k, l\} : l \in a\} \subseteq E$ and $\{\{k, l\} : l \in b\} \cap E = \emptyset$.*

Proof. Take $k \in (\bigcap_{i \in a} X_i) \setminus \bigcup_{j \in b} X_j$. Such k exists by independence. $\square$

Actually, this property gives an algorithm to construct the random graph in an alternative way: Let us define recursively an increasing family of initial segments $\{F_n : n < \omega\}$ of ω and an increasing family $\{E_n : n < \omega\}$ of subsets of $[\omega]^2$ as follows. Define $F_0 = \{0\}$ and $E_0^0 = \emptyset$. Let suppose defined F_n and E_n and let $\{a_j : j < 2^{|F_n|}\}$ be an enumeration of all subsets of F_n. Then, define $E_{n+1} = E_n \cup \big\{\{|F_n| + j, i\} : j < 2^{|F_n|} \wedge i \in a_j\big\}$. It is immediate that $E = \bigcup_{n<\omega} E_n$ satisfies the lemma.

Moreover, the random graph E satisfies the following property:

Lemma 1.6.29. *Given a graph $\langle\omega, G\rangle$, there is a subset $X \subseteq \omega$ such that $\langle\omega, G\rangle \cong \langle X, E \restriction X\rangle$.*

Proof. Such set X can be found by taking $x_n \in \omega$ as in lemma 1.6.28 for $a = \{x_j : j < n \wedge \{j, n\} \in G\}$ and $b = \{x_j : j < n \wedge \{j, n\} \notin G\}$. The map $n \mapsto x_n$ is the required isomorphism onto $X = \{x_n : n < \omega\}$. $\square$

By Ramsey theorem, $\mathcal{R}$ is tall. The function φ defined by

$$\varphi(A) = \min\{|\mathcal{X}| : (\forall X \in \mathcal{X})(([X]^2 \subseteq E \vee [X]^2 \cap E = \emptyset) \wedge A \subseteq \bigcup \mathcal{X})\}$$

is a lscsm such that $\mathcal{R} = Fin(\varphi)$. The random graph ideal $\mathcal{R}$ has a very important property with respect to the Ramsey properties studied in section 3.3. About cardinal invariants of $\mathcal{R}$ we have the following results.

Theorem 1.6.30. *The following holds.*

1. $\mathsf{add}^*(\mathcal{R}) = \mathsf{non}^*(\mathcal{R}) = \aleph_0$ *and*

2. $\mathsf{cov}^*(\mathcal{R}) = \mathsf{cof}(\mathcal{R}) = \mathfrak{c}$.

Proof. Lemma 3.3.3 claims that $\mathcal{R} \leq_K \mathsf{conv}$, and then, by theorem 1.5.2 we are done. $\square$

Ideals generated by MAD families

Given A, B infinite subsets of ω we say A and B are *almost disjoint* if $A \cap B$ is finite. A family $\mathcal{A}$ of infinite subsets of ω is an *almost disjoint family* if A and B are almost disjoint for any $A \neq B \in \mathcal{A}$. A *MAD family* is an almost disjoint family maximal with respect to almost disjoint property, that is, $\mathcal{A}$ is a MAD family if for every infinite set X there is an element A of $\mathcal{A}$ such that $A \cap X$ is infinite. Existence of MAD families is a consequence of Axiom of Choice.

Given an almost disjoint family A we can define the ideal $\mathscr{I}(\mathcal{A})$ generated by $\mathcal{A}$, that is, $I \in \mathscr{I}(\mathcal{A})$ if and only if there is a finite subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $I \subseteq \bigcup \mathcal{B}$. Note that $\mathscr{I}(\mathcal{A})$ is a tall ideal if and only if $\mathcal{A}$ is a MAD family. In [37], Adrian Mathias proved that ideals based on MAD families are meager but non analytic ideals.

Summable ideals

Let $f : \omega \to (0,\infty)$ be a function such that $\sum_{n=0}^{\infty} f(n) = \infty$. The *summable ideal of* f is defined by $\mathscr{I}_f = \{I \subseteq \omega : \sum_{n\in I} f(n) < \infty\}$. If $\liminf f(n) = 0$ then $\mathbf{Fin} \subsetneq \mathscr{I}_f$ and if $\lim f(n) = 0$ then $\mathscr{I}_f$ is a tall ideal. Note that the function $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ given by $\varphi(A) = \sum_{n\in A} f(n)$ is a lower semicontinuous submeasure (actually, a measure) on ω and $\mathscr{I}_f = Fin(\varphi) = Exh(\varphi)$, proving that $\mathscr{I}_f$ is an F_σ P-ideal. In subsection 3.1 we will see that Katětov order is very complex among summable ideals, and cardinal invariants possibly too.

Erdös-Ulam ideals

Given a function $f : \omega \to [0,\infty)$, the Erdös-Ulam ideal $\mathcal{EU}_f$ for f is defined as the family of subsets of ω with f-density zero, that is

$$\mathcal{EU}_f = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i<n} f(i)} = 0\right\}.$$

The function $\varphi : \mathcal{P}(\omega) \to [0,\infty]$ given by

$$\varphi(A) = \sup_n \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i<n} f(i)}$$

is a lscsm and $\mathcal{EU}_f = Exh(\varphi)$, and then it is an $F_{\sigma\delta}$ P-ideal. The most important of the Erdös-Ulam ideals is $\mathcal{Z}$, the *asymptotical density zero ideal*, defined by

$$\mathcal{Z} = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A\cap n|}{n} = 0\right\}.$$

An alternative expression for $\mathcal{Z}$ is:

$$\mathcal{Z} = \left\{A \subseteq \omega : \lim_{n\to\infty} \frac{|A\cap [2^n, 2^{n+1})|}{2^n} = 0\right\}.$$

Ilijas Farah (Claim 1.13.10 in [14]) has proved that every Erdös-Ulam ideal is Rudin-Blass equivalent to $\mathcal{Z}$. Then $\mathsf{cov}^*(\mathscr{I}) = \mathsf{cov}^*(\mathcal{Z})$ and $\mathsf{non}^*(\mathscr{I}) = \mathsf{non}^*(\mathcal{Z})$ for all Erdös-Ulam ideal $\mathscr{I}$. Hernández-Hernández and Hrušák have proved in Theorem 2.2 of [20] that $\min\{\mathfrak{b}, \mathsf{cov}(\mathcal{N})\} \leq \mathsf{cov}^*(\mathcal{Z}) \leq \max\{\mathfrak{b}, \mathsf{non}(\mathcal{N})\}$. Additionally, they proved that $\mathsf{add}^*(\mathscr{I}) = \mathsf{add}(\mathcal{N})$ and $\mathsf{cof}(\mathscr{I}) = \mathsf{cof}(\mathcal{N})$, for all tall Erdös-Ulam ideal $\mathscr{I}$.

1.7 Elementary facts about Katětov order

Position in Katětov order can be crucial for some combinatorial and topological properties but Katětov order is interesting in itself.

As an order type, Katětov order is complex. We prove that is both, downward and upward directed, with a minimal element, the ideal **Fin**. All non-tall ideals are Katětov-equivalent with **Fin** and among tall ideals, the family of ideals generated by MAD families is coinitial.

This order was introduced by Miroslav Katětov in [29]. It is a generalization of Rudin-Keisler order (remember that Rudin-Keisler order was first used by Katětov in the same paper).

1.7.1 Structural properties of Katětov order

Some immediate properties of Katětov order are listed here.

Proposition 1.7.1. *The following relations hold.*

1. *If* $\mathscr{I} \subseteq \mathscr{J}$ *then* $\mathscr{I} \leq_K \mathscr{J}$.
2. *If* $X \in \mathscr{I}^+$ *then* $\mathscr{I} \leq_K \mathscr{I} \restriction X$.
3. $\mathscr{I} \oplus \mathscr{J} \leq_K \mathscr{I}$ *and* $\mathscr{I} \oplus \mathscr{J} \leq_K \mathscr{J}$.
4. $\mathscr{I}, \mathscr{J} \leq_K \mathscr{I} \times \mathscr{J}$.

Proof. Identity in ω is a witness for (1); inclusion from X into ω is a witness for (2), inclusions of ω into $\omega \oplus \omega$ are witnesses for (3) and projection functions are witnesses for (4). □

Properties (3) and (4) establish that Katětov order is both, upward and downward directed. The following proposition lists some of the order-type properties of the Katětov order.

Proposition 1.7.2. *The following hold.*

(1) Every family $\mathcal{A}$ *of at most* $\mathfrak{c}$ *ideals has a* $\leq_K$*-lower bound,*

(2) The family of maximal ideals is cofinal in Katětov order,

(3) **Fin** *is minimal in Katětov order. Moreover, the following are equivalent:*

(a) $\mathscr{I} \leq_K \mathbf{Fin}$,

(b) $\mathscr{I} \cong_K \mathbf{Fin}$, *and*

(c) $\mathscr{I}$ *is not a tall ideal.*

(4) Ideals generated by MAD families are coinitial in Katětov order among tall ideals.

Proof. (1) Let $\mathcal{A} = \{\mathscr{I}_\alpha : \alpha < \mathfrak{c}\}$ be a family of ideals on ω and let $\{A_\alpha : \alpha < \mathfrak{c}\}$ a MAD family. For every $\alpha < \mathfrak{c}$ pick a bijection i_α from ω onto A_α and define

$$\mathscr{I} = \{I \subseteq \omega : (\forall \alpha < \mathfrak{c})(i_\alpha^{-1}(I \cap A_\alpha) \in \mathscr{I}_\alpha)\}.$$

It is easy to verify that $\mathscr{I}$ is an ideal on ω and i_α is a witness for $\mathscr{I} \leq_K \mathscr{I}_\alpha$.

(2) is a consequence of the Ulam and Tarski's Boolean Prime Ideal Theorem.

(3) $\mathbf{Fin} \subseteq \mathscr{I}$ for every ideal $\mathscr{I}$. It is clear that (a) and (b) are equivalent, and if f is a witness for $\mathscr{I} \leq_K \mathbf{Fin}$ then $X = f''\omega$ is an $\mathscr{I}$-positive set and $\mathscr{I} \upharpoonright X$ is the family of finite subsets of X. On the other hand, if $\mathscr{I} \upharpoonright X = \mathbf{Fin}(X)$ then $\mathscr{I} \leq_K \mathbf{Fin}$.

(4) Let $\mathscr{I}$ be a tall ideal and let $\mathcal{A}$ be an almost disjoint family which is maximal with respect to the property of be contained in $\mathscr{I}$. Actually $\mathscr{I}$ is a MAD family, because if $X \in \mathscr{I}^+$ is almost disjoint with every element of $\mathcal{A}$ then X is almost disjoint with every element of $\mathscr{I}$, contradicting the tallness of $\mathscr{I}$. Hence $\mathscr{I}(\mathcal{A}) \subseteq \mathscr{I}$. $\square$

1.7.2 Chains and antichains in Katětov order

The remain of this section will be dedicated to answer the following question: How long are chains and antichains in Katětov order? Some partial answers were done by Michael Hrušák and Salvador García-Ferreira in [19].

Theorem 1.7.3. *Let* $\mathscr{I}$ *be a tall ideal on* ω. *Then*

(1) there is a $\leq_K$*-antichain below* $\mathscr{I}$ *of cardinality* $\mathfrak{c}$ *and*

(2) there is a $\leq_K$*-decreasing chain with length* $\mathfrak{c}^+$ *below* $\mathscr{I}$.

Proof. (1) is an immediate consequence of proposition 1.7.2(4) and the following lemma

Lemma 1.7.4 (García-Ferreira and Hrušák, 2003 (Proposition 2.5 in [19])). *There is a collection of $\mathfrak{c}$-many pairwise Katětov incomparable MAD families $\leq_K$-below every MAD family $\mathcal{A}$.* □

In order to prove (2) we recursively construct a family $\{\mathcal{A}_\alpha : \alpha < \mathfrak{c}^+\}$ of MAD families such that $\mathscr{I}(\mathcal{A}_\alpha) \leq_K \mathscr{I}(\mathcal{A}_\beta)$ and $\mathscr{I}(\mathcal{A}_\beta) \not\leq_K \mathscr{I}(\mathcal{A}_\alpha)$ if $\beta < \alpha < \mathfrak{c}^+$ as follows: Let $\mathcal{A}_0$ be a MAD family such that $\mathscr{I}(\mathcal{A}_0) \leq_K \mathscr{I}$. If α is limit we use 1.7.2(1) in order to obtain $\mathcal{A}_\alpha$; and in successor step we will use the following lemma:

Lemma 1.7.5 (García-Ferreira and Hrušák, 2003 (Proposition 2.3 in [19])). *For every MAD family $\mathcal{A}$ there is a MAD family $\mathcal{B}$ such that $\mathcal{B}$ refines $\mathcal{A}$ and $\mathscr{I}(\mathcal{A}) \not\leq_K \mathscr{I}(\mathcal{B})$.* □

It is clear that if $\mathcal{B}$ refines $\mathcal{A}$ then $\mathscr{I}(\mathcal{B}) \leq_K \mathscr{I}(\mathcal{A})$. □

Question 1.7.6 (García-Ferreira and Hrušák [19]). *Is there (consistently) a MAD family $\leq_K$-maximal among MAD families?*

Chapter 2

Combinatorics of ideals and filters and Katětov order

The importance of Katětov order lies in the fact that many combinatorial properties of ideals and filters have a *critical* ideal with respect to $\leq_K$. In the first chapter we stated without proof that ideals like $\mathcal{R}$ and $\mathcal{ED}_{fin}$ are critical for properties such that fulfilling a Ramsey-type property or be an ω-splitting ideal. Hence, combinatorial properties can be stated by using Katětov order and critical ideals. In particular in subsection 2.8 we study the critical ideals for some classes of ultrafilters, and in next section we will see the relevance of critical ideals concerning their destructibility by forcing extensions.

2.1 Forcing with quotients

Jindřich Zapletal in [51] has studied the forcing notions P_I of I-positive Borel subsets of a Polish space X, ordered by inclusion, where I is a σ-ideal on X. P_I is a non-separative partial order whose separative quotient is the σ-algebra $Borel(X)/I$. Zapletal has given the following characterization of the proper forcing notions with this form: P_I is *proper* if and only if for every countable elementary submodel M of a large enough structure and every condition $B \in M \cap P_I$ the set $C = \{x \in B : x \text{ is } M\text{-generic}\}$ is not in the ideal I, where it is said that a point x is M-generic if the collection $\{A \in P_I \cap M : x \in A\}$ is a filter on $P_I \cap M$ which meets all open dense subset of the poset P_I that are elements of the model M. One of the most important properties about a forcing notion of the type P_I is the *Continuous Reading of Names* (CRN).

Definition 2.1.1 (Zapletal [51]). If P_I is a proper forcing then it *has the* CRN if for every Borel function $f : B \to 2^\omega$ with an I-positive Borel domain B there is an I-positive set $C \subseteq B$ such that $f \restriction C$ is continuous.

Zapletal and Hrušák in [24] studied the relationship between P_I forcing notions and quotients $P(\omega)/\mathscr{I}$, where $\mathscr{I}$ is an ideal on ω. The connection between this classes of posets is given by the following definition.

Definition 2.1.2 (Brendle). For a σ-ideal I on 2^ω or ω^ω is defined the *trace ideal* $tr(I)$ on $2^{<\omega}$ or $\omega^{<\omega}$ by $a \in tr(I)$ if and only if $\{r : \exists^\infty n(r \restriction n \in a)\} \in I$.

Hrušák and Zapletal proved the following theorem, completing the link between forcing notions with the form P_I and $\mathcal{P}(\omega)/\mathscr{I}$.

Theorem 2.1.3 (Hrušák and Zapletal [24]). *Let I be a σ-ideal on $X = 2^\omega$ or $X = \omega^\omega$. If P_I is a proper forcing with CRN then $\mathcal{P}(X)/tr(I)$ is a proper forcing as well and is naturally isomorphic to a two-step iteration of P_I and an $\aleph_0$-distributive forcing.*

2.1.1 Destructibility of ideals by forcing

In the same paper Hrušák and Zapletal characterized the *destructibility of ideals* in forcing extensions in terms of Katětov order. A forcing notion $\mathbb{P}$ *destroys* an ideal $\mathscr{I}$ on ω if it introduces a set $x \subseteq \omega$ such that every ground model element $A \in \mathscr{I}$ has a finite intersection with x.

Theorem 2.1.4 (Hrušák and Zapletal [24]). *If P_I is a proper forcing with CRN and $\mathscr{I}$ is an ideal on ω then the following conditions are equivalent:*

(1) there is a condition $B \in P_I$ such that $B \Vdash$ "the ideal $\mathscr{I}$ is destroyed", and

(2) there is a $tr(I)$-positive set a such that $\mathscr{I} \leq_K tr(I) \restriction a$.

Theorem 2.1.4 can be improved when the trace ideal is K-homogeneous (see section 2.1.2).

Theorem 2.1.5. *If P_I is a proper forcing with CRN and $\mathscr{I}$ is an ideal on ω, $tr(I)$ is K-homogeneous then the following conditions are equivalent:*

(1) there is a condition $B \in P_I$ such that $B \Vdash$ "the ideal $\mathscr{I}$ is destroyed"

(2) $\mathscr{I} \leq_K tr(I)$.

Some of the classical forcing notions have a critical ideal with respect to forcing destructibility.

Theorem 2.1.6 (Hrušák[21], Brendle and Yatabe [8]). *Let $\mathscr{I}$ be a tall ideal on ω. Then*

(1) $\mathscr{I}$ is Cohen-destructible iff $\mathscr{I} \leq_K \mathsf{nwd}$,

(2) $\mathscr{I}$ is Random-destructible iff $\mathscr{I} \leq_K tr(\mathcal{N})$ and

(3) $\mathscr{I}$ is Sacks destructible iff $\mathscr{I} \leq_K tr(\mathbb{S})$ iff $\mathscr{I}$ is $\mathbb{P}$-destructible for any forcing $\mathbb{P}$ which adds new real numbers. □

Remark 2.1.7. Concerning to (2) in previous theorem, Brendle and Yatabe proved that $\mathscr{I}$ is Random-indestructible if and only if there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \leq tr(\mathcal{N}) \restriction X$, but (2) follows from theorem 2.1.17.

In some cases, just knowing the cardinal invariants of the ideal suffices to show a Katětov relation.

Corollary 2.1.8. *Let $\mathscr{I}$ be a Borel ideal on ω. If $\mathsf{cov}(\mathcal{M}) \leq \mathsf{cov}^*(\mathscr{I})$ then $\mathscr{I} \leq_K \mathsf{nwd}$.*

Proof. Let assume $\mathsf{cov}(\mathcal{M}) \leq \mathsf{cov}^*(\mathscr{I})$. Since in Cohen model $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ we have that $\mathsf{cov}^*(\mathscr{I}) = \mathfrak{c}$ which proves that $\mathscr{I}$ is destroyed, and by theorem 2.1.6(1) we are done. □

2.1.2 Homogeneity and K-uniformity

We will say that an ideal $\mathscr{I}$ on ω is:

- *homogeneous* if $\mathscr{I} \simeq \mathscr{I} \restriction X$ for all $\mathscr{I}$-positive set X.
- *weakly homogeneous* if for any $\mathscr{I}$ positive set X there is an $\mathscr{I}$-positive subset Y of X such that $\mathscr{I} \restriction Y \simeq \mathscr{I}$.
- *K-uniform* if $\mathscr{I} \restriction X \leq_K \mathscr{I}$ for all $\mathscr{I}$-positive set X.

It is trivial that homogeneity implies weak homogeneity.

Proposition 2.1.9. *If $\mathscr{I}$ is a maximal ideal then $\mathscr{I}$ is K-uniform.*

Proof. Let X be an $\mathscr{I}$-positive set and note that $\omega \setminus X \in \mathscr{I}$. Then $f : \omega \to X$ such that $f \restriction X$ is the identity function in X and $f \restriction \omega \setminus X$ is constantly equal to $\min X$. Then f is a Katětov function. $\square$

Question 2.1.10.

Is every weakly homogeneous ideal K-uniform?

Proposition 2.1.11. *The ideal $\mathcal{Z}$ of asymptotic density zero is K-uniform.*

Proof. Let X be a $\mathcal{Z}$-positive set, and $e : \omega \to X$ the increasing enumeration of X. Then, $|e^{-1}(I) \cap n| = |I \cap \{e(0), \dots, e(n-1)\}|$ for any $I \subseteq X$, so if $I \in \mathcal{Z} \restriction X$ then $\lim_{n\to\infty} \frac{|e^{-1}(I) \cap n|}{n} = \lim_{n\to\infty} \frac{|I \cap \{e(0) \dots e(n-1)\}|}{n} = 0$ $\square$

We have a criterion for K-uniformity of trace ideals.

Definition 2.1.12. Let I and J be σ-ideals on $X = 2^\omega$ or $X = \omega^\omega$. We say I is *continuously lower than* J, $I \leq_c J$ if there is a continuous function $F : X \to X$ such that $F^{-1}[a] \in J$ for all $a \in I$.

Continuous order reflects on trace ideals in Katětov order.

Theorem 2.1.13. *Let I and J be σ-ideals on $X = 2^\omega$ or $X = \omega^\omega$. If $I \leq_c J$ then $tr(I) \leq_K tr(J)$.*

Proof. Let $F : X \to X$ be a witness of $I \leq_c J$. Inductively, we define a function $f : 2^{<\omega} \to 2^{<\omega}$ (or $f : \omega^{<\omega} \to \omega^{<\omega}$) by $f(\emptyset) = \emptyset$ and if $f \restriction 2^n$ is defined, $s \in 2^n$ and $k < \omega$ then

$$f(s^\frown k) = \begin{cases} \max\{r \supseteq f(s) : F''\langle s^\frown k\rangle \subseteq \langle r\rangle\} & \text{if } |F''\langle s^\frown k\rangle| \geq 2 \\ f(s)^\frown l & \text{otherwise and } F''\langle s^\frown k\rangle \subseteq \langle f(s)^\frown l\rangle. \end{cases}$$

Analogously $f : \omega^{<\omega} \to \omega^{<\omega}$ if $X = \omega^\omega$. Let us prove that f is a Katětov function. Let a be in $tr(I)$ and $x \in \pi(f^{-1}(a))$. Note that by the continuity of F in x, for each n such that $x \restriction n \in f^{-1}(a)$ there is $m > n$ such that

$$f(x \restriction n) \subsetneq f(x \restriction m) \in a.$$

Then, $F(x) \in \pi(a)$, proving that $\pi(f^{-1}(a)) \subseteq F^{-1}(\pi(a)) \in J$. $\square$

Definition 2.1.14. A forcing of the form P_I where I is a σ-ideal on $X = 2^\omega$ or $X = \omega^\omega$ is *continuously homogeneous* if for every I-positive Borel set b there is a continuous function $F : X \to b$ such that $F^{-1}(a) \in I$ for all $a \in I \restriction b$.

Corollary 2.1.15. *If I is a σ-ideal such that P_I is continuously homogeneous then $tr(I)$ is K-uniform.*

The σ-ideal $\mathcal{N}$ of null subsets of ω^ω is an example of a σ-ideal for which P_I is continuously homogeneous.

Lemma 2.1.16. *The forcing $P_\mathcal{N}$ is continuously homogeneous.*

Proof. Let X be a non-null subset of 2^ω. Then there is a non-null perfect set $C \subseteq X$. Let $T \subseteq 2^{<\omega}$ be a pruned tree such that $C = [T]$. Without lose of generality, we can assume that any open subset of C is non-null, because otherwise we can prune $[T]$ satisfying such condition. For each branching node $t \in T$, let $m_t = \mu(C \cap \langle t \rangle)$ and $m = m_\emptyset$. On the other hand, for each $s \in \omega^\omega$, let us denote $l_s = \lambda(\langle s \rangle) = 2^{-|s|s(|s|-1)}$, the product measure on ω^ω. For each $n < \omega$, we will define subsets P_t of ω^n for each branching node t of T such that there are exactly n branching nodes of T shorter than t, satisfying:

1. $P_t \cap P_{t'} = \emptyset$ if $t \neq t'$,
2. If $t' \in 2^{\leq |t|}$ and $t' \subseteq t$ is a branching node of T then $P_t \subseteq P_{t'}$ and
3. $\sum_{s \in P_t} l_s = \frac{m_t}{m}$

As we can see the sets P_t are uniquely defined. The continuous reduction that we are looking for is given by $\varphi(x) = y$ if and only if for all $n < \omega$ there is $k < \omega$ such that $y \restriction k$ is branching node of T having exactly n branching nodes of T contained in it and $x \restriction n \in P_{y \restriction k}$. It is clear that φ is continuous and carries the measure on C to ω^ω and for each null subset N of C the set $\varphi^{-1}[N]$ is a null set of ω^ω. $\square$

As an immediate consequence of we have the following theorem.

Theorem 2.1.17. *$tr(\mathcal{N})$ is a K-homogeneous ideal.* $\square$

2.2 Katětov order and Ramsey properties

Several Ramsey like properties are studied in this work. A *coloring* for unordered n-tuples of natural numbers in m colors is a function φ from the set $[\omega]^n$ of n-tuples of natural numbers into m. A subset A of ω is φ-*homogeneous* if $\varphi \restriction [A]^n$ is constant. Classical Infinite Ramsey Theorem claims that for every coloring φ of unordered n-tuples of natural numbers in m colors there is an infinite subset X of ω such that X is φ-homogeneous. Our first approach to ideal versions of Ramsey theorem concerns to colorings of pairs in two colors.

Definition 2.2.1. Let $\mathscr{I}$ be an ideal on ω. We will say that $\mathscr{I}$ satisfies

$$\omega \longrightarrow (\mathscr{I}^+)^2_2$$

if for every coloring $\varphi : \omega \to 2$ there is an $\mathscr{I}$-positive set X homogeneous with respect to φ. We will say that $\mathscr{I}$ satisfies

$$\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$$

if for every $\mathscr{I}$-positive set X and every coloring $\varphi : [X]^2 \to 2$ there is an $\mathscr{I}$-positive subset Y of X homogeneous with respect to φ.

Mathias' happy families [37] and Farah's semiselective coideals [13] give examples of ideals satisfying both properties. However, these ideals are not definable. In section 2.5 we will construct some examples of definable ideals satisfying $\omega \longrightarrow (\mathscr{I}^+)^2_2$, but failing $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$. The link between this properties and the ideal $\mathcal{R}$ is the following theorem which claims that the random graph ideal $\mathcal{R}$ is critical with respect to $\omega \longrightarrow (\mathscr{I}^+)^2_2$ property.

Theorem 2.2.2. *Let $\mathscr{I}$ be an ideal on ω. Then,*

$$\omega \longrightarrow (\mathscr{I}^+)^2_2 \quad \text{if and only if} \quad \mathscr{I} \not\geq_K \mathcal{R}.$$

Proof. Assume $\mathcal{R} \leq_K \mathscr{I}$. Let $f \in \omega^\omega$ such that for every $R \in \mathcal{R}$, $f^{-1}[R] \in \mathscr{I}$. Define a coloring φ by

$$\varphi(\{n, m\}) = \begin{cases} 0 & \text{if } \{f(n), f(m)\} \in E \\ 1 & \text{otherwise.} \end{cases}$$

If $A \subseteq \omega$ is φ-homogeneous, let say $\varphi''[A]^2 = \{0\}$ (the other case is analogous), we have that $\{f(n), f(m)\} \in E$ for all $n \neq m \in A$. Then, $f''A$ is E-homogeneous and consequently $f''A \in \mathcal{R}$, and so, $A \subseteq f^{-1}[f''A] \in \mathscr{I}$.

On the other hand, assume that $\omega \longrightarrow (\mathscr{I}^+)^2_2$ fails. Let G be a subset of $[\omega]^2$ such that every G-homogeneous is in $\mathscr{I}$, and take a subset X of ω and a function $f : \omega \to X$ such that $f : (\omega, G) \cong (X, E \cap [X]^2)$. Let see that f is a Katětov reduction. For a subbasic set $I \in \mathcal{R}$, assuming $[I]^2 \subseteq E$ (the other case is analogous), we have that for $n \neq m \in f^{-1}[I]$, $f(n) \neq f(m) \in I$ and so, $\{f(n), f(m)\} \in E$. Therefore, $\{n, m\} \in G$. That proves $[f^{-1}[I]]^2 \subseteq G$, and then, $f^{-1}[I] \in \mathscr{I}$. $\square$

Returning to the $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$ property, note that it implies the property $\omega \longrightarrow (\mathscr{I}^+)^2_2$, and actually, the following conditions are equivalent:

- $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$,
- $X \longrightarrow ((\mathscr{I} \restriction X)^+)^2_2$, for all $X \in \mathscr{I}^+$, and
- $\mathcal{R} \not\leq_K \mathscr{I} \restriction X$, for all $X \in \mathscr{I}^+$.

However, $\omega \longrightarrow (\mathscr{I}^+)^2_2$ is not equivalent to $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$ as will be proved in 2.5.11.

Definition 2.2.3. Let $\mathscr{I}$ be an ideal on ω. We will say that $\mathscr{I}$ is a P$^+$-*ideal* if for every decreasing sequence $\{X_n : n < \omega\}$ of $\mathscr{I}$-positive sets there is an $\mathscr{I}$-positive set X such that $X \subseteq^* X_n$, for all $n < \omega$. We will say that $\mathscr{I}$ is a Q$^+$-*ideal* if for every $\mathscr{I}$-positive set X and every partition $\{F_n : n < \omega\}$ of X in finite sets there is an $\mathscr{I}$-positive set $Y \subseteq X$ such that $|Y \cap F_n| \leq 1$, for all $n < \omega$.

Other related notions will be defined in section 3.2. The following result is a well known one.

Theorem 2.2.4 (Folklore). *If $\mathscr{I}$ is a P^+ and Q^+-ideal then $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$.*

Proof. Let X be an $\mathscr{I}$-positive subset of ω and $\varphi : [X]^2 \to 2$. For $i = 0, 1$ and $n \in X$, define $A^i_n = \{k \in X : \varphi\{k, n\} = i\}$. Inductively we will define an increasing function $g \in \omega^\omega$ and a $\subseteq$-decreasing sequence of $\mathscr{I}$-positive sets $\langle B_n : n < \omega \rangle$ such that $B_n \subseteq A^{g(n)}_n$ for all $n < \omega$. Such sequences can be constructed as follows: $B_0 = A_0^{g(0)}$ where $A_0^{g(0)} \in \mathscr{I}^+$. Since $B_n \in \mathscr{I}^+$ we note that either $B_n \cap A_{n+1}^0 \in \mathscr{I}^+$ or $B_n \cap A_{n+1}^1 \in \mathscr{I}^+$. Then take $g(n+1)$ and B_{n+1} such that $B_{n+1} = B_n \cap A_{n+1}^{g(n+1)} \in \mathscr{I}^+$. Given that $\mathscr{I}$ is P$^+$-ideal, there is an $\mathscr{I}$-positive set B such that $B \subseteq^* B_n$ for all $n \in X$. Since either $\{n \in B : g(n) = 0\} \in \mathscr{I}^+$ or $\{n \in B : g(n) = 1\} \in \mathscr{I}^+$, we can assume that $g(n) = i$ for a fixed $i \in 2$ and all $n \in B$. Let us define a sequence $\langle n_j : j < \omega\rangle$ as follows: $n_0 = \min B$; $n_1 = \min\{k \in B : B \setminus k \subseteq B_0\}$ and $n_{j+1} = \min\{k \in B : k > n_j \wedge B \setminus k \subseteq B_{n_j}\}$. Note that if $k \in B \cap [n_j, n_{j+1})$ and $l \in B \cap [n_{j+2}, \infty)$ then $\varphi\{k, l\} = i$. Define $C_0 = B \cap \bigcup_{j<\omega}[n_{2j}, n_{2j+1})$ and $C_1 = B \cap \bigcup_{j<\omega}[n_{2j+1}, n_{2j+2})$. Either C_0 or C_1 is $\mathscr{I}$-positive and intervals defined by $\langle n_j : j < \omega\rangle$ are a partition of such $\mathscr{I}$-positive set. Then, since $\mathscr{I}$ is a Q$^+$ ideal we have that there is a selector of this partition that is $\mathscr{I}$-positive. Such selector is an $\mathscr{I}$-positive subset of X which is φ-homogeneous in color i. $\square$

In order to investigate the validity of the converse of previous theorem, we have the following

Proposition 2.2.5. *If $\mathscr{I}$ satisfies $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$ then $\mathscr{I}$ is a Q$^+$ ideal.*

Proof. Suppose $\mathscr{I}$ is not Q$^+$ and let X be an $\mathscr{I}$-positive set having a partition $\{F_n : n < \omega\}$ in finite sets such that every selector is in $\mathscr{I}$. Define a coloring $\varphi : [X]^2 \to 2$ by $\varphi(\{k, m\}) = 0$ if and only if there is n such that $k, m \in F_n$. Any φ-homogeneous set is finite or is contained in a selector for $\langle F_n : n < \omega\rangle$ and so, the φ-homogeneous sets are in $\mathscr{I}$. $\square$

However we have an example witnessing that the previous theorem is not reversible.

Example 2.2.6. *A non* P$^+$*-ideal satisfying* $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$. Let suppose that for every $n < \omega$ we have a MAD family $\mathcal{A}_n$, satisfying $\mathcal{A}_{n+1} = \bigcup_{A \in \mathcal{A}_n} \mathcal{A}_{n+1} \restriction A$ and every $\mathcal{A}_{n+1} \restriction A$ is a MAD family in $\mathcal{P}(A)$. Let us define $\mathscr{I}_n = \mathscr{I}(\mathcal{A}_n)$ and $\mathscr{I} = \bigcap_{n<\omega} \mathscr{I}_n$. Let us prove that $\mathscr{I}$ satisfies $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$. First, note that $\mathscr{I}^+ = \bigcup_{n<\omega} \mathscr{I}_n^+$, and that (by a result of Mathias ([37])) $\mathscr{I}_n^+ \longrightarrow (\mathscr{I}_n^+)^2_2$. Let $\varphi : [X]^2 \to 2$ be a coloring. If $X \in \mathscr{I}^+$ then there is $n_0 < \omega$ such that $X \in \mathscr{I}_{n_0}^+$ and then there is $Y \in \mathscr{I}_{n_0}^+ \cap \mathcal{P}(X)$ (and so in $\mathscr{I}^+$) which is φ-homogeneous. Now, let us prove that $\mathscr{I}$ is not a P$^+$-ideal. Take a sequence $\langle A_n : n < \omega\rangle$ such that $A_n \in \mathcal{A}_n$, $A_n \supseteq A_{n+1}$, and

$A_n \in \mathscr{I}_n \setminus \mathscr{I}_{n+1}$ for all $n < \omega$. Hence, if $A \subseteq^* A_n$ for all $n < \omega$ then $A \in \mathscr{I}_n$ for all n and so, $A \in \mathscr{I}$.

A MAD family $\mathcal{A}$ is *completely separable* if for every $\mathscr{I}(\mathcal{A})$-positive set X there is an element A of $\mathcal{A}$ contained in X. Erdös and Shelah [12] asked whether completely separable MAD families exist in ZFC. It is well known that under CH and under MA there is a completely separable MAD family. Recently Shelah announced that if $2^{\aleph_0} < \aleph_\omega$ then there is a completely separable MAD family.

Remark 2.2.7. If the MAD families used in the previous example are completely separable then $\mathcal{P}(\omega)/\mathscr{I}$ is forcing equivalent with the collapse forcing[1] $Coll(\omega, 2^\omega)$.

Proof. Note that $\bigcup_{n<\omega} \mathcal{A}_n$ is a dense subset of $\mathscr{I}^+$, since $A \in \mathscr{I}^+$ implies there is $n < \omega$ such that $A \in \mathscr{I}_n^+$ and then there is $B \in \mathcal{A}_n$ contained in A. Then, below every $A \in \mathscr{I}^+$ there is an antichain of cardinality $\mathfrak{c}$ ($\mathcal{A}_{n+1} \upharpoonright A$ is such antichain) and by McAloon's theorem (see theorem 14.17 in [32]) we are done. □

The following theorem due to Adrian Mathias will be used in this section. In section 3.2 we will give a game theoretic proof of it.

Theorem 2.2.8 (Mathias [37], Theorem 2.12)**.** *Let $\mathcal{U}$ be a ultrafilter on ω. Then, $\mathcal{U}$ is selective if and only if $\mathcal{U} \cap \mathscr{I} \neq \emptyset$ for every analytic and tall ideal $\mathscr{I}$ on ω.* □

We want to know whether a definable ideal fulfils $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$. Unfortunately P^+ and Q^+ is not a criterion for analytic ideals, as the following remark shows.

Remark 2.2.9. There are no tall analytic ideals which are both P^+ and Q^+.

Proof. Let $\mathscr{I}$ be an analytic ideal on ω, and suppose that $\mathscr{I}$ is a P^+ and Q^+-ideal. Then, by P^+ condition $\mathscr{I}^+$ is a σ-closed forcing, and then it does not add new real numbers. Let G be an $\mathscr{I}^+$-generic ultrafilter. Let us prove that in $V[G]$, G is a P and Q ultrafilter. Given a sequence $\langle A_n : n < \omega \rangle \subseteq G$, since $\mathscr{I}$ is a P^+-ideal, there is $A \in G$ such than $A \subseteq^* A_n$ for all $n < \omega$. Let $\{I_n : n < \omega\}$ be a partition of ω in finite sets. Since $\mathscr{I}$ is a Q^+-ideal there

[1] For reference about Collapse Forcing see [26].

is an $\mathscr{I}$-positive selector of such partition and then there is $B \in G$ such that $|B \cap I_n| \leq 1$ for all $n < \omega$. Then, in $V[G]$, G is a selective ultrafilter and $\mathscr{I}^{V[G]} = \mathscr{I}$ is an analytic ideal disjoint from G, contradicting Mathias theorem 2.2.8 (in $V[G]$). $\square$

Among the maximal ideals, these conditions have been well studied, since a maximal P^+-ideal is the dual of a P-point and maximal Q^+-ideal is the dual ideal of a Q-point.

2.3 Q and Q^+-ideals

Definition 2.3.1. Let $\mathscr{I}$ be an ideal on ω. $\mathscr{I}$ is a Q-*ideal* if for every partition $\langle F_n : n < \omega \rangle$ of ω into finite sets there is an $\mathscr{I}$-positive set X such that $|X \cap F_n| \leq 1$ for all $n < \omega$.

The ideal $\mathcal{ED}_{fin}$ is critical for the property of be a Q-ideal with respect to the Katětov-Blass order, as the following theorem shows.

Theorem 2.3.2. *Let $\mathscr{I}$ be an ideal on ω. Then*

1. *$\mathscr{I}$ is a Q-ideal if and only if $\mathscr{I} \not\geq_{KB} \mathcal{ED}_{fin}$, and*
2. *$\mathscr{I}$ is a Q^+-ideal if and only if $\mathscr{I} \upharpoonright X \not\geq_{KB} \mathcal{ED}_{fin}$ for all $\mathscr{I}$-positive set X.*

Proof. (1) Let suppose that $\mathcal{ED}_{fin} \leq_{KB} \mathscr{I}$, and let $f : \omega \to \Delta$ be a Katětov-Blass reduction. Then, the family $\{f^{-1}[\{n\} \times (n+1)] : n < \omega\}$ is a partition of ω in finite sets and every selector of this partition belongs $\mathscr{I}$. On the other hand, if $\mathscr{I}$ is not a Q-ideal then there is a partition $\langle F_n : n < \omega \rangle$ of ω in finite sets such that every selector belongs $\mathscr{I}$. Note such a partition must have elements of arbitrarily large finite cardinality, because otherwise, $\mathscr{I}$ would be an improper ideal. Take a sequence $\langle G_n : n < \omega \rangle$ such that $|G_n| = n$, $G_n \subseteq F_{i_n}$ for some $i_n < \omega$ and $X = \bigcup_n G_n \in \mathscr{I}^+$. Then $\mathscr{I} \upharpoonright X$ is a copy of $\mathcal{ED}_{fin}$ inside $\mathscr{I}$. (2) Analogous to (1). $\square$

2.4 P and P^+-ideals

The P^+ property as such does not have a critical ideal, since the ideal $\mathscr{I} = (\mathbf{Fin} \times \mathbf{Fin}) \oplus \mathcal{ED}$ is Katětov-equivalent to $\mathcal{ED}$ and the second one is a P^+

ideal while the first one is not. Proofs of these facts are (1) by Proposition 1.7.1 we have $\mathscr{I} \leq \mathcal{ED}$, (2) up isomorphism, $\mathcal{ED} \subseteq \mathscr{I}$ and (3) $\mathscr{I}$ is not P^+ because it contains a copy of $\mathbf{Fin} \times \mathbf{Fin}$ (see Theorem 2.4.2). However, the ideals $\mathbf{Fin} \times \mathbf{Fin}$ and conv are critical for a slight strengthening of P$^+$. Let us introduce the following definition.

Definition 2.4.1. Let $\mathscr{I}$ be an ideal on ω. We will say that $\mathscr{I}$ is *decomposable* if there is an infinite partition $\{X_n : n < \omega\}$ of ω in $\mathscr{I}$-positive sets such that for every $X \subseteq \omega$

$$X \in \mathscr{I} \text{ if and only if } (\forall n < \omega)(X \cap X_n \in \mathscr{I}).$$

Such partition X will be called an *$\mathscr{I}$-decomposition.* We will say that $\mathscr{I}$ is *hereditarily decomposable* if for every $\mathscr{I}$-positive set X, $\mathscr{I} \restriction X$ is decomposable.

We will say that $\mathscr{I}$ is *indecomposable* if it is not decomposable.

Theorem 2.4.2. *Let $\mathscr{I}$ be an ideal. Then $\mathscr{I}$ is a P$^+$-ideal if and only if $\mathscr{I}$ is indecomposable and $\mathbf{Fin} \times \mathbf{Fin} \not\leq_K \mathscr{I} \restriction X$, for all $X \in \mathscr{I}^+$.*

Proof. Let $\mathscr{I}$ be a P$^+$-ideal. If $\{X_n : n < \omega\}$ is a partition of ω in $\mathscr{I}$-positive sets then there is $Y \in \mathscr{I}^+$ such that $Y \setminus \bigcup_{k \geq n} X_k$ is finite for all n, proving that $\{X_n : n < \omega\}$ is not a decomposition. Given $X \in \mathscr{I}^+$ and $f : X \to \omega \times \omega$, if Y is an $\mathscr{I}$-positive pseudointersection of the family $\{f^{-1}[(\omega \setminus n) \times \omega] : n < \omega\}$ then $f''Y \in \mathbf{Fin} \times \mathbf{Fin}$ but $Y \in \mathscr{I}^+$, proving that f is not a Katětov function.

On the other hand, if $\mathscr{I}$ is not P$^+$ then there is a decreasing sequence $\langle X_n : n < \omega \rangle$ of $\mathscr{I}$-positive sets whose pseudointersections are in $\mathscr{I}$ and $X_0 = \omega$. If there is N such that for all $n \geq N$, $X_n \setminus X_{n+1}$ belongs $\mathscr{I}$ then $\mathscr{I} \restriction X_N$ contains a copy of $\mathbf{Fin} \times \mathbf{Fin}$. Otherwise, by taking a sequence k_n such that $X_{k_n} \setminus X_{k_{n+1}} \in \mathscr{I}^+$ we have that this family is a decomposition of $\mathscr{I} \restriction X$. $\square$

The ideal conv is useful for study the property P$^+$. The following theorem characterizes those ideals which are Katětov above the ideal conv.

Theorem 2.4.3. *For any ideal $\mathscr{I}$ on ω the following are equivalent*

1. $\mathscr{I} \geq_K \mathsf{conv}$,

2. *there is a linear order $\sqsubseteq$ for ω such that $(\omega, \sqsubseteq)$ is order-isomorphic to $\mathbb{Q} \cap [0,1]$ and every increasing and every decreasing sequence with respect to $\sqsubseteq$ is in $\mathscr{I}$,*

3. *there is a topology τ on ω such that (ω, τ) is homeomorphic to $\mathbb{Q} \cap [0,1]$ and every τ-convergent sequence (having a limit in $[0,1]$) belongs to $\mathscr{I}$, and*

4. *there is a countable family $\mathcal{X} \subseteq [\omega]^\omega$ such that for every $Y \in \mathscr{I}^+$ there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_0$.*

Proof. $(2 \to 1)$ Such an isomorphism from ω into $\mathbb{Q}$ is a Katětov reduction between $\mathscr{I}$ and conv since every convergent sequence can be split as the union of an increasing with a decreasing sequences.

$(3 \to 2)$ Such homeomorphism between ω and $\mathbb{Q}$ carries an order to ω isomorphic to the order of $\mathbb{Q} \cap [0,1]$, and every increasing and every decreasing sequences are in $\mathscr{I}$ since they are τ-convergent.

$(4 \to 3)$ Let $\mathcal{X}$ be such family. We can suppose that $\mathcal{X}$ separates points (i.e., for each pair $\{n, m\}$ there is $X \in \mathcal{X}$ such that $|X \cap \{m, n\}| = 1$) and every Boolean combination of its elements is infinite. Let τ be the topology on ω whose basic clopen sets are the Boolean combinations of elements of $\mathcal{X}$. This topology is homeomorphic to the standard topology of $\mathbb{Q}$ and if $Y \in \mathscr{I}^+$ then Y can not be covered by finitely many τ convergent sequences because there is a family of clopen sets $\{C_s : s \in 2^{<\omega}\}$ such that $\{C_s : s \in 2^n\}$ is pairwise disjoint for all n, $C_{s^\frown 0} \cup C_{s^\frown 1} = C_s$ and $Y \cap C_s$ is infinite for all $s \in 2^{<\omega}$.

$(1 \to 4)$ Let $\mathcal{C}$ be a base of clopen sets of $\mathbb{Q}$ and f a witness of $\mathsf{conv} \leq_K \mathscr{I}$. Define $\mathcal{X} = \{f^{-1}[C] : C \in \mathcal{C}\}$. Let $I \subseteq \omega$ be such that for every $X \in \mathcal{X}$, $I \cap X$ is finite or $I \setminus X$ is finite. Then, for every basic set $C \in \mathcal{C}$, $f''I$ is almost contained in C or is almost contained in $\mathbb{Q} \setminus C$, then $f''I$ is covered by a convergent sequence of $\mathbb{Q}$, and hence $I \subseteq f^{-1}[f''I] \in \mathscr{I}$. $\square$

The decomposability of ideals gives a criterion for ideals which are Katetov-above the ideal conv.

Lemma 2.4.4. *If there is a family $\{\mathcal{X}_n : n < \omega\}$ of $\mathscr{I}$-decompositions such that (1) $\mathcal{X}_{n+1}$ refines $\mathcal{X}_n$ and (2) all pseudointersections of decreasing chains $\langle A_n : n < \omega\rangle$ of $\mathscr{I}$-positive sets such that $A_n \in \mathcal{X}_n$ are in $\mathscr{I}$, then $\mathscr{I} \geq_K \mathsf{conv}$.*

Proof. We will proof that $\mathcal{X} = \bigcup_{n<\omega} \mathcal{X}_n$ is a family as in 2.4.3(4). Let Y be an $\mathscr{I}$-positive set. Then for all $n < \omega$ and all $A \in \mathcal{X}_n$ either $Y \subseteq^* A$ or $|Y \cap A| < \aleph_0$ or $|Y \cap A| = |Y \setminus A| = \aleph_0$. If for some $n < \omega$ the third case holds, we are done. Let assume case 3 does not hold for all n. Since any $\mathcal{X}_n$ is an $\mathscr{I}$-decomposition, second case is not possible for some $A_n \in \mathcal{X}_n$. If there is $n < \omega$ and $B \in \mathcal{X}_n$ with $A_n \neq B$ such that $|B \cap Y| = \omega$ we are done. Let us assume not. Then Y is a pseudointersection of the sequence $\langle A_n : n < \omega \rangle$, and then $Y \in \mathscr{I}$, a contradiction. $\square$

Theorem 2.4.5. *Let $\mathscr{I}$ be an ideal on ω such that the quotient $\mathcal{P}(\omega)/\mathscr{I}$ is a proper forcing and adds a new real number. Then there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_K \mathsf{conv}$.*

We will work with the forcing $\mathscr{I}^+$ instead of $\mathcal{P}(\omega)/\mathscr{I}$ since the last one is the separative quotient of the first one.

Proof. Properness of $\mathscr{I}^+$ implies that for any countable family of maximal antichains $\{\mathcal{A}_n : n < \omega\}$ there exists an $\mathscr{I}^+$-condition X which is incompatible with all but countably many elements of $\mathcal{A}_n$, for all $n < \omega$. Let $\dot{r}$ be an $\mathscr{I}^+$ name for a new real number, and pick a family $\{\mathcal{A}_n : n < \omega\}$ of maximal antichains in $\mathscr{I}^+$ such that $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$ and any condition in $\mathcal{A}_n$ decides $\dot{r} \restriction n$, for all $n < \omega$. Now, pick $X \in \mathscr{I}^+$ such that $\mathcal{B}_n = \{A \in \mathcal{A}_n : A \cap X \in \mathscr{I}^+\}$ is countable for all $n < \omega$. Recursively, we can refine each $\mathcal{B}_n$ into a pairwise disjoint family of positive sets $\mathcal{X}_n$ which is a family of decompositions as in previous lemma. Note that if Y is an $\mathscr{I}$-positive subset of X then there is $C \in \mathcal{X}_n$ such that $C \cap Y \in \mathscr{I}^+$. If not, Y (and consequently X too) must be compatible with an element of $\mathcal{A}_n \setminus \mathcal{B}_n$, a contradiction. That proves $\mathcal{X}$ is an $\mathscr{I}$-decomposition. Condition (1) was required while constructing $\mathcal{X}$, and finally we will prove condition (2). Let Y be an $\mathscr{I}$-positive subset of X. We claim that there is $n < \omega$ and there are $A_0 \neq A_1 \in \mathcal{C}_n$ such that $|Y \cap A_0| = |Y \cap A_1| = \aleph_0$. That n can be found by taking a large enough $n < \omega$ such that Y does not decide $\dot{r} \restriction n$, and A_0 and A_1 must be compatible conditions with Y deciding different values for $\dot{r} \restriction n$. $\square$

2.5 Weaker partition properties

The partition properties defined below are weak versions of the previous properties.

Definition 2.5.1. Let $\mathscr{I}$ be an ideal on ω. We will say that $\mathscr{I}$ satisfies:

(a) $\omega \longrightarrow (\omega, \mathscr{I}^+)^2_2$ if for every coloring $\varphi : [\omega]^2 \to 2$ either there is an infinite 0-homogeneous set or there is an $\mathscr{I}$-positive 1-homogeneous set.

(b) $\omega \longrightarrow (<\omega, \mathscr{I}^+)^2_2$ if for every coloring $\varphi : [\omega]^2 \to 2$ either for every $m < \omega$ there is a 0-homogeneous set X with $|X| = m$ or there is an $\mathscr{I}$-positive 1-homogeneous set.

(c) $\mathscr{I}^+ \longrightarrow (\omega, \mathscr{I}^+)^2_2$ if for any $\mathscr{I}$-positive set Y and any coloring $\varphi : [Y]^2 \to 2$ either there is an infinite 0-homogeneous set or there is an $\mathscr{I}$-positive 1-homogeneous set.

(d) $\mathscr{I}^+ \longrightarrow (<\omega, \mathscr{I}^+)^2_2$ if for any $\mathscr{I}$ positive set Y and any coloring $\varphi : [Y]^2 \to 2$ either for every $m < \omega$ there is a 0-homogeneous set X with $|X| = m$ or there is an $\mathscr{I}$-positive 1-homogeneous set.

In the following diagram are shown the implications between these properties:

$$\begin{array}{ccccc}
\omega \longrightarrow (\mathscr{I}^+)^2_2 & \Longrightarrow & \omega \longrightarrow (\omega, \mathscr{I}^+)^2_2 & \Longrightarrow & \omega \longrightarrow (<\omega, \mathscr{I}^+)^2_2 \\
\Uparrow & & \Uparrow & & \Uparrow \\
\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2 & \Longrightarrow & \mathscr{I}^+ \longrightarrow (\omega, \mathscr{I}^+)^2_2 & \Longrightarrow & \mathscr{I}^+ \longrightarrow (<\omega, \mathscr{I}^+)^2_2
\end{array}$$

Definition 2.5.2 (Thümmel, [47])**.** We say that $\langle X_i : i \in X \rangle$ is an *$\mathscr{I}$-tower* if $X \in \mathscr{I}^+$, $X_i \subseteq X$ and $X \setminus X_i \in \mathscr{I}$ for all $i \in X$. A *diagonal* of an $\mathscr{I}$-tower $\langle X_i : i \in X \rangle$ is a subset D of X such that $D \setminus (i+1) \subseteq X_i$ for all $i \in D$. We say that an ideal $\mathscr{I}$ *contains an $\mathscr{I}$-tower* if there is an $\mathscr{I}$-tower such that all of its diagonals are in the ideal.

Theorem 2.5.3 (Thümmel)**.** *Let $\mathscr{I}$ be an ideal on ω. If $\mathscr{I}$ does not contain an $\mathscr{I}$-tower then* $\mathscr{I}^+ \longrightarrow (<\omega, \mathscr{I}^+)^2_2$.

Proof. Let $X \in \mathscr{I}^+$ and $\psi : [X]^2 \to 2$ be a coloring. Suppose by contradiction that for some $m < \omega$ there is no $Y \in [X]^m$ homogeneous in color 0 and there is no $Y \in \mathscr{I}^+$ homogeneous in color 1. Take an $\mathscr{I}$-positive $\bar{X} = \{x_i : i < \omega\} \subseteq X$ such that this m becomes minimal. Define $X_i = \{y \in \bar{X} : \psi(\{x_i, y\}) = 1\}$. Suppose that for some i, $Z = \bar{X} \setminus X_i \in \mathscr{I}^+$.

By minimality of m, there is a $Z' \in [Z]^{m-1}$ homogeneous in color 0, a contradiction. We have therefore $\bar{X} \setminus X_i \in \mathscr{I}$ for all i. Let $\bar{X}_i = \bigcap_{j \leq i} X_j$. $\langle \bar{X}_i : i \in \bar{X} \rangle$ is an $\mathscr{I}$-tower. Any diagonal is homogeneous in color 1, and by our assumption is in $\mathscr{I}$, i.e. the ideal contains an $\mathscr{I}$-tower. $\square$

An example of an ideal $\mathscr{I}$ which does not contain a $\mathscr{I}$-tower is nwd.

Lemma 2.5.4. nwd *does not contain a* nwd*-tower.*

Proof. Suppose that $\langle X_i : i \in X \rangle$ is a nwd-tower and fix a contable base $\{U_i : i < \omega\}$ of $\mathbb{Q}$ and a type-ω order $\sqsubset$ for $\mathbb{Q}$. We can assume that $X_{i+1} \subseteq X_i$. Choose $d_0 \in X$ arbitrary and suppose d_i was found. If there is $d \sqsupset d_i$ with $d \in X_{d_i} \cap U_i$ then take $d_{i+1} = d$, else take arbitrary $d_{i+1} \sqsupset d_i$ with $d_{i+1} \in X_{d_i}$. By the construction $D = \{d_i : i < \omega\}$ is a diagonal of the tower, but if X is dense in an open set U then D is dense in U too, hence $D \in \mathsf{nwd}^+$. $\square$

By 2.5.3 we have that $\mathsf{nwd}^+ \longrightarrow (<\omega, \mathsf{nwd}^+)^2_2$. Two other ideals satisfying that property are $\mathbf{Fin} \times \mathbf{Fin}$ and $\mathcal{ED}_{fin}$.

Lemma 2.5.5. $\mathcal{ED}_{fin}$ *does not contain an* $\mathcal{ED}_{fin}$*-tower and* $\mathbf{Fin} \times \mathbf{Fin}$ *does not contain a* $\mathbf{Fin} \times \mathbf{Fin}$*-tower.*

Proof. Let $\langle X_i : i \in X \rangle$ be a decreasing $\mathcal{ED}_{fin}$-tower and let e be a one-to-one function from X onto ω. For every $i \in X$, we can find $n_i > e(i)$ such that $|X_i \cap (\{n_i\} \times (n_i + 1))| \geq e(i)$. Hence, $D = \bigcup_{i \in X} (X_i \cap (\{n_i\} \times (n+1)))$ is an $\mathcal{ED}_{fin}$-positive diagonal of $\langle X_i : i \in X \rangle$. Analogously (but taking n_i such that $|X_i \cap (\{n_i\} \times \omega)| = \aleph_0$) we can prove $\mathbf{Fin} \times \mathbf{Fin}$ does not contain a $\mathbf{Fin} \times \mathbf{Fin}$-tower. $\square$

However, $\mathcal{ED}_{fin}$ does not satisfy the property $\omega \longrightarrow (\omega, \mathscr{I}^+)^2_2$.

Lemma 2.5.6.

$$\omega \nrightarrow (\omega, \mathcal{ED}^+_{fin})^2_2.$$

Proof. Let $\varphi : \Delta \to 2$ be given by $\varphi(\{(n_1, m_1), (n_2, m_2)\}) = 0$ if and only if $n_1 = n_2$. It is clear that every 0-homogeneous set is contained in a column, and every 1-homogeneous is a partial selector of columns and so 1-homogeneous are in $\mathcal{ED}_{fin}$. $\square$

In fact, every ideal $\mathscr{I} \geq_{KB} \mathcal{ED}_{fin}$ fails the property $\omega \longrightarrow (\omega, \mathscr{I}^+)^2_2$ since a Katětov-Blass reduction defines a partition of ω in finite arbitrarily large sets such that every selector of that partition is in $\mathscr{I}$, and so, we can define a coloring φ in analogous way of previous proof, witnessing $\mathscr{I}$ does not satisfy $\omega \longrightarrow (\omega, \mathscr{I}^+)^2_2$.

In order to define an ideal $\mathscr{I}$ satisfying $\omega \longrightarrow (<\omega, \mathscr{I}^+)^2_2$ we have studied some properties closely related. For every $n < \omega$ we will say that an ideal $\mathscr{I}$ is *n-Ramsey* if for any coloring $\varphi : [\omega]^2 \to 2$ either there is a 0-homogeneous set A of cardinality n or there is an $\mathscr{I}$-positive 1-homogeneous set A. We denote such property by

$$\omega \longrightarrow (n, \mathscr{I}^+)^2_2 .$$

Let us denote by $\mathcal{K}_n$ the complete graph with n vertices. We will say that a graph H *contains* $\mathcal{K}_n$ if H has a subgraph isomorphic to $\mathcal{K}_n$.

Lemma 2.5.7. *For every $3 \leq n < \omega$ there is a unique-up-isomorphisms graph $G_n = \langle \omega, E_n \rangle$ such that for every graph H on ω, H does not contain $\mathcal{K}_n$ if and only if there is a subset A of ω such that $H \cong \langle A, E_n \restriction [A]^2 \rangle$.*

Proof. Fix $n < \omega$, and let us define E_n as in the alternative construction of Random Graph below proof of 1.6.28 was done. We will define recursively an increasing family $\{F_k : k < \omega\}$ of initial segments of ω and a $\subseteq$-increasing family $\{E^k : k < \omega\}$ of subsets of $[\omega]^2$ as follows: Define $F_0 = \{0\}$ and $E^0 = \emptyset$. Let suppose defined F_k and E^k and Define $P(k) = \{a \subseteq F_k : (\forall b \in [a]^{n-1})(\exists l \neq m \in b)(\{l, m\} \notin E^k)\}$ and let $\{a_j : j < |P(k)|\}$ be an enumeration of $P(k)$. Then, we define $E^{k+1} = E^k \cup \{\{|F_k| + j, i\} : j < |P(n)| \wedge i \in a_j\}$, $F_{k+1} = F_k \cup \{|F_k| + j : j < |P(k)|$ and $E_n = \bigcup_{k<\omega} E^k$. It is clear that every subgraph of E_n does not contain $\mathcal{K}_n$, and given a graph H which does not contain $\mathcal{K}_n$ we can construct the isomorphism required as we have done in proof of 1.6.29. □

Definition 2.5.8. We define $\mathcal{R}_n$ as the ideal on ω generated by the free sets in G_n.

Let us note that $\mathcal{R}_n$ is not a trivial ideal for all $n \geq 3$. If there were a finite family $\{X_i : i \leq N\}$ of free sets with respect to G_n such that $\bigcup_{i<N} X_i = \omega$ then we can not embed the $(N+2)$-cycle in G_n, contradicting the universal property of G_n, since the $(N+2)$-cycle does not have complete subgraphs of order greater than 2 for $N > 2$.

The ideal $\mathcal{R}_n$ is critical with respect to n-Ramsey property.

Theorem 2.5.9. *Let $\mathscr{I}$ be an ideal on ω. Then $\mathscr{I}$ is n-Ramsey if and only if $\mathscr{I} \not\geq_K \mathcal{R}_n$.*

Proof. If $\mathscr{I}$ is not n-Ramsey then there is a coloring $\varphi : [\omega]^2 \to 2$ such that there is not 0-homogeneous sets of cardinality n and every 1-homogeneous set is in $\mathscr{I}$. Put $H = \varphi^{-1}\{0\}$. Note that H does not have complete subgraphs of cardinality n, and so, there is an isomorphism f from $\langle \omega, H\rangle$ to $\langle A, E_n \restriction A\rangle$ for some $A \subseteq \omega$. f is a Katětov reduction since for every E_n-free set B, $f^{-1}[B]$ is 1-homogeneous with respect to φ, and so, $f^{-1}[B] \in \mathscr{I}$. On the other hand, let f be a Katětov reduction, i.e., a function such that for every G_n-free set $f^{-1}[A] \in \mathscr{I}$, and define $\varphi : [\omega]^2 \to 2$ by $\varphi(\{m, n\}) = 0$ iff $f(m) \neq f(n)$ and $\{f(m), f(n)\} \in E_n$. If A is a 0-homogeneous set with $|A| = n$ then $f[A]$ is a complete subgraph of G_n, a contradiction. If A is a 1-homogeneous set then $f[A] \in \mathcal{R}_n$ and so, $A \subseteq f^{-1}[f[A]] \in \mathscr{I}$. $\square$

Given an ideal $\mathscr{I}$ on ω, the ideal $\widetilde{\mathscr{I}}$ is defined as follows.

$$\widetilde{\mathscr{I}} = \{A \subseteq \omega \times \omega : (\exists k, i_0)((\forall i \leq i_0\ (A)_i \in \mathscr{I}) \wedge (\forall i > i_0\ |(A)_i| < k))\}.^2$$

It is clear that if $\mathscr{I}$ is a Borel ideal then $\widetilde{\mathscr{I}}$ is a Borel ideal.

Lemma 2.5.10. *If $\mathscr{I}^+ \longrightarrow (<\omega, \mathscr{I}^+)^2_2$ then $\omega \longrightarrow (\widetilde{\mathscr{I}^+})^2_2$.*

Proof. Let a coloring $\psi : [\omega \times \omega]^2 \to 2$ be given. Assume by contradiction that there is no homogeneous $X \in \widetilde{\mathscr{I}}^+$. We construct inductively for every $i < \omega$:

- an increasing sequence $m_i < \omega$,
- a pairwise different finite sequence $n_{i,j}$ with $j < i$,
- a color $l_i < 2$ and
- a decreasing $A_i \in (\mathbf{Fin} \times \mathscr{I})^+$,

such that if $(m, n) \in A_i$ then $\psi(\{(m_i, n_{i,j})(m, n)\}) = l_i$, for all $i < \omega$ and all $j < i$. In step $i + 1$ we take $m_{i+1} > m_i$ such that $(A_i)_{m_{i+1}} \in \mathscr{I}^+$. Let $\{e^i_k : k < \omega\}$ be an enumeration of $(A_i)_{m_{i+1}}$. We find a decreasing sequence $A_{i,k} \in (\mathbf{Fin} \times \mathscr{I})^+$ for $k < \omega$ such that $\psi(\{(m_{i+1}, e^i_k), (m, n)\}$ is

[2] See the notation for columns in products given in the section of preliminaries.

constant for all $(m,n) \in A_{i,k}$. We find $A \in \mathscr{I}^+$ such that $A \subseteq (A_i)_{m_{i+1}}$ and $\psi(\{(m_{i+1}, e^i_k), (m,n)\}) = l_{i+1}$ for all $k \in A$ and $(m,n) \in A_{i,k}$. Our assumption and the partition relation give us now a set $\{n_{i+1,j} : j \leq i\}$ homogeneous in color l_{i+1}. We put $A_{i+1} = \bigcap_{j \leq i} A_{i,n_{i+1,j}}$. In this way, we fulfill the inductive assumption. Let $C \subseteq \omega$ be such that $l_i = l_j$ for all $i,j \in C$. Then $\{(m_i, n_{i,j}) : i \in C \wedge j < i\}$ is an $\widetilde{\mathscr{I}}$-positive homogeneous set for ψ. $\square$

An immediate consequence of the lemma is the following.

Corollary 2.5.11.

$$\omega \longrightarrow (\widetilde{\mathsf{nwd}}^+)^2_2 \quad \textit{and} \quad \omega \longrightarrow (\widetilde{\mathcal{ED}_{fin}}^+)^2_2$$

$\square$

Another (non immediate) consequence is

Corollary 2.5.12.

$$\omega \longrightarrow (\omega, \mathcal{ED}^+)^2_2.$$

Proof. We still using notation for columns in products defined in the section of preliminaries. Let a coloring $\psi : [\omega \times \omega]^2 \to 2$ be given and suppose that there are not infinite 0-homogeneous sets. We will construct, for $i < \omega$:

- an increasing sequence $m_i \in \omega$,
- a set $\{n_{i,j} : j < i\}$ of pairwise distinct elements of ω, and
- a decreasing $A_i \in (\mathbf{Fin} \times \mathbf{Fin})^+$

such that for any $i < \omega$, $\{(m_i, n_{i,j}) : j < i\}$ is 1-homogeneous and $\psi(\{(m_i, n_{i,j})(m,n)\}) = 1$ for all $(m,n) \in A_i$ and $j < i$.

In step 0 we can take $m_0 = 0$, and $A_0 = \omega \times \omega$. In step $i+1$ we take $m_{i+1} > m_i$ such that

1. $|(A_i)_{m_{i+1}}| = \aleph_0$ and
2. there is a 1-homogeneous subset B of $\{m_{i+1}\} \times (A_i)_{m_{i+1}}$ with $|B| = i$ and such that $\{(n,m) \in A_i : \psi(\{(n,m), (m_{i+1}, k)\}) = 1\} \in (\mathbf{Fin} \times \mathbf{Fin})^+$ for all pair $(m_{i+1}, k) \in B$.

We will prove by contradiction that it is possible to find such m_{i+1}. If not, let K the maximal i that we can do the construction below, and we now construct:

- an increasing sequence $\{m'_i : i < \omega\}$ with $m'_0 = m_K$ and
- a decreasing sequence of sets $B_i \in (\mathbf{Fin} \times \mathbf{Fin})^+$ with $B_0 \subseteq A_K$,

such that for every $i < \omega$ there is $k_i \in (B_i)_{m'_i}$ such that

$$B_{i+1} = \{(m,n) \in B_i : m > m'_i \wedge \varphi(\{(m'_i,k_i),(m,n)\}) = 0\} \in (\mathbf{Fin} \times \mathbf{Fin})^+.$$

We can find such sequences since for every $k \geq K$, if $(A_K)_k$ is infinite and $\{k\} \times (A_K)_k$ does not have infinite 0-homogeneous sets, then by Ramsey theorem, it has an infinite 1-homogeneous set. But 1-homogeneous infinite subsets of columns must have an element l such that $\{(n,m) \in A_K : n > k \wedge \psi(\{(k,l),(n,m)\}) = 0\} \in (\mathbf{Fin} \times \mathbf{Fin})^+$. Hence, by construction, $\{(m'_i,k_i) : i < \omega\}$ is an infinite 0-homogeneous set, a contradiction. Then $\{(m_i,n_{i,j}) : i < \omega \wedge j < i\}$ is an $\mathcal{ED}$-positive 1-homogeneous set. □

Example 2.5.13. Let $\mathscr{I}$ be the ideal on $\mathbb{Q}$ of all subsets A of $\mathbb{Q}$ that there is not an order embedding from $\mathbb{Q}$ into A. Such ideal is a co-analytic one, satisfies $\mathscr{I}^+ \longrightarrow (\omega, \mathscr{I}^+)^2_2$ and does not satisfy $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$. Details of this proof are given by Todorčević and Farah in [16]. James Baumgartner proved in [4] that for any ultrafilter $\mathcal{U}$, $\mathcal{U} \longrightarrow (\omega, \mathcal{U})^2_2$ if and only if $\mathcal{U} \longrightarrow (\mathcal{U})^2_2$. We would like to know which is the relationship between these properties among Borel ideals.

2.6 On a theorem of Farah

We now consider colorings into infinitely many colors. Of course, one can not ask for infinite homogeneous sets, but can ask for sets which are weakly homogeneous in the sense of using "few" colors.

Definition 2.6.1. Let $\mathscr{I}$ be an ideal on ω and $\varphi : [\omega]^n \to \omega$. We will say $A \subseteq \omega$ is *$\mathscr{I}$-homogeneous for φ* if $\varphi''[A]^n \in \mathscr{I}$. We will say that $\mathscr{I}$ *has the property*

$$\omega \longrightarrow (\omega)^n_{\omega,\mathscr{I}}$$

if every $\varphi : [\omega]^n \to \omega$ has an infinite $\mathscr{I}$-homogeneous set.

In [15] Ilijas Farah used this property with the aim of extending the characterization of non meager ideals done by Jalali-Naini - Talagrand theorem 1.2.3. Farah claimed that for every hereditary subset $\mathcal{J}$ of $\mathcal{P}(\omega)$, $\mathcal{J}$ is nonmeager if and only if for every $n \geq 1$ and every $h : [\omega]^n \to \omega$ either h has an infinite homogeneous set or there is an infinite $\mathcal{J}$-homogeneous set. Restricting this result to ideals, we have noted that this is not completely correct. Since singletons are in any ideal, Farah's claim becomes equivalent with the sentence: for all ideal on ω, $\mathscr{I}$ is nonmeager if and only if for all $n \geq 1$, $\omega \longrightarrow (\omega)^n_{\omega,\mathscr{I}}$. However, Farah's proof of necessity uses only case $n = 1$ and the fact that a function $h : [\omega]^1 \to \omega$ which does not have any infinite homogeneous set is essentially a finite-to-one function from ω into ω and then the case 1 is equivalent to a version of the Jalali-Naini - Talagrand theorem. That version must be (5) in theorem 1.2.3, which is immediately equivalent with $\mathscr{I} \not\leq_{RB}$ **Fin**. However, it is easy to see that case $n = 1$ is equivalent with $\mathscr{I} \not\leq_K$ **Fin**, and this condition is equivalent to $\mathscr{I}$ is tall.

The ideal $\mathcal{G}_c$ of *graphs without infinite complete subgraphs* is a critical ideal with respect to $\omega \longrightarrow [\omega]^2_{\omega,\mathscr{I}}$ property in Katětov order. The ideal $\mathcal{G}_c$ was defined in section 1.6 as the family of subsets I of $[\omega]^2$ such that for every infinite $X \subseteq \omega$ there is $n \neq m \in X$ such that $\{n, m\} \notin I$.

Proposition 2.6.2. *Let $\mathscr{I}$ be an ideal on ω. Then $\omega \longrightarrow (\omega)^2_{\omega,\mathscr{I}}$ if and only if $\mathscr{I} \not\leq_K \mathcal{G}_c$.*

Proof. Let $f : [\omega]^2 \to \omega$ be a witness of $\mathscr{I} \leq_K \mathcal{G}_c$. f itself is a coloring and given an infinite subset X of ω, $f''[X]^2$ is not in $\mathscr{I}$, because in the other case, $[X]^2 \subseteq f^{-1}[f''[X]^2] \in \mathcal{G}_c$, a contradiction. On the other hand, given a coloring $\varphi : [\omega]^2 \to \omega$ witnessing no $\omega \longrightarrow (\omega)^2_{\omega,\mathscr{I}}$, φ itself witnesses $\mathscr{I} \leq_K \mathcal{G}_c$. $\square$

Actually, the previous theorem can be extended for all $n \geq 2$ if we define the ideal $\mathcal{G}^n_c$ on $[\omega]^n$ generated by all subsets of $[\omega]^n$ which do not contain a set $[A]^n$ with $|A| = \aleph_0$. Then, $\omega \longrightarrow (\omega)^n_{\omega,\mathscr{I}}$ if an only if $\mathscr{I} \not\leq_K \mathcal{G}^n_c$. Farah's theorem would imply that for every meager (in particular Borel) ideal there is $n < \omega$ such that $\mathscr{I} \leq \mathcal{G}^n_c$. We will prove that this is not possible.

The ideal nwd satisfies the arrow property $\omega \longrightarrow (\omega)^2_{\omega,\mathscr{I}}$. We have a proof of this fact using cardinal invariants, forcing, and absoluteness. In theorem 1.6.26 we have proved that $\mathfrak{par}_2 \leq \mathsf{cov}^*(\mathcal{G}_c)$; Balcar, Hernández-Hernández and Hrušák [1] have proved that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and Blass

[5] has proved that $\mathfrak{h} \le \mathfrak{par}_2$. Mathias model satisfies $\mathsf{cov}(\mathcal{M}) = \aleph_1$ and $\mathfrak{h} = \aleph_2$, hence in this model $\mathsf{nwd} \not\le_K \mathcal{G}_c$. This fact is expressed by the Π^1_2-sentence

$$(\forall f : [\omega]^2 \to \mathbb{Q})(\exists A \in [\omega]^\omega)(f''[A]^2 \in \mathsf{nwd})$$

and by Shoenfield's absoluteness theorem this formula is true in any model of ZFC.[3] We can generalize this result for every analytic ideal $\mathscr{I}$ with $\mathsf{cov}^*(\mathscr{I}) \le \mathsf{cov}(\mathcal{M})$.

The following lemma plus a forcing argument followed by an absoluteness argument, shows the connection between CB_α ideals and the property $\omega \longrightarrow (\omega)^2_{\omega,\mathscr{I}}$.

Lemma 2.6.3 (ZFC + there exists a selective ultrafilter)**.** *Let f be a function from $[\omega]^2$ to $\mathbb{Q} \cap [0,1]$. Then there is an infinite subset A of ω such that $f''[A]^2 \in CB_3$.*

Proof. Let $\mathcal{U}$ be a selective ultrafilter on ω.

Claim. *There is a real number r such that for every $\varepsilon > 0$ there is $U_\varepsilon \in \mathcal{U}$ such that $f''[U_\varepsilon]^2 \subseteq B(r,\varepsilon)$.*

Proof. Define recursively a sequence I_n of subintervals of $I_0 = [0,1]$ by defining I^0_{n+1} and I^1_{n+1} both with length 2^{-n-1} and such that $I^0_{n+1} \cup I^1_{n+1} = I_n$. Let $I_{n+1} = I^0_{n+1}$ if there is a $U \in \mathcal{U}$ such that $f''[U]^2 \in I^0_{n+1}$, and $I_{n+1} = I^1_{n+1}$ otherwise. Note that since $\mathcal{U}$ is a Ramsey ultrafilter, if $I_{n+1} = I^1_{n+1}$ then there is $U \in \mathcal{U}$ such that $f''[U]^2 \subseteq I^1_{n+1}$. Let r be the unique element of $\bigcap_{n<\omega} \overline{I_n}$. Hence, for every $\varepsilon > 0$ take n such that $2^n > \varepsilon^{-1}$ and then take U_ε such that $f''[U_\varepsilon]^2 \subseteq I_n \subseteq B(r,\varepsilon)$. □

Claim. *For all $n \in \omega$ there is $V_n \in \mathcal{U}$ such that $\{f\{n,m\} : m \in V_n\}$ is a sequence in $[0,1]$ which converges to some $r_n \in [0,1]$.*

Proof. We only will use that $\mathcal{U}$ is a P-point. Analogously to first claim, using a sequence of nested two-part partitions, we can find r_n, and, for all $k \in \omega$, we can find $V_{(n,k)} \in \mathcal{U}$ such that $\left\{f\{n,m\} : m \in V_{(n,k)}\right\} \subseteq B(r_n, 2^{-k})$. By taking a pseudointersection V_n of $\{V_{(n,k)} : k \in \omega\}$ in $\mathcal{U}$, we are done. □

Let U be a pseudointersection of $\{U_{2^{-k}} : k \in \omega\}$. It is clear that $\{r_n : n \in U\}$ converges to r. We have two cases:

[3] See subsection 0.2.

I. $U' = \{n \in U : r_n = r\} \in \mathcal{U}$.

II. $U'' = \{n \in U : r_n \neq r\} \in \mathcal{U}$.

In the first case, we shall prove that for every $\varepsilon > 0$ the set $\{\{i,j\} \subseteq U' : f(\{i,j\}) \notin B(r,\varepsilon)\}$ is finite. If not, either there is $i \in U'$ such that for infinitely many $j \in U'$, $f(\{i,j\}) \notin B(r,\varepsilon)$ (but that is impossible, since $f(\{i,j\}) \xrightarrow[j\to\infty]{j\in U'} r_i = r$), or for infinitely many $i \in U'$ there is $j \in U'$ such that $i < j$ and $f(\{i,j\}) \notin B(r,\varepsilon)$) (but that is impossible too, because for every k, if i, j are sufficiently large $\{i,j\} \in U_{2^{-k}}$ and so, $f(\{i,j\}) \in B(r, 2^{-k})$). Therefore, in case I, $f''[U]^2 \in CB_2$. In the second case, let $\{n_k : k \in \omega\}$ be the increasing enumeration of U''. Since $\mathcal{U}$ is a Ramsey ultrafilter, we can suppose without loss of generality that $\{r_{n_k} : k \in \omega\}$ are pairwise distinct. Inductively, for every $k \in \omega$, define $\varepsilon_k > 0$ and $W_k \in \mathcal{U}$ such that $\overline{B(r_{n_k}, \varepsilon_k)} \cap \{r_m : k < m\} = \emptyset = \overline{B(r_{n_k}, \varepsilon_k)} \cap \overline{B(r_{n_i}, \varepsilon_i)}$, for all $i < k$, and $f''[W_k]^2 \subseteq B(r_{n_k}, \varepsilon_k)$. Let V be a pseudointersection of $\{W_k : k \in \omega\}$ in $\mathcal{U}$. Then, $f''[V]^2 \in CB_3$, because any $f(\{m,n\})$ (with $m, n \in V$) is a term of a convergent sequence isolated in some $B(r_k, \varepsilon_k)$. □

Lemma 2.6.4.

$$\omega \longrightarrow (\omega)^2_{\omega, CB_3} .$$

Proof. Let V be any model of ZFC^*. Let G be a $\langle [\omega]^\omega, \subseteq^* \rangle$-generic ultrafilter on V. By σ-closedness, $V[G]$ does not have new real numbers and G is a selective ultrafilter. Then in $V[G]$, $\omega \longrightarrow (\omega)^2_{\omega, CB_3}$ holds. By the absoluteness of Katětov order on Borel ideals $V \models \omega \longrightarrow (\omega)^2_{\omega, CB_3}$. □

Argument given in lemma 2.6.3 can be displayed in order to prove that $\omega \longrightarrow (\omega)^n_{\omega, CB_{n+1}}$ implies $\omega \longrightarrow (\omega)^{n+1}_{\omega, CB_{n+2}}$, and since $CB_\omega \geq_K CB_n$ for all n, we have that CB_ω (and every Katětov greater ideal, particularly nwd) is a counterexample to Farah's theorem.

2.7 Monotonicity and Ramsey property

In [17], Filipów, Mrożek, Recław and Szuca introduced the following definitions:

Definition 2.7.1. Let $\mathscr{I}$ be an ideal on ω. Then

1. $\mathscr{I}$ is Mon (monotone) if for any sequence $(x_n : n < \omega)$ of real (equivalently rational) numbers there is an $\mathscr{I}$-positive set X such that $(x_n) \restriction X$ is monotone.

2. $\mathscr{I}$ is $h-Mon$ (hereditarily monotone) if $\mathscr{I} \restriction A$ is Mon for all $A \in \mathscr{I}^+$.

3. $\mathscr{I}$ is $Fin - BW$ (finitely Bolzano-Weierstrass) if for any bounded sequence $(x_n : n < \omega)$ of real numbers there is an $\mathscr{I}$-positive set X such that $(x_n) \restriction X$ is convergent.

4. $\mathscr{I}$ is *locally selective* if for every partition $\{A_n : n < \omega\}$ of ω in sets from $\mathscr{I}$ there is an $\mathscr{I}$-positive set S such that $|A_n \cap S| \leq 1$.

It is easy to prove that $\omega \longrightarrow (\mathscr{I}^+)^2_2$ implies $\mathscr{I}$ is Mon and $\mathscr{I}$ is Mon implies $\mathscr{I}$ is $Fin-BW$. They remark that the summable ideal $\mathscr{I}_{\frac{1}{n}}$ is $Fin-BW$ but is not Mon and they ask if every Mon ideal satisfies $\omega \longrightarrow (\mathscr{I}^+)^n_2$, for all $n < \omega$. An immediate corollary of theorem 2.4.3 is

$$\mathscr{I} \text{ is } Fin - BW \text{ if and only if } \mathscr{I} \not\geq_K \mathsf{conv}$$

and it is immediate that

$$\mathscr{I} \text{ is locally selective if and only if } \mathscr{I} \not\geq_K \mathcal{ED}.$$

Proposition 22 in [17] claims that if $\mathscr{I}$ is Mon then it is locally selective. The following result is an easy consequence of their results.

Theorem 2.7.2. *If $\mathscr{I}$ is Mon then $\mathscr{I} \not\geq_K \mathsf{conv}$ and $\mathscr{I} \not\geq_K \mathcal{ED}$.* □

On the other hand, we have a criterion for Mon ideals.

Theorem 2.7.3. *Let $\mathscr{I}$ be an ideal on ω. If $\mathscr{I} \not\geq_K \mathsf{conv}$ and $\mathscr{I} \restriction X \not\geq_K \mathcal{ED}$ for all $\mathscr{I}$-positive set X then $\mathscr{I}$ is Mon.*

Proof. Let $f : \omega \to \mathbb{Q}$ be a sequence. Since $\mathscr{I} \not\geq_K \mathsf{conv}$ there is $A \in \mathscr{I}^+$ such that $f''A$ is the image of a convergent sequence. $\{A \cap f^{-1}\{m\} : m \in f''A\}$ is a partition of A. If there is $m \in f''A$ such that $f^{-1}\{m\} \in \mathscr{I}^+$ we are done. If not, since f is not a witness for $\mathscr{I} \restriction A \geq_K \mathcal{ED}$, there is an $\mathscr{I}$-positive subset B of A such that $f \restriction B$ is one to one. Let l be the limit of $f \restriction B$. Then either $B_0 = B \cap f^{-1}(-\infty, l) \in \mathscr{I}^+$ or $B_1 = B \cap f^{-1}(l, \infty) \in \mathscr{I}^+$. Let us suppose the first case (the other case is analogous). Let $\{b_k : k < \omega\}$ be the increasing enumeration of B_0. Let $k_0 = 0$ and k_1 be such that $f(b_k) > f(b_0)$ for all $k \geq k_1$; and for every $j \geq 1$ let k_{j+1} be such that $f(b_k) > \max\{f(b_i) : i < k_j\}$ for all $k \geq k_{j+1}$. For any $i < \omega$ define the family $C_i = \{b_k : k_i \leq k < k_{i+1}\}$. Then, $\{C_i : i < \omega\}$ is a partition of B in finite sets. Since $\mathscr{I} \restriction B \ngeq_K \mathcal{ED}_{fin}$ there is an $\mathscr{I}$-positive subset D of B such that $|C \cap C_i| \leq 1$ for all $i < \omega$. Now, either $D_0 = \bigcup_{i<\omega}(C \cap C_{2i}) \in \mathscr{I}^+$ or $D_1 = \bigcup_{i<\omega}(C \cap C_{2i+1}) \in \mathscr{I}^+$, and $f \restriction D_0$ and $f \restriction D_1$ are both increasing sequences. $\square$

Corollary 2.7.4. *Every ideal $\mathscr{I}$ on ω is $h - Mon$ if and only if $\mathscr{I} \restriction X \ngeq_K$ conv and $\mathscr{I} \restriction X \ngeq_k \mathcal{ED}$, for all $\mathscr{I}$-positive set X.*

Moreover, in [17] the authors asked about colorings with more than two colors.

Question 2.7.5 (Filipow et al, [17], problems 59 and 60)**.** *Let $\mathscr{I}$ be an ideal on ω. Is it true that if $\omega \longrightarrow (\mathscr{I}^+)^2_2$ then $\omega \longrightarrow (\mathscr{I}^+)^2_3$?*
Is it true that if $\omega \longrightarrow (\mathscr{I}^+)^2_2$ then $\omega \longrightarrow (\mathscr{I}^+)^2_n$ for all $n > 2$?

We have an example answering in negative both questions. Such example is $\widetilde{\mathcal{ED}}$.

Theorem 2.7.6. $\omega \longrightarrow (\widetilde{\mathcal{ED}}^+)^2_2$ *but* $\omega \nrightarrow (\widetilde{\mathcal{ED}}^+)^2_3$.

Proof. By corollary 2.5.12 we have that $\omega \longrightarrow (<\omega, \mathcal{ED}^+)^2_2$ and by 2.5.10, $\omega \longrightarrow (\widetilde{\mathcal{ED}}^+)^2_2$. Let us consider the following coloring $\varphi : [\omega \times \omega \times \omega]^2 \to 3$ given by

$$\varphi\{(n_1, n_2, n_3), (m_1, m_2, m_3)\} = \begin{cases} 0 & \text{if } n_1 = m_1 \text{ and } n_2 = m_2, \\ 1 & \text{if } n_1 = m_1 \text{ and } n_2 \neq m_2, \\ 2 & \text{if } n_1 \neq m_1. \end{cases}$$

Let A be a subset of $\omega \times \omega \times \omega$. If A is 0-homogeneous set then the first projection of A is in $\mathcal{ED}$ because second projection of A is contained in a column of $\omega \times \omega$; if A is 1-homogeneous then the first projection of A is in $\mathcal{ED}$ because second projection of A is contained in a selector of the columns of $\omega \times \omega$. Finally, if A is 2-homogeneous, then A is contained in a selector of the columns of the product $\omega \times (\omega \times \omega)$. In the three cases, $A \in \widetilde{\mathcal{ED}}$. □

2.8 Ultrafilters and Katětov order

In this section we study some classes of ultrafilters on ω, and their relation with Katětov order. Particularly, we study the critical ideals for well studied classes of ultrafilters: P-points, Q-points, selective ultrafilters and rapid ultrafilters. We conclude this section with the study of $\mathcal{S}$-ultrafilters, i.e. the ultrafilters which satisfy the Fubini property. We must emphasize that every one of the ideals which are critical with respect to classical properties on ultrafilters, is a Borel ideal.

First we give game theoretic proofs of a different kind of characterizations of selectivity and P-pointness due to Mathias [37] and Zapletal [50] respectively.

2.8.1 Game theoretic proof of Mathias' theorem

We present a new, game theoretic, proof of Mathias theorem 2.2.8 which say that a ultrafilter $\mathcal{U}$ is selective if and only if it intersects all the analytic tall ideals.

Game theoretic proof of Mathias theorem. Let us consider the following game $G(\mathcal{U}, \omega, \mathscr{I}^+)$ defined by the following clauses: In step k, Player I chooses an element U_k of $\mathcal{U}$; Player II chooses a natural number $n_k \in U_k$; and Player II wins a run of $G(\mathcal{U}, \omega, \mathscr{I}^+)$ if $\{n_k : k < \omega\} \in \mathscr{I}^+$. It is easy to see that

- Player I has a winning strategy if and only if there is a $\mathcal{U}$-branching tree $T \subseteq \omega^{<\omega}$ such that $ran(f) \in \mathscr{I}$ for all branch $f \in [T]$, and
- Player II has a winning strategy if and only if there is a $\mathcal{U}$-branching tree $T \subseteq \omega^{<\omega}$ such that $ran(f) \in \mathscr{I}^+$ for all branch $f \in [T]$.

Claim. *If player II has a winning strategy then $\mathscr{I}$ is not a tall ideal.*

Proof of the claim. Let $T = \{t_n : n < \omega\}$ be an enumeration of a $\mathcal{U}$-branching tree such that $ran(f) \in \mathscr{I}^+$ for all $f \in [T]$. Define $A_n =$

$\bigcap_{m\leq n} succ_T(t_m)$, for all $n<\omega$. In the first place, note that $\{A_n : n<\omega\}$ is a decreasing sequence of elements of $\mathcal{U}$. In second place, note that for all $I\in\mathscr{I}$ there is $n<\omega$ such that $I\cap A_n=\emptyset$, because in other case, there were a branch of T contained in an element of $\mathscr{I}$. We will define a branch $f\in[T]$ such that $\mathscr{I}\restriction ran(f)=\mathbf{Fin}(ran(f))$. Let k_0 be in A_0 and r_0 such that $t_0{}^\frown k_0=t_{r_0}$. For all $j<\omega$ let k_{j+1} be in A_{r_j} and r_{j+1} such that $t_{r_j}{}^\frown k_{j+1}=t_{r_{j+1}}$. It is clear that $f=\bigcup_{j<\omega} t_{r_j}$ is a branch of $[T]$ and $ran(f)\subseteq^* A_n$ for all $n<\omega$. Then $|I\cap ran(f)|<\infty$ for all $I\in\mathscr{I}$. □

Claim (Galvin, Shelah (see [2], theorem 4.5.3))**.** *$\mathcal{U}$ is selective if and only if for all $\mathcal{U}$-branching tree T there is a branch $f\in[T]$ such that $ran(f)\in\mathcal{U}$.*

Proof of the claim. Let suppose that $\mathcal{U}$ is a selective ultrafilter and T is a $\mathcal{U}$-branching tree. Since $\mathcal{U}$ is a P-point, there is a pseudointersection X of $\{succ_T(t) : t\in T\}$ in $\mathcal{U}$. Define $g:\omega\to\omega$ by $g(0)=0$, and $g(n+1)$ as the minimal $k>g(n)$ such that for every increasing $t\in T$ such that $ran(t)\subseteq g(n)$, $X\setminus k\subseteq succ_T(t)$. Since $\mathcal{U}$ is a Q-point, there is $Y\in\mathcal{U}$ such that $|Y\cap X\cap[g(n),g(n+1))|\leq 1$ for all $n<\omega$. Then either $Y_0=\bigcup_{n<\omega}(Y\cap X\cap[2n,2n+1))\in\mathcal{U}$ or $Y_1=\bigcup_{n<\omega}(Y\cap X\cap[2n+1,2n+2))\in\mathcal{U}$. If $Y_i\in\mathcal{U}$ then Y_i is a positive set contained in the image of a branch of $[T]$. On the other hand, let $\langle U_n : n<\omega\rangle$ be a decreasing sequence of elements of $\mathcal{U}$ and define T the tree such that $succ_T(t)=U_n$ for all $t\in T_n$. The image of a branch $f\in[T]$ is a pseudointersection of $\langle U_n : n<\omega\rangle$. □

By the first claim if $\mathscr{I}$ is tall then Player II does not have a winning strategy and by the second claim, if $\mathcal{U}$ is selective and $\mathscr{I}\cap\mathcal{U}=\emptyset$ then Player I does not have a winning strategy. Then $G(\mathcal{U},\omega,\mathscr{I}^+)$ is not determined and then, $\mathscr{I}$ is not analytic. □

2.8.2 Characterization of P-point ultrafilters

We will use the following result which is an immediate consequence of theorem 2.4.2.

Remark 2.8.1. The ideal $\mathbf{Fin}\times\mathbf{Fin}$ is a Borel ideal such that there are no P^+-ideals (and so, there are no F_σ-ideals) Katětov-above it.

In [50] (claim 2.4), Zapletal proved that if $\mathcal{U}$ is a P-point ultrafilter on ω then for every analytic ideal $\mathscr{I}$ disjoint from $\mathcal{U}$ there is an F_σ-ideal $\mathscr{J}$ disjoint from $\mathcal{U}$ and containing $\mathscr{I}$. We prove the inverse implication of Zapletal's result and integrate it in the following theorem. However, we will begin with a definition.

Definition 2.8.2 (Laflamme and Leary [36]). Let $\mathcal{X}$ be a set of infinite subsets of ω. A tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ is a *$\mathcal{X}$-tree of finite sets* if for each $s \in T$ there is an $X_s \in \mathcal{X}$ such that $s^\frown a \in T$ for each $a \in [X_s]^{<\omega}$.

Theorem 2.8.3 (Characterization of P-points; Zapletal 2007, Hrušák, Meza--Alcántara and Thümmel, 2008). *Let $\mathcal{U}$ be a ultrafilter on ω. Then $\mathcal{U}$ is a P-point if and only if for every Borel ideal $\mathscr{I}$ disjoint from $\mathcal{U}$ there is an F_σ-ideal $\mathscr{J}$ disjoint from $\mathcal{U}$ and containing $\mathscr{I}$.*

Proof. Let $\mathcal{U}$ be a P-point ultrafilter and let $\mathscr{I}$ be a Borel ideal disjoint from $\mathcal{U}$. Let us consider the following game G_2. In step n, Player I chooses a set $U_n \in \mathcal{U}$ and Player II chooses a finite subset a_n of U_n. Player II wins if $\bigcup_n a_n \in \mathscr{I}^+$. It is immediate that

Claim. *Player I has a winning strategy for G_2 if and only if there is a $\mathcal{U}$-tree of finite sets T such that $\bigcup_{n<\omega} f(n) \in \mathcal{I}$ for all $f \in [T]$.* □

However if $\mathcal{U}$ is a P-point then for all $\mathcal{U}$-tree of finite sets there is $f \in [T]$ such that $\bigcup_n f(n) \in \mathcal{U}$. Hence, if $\mathcal{U} \cap \mathscr{I} \neq \emptyset$ then Player I does not have a winning strategy for G_2. Since G_2 is a determined game there is a winning strategy for Player II. It is immediate that

Claim. *Player II has a winning strategy in G_2 if and only if there is a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ such that (a) for all $U \in \mathcal{U}$ and all $t \in T$ there is $a \in succ_T(t)$ such that $a \subseteq U$ and (b) $\bigcup_n f(n) \in \mathscr{I}^+$ for all $f \in [T]$.* □

Let T be as in the previous claim. For any $t \in T$ define

$$C_t = \{X \subseteq \omega : (\forall a \in succ_T(t))(a \setminus X \neq \emptyset)\}.$$

(1) It is immediate that C_t is hereditary.

(2) Note that C_t is also a closed set since for a given $a \in succ_T(t)$ and $Y \in \overline{C_t}$, there is $X \in C_t$ such that $X \cap (\max(a)+1) = Y \cap (\max(a)+1)$ and then $a \setminus Y \neq \emptyset$.

(3) Note that $\mathscr{I} \subseteq \bigcup_{t\in T} C_t$ since if $X \notin \bigcup_{t\in T} C_t$ then there is a branch f of T such that $\bigcup_n f(n) \subseteq X$, proving that $X \in \mathscr{I}^+$.

(4) By (a) it is clear that $\mathcal{U} \cap \bigcup_{t\in T} C_t = \emptyset$. The ideal $\mathscr{J}$ required is the ideal generated by $\bigcup_{t\in T} C_t$. It is clear that $\mathscr{J} \cap \mathcal{U} = \emptyset$. Then we will be

done if we prove that $\mathscr{J}$ is F_σ. However, note that if K_i $(i \leq n)$ are closed hereditary sets then $U(K_0, \ldots, K_n) = \{X : (\exists k_0 \in K_0) \ldots (\exists k_n \in K_n)(X \subseteq \bigcup_{i \leq n} k_i)\}$ is a closed hereditary set (since is the image of the compact set $K_0 \times \cdots \times K_n$ under the continuous function $(k_0, \ldots, k_n) \mapsto \bigcup_{i \leq n} k_i$), and then, $J = \bigcup_{n<\omega} \bigcup_{f \in [T]^n} U(C_{f(0)}, \ldots, C_{f(n)})$ is an F_σ ideal.

On the other hand, let us assume that $\mathcal{U}$ is not a P-point. Then let $\{X_n : n < \omega\}$ be a partition of ω in elements of $\mathcal{U}^*$, such that if $A \cap X_n$ is finite for all $n < \omega$ then $A \in \mathcal{U}^*$. Such partition can be chosen by defining $X_n = U_n \setminus U_{n+1}$ where $\langle U_n : n < \omega \rangle$ is a strictly $\subseteq^*$-decreasing sequence of elements of $\mathcal{U}$ without pseudointersections in $\mathcal{U}$, and with $U_0 = \omega$. By defining the ideal $\mathscr{I}$ generated by $\{X_n : n < \omega\} \cup \{A \subseteq \omega : (\forall n < \omega)(|A \cap I_n| < \aleph_0)\}$, we have that $\mathscr{I}$ is a copy of $\mathbf{Fin} \times \mathbf{Fin}$ contained in $\mathcal{U}^*$ and then, $\mathscr{I}$ can not be extended to an F_σ-ideal disjoint from $\mathcal{U}$, by lemma 2.8.1. $\square$

2.8.3 $\mathscr{I}$-ultrafilters

James Baumgartner has introduced the following definition in [3]. Let I be a family of subsets of a set X such that I contains all singletons and is closed under subsets. A ultrafilter $\mathcal{U}$ is an *I-ultrafilter* if for any function $F : \omega \to X$ there is $A \in \mathcal{U}$ such that $F''A \in I$. The I-ultrafilters have been studied by Baumgartner and some other authors like Jana Flašková [18], Saharon Shelah, Jörg Brendle and Christopher Barney. They have made emphasis on families I contained in $\mathbb{R}$, 2^ω and ω^ω. Particularly, they have studied I-ultrafilters when I consists of:

(1) all discrete subsets of 2^ω (discrete ultrafilters),

(2) all scattered subsets of 2^ω (scattered ultrafilters),

(3) all subsets of 2^ω with closure of measure zero (measure zero ultrafilters) and

(4) all nowhere dense subsets of 2^ω (nowhere dense ultrafilters).

Baumgartner noted that if I consists of all finite subsets of 2^ω then the I-ultrafilters are exactly the principal ultrafilters, and he also proved that every P-point ultrafilter is discrete and that the class defined in (n) is included in the class defined in $(n+1)$ for $1 \leq n \leq 3$. David Booth proved in [6] that if $I = \{Y \subseteq 2^\omega : Y$ is finite or has order type $\omega^*\}$ then the I-ultrafilters are exactly the P-points, where the order type of Y is calculated with respect to the lexicographical ordering. Shelah proved that it is consistent that there are no P-point ultrafilters [41] and that there are no nowhere dense ultrafilters [42].

We have noted that I-ultrafilters are characterized by Katětov order, when I is an ideal on ω.

Proposition 2.8.4. *Let $\mathscr{I}$ be an ideal on ω. Then a ultrafilter $\mathcal{U}$ on ω is an $\mathscr{I}$-ultrafilter if and only if $\mathscr{I} \not\leq_K \mathcal{U}^*$.*

Proof. Let us suppose that $\mathcal{U}$ is an $\mathscr{I}$-ultrafilter, and $f \in \omega^\omega$. Then there is a $U \in \mathcal{U}$ such that $f[U] \in \mathscr{I}$, but $U \subseteq f^{-1}[f[U]] \in \mathcal{U}$, proving f is not a witness for $\mathscr{I} \leq \mathcal{U}^*$. On the other hand, let us suppose that $\mathscr{I} \not\leq_K \mathcal{U}^*$. Given $f \in \omega^\omega$ there is $I \in \mathscr{I}$ such that $f^{-1}[I] \notin \mathcal{U}^*$, that is, $f^{-1}[I] \in \mathcal{U}$. $\square$

We have characterized some special classes of ultrafilters in Katětov order. The following theorem shows our first example.

Theorem 2.8.5. *Let $\mathcal{U}$ be a free ultrafilter on ω. The following conditions are equivalent:*

(1) $\mathcal{U}$ is a selective ultrafilter,

(2) $\mathcal{ED} \not\leq_K \mathcal{U}^*$, and

(3) $\mathcal{R} \not\leq_K \mathcal{U}^*$.

Proof. 1↔2 follows directly from the following lemma.

Lemma 2.8.6. *For any ideal $\mathscr{I}$ on ω, $\mathcal{ED} \leq_K \mathscr{I}$ if and only if there is a partition of ω in infinitely many infinite sets $\{A_n : n < \omega\}$ such that every A_n and every selector of such partition are in $\mathscr{I}$.*

Proof of lemma. If $f : \omega \to \omega \times \omega$ is a witness for $\mathcal{ED} \leq_K \mathscr{I}$ then $A_n = f^{-1}[\{n\} \times \omega]$ defines our partition requested.

On the other hand, doing $f : \omega \to \omega \times \omega$ such that $f \restriction A_n$ is a bijection between A_n and $\{n \times\}\omega$ we have that f is the Katětov function requested. $\square$

Now, 1↔3 is a consequence of 2.2.2, since a selective ultrafilter is the dual filter of a maximal Ramsey ideal. $\square$

Theorem 2.8.7. *Let $\mathcal{U}$ be a free ultrafilter on ω. The following conditions are equivalent:*

1. *$\mathcal{U}$ is a P-point ultrafilter,*

2. $\mathbf{Fin} \times \mathbf{Fin} \not\leq_K \mathcal{U}^*$ *and*

3. $\mathsf{conv} \not\leq \mathcal{U}^*$.

Proof. (2→1) In example 2.8.1 we discussed that there are no P$^+$-ideals Katětov-above $\mathbf{Fin} \times \mathbf{Fin}$ and note that $\mathcal{U}^*$ is a P$^+$-ideal if $\mathcal{U}$ is a P-point ultrafilter.

(3→ 2) Note that $f : \omega \to \mathbb{Q}$ defined by $f(n,m) = \frac{(n+1)(m+1)(n+3)-1}{(m+1)(n+2)(n+3)}$ is a witness for $\mathsf{conv} \leq_K \mathbf{Fin} \times \mathbf{Fin}$.

(1→3) Let us suppose that $\mathsf{conv} \leq_K \mathcal{U}^*$ and let $f : \omega \to \mathbb{Q} \cap [0,1]$ be a witness. For any interval I of rational numbers (open, closed or semi-closed) with rational end points let us denote by I^0, I^1 the intervals contained in I such that I^0 is delimited by $\inf I$ and the middle point of I not including it, and I_1 is delimited by the middle point of I including it and $\sup I$; and I^0 (resp. I^1) will include the left end point (resp. right) if and only if I includes it. By recursion we define two sequences, $\langle U_n : n < \omega \rangle$ of elements of $\mathcal{U}$ and $\langle I_n : n < \omega \rangle$ of intervals of rational numbers, as follows: $I_0 = [0,1]$, U_n is given by

$$U_n = \begin{cases} f^{-1}[I_n^0] & \text{if } f^{-1}[I_n^0] \in \mathcal{U} \\ f^{-1}[I_n^1] & \text{otherwise} \end{cases}$$

and $I_{n+1} = f''U_n$. Note that $\langle U_n : n < \omega \rangle$ is a decreasing sequence of elements of $\mathcal{U}$ and if A is a pseudointersection of $\langle U_n : n < \omega \rangle$ then $f \restriction A$ is a sequence in $\mathbb{Q} \cap [0,1]$ convergent to the unique element of $\bigcap_{n<\omega} cl_{\mathbb{R}}(I_n)$. Since A is a subset of the f-preimage of that sequence, we conclude that $A \in \mathcal{U}^*$. □

The Q-points are characterized by Katětov-Blass order.

Theorem 2.8.8. *Let $\mathcal{U}$ be a free ultrafilter on ω. Then $\mathcal{U}$ is a Q-point ultrafilter if and only if $\mathcal{ED}_{fin} \not\leq_{KB} \mathcal{U}^*$.*

Proof. Let us suppose that $\mathcal{U}$ is a Q-point ultrafilter and let $f : \omega \to \Delta$ be a finite-to-one function. Let us denote $\Delta_n = \{n\} \times (n+1)$. Then $\{f^{-1}[\Delta_n] : n < \omega\}$ is a partition on ω in finite sets. Let $A \in \mathcal{U}$ be a selector for that partition, and note that $|f''A \cap \Delta_n| \leq 1$ for all $n < \omega$ and so, $f''A \in \mathcal{ED}_{fin}$ but $A \subseteq f^{-1}[f''A] \in \mathcal{U}$. On the other hand, let $P = \{I_n : n < \omega\}$ be a partition of ω in finite sets. We can assume that $|I_n|$ increases to infinity since if not, then we can reorder that family in such way that $|I_n|$ is increasing and if such cardinalities were bounded, then we can find a partial selector in $\mathcal{U}$ without using Katětov order as follows. Let $N, M < \omega$ such that $|I_n| = N$ for all $n > M$. For any $i < N$ and $n > M$ define a_n^i the i-th element of I_n and define $C_i = \{a_n^i : n > M\}$. Note that $\bigcup_{i<N} C_i = \bigcup_{n>M} I_n \in \mathcal{U}$ and then there is $i < N$ such that $C_i \in \mathcal{U}$. Such C_i is a partial selector of P. Done that assumptions, define $\Delta' = \bigcup_{n<\omega}\{n\} \times I_n$ and define $f : \omega \to \Delta'$ given by $f(k) = (n, k)$ where n is the unique element of ω such that $k \in I_n$. Note that f is a finite-to-one function (actually is one-to-one) but since f is not a Katětov function, there are $n < \omega$ and $B \subseteq \Delta'$ such that $|B \cap \Delta'_k| = n$ for all $k > n$ and $f^{-1}[B] \in \mathcal{U}$. But $P \restriction f^{-1}[B]$ has bounded cardinalities and by some lines above, it has a partial selector in $\mathcal{U}$. □

Finally, we characterize the rapid ultrafilters in Katětov order.

Definition 2.8.9. A filter $\mathcal{F}$ on ω is a *rapid filter* if for any increasing function $f : \omega \to \omega$ there is $X \in \mathcal{F}$ such that $|X \cap f(n)| \leq n$ for all $n < \omega$.

Clearly every Q-point ultrafilter is a rapid ultrafilter, but the converse does not hold. Bartoszyński and Judah (lemma 4.6.2 in [2]) have proved that a filter $\mathcal{F}$ is a rapid filter if and only if for any sequence $\langle \varepsilon_n : n < \omega \rangle$ such that $\varepsilon_n \to 0$, there is $X \in \mathcal{F}$ such that $\sum_{n \in X} \varepsilon_n < \infty$.

The summable ideals $\mathscr{I}_g$ and Katětov-Blass order characterize the rapid ultrafilters.

Theorem 2.8.10. *Let $\mathcal{U}$ be a free ultrafilter on ω. Then, the following conditions are equivalent:*

1. *$\mathcal{U}$ is a rapid filter,*
2. *$\mathscr{I} \not\leq_{KB} \mathcal{U}^*$ for all tall summable ideal $\mathscr{I}$ and*
3. *$\mathscr{I} \not\leq_{KB} \mathcal{U}^*$ for all tall analytic P-ideal $\mathscr{I}$.*

Proof. (1→2). Let $g : \omega \to (0, \infty)$ be a function such that $\lim_{n\to\infty} g(n) = 0$ and $\sum_{n=0}^{\infty} g(n) = \infty$, and denote $\mathscr{I} = \mathscr{I}_g$. Let us suppose that there is a finite-to-one function $h : \omega \to \omega$ witnessing $\mathscr{I} \leq_{KB} \mathcal{U}^*$. Take an infinite subset $X = \{x_n : n < \omega\}$ of ω such that for any $k < \omega$, if $k > x_n$ then $g(k) \leq 2^{-n}$. Without loss of generality we can assume that $h[\omega] \cap [x_n, x_{n+1}) \neq \emptyset$ for all $n < \omega$. Now we define an increasing function $f : \omega \to \omega$ as follows: $f(0) = \max h^{-1}[0, x_0]$ and $f(n+1) = \max(\{f(n) + 1\} \cup h^{-1}[0, x_{n+1}])$. Note that if $k > f(n)$ then $h(k) > x_n$ and then $g(h(k)) < 2^{-n}$. Now, if $A = \{a_k :$

$k < \omega\} \subseteq \omega$ satisfies $|A \cap f(n)| \leq n$ for all n then $h(a_{n+1}) > x_n$ for all n. Then $\sum_{n \in A} g(h(n)) \leq g(h(a_0)) + \sum_{n=0}^{\infty} 2^n$. Hence $h[A]$ is a g-summable set and since h is a Katětov function, $A \in \mathcal{U}^*$.

(2→1). Let us suppose that $\mathscr{I} \not\leq_{KB} \mathcal{U}^*$ for all summable ideal $\mathscr{I}$. Let $f : \omega \to \omega$ be an increasing function such that $f(0) = 0$. Let us define $g : \omega \to \omega$ such that for $k \in [f(n), f(n+1))$, $g(k) = \frac{1}{n+1}$. Since identity is not a Katětov function, there is a g-summable set A in $\mathcal{U}$. By removing an initial segment of A we can find $B \subseteq A$ in $\mathcal{U}$ such that $\sum_{k \in B} g(k) < 1$. This condition implies that $|B \cap f(n)| \leq n$, proving that $\mathcal{U}$ is a rapid filter.
2→3. If $\mathscr{I}$ is a tall P-ideal then there exists a lscsm φ such that $\mathscr{I} = Exh(\varphi)$ and $\lim_{n\to\infty} \varphi(\{n\}) = 0$. By defining $g(n) = \varphi(\{n\})$ we have that $\mathscr{I}_g \subseteq \mathscr{I}$ and then, if $\mathscr{I}_g \not\leq_{KB} \mathcal{U}^*$ then $\mathscr{I} \not\leq_{KB} \mathcal{U}^*$.

(3→2). Every summable ideal is an F_σ P-ideal. □

2.8.4 $\mathcal{S}$-ultrafilters

This subsection is very related with the study of pathology of submeasures that is given in sections 3.5, 3.6 and 3.7. We will refer the reader to those sections for many definitions and results used in this subsection.

Let $\mathcal{U}$ be an ultrafilter on ω, and A_n a Borel subset of the Cantor space 2^ω, for all $n < \omega$. The $\mathcal{U}$-limit of that sequences of sets is the set defined by

$$\mathcal{U} - \lim A_n = \{x \in 2^\omega : \{n \in \omega : x \in A_n\} \in \mathcal{U}\}.$$

If $\langle x_n : n < \omega \rangle$ is a sequence of real numbers then $l \in \mathbb{R}$ is the $\mathcal{U}$-limit of that sequence provided that $\{n < \omega : |x_n - l| < \varepsilon\} \in \mathcal{U}$ for all $\varepsilon > 0$.

An $\mathcal{S}$-ultrafilter is, by proposition 2.8.4, a free ultrafilter that $\mathcal{U}$ is not Katětov above the Solecki's ideal $\mathcal{S}$.

Definition 2.8.11. We will say that $\mathscr{I}$ satisfies the *Fubini property* if for any Borel subset A of $\omega \times 2^\omega$ and any $\varepsilon > 0$, $\{n < \omega : \lambda(A_n) > \varepsilon\} \in \mathscr{I}^+$ implies $\lambda^*(\{x \in 2^\omega : A^x \in \mathscr{I}^+\}) \geq \varepsilon$ (here λ^* means the Lebesgue outer measure on 2^ω).

Theorem 2.8.12. *Let $\mathcal{U}$ be a free ultrafilter. Then the following conditions are equivalent:*

1. *$\mathcal{U}$ is an $\mathcal{S}$-ultrafilter,*

2. *$\mathcal{U}^*$ satisfies the Fubini property and*

3. for any sequence $\langle A_n : n < \omega \rangle$ *of Borel subsets of* 2^ω*,*

$$\textit{if}\; \mathcal{U} - \lim \lambda(A_n) > 0 \;\textit{then}\; \mathcal{U} - \lim A_n \neq \emptyset.$$

Proof. Theorem 3.7.1 claims that the ideals $\mathscr{I}$ which do not have $\mathscr{I}$-positive sets X such that $\mathscr{I} \restriction X \geq_K \mathcal{S}$, are exactly those ideals satisfying the Fubini property, and since every maximal ideal is K-uniform (lemma 2.1.9) we have that dual ideals of $\mathcal{S}$-ultrafilters are exactly the maximal ideals with the Fubini property. Now, Fubini property among maximal ideals (or ultrafilters) means: for any sequence $\langle A_n : n < \omega \rangle$ of Borel subsets of 2^ω and any $\varepsilon > 0$, if $\{n < \omega : \lambda(A_n) > \varepsilon\} \in \mathcal{U}$ then $\lambda^*(\{x \in 2^\omega : \{n < \omega : x \in A_n\} \in \mathcal{U}\}) \geq \varepsilon$. Hence, if $\mathcal{S} \not\leq_K \mathcal{U}^*$ and $\mathcal{U} - \lim \lambda(A_n) > 0$ then $\lambda^*(\mathcal{U} - \lim A_n) > 0$ and then $\mathcal{U} - \lim A_n \neq \emptyset$. On the other hand, let suppose that $\mathcal{U} - \lim \lambda(A_n) > \varepsilon$ and $\lambda^*(\mathcal{U} - \lim A_n) = \delta < \varepsilon$, for some sequence $\langle A_n : n < \omega \rangle$ and some $\varepsilon > 0$. For any $k < \omega$ let us choose a Borel set $A'_k \subseteq A_k \setminus \mathcal{U} - \lim A_n$, with $\lambda(A'_k) = \varepsilon - \delta$. Then, $\mathcal{U} - \lim \lambda(A'_n) \geq \varepsilon - \delta$ but $\mathcal{U} - \lim A'_n = 0$, since for any $x \in 2^\omega$, $\{n : x \in A_n\} \in \mathcal{U}^*$. □

Chapter 3

Katětov order on definable ideals

As we saw in the previous chapter, in most of the cases, actually in all cases considered here, critical ideals are definable ideals. We now restrict our study of Katětov order to definable (basically Borel and analytic) ideals.

Even among definable ideals Katětov order is interesting. Summable ideals are a very complicated segment of Borel ideals in Katětov order. In theorem 3.1.1 we prove that there is an order embedding from the algebra $\mathcal{P}(\omega)/fin$ into the family of summable ideals.

In section 3.3 we investigate for which classes of Borel ideals, the ideal $\mathcal{R}$ generated by the homogeneous sets of the random graph is *locally minimal* among Borel (analytic) tall ideals, that is, for any Borel (analytic) ideal $\mathscr{I}$ there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_K \mathcal{R}$. We prove that there is a large class of definable ideals having such property: the class of ideals $\mathscr{I}$ such that $\mathcal{P}(\omega)/\mathscr{I}$ is a proper forcing or does not add new real numbers (see theorems 3.3.1 and 3.3.4). All definable tall ideals that we know are not strictly Katětov below $\mathcal{R}$. In theorem 2.2.2 we showed that $\mathcal{R}$ is critical with respect to the Ramsey property $\omega \longrightarrow (\mathscr{I}^+)^2_2$. That is, $\mathscr{I}$ satisfies this property if and only if $\mathscr{I} \not\geq_K \mathcal{R}$. Then, this makes this question doubly interesting.

In subsection 2.1 we studied the Q and Q$^+$-ideals and we showed that $\mathcal{ED}_{fin}$ is a critical ideal with respect to Q property in the Katětov-Blass order. Adding definability, this property is equivalent to be non ω-spliting, to be non ω-hitting, and to have countable non^* number.

We show that the P^+-ideals are very closely related with F_σ-ideals. In subsection 3.2 we study the ideals which are extendable to F_σ, P^+ and maximal P-ideals. This property is very relevant since by a theorem of Lafflame (see [35]), every ideal contained in an F_σ-ideal is destructible by an ω^ω-bounding forcing.

Hrušák's Category Dichotomy splits ideals in two classes, the first one consisting in all the Cohen destructible ideals, and the second one split in two classes. Ideals which are not P^+ and ideals which are not Q^+.

Sławomir Solecki probably gave the first theorem showing a critical ideal for some property about ideals in terms of the Katětov order. He proved that ideal the $\mathcal{S}$ is critical with respect to fulfil the Fatou's lemma. In section 3.5 we answer a question of Solecki about criticality of $\mathcal{G}_{fc}$ with respect to Fatou's lemma.

Pathology of submeasures is closely related with the Fubini property. Hrušák's Measure Dichotomy splits all the analytic P-ideals in two classes. The first one, consisting of those ideals which have a non too pathological submeasure, is determined by $\mathcal{Z}$, the ideal of asymptotical density zero; and on the other hand, the class of ideals corresponding with locally strongly pathological submeasures, determined by the Solecki's ideal $\mathcal{S}$.

Finally, by using the Measure Dichotomy we prove the equivalence between the classes of ideals such that (1) fulfil de Fatou's lemma, (2) are not pathological and (3) satisfy the Fubini property.

3.1 Summable ideals in Katětov order

The following theorem has as a consequence that Katětov order on Borel (even F_σ) ideals has chains of length $\mathfrak{b}$ and antichains of cardinality $\mathfrak{c}$. We will denote by Σ the family of summable ideals on ω. A similar but weaker result was obtained by Ilijas Farah (theorem 1.12.1(c) in [14]) about Rudin-Blass order.

Theorem 3.1.1. *There is an order embedding φ from $\mathcal{P}(\omega)/\mathbf{Fin}$ into Σ.*

Proof. It is easy to construct by recursion a partition of ω in finite intervals $\langle I_n : n < \omega\rangle$ such that $\min(I_{n+1}) = \max(I_n) + 1$, and a sequence $\langle r_n : n < \omega\rangle$ of real numbers in $(0, 1]$ such that:

(1) $|I_n| r_n \geq |\bigcup_{j<n} I_j|$ and

(2) $|I_n| r_{n+1} \leq 2^{-n-1}$.

For each infinite subset A of ω, define a function $f_A : \omega \to (0,1]$ such that for every $k < \omega$

$$f_A(k) = \begin{cases} r_n & \text{if } k \in I_n \text{ and } n \notin A \\ r_{n+1} & \text{if } k \in I_n \text{ and } n \in A \end{cases}$$

Theorem follows immediately from claims below.

Claim. *For every infinite and coinfinite subset A of ω, $\mathscr{I}_{f_A}$ is a non-trivial tall ideal.*

Proof of claim. Note that

$$\sum_{n<\omega} f_A(n) = \sum_{j<\omega} \sum_{n \in I_j} f_A(n) \geq \sum_{j \in \omega \setminus A} r_j |I_j| \geq \sum_{j \in \omega \setminus A} |\bigcup_{i<j} I_i| = \infty.$$

□

Claim. *If $A, B \in [\omega]^\omega$ and $A \subseteq^* B$ then $\mathscr{I}_{f_A} \leq_K \mathscr{I}_{f_B}$.*

Proof of claim. Note that if $A \subseteq^* B$ then $f_B \leq^* f_A$ and then $\mathscr{I}_{f_A} \subseteq \mathscr{I}_{f_B}$, and so $\mathscr{I}_{f_A} \leq_K \mathscr{I}_{f_B}$. □

Claim. *If $A, B \in [\omega]^\omega$ and $|A \setminus B| = \aleph_0$ then $\mathscr{I}_{f_A} \not\leq_K \mathscr{I}_{f_B}$.*

Proof of claim. Let φ be in ω^ω and let us prove that φ is not a witness for $\mathscr{I}_{f_A} \leq_K \mathscr{I}_{f_B}$. First note that for any $n < \omega$ there is $F_n \subseteq I_n$ such that $|F_n| \geq \frac{1}{2}|I_n|$ and, either $\varphi(x) < \min(I_n)$ for all $x \in F_n$ or $\varphi(x) \geq \min(I_n)$ for all $x \in F_n$. Then we have two cases:

Case 1. *The family $C = \{n \in A \setminus B : x \in F_n$ implies $\varphi(x) < \min(I_n)\}$ is infinite.* Note that by condition (1) and the pigeonhole principle, for any $n \in C$ there is $k_n \in \bigcup_{j<n} I_j$ such that $|\varphi^{-1}[\{k_n\}] \cap F_n| \geq \frac{1}{2r_n}$. Note that for any $n \in A \setminus B$, $\sum_{i \in \varphi^{-1}(k_n)} f_B(i) \geq r_n \cdot \frac{1}{2r_n} = \frac{1}{2}$. If $\{k_n : n \in C\}$ is finite, then it belongs $\mathscr{I}_{f_A}$. In other case we can take an infinite $C' \subseteq C$ such that for every $j < \omega$, $|\{k_n : n \in C'\} \cap I_j| \leq 1$, Then, we have that $\bigcup_{n \in C'} \varphi^{-1}[\{k_n\}] \notin \mathscr{I}_{f_B}$ but $\{k_n : n \in C'\} \in \mathscr{I}_{f_A}$. Hence in this case, φ is not a witness for $\mathscr{I}_{f_A} \leq_K \mathscr{I}_{f_B}$.

Case 2. *The family* $D = \{n \in A \setminus B : x \in F_n$ *implies* $\varphi(x) \geq \min(I_n)\}$ *is infinite.* Note that $Y = \bigcup_{n \in D} F_n$ is an $\mathscr{I}_{f_B}$ positive set and $J = \varphi'' Y \in \mathscr{I}_{f_A}$ since

$$\sum_{n \in J} f_A(n) \leq \sum_{y \in Y} f_A(\varphi(y)) = \sum_{n \in D} \sum_{y \in F_n} f_A(\varphi(y)) \leq$$

$$\sum_{n \in D} r_{n+1} |F_n| \leq \sum_{n \in D} 2^{-n-1}.$$

Hence in case 2, φ is not a witness for $\mathscr{I}_{f_A} \leq_K \mathscr{I}_{f_B}$. □

□

3.2 Q$^+$, P$^+$ and F_σ-ideals

In this section we continue the study of P$^+$, Q$^+$ and related properties from chapter 2, in the definable context. Among definable Q-ideals, theorem 2.3.2 is extended as the following theorem shows.

Theorem 3.2.1. *For any Borel ideal* $\mathscr{I}$ *the following are equivalent:*

1. $\mathscr{I}$ *is a Q-ideal,*
2. $\mathcal{ED}_{fin} \not\leq_{KB} \mathscr{I}$*,*
3. $\mathscr{I}$ *is not* ω*-hitting ideal,*
4. $\mathscr{I}$ *is not* ω*-splitting ideal and*
5. $\mathsf{non}^*(\mathscr{I}) = \omega$.

Proof. $(1 \leftrightarrow 2)$ was proved in section 2.3. Equivalence between 3,4 and 5 is very easy (and it does not need the hypothesis of Borelness).

$(3 \to 1)$ Assume $\mathscr{I}$ is not a Q-ideal, and let $\{F_n : n < \omega\}$ be a partition of ω in finite sets such that every selector belongs to $\mathscr{I}$. Given a partition $\{X_n : n < \omega\}$ of ω in infinite sets, there is a partial selector X of $\{F_n : n < \omega\}$ which intersects every X_k in an ifinite set. In order to find this partial selector, let $h, k : \omega \to \omega$ be two functions such that $h^{-1}\{n\}$ is infinite for all $n < \omega$, and k is an increasing function such that $F_{k(n)} \cap X_{h(n)} \neq \emptyset$. Now, for every n, take $x_n \in F_{k(n)} \cap X_{h(n)}$ and then, $X = \{x_n : n < \omega\}$ is a partial selector of $\{F_n : n < \omega\}$ and $X_n \cap X$ in infinite for all n.

(2 → 4) Let us play the following game G_1: In step k, Player I chooses a finite subset F_k of ω and Player II chooses $n_k \in \omega \setminus F_k$. Player I wins if $\{n_k : k < \omega\} \in \mathscr{I}$. Since $\mathscr{I}$ is analytic this game is determined and

Claim. *Player I has a winning strategy if and only if $\mathcal{ED}_{fin} \leq_{KB} \mathscr{I}$.*

Proof of the Claim. If $\mathcal{ED}_{fin} \leq_{KB} \mathscr{I}$ then there is a partition $\{H_n : n < \omega\}$ in finite sets such that every selector belongs $\mathscr{I}$. We will describe a winning strategy for I. In step 0, I plays $F_0 = \emptyset$ and in step $k+1$, I must play $F_{k+1} = \bigcup\{H_i : (\exists j \leq k)(n_j \in H_i)\}$. Then II must play a partial selector of $\{H_n : n < \omega\}$ and then I wins.

On the other hand, a winning strategy for I could be seen as a cofinite-branching tree $T \subseteq \omega^{<\omega}$ such that $ran(f) \in \mathscr{I}$ for all branch $f \in [T]$. Without loss of generality, we can assume that T consists of strictly increasing sequences, because otherwise we can refine it. Let $g \in \omega^\omega$ such that $g(0) = 0$, $g(1) > 0$ and

$$g(n+1) = \min\{k : (\forall t \in T)(ran(t) \subseteq g(n) \text{ implies } succ_T(t) \cup k = \omega)\}.$$

Define $C_n = [g(n), g(n+1))$ for all $n < \omega$ and consider $C = \bigcup_{k<\omega} C_{2k}$ and $D = \bigcup_{k<\omega} C_{2k+1}$ partitioned by the sets C_k. Every selector in C follows a branch in T and the same for D. Hence, every selector of $\{C_n : n < \omega\}$ belongs $\mathscr{I}$. $\square$

Now it will be enough to prove that if $\mathscr{I}$ is an ω-splitting ideal then Player II does not have a winning strategy. In order to see this, note that a winning strategy for II produces an infinitely branching tree $T \subseteq \omega^{<\omega}$ such that $ran(f) \in \mathscr{I}^+$ for all branch f of T. However, being $\mathscr{I}$ an ω-splitting ideal that is not possible, since for $\{succ_T(t) : t \in T\}$ we have an element I of $\mathscr{I}$ such that $|I \cap succ_T(t)| = \aleph_0$ for all $t \in T$, and then we can construct a branch $f \in [T]$ in $\mathscr{I}$ by taking $x_0 \in succ_T(\emptyset)$ and $x_{n+1} \in I \cap succ_T(f \restriction n)$, for all $n < \omega$. $\square$

In 1.6.6 was proved that $\mathsf{non}^*(\mathcal{ED}_{fin}) \geq \mathsf{cov}(\mathcal{M})$ (in fact, it was proved that $\mathsf{cov}(\mathcal{M}) = \min\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{fin})\}$) and from previous theorem, we have immediately the following corollary.

Corollary 3.2.2. *If $\mathscr{I}$ is an analytic ideal on ω then either $\mathsf{non}^*(\mathscr{I}) = \omega$ or $\mathsf{non}^*(\mathscr{I}) \geq \mathsf{non}^*(\mathcal{ED}_{fin})$.*

In [45], Otmar Spinas constructed an F_σ splitting family that does not contain a closed family. On the other hand, he also proved that every analytic ω-splitting family contains a closed ω-splitting family. Previous theorem make us able to prove in a very easy way this result for ideals.

Corollary 3.2.3. *If $\mathscr{I}$ is an ω-splitting ideal then $\mathscr{I}$ contains a perfect ω-splitting subset.*

The P^+ property has a very important link with F_σ ideals. In order to prove this we will begin with some preliminary results The first one is probably due to Phil Olin of Ilijas Farah.

Lemma 3.2.4 (Folklore)**.** *Every F_σ-ideal is a P^+-ideal.*

Proof. Let φ be a lower semicontinuous submeasure on ω such that $\mathscr{I} = Fin(\varphi)$, and $\langle X_n : n < \omega\rangle$ a decreasing sequence of $\mathscr{I}$-positive sets. Then we can find finite sets $F_n \subseteq X_n$ such that $\varphi(F_n) \geq n$ for all $n < \omega$. Then $X = \bigcup_{n<\omega} F_n$ is a pseudointersection of $\langle X_n : n < \omega\rangle$ and $\varphi(X) = \infty$, hence $X \in \mathscr{I}^+$. □

Corollary 3.2.5. *If $\mathscr{I}$ is a tall F_σ-ideal then there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X$ is ω-splitting.*

Proof. Put 3.2.4, 2.3.2 and 2.2.9 together. □

Remember that $\mathcal{R}$, the ideal generated by the homogeneous sets of the random graph, is an F_σ-ideal. We conjecture that is locally minimal in Katětov order among Borel (analytic) ideals, but it could be interesting to weaken this conjecture and ask if $\mathcal{R}$ is locally minimal among F_σ-ideals. About extendability to maximal P-ideals we have the following trivial observation.

Lemma 3.2.6 (CH)**.** *If $\mathscr{I}$ is a P^+ ideal then there is a maximal P-ideal $\mathscr{J}$ containing $\mathscr{I}$.*

Proof. We will find a P-point $\mathcal{U}$ such that $\mathscr{I}^* \subseteq \mathcal{U}$. Let $\langle X_\alpha : \alpha < \omega_1\rangle$ be an enumeration of $\mathcal{P}(\omega)$. For $\alpha < \omega$ we define $U_\alpha \in \mathscr{I}^+$ such that $U_\alpha \subseteq X_\alpha$ or $U_\alpha \cap X_\alpha =^* \emptyset$ and if $\beta < \alpha$ then $U_\beta \supseteq^* U_\alpha$. Let suppose defined U_β for all $\beta < \alpha$, and let $V \in \mathscr{I}^+$ such that $V \subseteq^* U_\beta$ for all $\beta < \alpha$ and define $U_\alpha = V \cap X_\alpha$ if this set belongs to $\mathscr{I}^+$ and $U_\alpha = V \setminus X_\alpha$ otherwise. The family $\langle U_\alpha : \alpha < \omega_1\rangle$ is a base for a ultrafilter contained in $\mathscr{I}^+$ and since this sequence is $\subseteq^*$-decreasing, the filter generated is a P-point. □

We characterize those ideal which are extendable to F_σ, P^+ and maximal P-ideals as the following theorem shows.

Theorem 3.2.7. *Let $\mathscr{I}$ be a Borel ideal on ω. Then the following conditions are equivalent*

1. *there is an F_σ-ideal $\mathscr{J}$ containing $\mathscr{I}$,*
2. *there is a P^+-ideal $\mathscr{K}$ containing $\mathscr{I}$*

and assuming CH, 1 and 2 are equivalent to

3. *there is a maximal P-ideal $\mathscr{L}$ containing $\mathscr{I}$.*

Proof. By lemma 3.2.4, every F_σ ideal is P^+, hence $1 \to 2$. Assuming CH, lemma 3.2.6 proves $2 \to 3$, and since every maximal P-ideal is a P^+-ideal we have proved $3 \to 2$. Let us prove $2 \to 1$. Let G be an $\mathscr{I}^+$-generic ultrafilter. Since $\mathscr{I}$ is P^+, $\mathscr{I}^+$ is a σ-closed forcing and then it does not add new real numbers, sequences of real numbers and Borel sets. Note that, in $V[G]$, G is a P-point ultrafilter, because given a decreasing sequence $\mathcal{X} = \langle X_n : n < \omega\rangle$ of sets in G, we can define a set $D_{\mathcal{X}} = \{Y \in \mathscr{I}^+ : (\forall n Y \subseteq^* X_n) \vee (\exists n Y \cap X_n = \emptyset)\}$ which is dense, but since there is no $Z \in G$ such that $Z \cap X_n = \emptyset$, G must contain a set Y which is almost contained in every X_n. It is clear that $\mathscr{I} \cap G = \emptyset$. Then, by theorem 2.8.3, there is an F_σ ideal $\mathcal{K}$ containing $\mathscr{I}$ and disjoint from G. Since $\mathscr{I}^+$ does not add new real numbers, such ideal $\mathcal{K}$ was already defined in V by a lower semicontinuous submeasure. □

Definition 3.2.8 (Laflamme and Leary [36])**.** An ideal $\mathscr{I}$ on ω is a P^+*(tree)-ideal* if every $\mathscr{I}^+$-tree of finite sets has a branch whose union is in $\mathscr{I}^+$.

Laflamme and Leary proved that an ideal $\mathscr{I}$ is not P^+(tree) if and only if Player I has a winning strategy for the following game G_3: In step n, Player I chooses an $\mathscr{I}$-positive set X_n and Player II chooses a finite set $F_n \subseteq X_n$. Player II wins if $\bigcup_{n<\omega} F_n \in \mathscr{I}^+$. In fact that game characterizes the F_σ ideals, as the following theorem proves.

Lemma 3.2.9. *Let $\mathscr{I}$ be a Borel ideal. Then, Player II has a winning strategy in G_3 if and only if $\mathscr{I}$ is an F_σ-ideal.*

Proof. If $\mathscr{I}$ is an F_σ ideal then there is a lower semicontinuous submeasure φ such that $\mathscr{I} = Fin(\varphi)$. In step n, II plays a finite subset F_n of X_n with $\varphi(F_n) \geq n$. That is possible since $\varphi(X_n) = \infty$.

On the other hand, we will prove that Player I has a winning strategy in G_3 if $\mathscr{I}$ is not an F_σ ideal. Recall the following result.

Lemma 3.2.10 (Kechris-Louveau-Woodin [30], theorem 21.22)**.** *Let X be a Polish space, let $A \subseteq X$ be analytic, and let $B \subseteq X$ be arbitrary with $A \cap B = \emptyset$. Then either there is an F_σ set $K \subseteq X$ separating A from B or there is a perfect set $C \subseteq A \cup B$ such that $C \cap B$ is countable dense in C.* □

By Kechris-Louveau-Woodin theorem there is a perfect set $C \subseteq \mathcal{P}(\omega)$ such that $C \cap \mathscr{I}^+$ is countable dense in C. In the Banach-Mazur Game [1] G_0 for $C \cap \mathscr{I}^+$ Player I has a winning strategy, since $\mathscr{I}$ is comeager in C and the game is determined. Now, we will prove that if Player I has a winning strategy for $G_0(C \cap \mathscr{I}^+)$ then Player I has a winning strategy for G_3. Let σ be a winning strategy for Player I in $G_0(C \cap \mathscr{I}^+)$. In step 0, let $\tau(\emptyset) = X_0 \in V_0 = \sigma(\emptyset)$ be an $\mathscr{I}$-positive set. Such set exists since V_0 is an open non-empty subset of C and $\mathscr{I}^+ \cap C$ is dense in C. Let us assume that we have defined our strategy τ until step n joint with a sequence of σ-legal positions. We will define it for step $n+1$. Given an answer $F \subseteq X_n$ of Player II for a τ-legal sequence $\langle X_0, F_0, \dots, X_{n-1}, F_{n-1}, X_n \rangle$, σ entails F as the clopen set U of all subsets A of ω such that $A \cap (\max(F) + 1) = F$, and if $\langle V_0, U_0, \dots, V_{n-1}, U_{n-1}, V_n \rangle$ is the σ-legal position associated to $\langle X_0, F_0, \dots, X_{n-1}, F_{n-1}, X_n \rangle$, then $U = U_n$, $V_{n+1} = \sigma(\langle V_0, U_0, \dots, V_{n-1}, U_{n-1}, V_n, U_n \rangle)$ and let

$$\tau(\langle X_0, F_0, \dots, X_{n-1}, F_{n-1}, X_n, F \rangle) = X_{n+1} \in V_n$$

be an $\mathscr{I}$-positive set. By density of $\mathscr{I}^+$ in C we have the existence of such set again. Finally, note that τ is a winning strategy for I, since for every τ-legal run of G_3 $\langle X_0, F_0, X_1, F_1, \dots \rangle$, $\bigcup_{n<\omega} F_n \subseteq^* \bigcap_{n<\omega} U_n \in \mathscr{I}$. □

From this results and Borel determinacy follows immediately the following theorem.

Theorem 3.2.11. *Let $\mathscr{I}$ be a Borel ideal. Then $\mathscr{I}$ is a P^+(tree)-ideal if and only if $\mathscr{I}$ is an F_σ-ideal.* □

At this moment we do not have an example of a tall P^+-ideal non-F_σ-ideal $\mathscr{I}$. The ideal $\mathscr{I}_0$ isolated in our study of the *comparison game* on Borel ideals (see [22] or chapter 4) is presented below.

[1] Banach-Mazur Game $G_0(C \cap \mathscr{I}^+)$ is defined as follows: In step 0, Player I chooses an open set V_0 and Player II chooses an open subset U_0 of V_0. In step $n+1$, Player I chooses an open set $V_{n+1} \subseteq U_n$ and Player II chooses an open set $U_{n+1} \subseteq V_{n+1}$. Player II wins if $\bigcap_{n<\omega} \overline{U_n} = \bigcap_{n<\omega} \overline{V_n} \subseteq \mathscr{I}^+$.

Example 3.2.12. We define the ideal $\mathscr{I}_0$ as the minimal ideal such that there is an $\mathscr{I}_0^+$-tree of finite sets which does not have an $\mathscr{I}_0$-positive branch. Such ideal has an incarnation in $2^{<\omega}$ as the ideal generated by all the non-eventually zero branches. It is clear that such ideal is not a tall ideal and in theorem 3.1 of the same paper was proved that is not an F_σ ideal. We now prove that $\mathscr{I}_0$ is a P$^+$ ideal. Let $\langle A_n : n < \omega \rangle$ be a decreasing sequence of $\mathscr{I}_0$-positive sets. If for every $n < \omega$, A_n contains an infinite antichain then we can select $a_n \in A_n$ incompatible with a_k for all $k < n$, and then $\{a_n : n < \omega\}$ is an $\mathscr{I}_0$-positive pseudointersection of $\langle A_n : n < \omega \rangle$. If for some $N < \omega$, A_N does not have an infinite antichain then there is an eventually zero branch B such that $|B \cap A_n| = \aleph_0$ for all $n \geq N$, and then we can choose $a_n \in A_n \cap B$ such that $a_k \subsetneq a_n$ for all $k < n$. Hence $\{a_n : n \geq N\}$ is an $\mathscr{I}_0$ pseudointersection of $\langle A_n : n < \omega \rangle$.

One of our principal problems is to characterize (in Katětov order) whether an ideal $\mathscr{I}$ can be extended to an F_σ-ideal. About this, we have conjectured the following.

Conjecture 3.2.13. *If $\mathscr{I}$ is a Borel ideal then either there is an $\mathscr{I}$-positive subset X of ω such that $\mathscr{I} \upharpoonright X \geq_K \mathsf{conv}$ or there is an F_σ-ideal $\mathcal{J}$ containing $\mathscr{I}$.*

An approximation to this conjecture is the following result.

Theorem 3.2.14. *Let $\mathscr{I}$ be a Borel ideal such that the forcing quotient $\mathcal{P}(\omega)/\mathscr{I}$ is proper. Then, either there is an $\mathscr{I}$-positive set X such that $\mathsf{conv} \leq_K \mathscr{I} \upharpoonright X$ or there is an F_σ-ideal $\mathscr{J}$ containing $\mathscr{I}$.*

Proof. **Case 1.** *$\mathcal{P}(\omega)/\mathscr{I}$ adds new real numbers.* This is just theorem 2.4.5.
Case 2. *$\mathcal{P}(\omega)/\mathscr{I}$ does not add new real numbers.* Let G be a $\mathcal{P}(\omega)/\mathscr{I}$-generic ultrafilter and define $\mathcal{U} = \{U \subseteq \omega : [U] \in G\}$. If $\mathcal{U}$ were a P-point (in $V[G]$) then, by Zapletal's theorem 2.8.3, there is an F_σ-ideal $\mathscr{J} \supseteq \mathscr{I}$ disjoint from $\mathcal{U}$. Then, in $V[G]$, there is a lower semicontinuous submeasure φ such that $\mathscr{J} = Fin(\varphi)$. Since φ is defined by its values in the finite sets, and $\mathcal{P}(\omega)/\mathscr{I}$ does not add new real numbers, φ is actually defined in V and $\mathscr{I} \subseteq \mathscr{J}$. If $\mathcal{U}$ is not a P-point then there is a strictly $\subseteq^*$-decreasing sequence $\mathcal{X} = \langle U_n : n < \omega \rangle$ of elements of $\mathcal{U}$ without pseudointersections in $\mathcal{U}$. Define $D = \{Y \in \mathscr{I}^+ : (\forall n < \omega)(Y \subseteq^* U_n) \vee (\exists n < \omega)(Y \cap U_n = \emptyset)\}$. Since $\mathcal{X}$ does not have pseudointersections in $\mathcal{U}$, D is not dense, and so, there is $Z \in \mathscr{I}^+$ such that every $\mathscr{I}$-positive subset Y of Z is not in D. Let us see

that $\mathscr{I} \restriction Z \geq_K \mathbf{Fin} \times \mathbf{Fin}$. Note that for any $n < \omega$, $Z \setminus U_n \in \mathscr{I}$, but $Z \nsubseteq^* U_n$. For every $n < \omega$ define a set $I_n \in \mathscr{I}$ as follows: $I_0 = Z \setminus U_0$, $I_{n+1} = Z \cap U_n \setminus U_{n+1}$. Each I_n is an infinite element of $\mathscr{I} \restriction Z$, $\bigcup_{n<\omega} I_n = Z$ and if $A \subseteq Z$ is such that $A \cap I_n$ is finite for all $n < \omega$ then $A \in \mathscr{I}$. Hence $\mathscr{I} \restriction Z \geq_K \mathbf{Fin} \times \mathbf{Fin}$. Since $\mathbf{Fin} \times \mathbf{Fin} \geq_K \mathsf{conv}$ we are done. $\square$

3.3 Looking for Katětov-locally minimal ideals

We do not know whether there is a tall Borel ideal minimal among tall Borel ideals. We conjecture that the answer is negative. However, a weaker question i whether there is a Borel tall ideal $\mathscr{J}$ such that for every Borel tall ideal $\mathscr{I}$ there is an $\mathscr{I}$-positive set X such that $\mathscr{J} \leq_K \mathscr{I} \restriction X$ (we call such $\mathscr{J}$ *locally minimal*). There is a natural candidate, the ideal $\mathcal{R}$ introduced in chapter 1, and for which we proved that is critical with respect to the Ramsey property $\mathscr{I}^+ \to (\mathscr{I}^+)^2_2$ (see theorem 2.2.2).

Theorem 3.3.1. *Let $\mathscr{I}$ be an analytic tall ideal on ω such that $\mathcal{P}(\omega)/\mathscr{I}$ does not add new real numbers. Then there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_K \mathcal{R}$.*

Proof. Assume that theorem is false. Then for every $\mathscr{I}$-positive set X and for every $f : [\omega]^2 \to 2$ exists an $\mathscr{I}$-positive and f-homogeneous subset Y of X. Let G be a $\mathcal{P}(\omega)/\mathscr{I}$-generic ultrafilter. $\mathcal{U} = \{U \subseteq \omega : [U]_I \in G\} \subset \mathscr{I}^*$ is a selective ultrafilter. Since G does not add new partitions of $[\omega]^2$, G is also selective in $V[G]$. But $G \cap \mathscr{I} = \emptyset$, contradicting Mathias theorem 2.2.8. $\square$

As a corollary, we have that $\mathcal{R}$ is locally minimal among F_σ ideals.

Corollary 3.3.2. *For every F_σ tall ideal $\mathscr{I}$ there is an $\mathscr{I}$-positive X such that $\mathscr{I} \restriction X \geq_K \mathcal{R}$.*

Proof. Immediate from 3.2.4 and P^+ implies σ-closed. $\square$

We will prove that the class of ideals $\mathscr{I}$ such that $\mathcal{P}(\omega)/\mathscr{I}$ is proper is included in the class of ideals $\mathscr{I}$ failing the $\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2$ property, i.e. the class of ideals for which $\mathcal{R}$ is locally minimal.

Lemma 3.3.3.

$$\mathcal{R} \leq_K \mathsf{conv}.$$

Proof. It will be equivalent if we prove

$$\omega \not\rightarrow (\mathsf{conv}^+)^2_2.$$

We will use a trick due to Wacław Sierpiński. Let $\sqsubseteq$ be a type-ω ordering for $\mathbb{Q}$, and define $\psi : [\mathbb{Q}]^2 \to 2$ such that $\psi(\{q, r\}) = 0$ if and only if $q \sqsubseteq r \leftrightarrow q < r$. Hence, every infinite 0-homogeneous set has order type ω and every infinite 1-homogeneous set has order type ω^*, and then, all the ψ-homogeneous sets are in conv. □

From lemma 3.3.3 and theorems 2.4.5 and 3.3.1, it follows immediately one of the main results of this chapter:

Theorem 3.3.4. *If $\mathscr{I}$ is a Borel ideal and $\mathcal{P}(\omega)/\mathscr{I}$ is proper then there is an $\mathscr{I}$-positive set X such that $\mathcal{R} \leq_K \mathscr{I} \restriction X$.*

Proof. Assume $\mathcal{P}(\omega)/\mathscr{I}$ is proper. If it adds a new real number then by 2.4.5 there is an $\mathscr{I}$-positive set X such that $\mathsf{conv} \leq_K \mathscr{I} \restriction X$, and then, by lemma 3.3.3 $\mathcal{R} \leq_K \mathscr{I} \restriction X$. If not, then immediately by 3.3.1 we are done. □

Another consequence of theorem 2.4.3 is the following result.

Corollary 3.3.5. *If $\mathscr{I}$ is an F_σ ideal on ω then $\mathscr{I} \not\geq_K \mathsf{conv}$.*

Proof. Let $\{X_n : n < \omega\}$ be a family of infinite subsets of ω. Recursively define Y_n $(n < \omega)$ as follows: $Y_0 = X_0$ if $X_0 \in \mathscr{I}^+$, else $Y_0 = \omega \setminus X_0$; $Y_n{+}1 = X_{n+1} \cap Y_n$ if $X_{n+1} \cap Y_n \in \mathscr{I}^+$, else $Y_{n+1} = Y_n \setminus X_{n+1}$. For every $n < \omega$ let F_n a finite subset of $Y_n \setminus Y_{n+1}$ such that $\varphi(F_n) \geq n$, where φ is a lower semicontinuous submeasure on ω such that $\mathscr{I} = Fin(\varphi)$. Hence, $\bigcup_{n<\omega} F_n$ is an $\mathscr{I}$-positive set which is not split by the family $\{X_n : n < \omega\}$. □

3.4 Category dichotomy

In this section we will prove the following fundamental structural theorem for Borel ideals.

Theorem 3.4.1 (Category Dichotomy, Hrušák)**.** *Let $\mathscr{I}$ be a Borel ideal. Then either $\mathscr{I} \leq_K \mathsf{nwd}$ or there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_K \mathcal{ED}$.*

Proof. Let us play the following game $G_3(\mathscr{I})$: In step k, Player I chooses a set I_k in $\mathscr{I}$ and then Player II chooses n_k in ω but not in I_k. Player I wins if $\{n_k : k < \omega\} \in \mathscr{I}$.

Claim. *(a) Player I has a winning strategy in $G_3(\mathscr{I})$ if and only if there is an $\mathscr{I}^*$-branching tree $T \subseteq \{s \in \omega^\omega : s$ is increasing$\}$ such that $rng(x) \in \mathscr{I}$ for all branch x of T, and*

(b) The following conditions are equivalent:

- *Player II has a winning strategy in $G_3(\mathscr{I})$,*
- *there is an $\mathscr{I}^+$-branching tree $S \subseteq \omega^{<\omega}$ such that $rng(x) \in \mathscr{I}^+$ for all branch x of S, and*
- *there is a pairwise disjoint family $\{X_n : n < \omega\}$ of $\mathscr{I}$-positive sets such that for every $I \in \mathscr{I}$ there is $n < \omega$ such that $X_n \cap I = \emptyset$.*

Proof of claim. Let us prove (a). If σ is a winning strategy for Player I then it is easy to define recursively a tree $T' \subseteq (\omega \times \mathscr{I})^{<\omega}$ such that $\langle (n_0, I_0), \ldots (n_k, I_k)\rangle \in T'$ if and only if

- $I_0 = \sigma(\emptyset)$,
- $n_0 \notin I_0$,
- $I_{j+1} = \sigma(\langle I_0, n_0, \ldots, I_j, n_j\rangle)$ for all $j < k$, and
- $n_{j+1} > n_j$.

Hence, put $T \cap \omega^1 = \{\langle n\rangle : n \in \omega \setminus \sigma(\emptyset)\}$ and inductively assume that for any sequence $\langle n_0 < \cdots < n_{k-1}\rangle \in T \cap \omega^k$, there is $\langle I_0, \ldots, I_{k-1}\rangle$ such that $\langle (n_0, I_0), \ldots, (n_{k-1}, I_{k-1})\rangle \in T'$, and we define $succ_T(\langle n_0, \ldots, n_k\rangle) = \omega \setminus \sigma(\langle I_0, n_0, \ldots, I_k, n_k\rangle)$. Hence, any branch of T is following the strategy σ and then the images of branches of T are in $\mathscr{I}$.

On the other hand, given the tree T let us define recursively the strategy σ as follows: $\sigma(\emptyset) = \omega \setminus \{n < \omega : \langle n\rangle \in T\}$ and if $k < \omega$ and the sequence $\langle I_0, n_0, \ldots, n_{k-1}, I_k\rangle$ is following σ then we define for all $l \notin I_k$ $\sigma(\langle I_0, n_0, \ldots, I_k, l\rangle) = \omega \setminus succ_T(\langle n_0, \ldots, n_{k-1}, l\rangle)$. Since T is $\mathscr{I}^*$-branching, for all σ-legal position $s = \langle n_0, I_0, \ldots, n_k, I_k, l\rangle$, we have that $\sigma(s) \in \mathscr{I}$ and $\langle n_0, \ldots, n_{k-1}, l\rangle \in T$. Hence, if $\langle I_0, n_0, I_1, n_1, \ldots\rangle$ follows σ then $\langle n_k : k < \omega\rangle$ follows a branch of T, and then it is in $\mathscr{I}$.

Now we will prove (b). Let τ be a winning strategy for Player II. Let us consider the tree $S' \subseteq (\omega \times \mathscr{I})^{<\omega}$ such that $\langle (n_0, I_0), \dots, (n_k, I_k)\rangle \in S'$ if and only if $n_j = \tau(\langle I_0, n_0, \dots, n_{j-1}, I_j\rangle)$ for all $j \leq k$. We can define recursively by levels, a subtree S'' of S' such that for every $k < \omega$, if $\langle (n_0, I_0), \dots, (n_k, I_k)\rangle$ and $\langle (n_0, I_0'), \dots, (n_k, I_k')\rangle$ are in S'' then $I_j = I_j'$ for all $j \leq k$. Then $S = \{s \in \omega^{<\omega} : \exists t \in \mathscr{I}^{<\omega} : |s| = |t| \wedge \langle (s_i, t_i) : i < |s|\rangle \in S''\}$ is the required tree. First, if $s \in S \cap \omega^k$ and $I = succ_S(s) \in \mathscr{I}$ then we will be in contradiction because $\sigma(I_0, s_0, \dots, I_{k-1}, s_{k-1}, I) = l$ for some $l \in I$. Moreover, every branch of S follows τ, and then, $rng(x)$ is an $\mathscr{I}$-positive set for all branch x of S.

Let suppose the tree S given and let $\langle s_n : n < \omega\rangle$ be an enumeration of all branching nodes of S and $Y_n = succ_S(s_n)$. Then, if A intersects every Y_n then A contains a branch of T and then A is not in $\mathscr{I}$. By the Disjoint Refinement Lemma 1.2.5 we can get the family $\{X_n : n < \omega\}$ from $\{Y_n : n < \omega\}$.

Finally, if the family $\{X_n : n < \omega\}$ satisfies that for any $I \in \mathscr{I}$ there is a X_n disjoint from I, then Player II can choose an element n_k in $X_k \setminus I_k$ and then she will win. □

Claim. *If for every $\mathscr{I}$-positive set X Player II has a winning strategy in $G_3(\mathscr{I} \restriction X)$ then $\mathscr{I} \leq_K$ nwd.*

Proof of claim. Define $\{X_s : s \in \omega^{<\omega}\}$ such that $X_\emptyset = \omega$ and $\{X_{s^\frown n} : n < \omega\}$ is a partition of X_s in $\mathscr{I}$-positive sets such that for any $I \in \mathscr{I} \restriction X_s$ there is $n < \omega$ such that $I \cap X_{s^\frown n} = \emptyset$. We can refine that partitions in order to separate every pair $\{n, m\}$ by enumerating such pairs and making sure that k-th pair is separated in level k, that is, refine if necessary the disjoint family $\{X_s : s \in \omega^k\}$ in such way that the k-th pair-set of our enumeration would be separated by two elements of level k. Let τ be the topology on ω generated by $\{X_s : s < \omega\}$. It is clear that $\langle \omega, \tau\rangle$ is a countable Hausdorff second countable and zero-dimensional topological space without isolated points, and then, by Sierpinski's theorem (see [30]), it is isomorphic with the space of rational numbers. Note that any element I of $\mathscr{I}$ is nowhere dense in such topology since for any basic set X_s there is $n < \omega$ such that $I \cap X_{s^\frown n} = \emptyset$. Hence, a homeomorphism φ between $\langle \omega, \tau\rangle$ and $\mathbb{Q}$ is the Katětov function requested. □

Claim. *If there is an $\mathscr{I}$-positive set Y such that Player I has a winning strategy for $G_3(\mathscr{I} \restriction Y)$ then there is an $\mathscr{I}$-positive set $X \subseteq Y$ such that $\mathscr{I} \restriction X \geq_K \mathcal{ED}$.*

Proof. By the first claim, there is an $(\mathscr{I} \restriction Y)^*$-branching tree T such that any (image of a) branch is in $\mathscr{I}$. Let $\langle s_n : n < \omega \rangle$ be an enumeration of T and let us denote by $Y_n = succ_T(s_n)$, for each $n \in \omega$.
Case 1. If the family $\{Y_n : n \in \omega\}$ does not have $\mathscr{I}$-positive pseudointersections then define $I_0 = Y \setminus Y_0$ and $I_{n+1} = (\bigcap_{k \leq n} Y_k) \setminus Y_{n+1}$ for all $n < \omega$. Note that every I_n is an element of $\mathscr{I}$ since every Y_n is in $(\mathscr{I} \restriction X)^*$. We take Y as our X requested and note that $\{I_n : n < \omega\}$ is a partition of X in elements of $\mathscr{I} \restriction X$, and any $I \subseteq X$ such that $|I \cap I_n| < \omega$ for all n is necessarily in $\mathscr{I}$ since such I is a pseudointersection of the family $\langle Y_n : n < \omega \rangle$. Then in Case 1, we have that $\mathscr{I} \restriction X \geq_K \mathbf{Fin} \times \mathbf{Fin} \geq_K \mathcal{ED}$.
Case 2. If the family $\{Y_n : n < \omega\}$ has an $\mathscr{I}$-positive pseudointersection, let X be such $\mathscr{I}$-positive pseudointersection and we define a strictly increasing function $g : \omega \to X$ such that for any $t \in T$, if $rng(t) \subseteq X \cap g(n)$ then $X \setminus g(n+1) \subseteq succ_T(t)$. Define $g(0) = \min X$ and for every $n < \omega$ define $W_n = \bigcap\{succ_T(t) : rng(t) \subseteq X \cap g(n)\}$. Clearly $W_n \in \mathscr{I}^*$ and $X \subseteq^* W_n$. Define $g(n+1) = \min\{k < \omega : X \setminus k \subseteq W_n\}$. We define $A_n = X \cap [g(n), g(n+1))$ and we split $X = A \cup B$ where $A = \bigcup_{n<\omega} A_{2n}$ and $B = \bigcup_{n<\omega} A_{2n+1}$. Clearly $\{A_n : n < \omega\}$ is a partition of X and if S is a selector of this partition, we split $S = (S \cap A) \cup (S \cap B)$ and note that $S \cap A$ and $S \cap B$ are in $\mathscr{I}$ since both sets respectively follow a branch of T. Since $\mathscr{I} \geq_K \mathcal{ED}_{fin}$ is equivalent to the existence of a partition $\{I_n : n < \omega\}$ in finite sets such that every selector is in $\mathscr{I}$, we have proved the claim. □

Hence, by Martin's theorem 0.3.2 we are done. □

Note that from Category Dichotomy's proof we have actually deduced a trichotomy: For every Borel ideal $\mathscr{I}$ either $\mathscr{I} \leq_K \mathsf{nwd}$ or there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_K \mathbf{Fin} \times \mathbf{Fin}$ or there is an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_{KB} \mathcal{ED}_{fin}$. But note that $\mathscr{I} \restriction X \geq_{KB} \mathcal{ED}_{fin}$ is equivalent to not be a Q^+-ideal; and by theorem 2.4.2, $\mathbf{Fin} \times \mathbf{Fin} \leq_K \mathscr{I}$ implies $\mathscr{I}$ is not P^+. Hence we conclude that Category Dichotomy implies that for every Borel ideal, either $\mathscr{I} \leq_K \mathsf{nwd}$ or $\mathscr{I}$ is not a Q^+ ideal or $\mathscr{I}$ is not a P^+ ideal.

Recall that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and $\mathsf{cov}^*(\mathcal{ED}) = \mathsf{non}(\mathcal{M})$, hence the name of *Category Dichotomy.*

3.5 Fatou's lemma and Solecki's ideal $\mathcal{S}$

Another property having a critical ideal with respect to Katětov order was isolated by Sławomir Solecki in [44], where it was shown that the ideal $\mathcal{S}$ is critical with respect to fulfilling the Fatou's Lemma. We will explain the property of fulfilling Fatou's Lemma and answer a question of Solecki.

Given a sequence $(a_n)_{n\in\omega}$ of real numbers and an ideal $\mathscr{I}$ on ω, the *lower $\mathscr{I}$-limit* of this sequence is defined by

$$\liminf_{\mathscr{I}} a_n = \sup\{r \in \mathbb{R} : \{n \in \omega : a_n < r\} \in \mathscr{I}\}.$$

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with μ defined on a σ-algebra $\mathcal{B}$. Let $f_n : X \to [0, \infty]$ be a sequence of μ-measurable functions and let $\mathscr{I}$ be an ideal on ω. We say that *Fatou's lemma holds on $\langle f_n : n \in \omega\rangle$ with respect to $\mathscr{I}$* if

$$\underline{\int} \liminf_{\mathscr{I}} f_n d\mu \leq \liminf_{\mathscr{I}} \int f_n d\mu$$

where $\underline{\int}$ is the lower integral, i.e., if $g \geq 0$, then

$$\underline{\int} g d\mu = \sup\left\{\int f d\mu : f \leq g \text{ and } f \text{ is } \mu\text{-measurable}\right\}.$$

Let $\mathscr{I}$ be an ideal on ω. We say that *Fatou's lemma holds for$\mathscr{I}$* if Fatou's lemma holds with respect to $\mathscr{I}$ for any sequence $\langle f_n : n \in \omega\rangle$ of measurable functions from X to $[0, \infty)$ on any σ-finite measure space.

The following result due to Solecki shows in which sense the ideal $\mathcal{S}$ is critical with respect to fulfil Fatou's lemma.

Theorem 3.5.1 (Solecki, [44]). *Let $\mathscr{I}$ be a Borel ideal on ω. Then $\mathscr{I}$ does not satisfy Fatou's lemma if and only if there exists $X \in \mathscr{I}^+$ such that $\mathcal{S} \leq_K \mathscr{I} \restriction X$.*

Concerning this theorem, Solecki asked the following question.

Question 3.5.2 (Solecki, [44]). *Can $\mathcal{S}$ be replaced by $\mathcal{G}_{fc}$?*

Recall in chapter 1 was proved that $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N})$ and $\mathsf{cov}^*(\mathcal{G}_{fc}) = \mathfrak{s}_2$. In [44], Solecki proved that $\mathcal{S} \leq_K \mathcal{G}_{fc}$, but we have the following result.

Theorem 3.5.3.

$$\mathcal{G}_{fc} \not\leq_K \mathcal{S}.$$

Proof. In the Cohen model, $\mathsf{cov}^*(\mathcal{G}_{fc}) < \mathsf{cov}^*(\mathcal{S})$ since $\mathfrak{s}_2 \leq \mathsf{non}(\mathcal{M})$. By proposition 1.5.2, $\mathcal{G}_{fc} \not\leq_K \mathcal{S}$ in the Cohen model. By the absoluteness of Katětov order on Borel ideals, $ZFC \vdash \mathcal{G}_{fc} \not\leq_K \mathcal{S}$. □

In order to answer Solecki's question, we need to find a Borel ideal $\mathscr{I}$ such that $\mathscr{I} \geq_K \mathcal{S}$ but for every $X \in \mathscr{I}^+$, $\mathscr{I} \upharpoonright X \not\geq_K \mathcal{G}_{fc}$. The ideal nwd of nowhere dense subsets of $\mathbb{Q}$ is such ideal. Recall that nwd is a K-uniform ideal.

The ideal nwd is important when we think which ideals on ω are Cohen-destructible. Theorem 1.3 in [19]] (see 2.1.6(1)) claims that $\mathscr{I}$ is Cohen-destructible if and only if $\mathscr{I} \leq_K \mathsf{nwd}$. By this theorem, we can decide the Katětov order between $\mathcal{G}_{fc}$ and nwd and between $\mathcal{S}$ and nwd.

Theorem 3.5.4. *The following relations hold.*

1. $\mathcal{S} \leq_K \mathsf{nwd}$.

2. $\mathcal{G}_{fc} \not\leq_K \mathsf{nwd}$.

Proof. Since adding $\mathfrak{c}^+$-many Cohen reals enlarges $\mathsf{cov}^*(\mathcal{S}) = \mathsf{non}(\mathcal{N}) \geq \mathsf{cov}(\mathcal{M})$, $Fn(\mathfrak{c}^+, 2)$ destroys $\mathscr{I}$ for all F_σ-ideal $\mathscr{I}$, but by homogeneity and c.c.c this implies that $Fn(\omega, 2)$ destroys $\mathscr{I}$, and hence, Cohen forcing destroys $\mathcal{S}$. By Theorem 2.1.6,(1) $\mathcal{S} \leq_K \mathsf{nwd}$. However, $Fn(\mathfrak{c}^+, 2)$ forces that $\mathsf{cov}^*(\mathcal{G}_{fc}) = \mathfrak{s}_{pair} \leq \mathsf{non}(\mathcal{M}) = \omega_1$, so, $\mathcal{G}_{fc}$ is Cohen-indestructible. Then, if G is a $Fn(\mathfrak{c}^+, 2)$-generic ultrafilter on V then $V[G] \models \mathsf{cov}^*(\mathcal{G}_{fc}) < \mathsf{cov}^*(\mathsf{nwd})$ and then by theorem 1.5.2 $V[G] \models \mathcal{G}_{fc} \not\leq_K \mathsf{nwd}$. Hence by the absoluteness of the Katětov order $V \models \mathcal{G}_{fc} \not\leq_K \mathsf{nwd}$. □

By Theorems 3.5.4 and 3.5.3, $\mathcal{S}$ can not be replaced by $\mathcal{G}_{fc}$ in Theorem 3.5.1. Hence, the answer of Solecki's Question is in the negative.

3.6 Measure dichotomy

Pathological submeasures have been studied by many authors like Christensen, Farah, Kanovei, Reeken and Solecki.

Definition 3.6.1. Let φ be a submeasure supported by a set X. We will say that φ is a *pathological submeasure* if there is $A \subseteq X$ such that

$$\sup\{\mu(A) : \mu \text{ is a measure on } X \text{ dominated by } \varphi\} < \varphi(A).$$

Recall that Solecki's theorem 1.3.2 claims that for every analytic P-ideal $\mathscr{I}$ there is a lower semicontinuous submeasure (lscsm) φ such that $\mathscr{I} = Exh(\varphi)$. The main theorem of this section splits the family of analytic P-ideals in two classes: the first one contains all ideals whose submeasures are sufficiently non-pathological and the other one contains all ideals with very pathological submeasures. The ideals $\mathcal{Z}$ of sets with asymptotical density zero and Solecki's ideal $\mathcal{S}$ are the critical ideals with respect to these classes. In next section we will prove that the pathology of submeasures is closely related with the Fubini property defined in such section. Since the correspondence between analytic P-ideals and lower semicontinuous submeasures is far from being one-to-one, we need to chooses an adequate lower semicontinuous submeasure for any ideal given.

For each lscsm given φ, Farah defined another lscsm $\hat{\varphi}$ as follows: for all $A \subseteq X$

$$\hat{\varphi}(A) = \sup\{\mu(A) : \mu \text{ is a measure on } X \text{ dominated by } \varphi\},$$

and then, φ is non-pathological if and only if $\varphi = \hat{\varphi}$. Moreover, for any analytic P-ideal $\mathscr{I}$ the following conditions are equivalent:

- there is a lscsm φ such that $\mathscr{I} = Exh(\varphi) \subsetneq Exh(\hat{\varphi})$, and
- for any lscsm φ, if $\mathscr{I} = Exh(\varphi)$ then $\mathscr{I} \subsetneq Exh(\hat{\varphi})$.

Then we can say that an analytic P-ideal $\mathscr{I}$ on ω is a *non-pathological ideal* if there is a lscsm φ such that $\mathscr{I} = Exh(\varphi) = Exh(\hat{\varphi})$.

We will do a little change in the following definition with respect to Farah's definition. The *degree of pathology* of a submeasure φ on X is defined by

$$P(\varphi) = \frac{\varphi(X)}{\sup\{\mu(X) : \mu \text{ is a measure dominated by } \varphi\}},$$

provided that $\varphi(X) < \infty$.

J. L. Kelley defined in [31] what now is known as *Kelley's covering number* as follows: let F be a set and $\mathcal{B} \subseteq \mathcal{P}(F)$. For any finite sequence $S = \langle S_0, \dots S_n \rangle$ of (possibly non distinct) elements of $\mathcal{B}$:

$$m(S) = \min\left\{|\{i \in n+1 : x \in S_i\}| : x \in F\right\}.$$

And the *covering number* $C(\mathcal{B})$ is defined by:

$$C(\mathcal{B}) = \sup\left\{\frac{m(S)}{|S|} : S \in \mathcal{B}^{<\omega}\right\}.$$

Remark 3.6.2. If $\mathcal{B} \subseteq \mathcal{P}(F)$ and $\delta > 0$ then $C(\mathcal{B}) > \delta$ means that there is $N < \omega$ and there is a sequence $\langle A_0, \ldots, A_{N-1}\rangle$ of elements of $\mathcal{B}$ such that $|\{i < N : x \in A_i\}| \geq N\delta$, for all $x \in F$.

The fundamental theorem of Kelley which links the covering number with submeasures is the following. We say that a (sub)measure φ on a set X is *normalized* if $\varphi(X) = 1$.

Theorem 3.6.3 (Kelley; [31], corollary 6)**.** *For each non-void subclass $\mathcal{B}$ of $\mathcal{P}(F)$ the covering number $C(\mathcal{B})$ is the minimum of the numbers* $\sup\{\mu(A) : A \in \mathcal{B}\}$*, where the minimum is taken over all normalized measures μ on $\mathcal{P}(F)$.* □

Working on the Maharam's problem (equivalent with the Control measure problem), Jens Peter Reus Christensen defined in [10] the concept of pathological submeasure for Boolean algebras as follows: A submeasure φ on a Boolean algebra is *pathological* if it does not dominate any bounded finitely additive positive measure. This is a good definition for pathological submeasures in atomless Boolean algebras. However in the finite case it is not applicable since every positive submeasure on a finite set admits a positive measure bounded by it. Theorem 2 in [10] is very useful when we work with atomless measures, and we have a quantitative version of that result which is useful working with submeasures on finite sets.

Lemma 3.6.4 (Quantitative version of Christensen's lemma)**.** *Let F be a finite set, $\varepsilon > 0$, φ a normalized submeasure on $\mathcal{P}(F)$ and $\mathcal{A}_\varepsilon = \{A \subseteq F : \varphi(A) < \varepsilon\}$. Then*

$$C(\mathcal{A}_\varepsilon) \geq 1 - \frac{1}{\varepsilon P(\varphi)}.$$

Proof. By Kelley's theorem, it will be sufficient to proof that for all normalized measure on F there is a set $A \in \mathcal{A}_\varepsilon$ such that $\mu(A) \geq 1 - \frac{1}{\varepsilon P(\varphi)}$, i.e., it will be sufficient to show that for a given normalized measure μ on F there is a set $A \in \mathcal{A}_\varepsilon$ such that $\mu(F \setminus A) \leq \frac{1}{\varepsilon P(\varphi)}$. Fix a normalized measure μ on F and define $\psi = \varphi - \varepsilon\mu$. Note that $\psi(A \cup B) \leq \psi(A) + \psi(B)$

if A and B are disjoint subsets of F. Let $\mathcal{F}$ be a maximal disjoint family of subsets of F such that $\psi(B) < 0$ for all $B \in \mathcal{F}$. Define $A = \bigcup \mathcal{F}$. Note that (1) $\varepsilon\mu \restriction \mathcal{P}(F \setminus A) \leq \varphi \restriction \mathcal{P}(F \setminus A)$ and (2) $\varphi(B) < \varepsilon\mu(B)$ for all $B \in \mathcal{F}$. We denote by $\widehat{\varepsilon\mu}$ the measure on F supported by $F \setminus A$. Hence, $\widehat{\varepsilon\mu} \leq \sup\{\nu(F) : \nu \text{ is a measure dominated by } \varphi\}$. Then, by (1) $\varepsilon\mu(F \setminus A) \leq \frac{1}{P(\varphi)}$. On the other hand, as a consequence of (2) we have:

$$\varphi(A) \leq \sum_{B \in \mathcal{F}} \varphi(B) < \sum_{B \in \mathcal{F}} \varepsilon\mu(B) = \varepsilon\mu(A) \leq \varepsilon.$$

□

Theorem 3.6.5 (Measure Dichotomy, Hrušák)**.** *Let $\mathscr{I}$ be an analytic P-ideal. Then, either $\mathscr{I} \leq_K \mathcal{Z}$ or there is $X \in \mathscr{I}^+$ such that $\mathcal{S} \leq_K \mathscr{I} \restriction X$.*

Proof. By Solecki theorem 1.3.2, there is a lscsm φ such that $\mathscr{I} = Exh(\varphi)$. The remainder of this paragraph is a sketch of the remainder of this proof. We will consider two cases. The first one is when φ is *very-pathological*. In such case, we will take a sequence of intervals whose union is a positive set and the degrees of pathology of the restrictions of φ in these intervals increases to infinity. For each interval we will define a partition of the Cantor set 2^ω in clopen sets, and by joining some elements of that partition we will form elements of Ω which are going to be the images of the Katětov function requested. In the *not-so pathological* case we will use the previous theorem in order to define a measure which approximates φ, and then we will be able to define a Katětov function from ω to a restriction of $\mathcal{Z}$. As $\mathcal{Z}$ is K-uniform, this will show that $\mathscr{I} \leq_K \mathcal{Z}$.

Without loss of generality we can assume that $|\varphi| = \lim_{n\to\infty} \varphi(\omega \setminus n) > 1$. Since there are no φ-exhaustive final segments of ω, there exists a partition $\langle F_n : n < \omega \rangle$ of ω in intervals such that for any $n < \omega$, $\min(F_{n+1}) = \max(F_n) + 1$, $\varphi(F_n) \geq 1$ and $\varphi(F_n \setminus \{\max F_n\}) < 1$. Let φ_n be the normalization (by multiplying by $\frac{1}{\varphi(F_n)}$) of $\varphi \restriction F_n$ and let $r_n = P(\varphi_n) = P(\varphi \restriction F_n)$ for all $n < \omega$.

Case 1. *$\langle r_n : n < \omega \rangle$ is an unbounded sequence.*
Let $\langle r_{n_k} : k < \omega \rangle$ be a subsequence such that $r_{n_k} \geq 3 \cdot 2^{k+1}$ and define $X = \bigcup_{k<\omega} F_{n_k}$. It is clear that $X \in \mathscr{I}^+$. We will see that $\mathcal{S} \leq_K \mathscr{I} \restriction X$. By lemma 3.6.4, for any $k < \omega$, by taking $\varepsilon_k = 2^{-k-1}$, we have that $C(\mathcal{A}_{\varepsilon_k}) \geq \frac{2}{3}$, where $\mathcal{A}_{\varepsilon_k} = \{A \subseteq F_{n_k} : \varphi_{n_k}(A) < \varepsilon_k\}$. For any $k < \omega$, take a sequence

$A^k = \langle A_0^k, \ldots, A_{N_k-1}^k \rangle \subseteq \mathcal{A}_{\varepsilon_k}$ such that $|\{i < N_k : x \in A_i\}| \geq \frac{2}{3}N_k$, for all $x \in F_{n_k}$; and choose a sequence $\langle U_i^k : i < N_k \rangle$ of open subsets of 2^ω with $\lambda(U_i^k) = \frac{1}{N_k}$. For any $x \in F_{n_k}$ define $W_x = \bigcup\{U_i^k : x \in A_i^k\}$. Note that $\mu(W_x) \geq \frac{2}{3}$. Then, every W_x includes infinitely many elements of Ω. For every $x \in X$ choose an element U_x of Ω contained in W_x, such that $U_x \neq U_y$ for all $y < x$. $f(x) = U_x$ is a one-to-one function from X to Ω such that for all $z \in 2^\omega$ and for all $k < \omega$ there exists at most one $i < N_k$ such that $z \in U_i^k$. Hence, given $z \in 2^\omega$, we have that

$$\varphi_{n_k}(\{x \in F_{n_k} : z \in f(x)\}) \leq \varphi_{n_k}(\{x \in F_{n_k} : z \in W_x\}) \leq \varphi_{n_k}(A_i^k) < \frac{\varepsilon_k}{\varphi(F_{n_k})},$$

and then, for the subbasic set $I_z = \{C \in \Omega : z \in C\}$, we have that

$$\varphi(f^{-1}[I_z]) \leq \sum_{k<\omega} \varphi(f^{-1}[I_z] \cap F_{n_k}) =$$
$$= \sum_{k<\omega} [\varphi(F_{n_k}) \cdot \varphi_{n_k}(f^{-1}[I_z] \cap F_{n_k})] < \sum_{k<\omega} \frac{1}{2^{k+1}} < \infty.$$

Actually, the previous formula shows that $f^{-1}[I_z]$ is φ-exhaustive, and then we conclude that f witnesses $\mathcal{S} \leq_{KB} \mathscr{I} \restriction X$.

Case 2. *$\langle r_n : n < \omega \rangle$ is bounded by some $r < \infty$.*

For any $n < \omega$ there is a measure μ_n on F_n bounded by φ and such that $\frac{\varphi(F_n)}{\mu_n(F_n)} \leq r$. Define a submeasure ψ on $\mathcal{P}(\omega)$ such that for any $A \subseteq \omega$:

$$\psi(A) = \sup\{\mu_n(A \cap F_n) : n < \omega\}.$$

It is clear that $\psi \leq \varphi$, and so, $Exh(\varphi) \subseteq Exh(\psi)$. Moreover, $\omega \notin Exh(\psi)$ since $\psi(\omega \setminus n) \geq \frac{1}{r}$ for all $n < \omega$. We will see that there is $Y \in \mathcal{Z}^+$ such that $Exh(\psi) \leq_K \mathcal{Z} \restriction Y$. Let $\langle M_n : n < \omega \rangle$ be a sequence of natural numbers such that $2^{M_n-n-2} > |F_n|$ and let $\{A_x : x \in F_n\}$ be a family of pairwise disjoint subsets of $[2^{M_n}, 2^{M_n+1})$ such that for any $x \in F_n$:

$$2^{M_n} \frac{\mu_n(\{x\})}{\mu_n(F_n)} - \frac{2^{M_n-n-1}}{|F_n|} < |A_x| < 2^{M_n} \frac{\mu_n(\{x\})}{\mu_n(F_n)} + \frac{2^{M_n-n-1}}{|F_n|}.$$

It is always possible to find such pairwise-disjoint family since $2^{M_n-n-2} > |F_n|$. Hence, we have that for any $x \in F_n$,

$$\left| \frac{|A_x|}{2^{M_n}} - \frac{\mu_n(\{x\})}{\mu_n(F_n)} \right| \leq 2^{-n-1},$$

and so, the normalized counting-measure in $[2^{M_n}, 2^{M_n+1})$ is a 2^{-n-1}-approximation to μ_n in F_n for all $n < \omega$. Define $Y = \bigcup_{n<\omega} \bigcup_{x \in F_n} A_x$. Y is a $\mathcal{Z}$-positive set since $Y \setminus k$ is 2^{-m}-approximated to $\bigcup_{n \geq m} [2^{M_n}, 2^{M_n+1})$, where $m = \min\{l : k < 2^{M_l}\}$. Hence, defining $f : Y \to \omega$ by

$$f(y) = x \text{ iff } y \in A_x$$

we have that f witnesses $\mathscr{I} \leq_K \mathcal{Z} \restriction Y$. In order to prove this, take $B \in Exh(\psi)$. Then $f^{-1}[B] = \bigcup_{x \in B} A_x$ intersects every interval $[2^{M_n}, 2^{M_n+1})$ in a set whose cardinality is 2^{-n-1}-approximated to $\varphi(B \cap F_n)$. Hence

$$\lim_{n \to \infty} \frac{|f^{-1}[B] \cap [2^{M_n}, 2^{M_n+1})|}{2^{M_n}} = 0.$$

Since $\mathcal{Z}$ is a K-uniform ideal (see 2.1.11) we have that

$$\mathscr{I} \leq_K \mathcal{Z} \restriction Y \leq_K \mathcal{Z}.$$

$\square$

3.7 Fubini property

Recall that definition 2.8.11 says that $\mathscr{I}$ satisfies the *Fubini property* if for any Borel subset A of $\omega \times 2^\omega$ and any $\varepsilon > 0$, $\{n < \omega : \lambda((A)_n) > \varepsilon\} \in \mathscr{I}^+$ implies $\lambda^*(\{x \in 2^\omega : (A)^x \in \mathscr{I}^+\}) \geq \varepsilon$ (here λ^* means the Lebesgue outer measure on 2^ω). In [27], Kanovei and Reeken claimed without a proof that Fubini property is equivalent with fulfil Fatou's lemma. We will prove this as a corollary of Solecki's theorem 3.5.1 and theorem 3.7.1. By paraphrasing Solecki's proof of 3.5.1 we will prove the following theorem.

Theorem 3.7.1. *If $\mathscr{I}$ is an ideal on ω then the following conditions are equivalent:*

1. *there exists an $\mathscr{I}$-positive set X such that $\mathscr{I} \restriction X \geq_K \mathcal{S}$ and*

2. *$\mathscr{I}$ does not satisfy the Fubini property.*

Proof. Let $f : X \to \Omega$ be a witness of $\mathscr{I} \restriction X \geq_K \mathcal{S}$, and define $A = \{(n, x) : x \in f(n)\}$. Note that $(A)_n = f(n)$ and then $\lambda((A)_n) = \frac{1}{2}$ for all $n \in X$. However, for any $x \in 2^\omega$, $\{S \in \Omega : x \in S\} \in \mathcal{S}$ and then $\{n < \omega : x \in (A)_n\} \in \mathscr{I}$ for all $x \in 2^\omega$.

On the other hand, assume that $\mathscr{I}$ does not satisfy the Fubini property, and take a Borel set $A \subseteq \omega \times 2^\omega$ such that for some $\varepsilon > 0$,

$$X = \{n < \omega : \lambda((A)_n) > \varepsilon\} \in \mathscr{I}^+$$

and if $R = \{x \in 2^\omega : (A)^x \in \mathscr{I}^+\}$ then $\lambda^*(R) < \varepsilon$. We can assume that
(1) $R = \emptyset$,
(2) for any $n \in X$, A_n is closed and
(3) for any $n \in X$, $\lambda(A_n) = \varepsilon$.

If not we could replace (a) R with a G_δ set $R' \supseteq R$ with $\lambda(R') = \lambda^*(R)$, (b) ε with $\varepsilon - \lambda^*(R)$ and (c) A_n with a closed subset of $A_n \setminus R$ with measure $\varepsilon - \lambda(R)$. Let $k < \omega$ be such that $(1-\varepsilon)^k < \frac{1}{3}$. We recall that the power of Cantor space $(2^\omega)^k$ with the product measure λ^k is isomorphic with Cantor space 2^ω with Lebesgue measure λ, via a homeomorphism between those spaces. For any $n < \omega$, we will define a subset A'_n of $(2^\omega)^k$ by $A'_n = \bigcup_{i=1}^k proj_i^{-1}[A_n]$. Then $(2^\omega)^k \setminus A'_n = \prod_{i=1}^k (2^\omega \setminus A_n)$ and then $\lambda^n(A'_n) = 1 - \lambda^n(2^\omega \setminus A'_n) > 1 - \frac{1}{3} = \frac{2}{3}$. We note that the family $\{A'_n : n \in X\}$ fulfils that

$$R' = \{x \in (2^\omega)^k : \{n < \omega : x \in A'_n\} \in \mathscr{I}^+\} = \emptyset,$$

since if $x = \langle x_i : 1 \leq i \leq k\rangle$ then

$$\begin{aligned}\{n < \omega : x \in A'_n\} &= \{n < \omega : \exists i(1 \leq i \leq k \;\&\; x_i \in A_n)\}\\ &= \bigcup_{i=1}^k \{n < \omega : x_i \in A_n\} \in \mathscr{I}.\end{aligned}$$

Now, for $n \in X$ choose a clopen subset U_n of $(2^\omega)^k$ such that $\lambda^k(U_n) \geq \frac{7}{12}$ and $\lambda^k(U_n \setminus A_n) < \frac{1}{3 \cdot 2^{n+2}}$. If $S = \{x \in (2^\omega)^k : \{n \in \omega : x \in U_n\} \in \mathscr{I}^+\}$ then $S \subseteq \bigcap_{n<\omega} \bigcup_{m \geq n} (U_m \setminus A'_m)$, proving that $\lambda^k(S) = 0$. Let $\{C_n : n < \omega\}$ be an increasing family of clopen sets such that $S \subseteq \bigcup_{n<\omega} C_n$ and $\lambda^k(\bigcup_{n<\omega} C_n) \leq \frac{1}{12}$. Finally, by taking for any $n \in X$ a clopen subset V_n of $U_n \setminus C_n$ with $\lambda^k(V_n) = \frac{1}{2}$ we get the Katětov function wanted, since for any $x \in 2^\omega = (2^\omega)^k$, if $\{n \in X : x \in V_n\}$ is infinite then $x \notin \bigcup C_n$ and then $x \notin S$. Hence $\{n \in X : x \in V_n\} \in \mathscr{I}$ for all $x \in 2^\omega$. □

We have immediately from Solecki's theorem and the previous theorem the following result.

Corollary 3.7.2. *If $\mathscr{I}$ is a universally measurable ideal on ω then $\mathscr{I}$ has the Fubini property if and only if $\mathscr{I}$ fulfils Fatou's lemma.* □

Corollary 3.7.3. **Fin** *and* $\mathcal{Z}$ *have the Fubini property.*

Proof. Since $\mathcal{S}$ is a tall ideal and **Fin** is K-uniform we have that $\mathcal{S} \not\leq_K \mathbf{Fin} \restriction X$, for all infinite subset X of ω. Now, let $f : \omega \to \Omega$ be a function. By Fubini's theorem, there is $A_n \in \Omega$ such that $|\{m \in [2^n, 2^{n+1}) : x \in f(m)\}| \geq 2^{n-1}$. Since **Fin** has the Fubini property, there is $x \in 2^\omega$ and there is a sequence $\langle n_k : k \in \omega \rangle$ such that $x \in A_{n_k}$. Then, for any $k < \omega$,

$$\limsup_{n\to\infty} \frac{|f^{-1}[I_x] \cap [2^n, 2^{n+1})|}{2^n} \geq \lim_{k\to\infty} \frac{|f^{-1}[I_x] \cap [2^{n_k}, 2^{n_k+1})|}{2^{n_k}} \geq \frac{1}{2}$$

proving that f is not a witness for $\mathcal{S} \leq_K \mathcal{Z}$. $\square$

Definition 3.7.4. Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω. We say that $\mathscr{I}$ is *completely Katětov below* $\mathscr{J}$ (in symbols $\mathscr{I} <<_K \mathscr{J}$) if $\mathscr{I} \restriction X \leq_K \mathscr{J}$ for all $\mathscr{I}$-positive set X.

Theorem 3.7.5. *If* $\mathscr{I}$ *is an analytic P-ideal then the following conditions are equivalent:*

(a) $\mathscr{I} <<_K \mathcal{Z}$,

(b) $\mathscr{I}$ *fulfills Fatou's lemma,*

(c) $\mathscr{I}$ *has the Fubini property and*

(d) $\mathscr{I}$ *is non-pathological.*

Ilijas Farah claimed (without a proof) that (c)→(d) follows from arguments of Christensen in [10] and (d)→(c) was proved by Kanovei and Reeken (Corollary 25 in [28]) by using an idea of Christensen. We will prove the theorem by using arguments contained in the proof of Measure Dichotomy Theorem 3.6.5, where the cornerstone was the quantitative version of Christensen's theorem 3.6.4.

Proof. (a)↔(b) is Solecki's theorem 3.5.1 plus measure dichotomy 3.6.5.

(b)↔(c) is corollary 3.7.2.

(d)→(a) If $\mathscr{I}$ is a non-pathological ideal then there is a lscsm φ such that $\mathscr{I} = Exh(\varphi) = Exh(\hat{\varphi})$. We can suppose that $\hat{\varphi}(\omega) > 1$ and we can find a partition of ω into finite intervals F_n with $n < \omega$ as in the proof of 3.6.5. Since $P(\hat{\varphi} \restriction F_n) = 1$ for all $n < \omega$, we have fallen in case 2 of theorem 3.6.5, and then $\mathscr{I} <<_K \mathcal{Z}$.

(a)→(d) Let suppose that $\mathscr{I}$ is a pathological ideal and let φ be a lscsm such that $\mathscr{I} = Exh(\varphi)$. Then there is a subset A of ω such that $A \in Exh(\hat{\varphi}) \setminus Exh(\varphi)$. Put $\varepsilon = \lim_{n\to\infty} \varphi(A \setminus n)$. We will define a partition of A in infinitely many intervals F_n $(n < \omega)$, such that for $n \geq 1$, $\varphi(F_n) > \frac{\varepsilon}{2}$ and $\hat{\varphi}(F_n) < \frac{\varepsilon}{2n}$. We can define that sequence of intervals, by taking $k_0 = 0$, $k_1 < \omega$ such that $\hat{\varphi}(A \setminus k_1) < \frac{\varepsilon}{2}$, and $k_{n+1} > k_n$ such that $\varphi(A \cap [k_n, k_{n+1})) > \frac{\varepsilon}{2}$ and $\hat{\varphi}(A \setminus k_{n+1}) < \frac{\varepsilon}{2(n+1)}$, for all $n \geq 1$. Then $F_n = A \cap [k_n, k_{n+1})$. With that partition, we have fallen in case 1 of the proof of theorem 3.6.5 ($P(\varphi \restriction A \cap [k_n, k_{n+1}))$ increasing to infinity), and then $\mathscr{I}$ is not completely Katětov below $\mathcal{Z}$. $\square$

Chapter 4

Comparison game on Borel ideals

This chapter deals with the Comparison Game on Borel ideals. Motivated by Tukey order with monotone functions we propose a natural game involving pairs of ideals, called comparison game, which defines an order relation $\sqsubseteq$, and its corresponding equivalence relation $\simeq$. These relations are closely related with the Wadge order on Borel sets (see [30]). We studied the classes of F_σ and $F_{\sigma\delta}$ ideals. We prove that in this order the ideal **Fin** is the minimal ideal, all F_σ ideals are equivalent with **Fin**, I_0 is the minimal ideal among Borel non-F_σ ideals (see its definition in example 3.2.12), all the analytic P-ideals are below $\emptyset \times \mathbf{Fin}$, and $\emptyset \times \mathbf{Fin}$ is below $\mathbf{Fin} \times \mathbf{Fin}$.

4.1 Comparison Game Order

Definition 4.1.1. Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω. The *Comparison Game* for $\mathscr{I}$ and $\mathscr{J}$ denoted by $G(\mathscr{I}, \mathscr{J})$ is defined so that in step n, Player I chooses an element I_n of $\mathscr{I}$ and Player II chooses an element J_n of $\mathscr{J}$. Player II wins if $\bigcup_n I_n \in \mathscr{I}$ if and only if $\bigcup_n J_n \in \mathscr{J}$; otherwise, Player I wins.

Comparison game allows to define an order between ideals on ω.

Definition 4.1.2. Let $\mathscr{I}$ and $\mathscr{J}$ be ideals on ω. We say $\mathscr{I} \sqsubseteq \mathscr{J}$ if Player II has a winning strategy in the comparison game $G(\mathscr{I}, \mathscr{J})$. We say that $\mathscr{I} \simeq \mathscr{J}$ if $\mathscr{I} \sqsubseteq \mathscr{J}$ and $\mathscr{J} \sqsubseteq \mathscr{I}$.

Let us note that the relation $\sqsubseteq$ is reflexive and transitive, but not anti-symmetric; and the relation $\simeq$ is an equivalence relation.

Recall that a function f from $\mathscr{I}$ to $\mathscr{J}$ is a *Tukey function* if for each $A \in \mathscr{J}$ there is $B \in \mathscr{I}$ such that $I \subseteq B$ if $f(I) \subseteq A$. Tukey order is defined by $\mathscr{I} \leq_T \mathscr{J}$ if there is a Tukey function from $\mathscr{I}$ to $\mathscr{J}$; and let us denote by $\mathscr{I} \leq_{MT} \mathscr{J}$ when there is a monotone (with respect to inclusion) Tukey function from $\mathscr{I}$ to $\mathscr{J}$. The order $\sqsubseteq$ refines the monotone Tukey order.

First, we will prove that the comparison game among Borel ideals is determined. To that end we define the following game

Definition 4.1.3. The game $G'(\mathscr{I}, \mathscr{J})$ is defined for ideals $\mathscr{I}$ and $\mathscr{J}$ on ω as follows: In step n Player I has to choose a natural number k_n and Player II has to choose a natural number l_n too. Player II wins if $\{k_n : n < \omega\} \in \mathscr{I}$ if and only if $\{l_n : n < \omega\} \in \mathscr{J}$.

Let us note that by defining a set $\tilde{\mathcal{X}} = \{x \in \omega^\omega : rng(x) \in \mathcal{X}\}$ for a subset $\mathcal{X}$ of $\mathcal{P}(\omega)$, we have that game $G'(\mathscr{I}, \mathscr{J})$ is equivalent to the Wadge game $W(\tilde{\mathscr{I}}, \tilde{\mathscr{J}})$ (see [30]).

Theorem 4.1.4. *Player I has a winning strategy for $G(\mathscr{I}, \mathscr{J})$ if and only if Player I has a winning strategy for $G'(\mathscr{I}, \mathscr{J})$, and the same for Player II.*

Proof. First, let us assume that Player I has a winning strategy σ for the game $G(\mathscr{I}, \mathscr{J})$, and take a bijective function f from ω into $\omega \times \omega$ such that if $f(n) = \langle k, l\rangle$ then $\max\{k, l\} \leq n$. A winning strategy for Player I in $G'(\mathscr{I}, \mathscr{J})$ can be done by playing in parallel $G(\mathscr{I}, \mathscr{J})$. In step 0, Player I plays with the first element k_0 of I_0, where $I_0 = \sigma(\emptyset)$. In step $n+1$, we suppose played a sequence $\langle k_0, l_0, \ldots, k_n, l_n\rangle$ of legal positions for $G'(\mathscr{I}, \mathscr{J})$ and attached with this sequence, we have another sequence $\langle I_0, \{l_0\}, I_1, \{l_1\}, \ldots, I_n, \{l_n\}\rangle$ of legal positions of $G(\mathscr{I}, \mathscr{J})$ which follows σ. Then, by taking k_{n+1} as the k-th element of I_l, where $f(n+1) = \langle k, l\rangle$, we have defined the winning strategy required for Player I. That is true since $\{k_n : n < \omega\} = \bigcup_n I_n$ and the sequence $\langle I_0, \{l_0\}, I_1, \{l_1\}, \ldots\rangle$ follows a winning strategy for Player I in $G(\mathscr{I}, \mathscr{J})$, that is $\{k_n : n < \omega\} \in \mathscr{I}$ if and only if $\{l_n : n < \omega\} \notin \mathscr{J}$. Then we have that Player I will win this stage in $G'(\mathscr{I}, \mathscr{J})$.

On the other hand, let us assume that Player I has a winning strategy τ for $G'(\mathscr{I}, \mathscr{J})$. In step 0, Player I has to play $\{k_0\}$, where $k_0 = \tau(\emptyset)$, and in step $n+1$ Player I has to play $\{k_{n+1}\}$ where k_{n+1} is the answer given by

Player I in $G'(\mathscr{I}, \mathscr{J})$ following τ when Player II has played the l-th element l_{n+1} of J_k where $f(n+1) = \langle k, l\rangle$, if J_k has at least l elements, and 0 if not. Then, $\bigcup_n \{k_n\} \in \mathscr{I}$ if and only if $\{k_n : n < \omega\} \in \mathscr{I}$ if and only if $\bigcup_n J_n = \{0\} \cup \{l_n : n < \omega\} \notin \mathscr{J}$.

Analogously can be proved that Player II has a winning strategy for $G(\mathscr{I}, \mathscr{J})$ if and only if Player II has a winning strategy for $G'(\mathscr{I}, \mathscr{J})$. □

By previous theorem we can conclude that $\mathscr{I} \sqsubseteq \mathscr{J}$ if and only if $\tilde{\mathscr{I}} \leq_W \tilde{\mathscr{J}}$.

Lemma 4.1.5. *If $\mathscr{I} \leq_{MT} \mathscr{J}$ then $\mathscr{I} \sqsubseteq \mathscr{J}$.*

Proof. Let $f : \mathscr{I} \to \mathscr{J}$ a monotone Tukey function. Then Player II only has to answer $f(I_n)$ for any I_n given by Player I. If $\bigcup_n I_n \in \mathscr{I}$ then by monotonicity, $\bigcup_n f(I_n) \subseteq f(\bigcup_n I_n) \in \mathscr{J}$. If $\bigcup_n I_n \notin \mathscr{I}$ then by Tukeyness $\bigcup_n f(I_n) \notin \mathscr{J}$. □

We were interested in knowing how different monotone Tukey order and Tukey order are. In section 4.2 we will conclude those orders are quite different.

Let us note that on Borel ideals, the comparison game order is "almost" linear.

Lemma 4.1.6. *If $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{K}$ are Borel ideals, $\mathscr{I} \not\sqsubseteq \mathscr{J}$ and $\mathscr{J} \not\sqsubseteq \mathscr{K}$ then $\mathscr{K} \sqsubseteq \mathscr{I}$.*

Proof. Hypothesis means that Player I has a winning strategy in games $G(\mathscr{I}, \mathscr{J})$ and $G(\mathscr{J}, \mathscr{K})$. Then Player II is going to follow those strategies. First, in both games $G(\mathscr{I}, \mathscr{J})$ and $G(\mathscr{J}, \mathscr{K})$, Player I follows her own strategies, producing I_0 and J_0. Given the first choose K_0 of Player I in $G(\mathscr{K}, \mathscr{I})$, let us consider K_0 as the answer of Player II in $G(\mathscr{J}, \mathscr{K})$, and then let J_1 be the answer of Player I in the same game, given by her winning strategy. Let us consider J_1 as the answer of Player II in $G(\mathscr{I}, \mathscr{J})$ and let I_1 be the answer of Player I given by her winning strategy and then I_1 will be the answer of Player II in $G(\mathscr{K}, \mathscr{I})$. Let supposed that in step n, Player I chooses a set K_n and that set can be considered as the answer of Player II in $G(\mathscr{J}, \mathscr{K})$ for the sequence $\langle J_0, K_0, J_1, \ldots, J_n\rangle$, and then the winning strategy for Player I in this game makes her choose a set J_{n+1}. Such set J_{n+1} can be considered as the answer of Player II in $G(\mathscr{I}, \mathscr{J})$ for the sequence $\langle I_0, J_1, I_1, \ldots, I_n\rangle$ and then the winning strategy for Player I makes her choose a set I_{n+1}. Such set will be what Player II plays in $G(\mathscr{K},\mathscr{I})$ in step n. Hence, since the sequences $\langle J_0, K_0, J_1, K_1, \dots\rangle$ and $\langle I_0, J_1, I_1, J_2, \dots\rangle$ follow the winning strategies for Player I in $G(\mathscr{J},\mathscr{K})$ and $G(\mathscr{I},\mathscr{J})$ respectively, we have that $\bigcup_n J_n \in \mathscr{J}$ if and only if $\bigcup_n K_n \notin \mathscr{K}$, and $\bigcup_{n\geq 1} J_n \in \mathscr{J}$ if and only if $\bigcup_n I_n \notin \mathscr{I}$ and then we are done. □

An immediate consequence of the previous lemma is that if we have two incomparable ideals then every other ideal has the same order relation with both ideals of the incomparable pair.

Corollary 4.1.7. *Let $\mathscr{I}$ and $\mathscr{J}$ be two $\sqsubseteq$-incomparable ideals. Then, for any ideal $\mathscr{K}$ on ω, $\mathscr{K} \sqsubseteq \mathscr{I}$ iff $\mathscr{K} \sqsubseteq \mathscr{J}$ or $\mathscr{I} \sqsubseteq \mathscr{K}$ iff $\mathscr{J} \sqsubseteq \mathscr{K}$.* □

It is natural to ask if $\sqsubseteq$ is a linear order or moreover, if is a well order. Recall that comparison game can be seen as a game on integers which looks like a Wadge game.

Lemma 4.1.8. *If $\mathscr{I}$ is a Σ^0_α (respectively Π^0_α) ideal then $\tilde{\mathscr{I}}$ is a $\Sigma^0_{\alpha+1}$ (resp $\Pi^0_{\alpha+1}$) set.*

Proof. Let us define a function $rng_n : \omega^\omega \to \mathcal{P}(\omega)$ by $rng_n(x) = \{x(k) : k < n\}$ for all $x \in \omega^\omega$. Note that rng_n is a continuous function and $rng(x) = \lim_{n\to\infty} rng_n(x)$ for all $x \in \omega^\omega$. Hence, preimages under rng of clopen sets are Δ^0_2 sets, and inductively we can get the result. □

Previous lemma shows that equivalent classes modulus comparison game order could contain ideals with different complexities, but not too different; i.e., if $\mathscr{I} \simeq \mathscr{J}$ then complexities of $\mathscr{I}$ and $\mathscr{J}$ differ at most by 1. Another consequence is that comparison game order is at least as long as Borel hierarchy.

Corollary 4.1.9.

- *The game $G(\mathscr{I},\mathscr{J})$ is determined for every pair $\mathscr{I},\mathscr{J}$ of Borel ideals.*
- *The order $\sqsubseteq$ is well-founded.*
- *The equivalence classes of $\simeq$ are unions of "intervals" of Wadge degrees of ideal.*
- *There are uncountable many $\simeq$-classes.*

Another consequence is that $\mathscr{I} \leq_{MT} \mathscr{J}$ implies that the complexity of $\mathscr{I}$ is not so different with the complexity of $\mathscr{J}$, while there is an F_σ ideal $\mathscr{I}_{max}$ which is maximal in Tukey order, since classes of monotone Tukey order are included in classes of comparison game equivalence. Such ideal $\mathscr{I}_{max}$ is defined as the ideal on $2^{<\omega}$ generated by all sets $\{f \restriction n : n < \omega\}$ with $f \in 2^\omega$, making monotone Tukey order look very different with Tukey order.

Question 4.1.10. *Is the order $\sqsubseteq$ linear (a well order)?*

4.2 F_σ-ideals in the comparison game order

About the ideal **Fin** we have that

Lemma 4.2.1. *Let $\mathscr{J}$ be an ideal on ω. Then* **Fin** $\sqsubseteq \mathscr{J}$.

Proof. A winning strategy for Player II in $G(\mathbf{Fin}, \mathscr{J})$ is the following. Player II has to answer the initial interval $J_n = \left[0, \max\left(\bigcup_{i\leq n} I_i\right)\right]$, given that $I_i, (i \leq n)$ are the finite sets played by Player I until step n. Then, $\bigcup_n I_n \in \mathbf{Fin}$ implies $\bigcup_n J_n$ is a finite set and then an element on $\mathscr{J}$. On the other hand, if $\bigcup_n I_n \notin \mathbf{Fin}$ then $\bigcup_n J_n = \omega \in \mathscr{J}^+$. □

A criterion for equivalence with the ideal **Fin** is the following result.

Lemma 4.2.2. *If $\mathscr{I}$ is an ideal on ω then $\mathscr{I} \sqsubseteq$* **Fin** *if and only if Player II has a winning strategy in the game $G''(\mathscr{I})$ defined as follows: In step n, Player I chooses an element I_n of $\mathscr{I}$ and Player II chooses a natural number k_n. Player II wins if $\bigcup_n I_n \in \mathscr{I}$ if and only if the sequence $\{k_n : n < \omega\}$ is bounded.*

Proof. If Player II has a winning strategy for $G(\mathscr{I}, \mathbf{Fin})$ then in step n Player II of $G''(\mathscr{I})$ has to play $k_n = \max J_n$ where J_n is the finite set played by Player II following a fixed winning strategy for her in $G(\mathscr{I}, \mathbf{Fin})$, under the same sets played by Player I. On the other hand, the winning strategy for Player II in $G(\mathscr{I}, \mathbf{Fin})$ consists in to play $\{k_n\}$ in step n, where k_n is the answer given in step n for a fixed winning strategy for Player II in $G''(\mathscr{I})$. □

By Mazur's theorem 1.3.2, **Fin** can win when he plays vs. any F_σ-ideal.

Lemma 4.2.3. *If $\mathscr{I}$ is an F_σ-ideal then $\mathscr{I} \sqsubseteq$* **Fin**.

Proof. By Mazur's theorem, there is a lower semicontinuous submeasure φ such that $\mathscr{I} = Fin(\varphi)$. Let us play the game $G''(\mathscr{I})$. Then in step n Player II has to play k_n, the minimal $k \in \omega$ such that $\varphi(\bigcup_{j \leq n} I_j) < k$. Then $\varphi(\bigcup_n I_n) < \infty$ if and only if $\{k_n : n < \omega\}$ is bounded. □

Recall that in 3.2.11 we have proved that $\mathscr{I}$ is an F_σ-ideal if and only if $\mathscr{I}$ is a P$^+$-tree ideal, for any Borel ideal $\mathscr{I}$.

Lemma 4.2.4. *If $\mathscr{I}$ is not a P$^+$(tree) ideal then Player I has a winning strategy for $G''(\mathscr{I})$.*

Proof. Let T be an $\mathscr{I}^+$-tree of finite sets with all branches in $\mathscr{I}$. In her first steps, Player I has to play with the ordered elements of $\bigcup succ_T(\emptyset)$ until Player II increases her answers. If in step n, Player II chooses a number bigger than all of her previous plays then Player II collects the (finite) set F_0 of answers given until that step and then he begins taking the elements of $succ_T(F_0)$ in its order and he will not move of that positive set unless Player II increases her numbers picked. Hence, if eventually Player II does not increase her picks then Player I will choose every element of $succ_T(t)$ for some $t \in T$ and then he will collect an $\mathscr{I}$-positive set. In the other case Player II will collect a set which follows a branch of T and then its union will be in $\mathscr{I}$. □

Theorem 4.2.5. *For any Borel ideal $\mathscr{I}$, $\mathscr{I} \simeq$ **Fin** if and only if $\mathscr{I}$ is F_σ.*

Proof. It follows from two facts: if $\mathscr{I}$ is a Borel ideal then $G''(\mathscr{I})$ is determined, and by theorem 3.2.11, $\mathscr{J}$ is a P$^+$(*tree*)-ideal if and only if $\mathscr{J}$ is an F_σ-ideal, for all Borel ideal $\mathscr{J}$. □

It is well known that $\mathcal{Z} \leq_T \mathscr{I}_{\frac{1}{n}}$ and since $\mathcal{Z}$ is not an F_σ-ideal, we have that $\mathcal{Z} \not\sqsubseteq \mathscr{I}_{\frac{1}{n}}$ and by lemma 4.1.5, $\mathcal{Z} \not\leq_{MT} \mathscr{I}_{\frac{1}{n}}$, proving that Tukey order and monotone Tukey order are different orders.

4.3 $F_{\sigma\delta}$-ideals in the Comparison Game Order

The ideal $\mathscr{I}_0$ was defined as the minimal ideal $\mathscr{I}$ such that there is an $\mathscr{I}^+$-tree of finite sets which does not have an $\mathscr{I}$-positive branch (see 3.2.12). Let us denote $A_f = \{f \restriction n : n < \omega\}$ for a given $f \in 2^\omega$.

Theorem 4.3.1. *If $\mathscr{I}$ is not an F_σ ideal then $\mathscr{I}_0 \sqsubseteq \mathscr{I}$.*

Proof. By Kechris-Louveau-Woodin theorem 3.2.10 there is a Cantor set $C \subseteq \mathcal{P}(\omega)$ such that $D = C \setminus \mathscr{I}$ is countable dense in C. Let $T \subseteq 2^{<\omega}$ be a perfect tree such that $[T] = C$. On the other hand, there is a homeomorphism $\varphi : 2^\omega \to C$ such that if $F = \{f \in 2^\omega : (\forall^\infty n) f(n) = 0\}$ then $\varphi'' F = D$, since F is a countable dense subset of 2^ω. Such φ induces an embedding $\Phi : 2^{<\omega} \to [\omega]^{<\omega}$ which is monotone (i.e. $s \subseteq t$ implies $\Phi(s) \subseteq \Phi(t)$) and such that $\bigcup_n \Phi(f \restriction n) \in \mathscr{I}$ if and only if f is not eventually zero. Now we will describe a winning strategy for Player II in $G(\mathscr{I}_0, \mathscr{I})$. In step n, if Player I plays $I_n \in \mathscr{I}_0$ then Player II plays $J_n = [0, k_n] \cup \bigcup\{\Phi(s) : (\exists k \leq n)(\exists t \in I_k)(s \subseteq t)\}$, where k_n is the maximal cardinality of an antichain in $\bigcup_{k \leq n} I_k$. We argue why is this a winning strategy for Player II. If $I = \bigcup_n I_n \in \mathscr{I}_0$ then there are $m < \omega$ and $f_0, \dots, f_m \in 2^\omega \setminus F$ such that $I \subseteq \bigcup_{j \leq m} A_{f_j}$. Then m is an upper bound for k_n and $\bigcup\{\Phi(s) : (\exists k < \omega)(\exists t \in I_k)(s \subseteq t)\} \subseteq \bigcup_{j \leq m} \bigcup_n \Phi(f_j \restriction n) \in \mathscr{I}$, and then $\bigcup_n J_n \in \mathscr{I}$. On the other hand, if $I \notin \mathscr{I}_0$ then either $\langle k_n : n < \omega \rangle$ is unbounded, and then $J = \bigcup_n J_n \notin \mathscr{I}$, or there is an eventually zero function f such that $f \restriction n \in I$ for infinitely many $n < \omega$, and in that case,

$$\bigcup_n \{\Phi(s) : (\exists t \in I_n) s \subseteq t\} \supseteq \bigcup_n \{\Phi(f \restriction n) : n < \omega\} \notin \mathscr{I}.$$

□

Borel complexity of $\mathscr{I}_0$ is $F_{\sigma\delta}$. The family F of all sets which can be covered by finitely many branches is an F_σ-set and the family F' of all sets which have finite intersection with all eventually zero branches is an $F_{\sigma\delta}$-set, and $\mathscr{I}_0 = F \cap F'$.

We know that $\mathscr{I}_0 \sqsubseteq \emptyset \times \mathbf{Fin}$, since $\emptyset \times \mathbf{Fin}$ is not an F_σ-ideal but actually such inequality is strict.

Theorem 4.3.2.

$$\emptyset \times \mathbf{Fin} \not\sqsubseteq \mathscr{I}_0.$$

Proof. For every $1 \leq n < \omega$ we define a game G_n as follows. In step k, Player I picks a finite subset I_k of $\omega \times \omega$ and Player II picks an antichain J_k of cardinality n in $\mathscr{I}_0$, and such that for all $i < k$ and all t in J_i there is a unique $s \in J_k$ such that $s \supseteq t$. Player II wins if $\bigcup_n I_n \in \emptyset \times \mathbf{Fin}$ if and only if $\bigcup_n J_n \in \mathscr{I}_0$. Inductively, we will prove that Player I has a winning strategy in game G_n, for all n, but now we are going to prove how this fact implies that Player I has a winning strategy for $G(\emptyset \times \mathbf{Fin}, \mathscr{I}_0)$ by describing it.

Let $\{X_r : r < \omega\}$ be a partition of $\omega \setminus \{0\}$ in infinite sets. The main idea is based in the following trick: Player I is going to play the game G_n but in $X_n \times \omega$ instead of $\omega \times \omega$. In step 0, Player I plays $\emptyset$ and in step $k > 0$, define $N(k) = \min\{\sum h(l) : h \in J_k \wedge l \in dom(h)\}$ and let $M(k)$ be the maximal cardinality of an antichain in $\bigcup_{i<k} J_i$. If $M(k) = M(k-1)$ then Player I has to play the game $G_{M(k)}$ in $X_{M(k-1)} \times \omega$ instead of $\omega \times \omega$, and if $M(k) > M(k-1)$, then Player I has to abandon what he has played and begin a new game of $G_{M(k)}$ inside the copy $X_{M(k)} \times \omega$, and in both cases, Player I has to add $\{\min X_{M(k)}\} \times N(k)$ to the sets defined above.

If Player II makes $M(k)$ increase in infinitely many steps, then $\bigcup_n J_n \notin \mathscr{I}_0$, but Player I will abandon all pieces where he played, and the $\bigcup_n I_n \in \emptyset \times \mathbf{Fin}$.

If there is K such that $M(k) = M(K)$ for all $k > K$ then the winning strategy for Player I in $G_{M(K)}$ makes Player I win in $G(\emptyset \times \omega, \mathscr{I}_0)$.

Now, we prove that Player I has a winning strategy for the game G_1. In step 0, Player I plays $\{(0,0)\}$. In step k, Player I just play a doubletons with the form $\{(0, N(k)), (n_k, m_k)\}$, as we will define. Let $N(k)$ be defined as few lines above, and then in step k Player I has to play $\{0\} \times N(k) \bigcup \{(n_{K-1}, m_{K-1}+1)\}$ if $J_k \supsetneq J_{k-1}$ and there is $m \in domJ_k \setminus domJ_{k-1}$ such that $J_k(m) = 1$; and $\{0\} \times N(k) \bigcup \{(n_{K-1}+1, m_{K-1})\}$ otherwise. If Player II plays an infinite set then he will play along a branch and then Player I know that he has won because he just will fill the column $\{0\} \times \omega$. Let us assume that Player II plays finite sets only. Then if there is K such that $J_k = J_K$ for all $k \geq K$ then $\bigcup_n J_n \in \mathscr{I}_0$ but Player I will fill the column $\{m_K\} \times (\omega \setminus n_K)$ for K minimal; and if Player II increases the length of J_k for infinitely many steps k then, if there is K such that the increasing of J_k is just with 0's then column $\{0\} \times N(k)$ will not increase and playing of Player I will follow a row; but if Player II increases the length of J_k and he adds a new 1 in infinitely many steps then Player I will make the column $\{0\} \times N(k)$ increase to $\{0\} \times \omega$ and then $\bigcup_n I_n \notin \emptyset \times \mathbf{Fin}$.

Inductively assume that Player I has a winning strategy for G_n and let us prove that he has a winning strategy for G_{n+1}. Fix a partition $\{X_i^j : j \leq n \wedge i < \omega\}$ of $\omega \setminus \{0\}$. In step 0, Player I plays $\emptyset$ and then assume that Player II has been played with an antichain J_k with cardinality $n+1$. Let us enumerate this antichain as $\{a_r^0 : r \leq n\}$ and for each $r \leq n$, we enumerate $J_k = \{a_r^k : r \leq n\}$ in such way that $a_r^k \supseteq a_r^0$ for all $r \leq n$. Then, Player I will play simultaneously the game G_n in $X_i^r \times \omega$ for some i (depending of k and r), where answers of Player I are given by the winning strategy for her when Player II plays $J_k \setminus a_r^k$; and following this rule: If $a_r^k \supsetneq a_r^{k-1}$ and Player I is playing in the copy $X_i^r \times \omega$ then she abandon this copy and begins playing G_n in $X_{i+1}^r \times \omega$ and if not, he stills playing in the same $X_i^r \times \omega$, i.e., $i(k,r) = i(k-1,r)$. In both cases Player I adds the column $\{0\} \times N(k)$. Now we will prove that this is a winning strategy for Player I.

If all the sequences a_r^k are eventually increasing then we have two cases:

(1) For each $k \leq n$ the sequence $\bigcup_r a_r^k$ is not eventually-zero. Then, Player I will increase the column $\{0\} \times N(k)$ to $\{0\} \times \omega$, making $\bigcup_n J_n \in \emptyset \times \mathbf{Fin}^+$.

(2) There is $k \leq n$ such that $\bigcup_r a_r^k$ is an eventually-zero branch. Then, the column $\{0\} \times N(k)$ will not increase and in all the pieces of the partition will be played the game G_n and since all increase, all pieces are abandoned and then, $\bigcup_n J_n \in \emptyset \times \mathbf{Fin}$.

If for some k, the sequence a_r^k does not increase then Player I will be playing the game G_n and since she has a winning strategy for this game, we are done, because the column $\{0\} \times N(k)$ will not increase. □

Now we give a criterion for ideals $\sqsubseteq$-below $\emptyset \times \mathbf{Fin}$.

Lemma 4.3.3. *Let $\mathscr{I}$ be an ideal on ω. Then $\mathscr{I} \sqsubseteq \emptyset \times \mathbf{Fin}$ if and only if Player II has a winning strategy for the following game $G'''(\mathscr{I})$: In step n, Player I chooses an element I_n of $\mathscr{I}$ and then Player II chooses an increasing function $f_n \in \omega^\omega$. Player II wins if $\bigcup_n I_n \in \mathscr{I}$ if and only if the sequence $\{f_n : n < \omega\}$ is bounded.*

Proof. Let us assume that Player II has a winning strategy σ for $G(\mathscr{I}, \emptyset \times \mathbf{Fin})$. For every element $J \in \emptyset \times \mathbf{Fin}$, let $f_J : \omega \to \omega$ given by $f_J(n) = \min\{k > f_J(n-1) : (\forall m > k)\ (n,m) \notin J\}$. Then we describe a winning strategy for Player II in $G'''(\mathscr{I})$ as follows: Given $I_0 \in \mathscr{I}$, let f_0 be the function $f_{\sigma(I_0)}$. Assume that the legal position $\langle I_0, f_0, \dots, I_n, f_n\rangle$ follows the strategy which we are defining. Then in parallel we have a legal position $\langle I_0, J_0, \dots, I_n, J_n\rangle$ of $G(\mathscr{I}, \emptyset \times \mathbf{Fin})$ following σ. Then, given I_{n+1}, define $J_{n+1} = \sigma(\langle I_0, J_0, \dots, I_n, J_n, I_{n+1}\rangle)$ and the function $f_{n+1} = f_{J_{n+1}}$. It is easy to check that this is a winning strategy for Player II in $G'''(\mathscr{I})$. On the other hand, for any function $f \in \omega^\omega$ define $J_f = \{(n,m) \in \omega \times \omega : m \leq f(n)\}$. Analogous to first part, Player II in $G(\mathscr{I}, \emptyset \times \mathbf{Fin})$ has to play with J_f where f is the answer given by Player II in $G'''(\mathscr{I})$. □

Ilijas Farah asked in [14] if every $F_{\sigma\delta}$-ideal $\mathscr{I}$ satisfies that there is a family of compact hereditary sets $\{C_n : n < \omega\}$ such that

$$\mathscr{I} = \{A \subseteq \omega : (\forall n < \omega)(\exists m < \omega)(A \setminus [0,m) \in C_n)\}.$$

We will say $\mathscr{I}$ is a *Farah ideal* if $\mathscr{I}$ fulfils that property. Note that every Farah ideal $\mathscr{I}$ is an $F_{\sigma\delta}$ ideal. We have a characterization of Farah ideals by using F_σ hereditary and closed under finite changes sets.

Theorem 4.3.4. *Let $\mathscr{I}$ be an ideal on ω. Then, $\mathscr{I}$ is Farah if and only if there is a sequence $\{F_n : n < \omega\}$ of hereditary and closed under finite changes F_σ-sets such that $\mathscr{I} = \bigcap_n F_n$.*

Proof. Let $\langle C_n : n < \omega\rangle$ be a family of compact hereditary sets such that $\mathscr{I} = \{A \subseteq \omega : (\forall n)(\exists k)(A \setminus k \in C_n)\}$. For any n, define F_n as the closure of C_n under finite changes. It is clear that F_n is hereditary, closed under finite changes, F_σ and containing $\mathscr{I}$. If $A \in F_n$ then there is a finite set F such that $A \triangle F \in C_n$ and by taking and an adequate $k > \max(F)$ we have that $A \setminus k \in C_n$.

Now, let $\{F_n : n < \omega\}$ be an increasing sequence of hereditary and closed under finite changes F_σ-sets such that $\mathscr{I} = \bigcap_n F_n$. Let us write $F_n = \bigcup_k E^n_k$ where $\langle E^n_k : k < \omega\rangle$ is an increasing sequence of closed sets. We can assume that each E^n_k is a hereditary sets, and we can define

$$\tilde{E}^n_k = \{A \setminus (k+1) \cup \{k\} : A \in E^n_k\}$$

and $C_n = \{\emptyset\} \cup \bigcup_k \tilde{E}^n_k$. Note that each C_n is a closed hereditary set, and if $A \setminus k \in C_n$ we can assume $k \in A$ and then $A \in \tilde{E}^n_k \subseteq F_n$ for all n. Finally, if A is an infinite set in $\mathscr{I}$ (the finite case is trivial) then for each n take k such that $A \setminus k \in E_k$ and $k \in E_k$ (this is possible since the E^n_k is an increasing family). Hence $A \setminus k \in C_n$. □

Corollary 4.3.5. *The ideal* nwd *is Farah.*

Proof. Let $\{U_n : n < \omega\}$ be a base of the topology of $\mathbb{Q}$, and define $F_n = \{A \subseteq \mathbb{Q} : (\exists m)(U_m \subseteq U_n \wedge A \cap U_m = \emptyset)\}$. Note that $\mathsf{nwd} = \bigcap_n F_n$ and each F_n is F_σ hereditary and closed under finite changes. □

We have refined theorem 4.3.4 as the following shows.

Theorem 4.3.6. *Let $\mathscr{I}$ be an ideal on ω. Then, $\mathscr{I}$ is Farah if and only if there is a sequence $\{F_n : n < \omega\}$ of F_σ closed under finite changes sets such that $\mathscr{I} = \bigcap_n F_n$.*

Proof. Without loss of generality, we can assume that all F_n are meager, because if F_n is nonmeager then there is a non-empty clopen set contained in F_n and by closedness under finite changes, $F_n = 2^\omega$.

Sufficiency is a consequence of theorem 4.3.4, and by the same theorem, it will be enough to prove that if F is a meager F_σ closed under finite changes set then there is an F_σ hereditary set E such that $\mathscr{I} \subseteq E \subseteq F$. Let us consider the game H defined such that in step k, Player I chooses a set $B_k \notin F$ and Player II picks a finite subset a_k of B_k. Player I wins if $\bigcup_k a_k \in A$. Note that H is determined since A is Borel.

Claim. *Player II has a winning strategy in H.*

Proof of claim. Let $\{E_n : n < \omega\}$ be an increasing sequence of closed sets such that $F = \bigcup_n E_n$ and for each n, let T_n be a pruned tree such that $E_n = [T_n]$. Since each E_n is a nowhere dense set, in step k, if Player I plays B_k then there is $m_k < \omega$ such that $m_{k-1} < m_k$ ($m_{-1} = 0$) and $\chi_{B_k} \restriction m_k \notin T_k$. Then, Player II has to play $a_k = B_k \cap m_k$. It is clear that $\bigcup_k a_k \notin F$ and then $\bigcup_k a_k \notin \mathscr{I}$. □

It is very easy to see that

Claim. *Player II has a winning strategy in H if there is a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ such that (a) for all $A \notin F$ and all $t \in T$ there is $a \in succ_T(t)$ such that $a \subseteq A$ and (b) $\bigcup_n f(n) \in \mathscr{I}^+$ for all $f \in [T]$.*

Hence, by defining $C_t = \{A \subseteq \omega : (\forall a \in succ_T(t))(a \nsubseteq A)\}$, for all $t \in T$, we have immediately that C_t is closed and hereditary and $\mathscr{I} \subseteq C_t$ (details can be checked in proof of 2.8.3). Finally, (a) is the contrapositive of $\bigcup_{t\in T} C_t \subseteq F$. Hence, $\bigcup_t C_t$ is the F_σ hereditary set required. □

Question 4.3.7. *Given an $F_{\sigma\delta}$-ideal $\mathscr{I}$ and an F_σ-set such that $\mathscr{I} \subseteq F$, is there a hereditary F_σ-set E such that $\mathscr{I} \subseteq E \subseteq F$?*

Note that an affirmative answer implies each $F_{\sigma\delta}$ is a Farah ideal.

By theorem 4.3.4 it is clear that any Farah ideal fulfils the following definition.

Definition 4.3.8. An ideal $\mathscr{I}$ is *weakly Farah* if there is a sequence $\langle F_n : n < \omega\rangle$ of F_σ and hereditary sets such that $\mathscr{I} = \bigcap_n F_n$.

Without loss of generality, the sequence in previous definition is increasing, and it is clear that any weakly Farah ideal is $F_{\sigma\delta}$. Weakly Farah ideals are below $\emptyset \times \mathbf{Fin}$ in the comparison game order.

Theorem 4.3.9. *If $\mathscr{I}$ is a weakly Farah ideal then $\mathscr{I} \sqsubseteq \emptyset \times \mathbf{Fin}$.*

Proof. Let $\{F_n : n < \omega\}$ be a family of hereditary F_σ sets such that $\mathscr{I} = \bigcap_n F_n$. Without loss of generality, we can assume that for any n, $F_n = \bigcup_k E^n_k$ where each E^n_k is a closed hereditary set. Then, for any $A \subseteq \omega$

$$A \in \mathscr{I} \quad \text{if and only if} \quad (\exists f_A \in \omega^\omega)(\forall n < \omega)(A \notin E^n_k \leftrightarrow k < f_A(n)). \quad (4.1)$$

Hence, playing the game $G'''(\mathscr{I})$, for any step n, Player II has to play $f_{\bigcup_{j<n} I_j}$. So, if $I = \bigcup_n I_n \in \mathscr{I}$ then f_I bounds all the f_{I_n} functions; and if $I \notin \mathscr{I}$ then there is k such that $I \notin E^k_j$ for all $j < \omega$ and then, $\langle f_{I_n}(k) : n < \omega \rangle$ increases to infinity, because in other case, there were j such that $I_n \in E^k_j$ for all n and $I \notin E^k_j$, contradicting the closedness of E^j_k. $\square$

A positive answer to Farah's question would imply that every $F_{\sigma\delta}$ ideal is $\sqsubseteq$ below $\emptyset \times \mathbf{Fin}$.

Note that by Solecki's theorem, any analytic P-ideal is a Farah ideal.

Corollary 4.3.10. *If $\mathscr{I}$ is an analytic P-ideal then $\mathscr{I} \sqsubseteq \emptyset \times \mathbf{Fin}$.* $\square$

Theorem 4.3.11. *The following ideals on ω are comparison game equivalent:*

(1) $\mathcal{Z}$,

(2) nwd and

(3) $\emptyset \times \mathbf{Fin}$.

Proof. (1) $\sqsubseteq$ (3) use $\mathcal{Z}$ is an analytic P-ideal.

(2) $\sqsubseteq$ (3) use nwd is a Farah ideal.

(3) $\sqsubseteq$ (1). Let g be a bijective function from $\omega \times \omega$ on to ω such that $g(n, m) \geq \max\{n, m\}$. Given $n, m \in \omega$, we define $I_{(j,k)} = [2^{g(j,k)}, 2^{g(j,k)+1})$. Given a function $f \in \omega^\omega$ and $k \in \omega$, we define $B^f_{(j,k)} \subseteq I_{(j,k)}$ as the interval $[2^{g(j,k)}, 2^{g(j,k)}+2^{g(j,k)-k})$ if $j \leq f(k)$, and the interval $[2^{g(j,k)}, 2^{g(j,k)}+2^{g(j,k)-k-j})$ if $j > f(k)$. Finally, we define $B_f = \bigcup_{j,k<\omega} B^f_{(j,k)}$. For any $I \in \emptyset \times \mathbf{Fin}$, let us take f_I as we defined in proof of lemma 4.3.3, and now we describe the winning strategy for Player II in $G(\emptyset \times \mathbf{Fin}, \mathcal{Z})$. In step n, if $I_0, \ldots I_n$ are the

elements of $\emptyset \times \mathbf{Fin}$ played by Player I, then define $I'_n = \bigcup_{l \leq n} I_l$ and define $f_n = f_{I'_n}$. Player II has to play B_{f_n}. Now we will argue why is this a winning strategy for Player II. If $I = \bigcup_n I_n \in \mathscr{I}$ then $\{f_n : n < \omega\}$ is bounded by a function f, and then, given $\varepsilon > 0$ we can find $N = \max\{g(j,k) : 2^{-k} \geq \varepsilon \wedge j \leq f(k)\}$. Then, note that for all $m \geq N$, $|I \cap [2^m, 2^{m+1})| \leq 2^{m-N} < 2^m\varepsilon$. On the other hand, if $I \notin \mathscr{I}$ then there is $K < \omega$ such that $\{f_n(K) : n < \omega\}$ is not bounded and then $|I \cap I_{j,K}| = 2^{g(j,K)-K} = 2^{-K}|I_{(j,K)}|$ for all $j < \omega$, and then $I \notin \mathcal{Z}$.

(3) $\sqsubseteq$ (2) Let $\{V_n : n < \omega\}$ be a sequence of pairwise disjoint open subsets of $\mathbb{Q}$ and for each n, let $\{q^n_k : k < \omega\}$ be an enumeration of V_n. Let us play the $G(\emptyset \times \mathbf{Fin}, \mathsf{nwd})$ game. In step n, if Player I has played $I_n \in \times\mathbf{Fin}$, take a function $f \in \omega^\omega$ such that for all k, m, $(k,m) \in I_n$ implies $m \leq f(k)$, and then Player II must play $J_n = \{q^k_s : s < g(k) \wedge k < \omega\}$. J_n is a nowhere dense subset of $\mathbb{Q}$ since it intersects each V_n in a finite set, and if $I = \bigcup_n I_n \in \emptyset \times \mathbf{Fin}$ then $J = \bigcup_n J_n$ intersects each V_n in a finite set, and then, $J \in \mathsf{nwd}$; and if for some k, $I \cap (\{k\} \times \omega)$ is infinite, then J will contain V_k, and then $J \in \mathsf{nwd}^+$. □

Concerning to analytic P-ideals, we have noted that every one of them is either equivalent with $\mathbf{Fin}$ (i.e., is F_σ) or equivalent with $\emptyset \times \mathbf{Fin}$. Then the class of P-ideals skips the intermediate class of $\mathscr{I}_0$.

Theorem 4.3.12. *Let $\mathscr{I}$ be an analytic P-ideal. Then either $\mathscr{I} \simeq \mathbf{Fin}$ or $\mathscr{I} \simeq \emptyset \times \mathbf{Fin}$.*

Proof. Let φ be a lscsm such that $\mathscr{I} = Exh(\varphi)$. We will consider two cases:
Case 1. There is $\varepsilon > 0$ such that for any set X, $\varphi(X) < \varepsilon$ implies $X \in \mathscr{I}$. Note than in such case $\mathscr{I}$ is an F_σ ideal, because $C = \{A \subseteq \omega : \varphi(X) \leq \varepsilon\}$ is a closed set and $\mathscr{I} = \bigcup_n \{A \subseteq \omega : A \setminus n \in C\}$.
Case 2. For all $\varepsilon > 0$ there is an $\mathscr{I}$-positive set X such that $\varphi(X) < \varepsilon$. Take a family Y_n of $\mathscr{I}$-positive sets such that $\varphi(Y_n) \leq 2^{-n}$ and by the Disjoint Refinement Lemma for hereditary meagre ideals 1.2.5, there is a disjoint family of positive sets $\{X_n : n < \omega\}$ such that $\varphi(X_n) \leq 2^{-n}$. Let $\{x^n_k : k < \omega\}$ be an enumeration of X_n. Let us describe a winning strategy for Player II in $G(\emptyset \times \mathbf{Fin}, \mathscr{I})$. In step n, if Player I plays I_n, we consider the function f_n given by $f_n(k) = \max\{0\} \cup \{j : (\exists l \leq n)((k,j) \in I_l)\}$ and then Player II has to play $J_n = \{x^k_j : j \leq f_n(k)\}$. Hence, if $I = \bigcup_n I_n \in \mathscr{I}$ then the family $\langle f_n : n < \omega\rangle$ is bounded by f and then $J = \bigcup_n J_n$ intersects each X_n in a finite set F_n which has submeasure smaller than 2^{-n} and so, J

is a φ-exhaustive set. On the other hand, if $I \notin \emptyset \times \mathbf{Fin}$ then there is k such that $f_n(k)$ increases to infinity, and so, $J \cap X_k = X_k \in \mathscr{I}^+$. $\square$

Finally, we will show that the ideal $\mathbf{Fin} \times \mathbf{Fin}$ belongs a higher class than $\emptyset \times \mathbf{Fin}$. It is very easy to see that $\emptyset \times \mathbf{Fin} \sqsubseteq \mathbf{Fin} \times \mathbf{Fin}$.

Proposition 4.3.13. $\emptyset \times \mathbf{Fin} \sqsubseteq \mathbf{Fin} \times \mathbf{Fin}$.

Proof. Let $\{X_n : n < \omega\}$ be an infinite partition of ω in infinite pieces. Given I in $\emptyset \times \mathbf{Fin}$, we define another element J_I of $\emptyset \times \mathbf{Fin}$ by

$$J_I = \{(k,l) : (\exists n < \omega)(k \in X_n \wedge (n,l) \in I)\}.$$

The winning strategy for Player II consists in play J_{I_n} for a set I_n played by Player I in step n. If $I = \bigcup_n I_n \in \emptyset \times \mathbf{Fin}$ then $J = \bigcup_n J_{I_n} \in \emptyset \times \mathbf{Fin}$, and if for some $k < \omega$, $I \cap \{k\} \times \omega$ is infinite then $J \cap \{l\} \times \omega$ will be infinite for all $l \in X_k$, and then $J \notin \mathbf{Fin} \times \mathbf{Fin}$. $\square$

Theorem 4.3.14. $\mathbf{Fin} \times \mathbf{Fin} \not\sqsubseteq \emptyset \times \mathbf{Fin}$.

Proof. We will describe a winning strategy for Player I in $G'''(\mathbf{Fin} \times \mathbf{Fin})$. Without loss of generality, we can assume that Player II plays in such way that $f_k(n) \geq f_{k-1}(n)$ for all n. First, take an infinite partition $\{X_n : n < \omega\}$ of ω in infinite pieces, and let $\{x_r^n : r < \omega\}$ be an enumeration of X_n. Player I will play just with selectors of the family $\{X_n \times \omega : n < \omega\}$. In step 0, Player I plays $(0,0)$. In step k, if $f_k = f_{k-1}$ ($f_{-1} \equiv 0$) and $J_{k-1} = \{(x_r^n, m_r^n) : r < \omega\}$ then $J_{k+1} = \{(x_r^n, m_r^n + 1) : r < \omega\}$, and otherwise, if $l = min\{n : f_k(n) > f_{k-1}(n)$ then $J_{k+1} = \{(x_r^n, m_r^n + 1) : r \leq l\} \cup \{(x_{r+1}^n, m_r^n) : r > l\}$.

If there is N such that $\{f_k(N) : k < \omega\}$ increases in infinitely steps then $\bigcup_n J_n \in \mathscr{I}$ since all but finitely many pieces X_r are turning to the right in infinitely many steps and if $\{f_k : k < \omega\}$ is bounded by a function f then eventually in all the pieces Player I will be adding point in a same column, making $\bigcup_n J_n \notin \mathbf{Fin} \times \mathbf{Fin}$. $\square$

4.4 Questions

Our principal questions about Katětov order are:

Question 4.4.1. *Is there a $\leq_K$-minimal ideal $\mathscr{I}$ among Borel tall ideals?*

Question 4.4.2. *Is there a locally $\leq_K$-minimal ideal $\mathscr{I}$ among Borel tall ideals?*

Question 4.4.3. *Is $\mathcal{R}$ locally $\leq_K$-minimal among Borel tall ideals?*

Or in the contrary sense,

Question 4.4.4. *Is there a Borel tall ideal $\mathscr{I}$ on ω satisfying*

$$\mathscr{I}^+ \longrightarrow (\mathscr{I}^+)^2_2\ ?$$

Question 4.4.5. *Is there a decreasing (increasing) chain of length $\mathfrak{c}$ in Katětov order among Borel ideals?*

Question 4.4.6. *Is it true that for every Borel ideal $\mathscr{I}$ either there is an $\mathscr{I}$-positive set X such that $\mathsf{conv} \leq_K \mathscr{I} \restriction X$ or there is an F_σ-ideal $\mathscr{J}$ containing $\mathscr{I}$.*

A result of M. Laczkovich and I. Recław [34] shows that for every ideal $\mathscr{I}$ either $\mathscr{I} \geq_K \mathbf{Fin} \times \mathbf{Fin}$ or there is an F_σ set E such that $\mathscr{I} \subseteq E$ and $E \cap \mathscr{I}^* = \emptyset$. We wish to know if F_σ hypothesis could be weaken in order to replace "set" with "ideal".

Question 4.4.7. *Must every ideal $\mathscr{I}$ satisfy $\mathscr{I} \geq_K \mathbf{Fin} \times \mathbf{Fin}$ or satisfy that there is an $F_{\sigma\delta}-$ideal $\mathscr{J}$ such that $\mathscr{I} \subseteq \mathscr{J}$?*

About F_σ and P^+ ideals, in example 3.2.12 we have a Borel P^+ ideal which is not an F_σ-ideal, but such ideal is not tall.

Question 4.4.8. *Does every Borel tall ideal $\mathscr{I}$ contain an F_σ tall ideal?*

Question 4.4.9. *Is there a Borel tall P^+-ideal which is not an F_σ-ideal?*

Question 4.4.10. *Does Measure dichotomy hold for F_σ-ideals?*

About cardinal invariants of the continuum we have the following questions

Question 4.4.11. *Is $\mathsf{non}^*(\mathcal{G}_{fc}) = \mathfrak{r}_2$?*

Question 4.4.12. *Is consistently strict some inequality of $\mathfrak{r}_2 \leq \mathsf{non}^*(\mathcal{G}_{fc}) \leq \mathfrak{r}$?*

Question 4.4.13. *Is* $\mathsf{cov}^*(\mathcal{G}_c) = \min\{\mathfrak{b}, \mathfrak{s}_2\}$*?*

Question 4.4.14. *Is some inequality consistently strict in* $\mathfrak{par}_2 \leq \mathsf{cov}^*(\mathcal{G}_c) \leq \mathfrak{b}, \mathfrak{s}_2$*?*

Concerning with the comparison game order we have the following questions.

Question 4.4.15. *Is there a pair of incomparable Borel ideals in comparison game?*

Question 4.4.16. *Is there a Borel ideal* $\mathscr{I}$ *satisfying the inequalities* $\mathscr{I}_0 \sqsubseteq \mathscr{I} \sqsubseteq \emptyset \times \mathbf{Fin}$ *strictly?*

Question 4.4.17. *Are all* $F_{\sigma\delta}$ *ideals below* $\emptyset \times \mathbf{Fin}$*? Are they below* $\mathbf{Fin} \times \mathbf{Fin}$*?*

Question 4.4.18. *Does weakly Farah property imply Farah property?*

Question 4.4.19. *Are there* $F_{\sigma\delta}$ *ideals which are not weakly Farah? and not Farah?*

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