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On Separoids

T E S I S

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P R E S E N T A:

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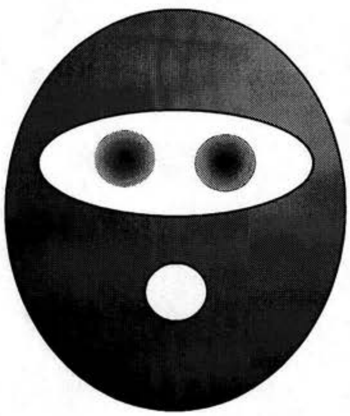
On Separoids

Ricardo Strausz UNAM



I want to thank to the Alfréd Rényi Institute of Mathematics, the Hungarian Academy of Sciences, the Central European University and the Soros Foundation for their support during the time I wrote this Ph.D. thesis. *2001-2002*

Nagyszüleim emlékére

Y a la memoria de
mi abuelo huasteco
Victor Neumann-Lara

Prólogo

El material que se desarrolla en las páginas de esta tesis, es la culminación de la investigación que comencé, junto con mi tutor y otros colaboradores, en el último lustro del siglo pasado. Si bien parte de este material ya ha aparecido en otros textos, la recopilación final fue hecha durante una estancia en Budapest, Hungría. Dicha estancia fué financiada por la Fundación Soros (a través de la Universidad de Europa Central) que, a manera de intercambio, me comprometió a terminar mi tesis y dejar ahí una copia de ésta; es por esto que el cuerpo principal de esta tesis se presenta en inglés. Sin embargo, incluye también un amplio resumen en español (con referencias a los resultados principales) para facilitar su lectura.

El texto, además de ser un tratado de la teoría de los separoides, pretende ser autocontenido y explicativo; la teoría es muy nueva asi que no supuse ningún conocimiento previo —salvo, por supuesto, una formación sólida en las ramas más comunes de la Matemática. Esto me llevó a organizar el material en un orden "que se llevara bien" con la lógica que surge detrás de los resultados, más que con un orden histórico o, quizás, didáctico. Sin embargo, en un intento de fortalecer la intuición que va detrás de la formalidad, incluí una amplia introducción (en inglés) que recorre con ejemplos concretos las definiciones más importantes, y en el resumen (en español) seguí un orden levemente diferente al texto central.

Al momento de organizar todo el material, me encontré que había parte de éste que, si bien no era lógicamente necesario para los resultados principales de la tesis, tenía que incluirlo si pretendía dar una exposición completa de la teoría. Así que decidí incluir este material en parrafos como éste. El lector, si así lo quiere, puede saltárselos sin perder información fundamental.

Dado que hay muchas referencias cruzadas, para facilitar la navegación dentro del texto, he decidido darles la siguiente forma: cuando se hace referencia a un resultado dentro de la misma sección, se refiere simplemente con un número; si el resultado está en otra sección, pero en el mismo capítulo, se refiere con dos números (sección.parrafo); y, finalmente, si está en otro capítulo, se refiere con tres números (capítulo.sección.parrafo).

También, para enfatizar el contexto histórico que rodea a la teoría, la bibliografía es referida con el nombre de los autores y el año de la publicación.

Finalmente, he incluido una amplia colección de imágenes al margen que, además de revestir el texto, ayudan a explotar la intuición que surge de la geometría intrínseca a la teoría.

Resumen

El origen de la teoría de los separoides puede ser rastreada a los principios del siglo XX cuando Radon demuestra el teorema de Helly usando que

Teorema (Radon 1921). *Dada una familia de puntos $\mathcal{P} \subseteq \mathbb{E}^d$, si el cardinal de $\mathcal{P}$ es suficientemente grande, a saber $|\mathcal{P}| \geq d+2$, entonces existen dos subconjuntos ajenos $A, B \subset \mathcal{P}$ cuyos cascos convexos se intersectan:*

$$A \cap B = \phi \quad \text{y} \quad \langle A \rangle \cap \langle B \rangle \neq \phi.$$

Sin embargo el nombre, y la axiomática que aquí se estudiará, no se acuñó sino hasta finales de los años 90 (véase Strausz 1998) cuando se describió la topología de la familia de hiperplanos transversales a una familia de conjuntos convexos (cf. Arocha, Bracho, Montejano, Oliveros & Strausz 2002).

Los separoides son simplemente una abstracción del teorema de Radon: un *separoide* es una relación simétrica $\dagger \subset \binom{2^S}{2}$ en una familia de subconjuntos —léase "...no se separa de..."— que tiene dos propiedades: si $A, B \subseteq S$ entonces

$$\begin{array}{ll} \bullet & A \dagger B \implies A \cap B = \phi, \\ \bullet\bullet & A \dagger B \text{ y } C \subseteq S \setminus A \implies A \dagger B \cup C. \end{array} \qquad (*)$$

El separoide se identifica con el conjunto S. El *orden* y el *tamaño* son los cardinales $|S|$ y $|\dagger|$, respectivamente.

Asi, dada una familia de puntos $\mathcal{P} \subset \mathbb{E}^d$ se puede definir un separoide $S = S(\mathcal{P})$ con la relación

$$A \dagger B \iff A \cap B = \phi \;\text{ y }\; \langle A \rangle \cap \langle B \rangle \neq \phi,$$

y claramente las dos condiciones $(*)$ se cumplen. La noción de "la dimensión del generado afín" de $\mathcal{P}$ se traduce en términos puramente combinatorios a la noción de *dimensión (combinatoria)* $\mathrm{d}(S)$, viz. el orden (menos uno) del máximo subseparoide de tamaño cero, donde todo par de subconjuntos disjuntos se separan. A los separoides de tamaño cero los llamaremos *simploides* por ser los asociados al conjunto de vértices de un simplejo. Entonces, si σ es un simploide maximal de S, $\mathrm{d}(S) = |\sigma| - 1$.

La motivación principal de esta tesis fué la pregunta: ¿cuándo se puede "realizar" un separoide con puntos? Es decir, dado un separoide "en abstracto" S, ¿cuándo podemos garantizar que existe una familia de puntos $\mathcal{P}$ tal que $S \approx S(\mathcal{P})$? Llamaremos a éstos, *separoides de puntos*.

1. Convexidad Abstracta

Como sugieren Danzer et al. (1963), la interrelación entre los teoremas de Radon, Helly y Carathéodory "podrán ser entendidos mejor formulando varios conjuntos de axiomas para la teoría de convexidad". La primera aproximación axiomática a la convexidad fue hecha por Levi (1951) y la teoría de los separoides puede ser vista como un nuevo intento en esta dirección.

Así como cada configuración de puntos tiene asociada un separoide —y de hecho, la configuración puede ser "recuperada" de esta información combinatoria (cf. Goodman & Pollack 1983)— a cada familia de conjuntos convexos $\mathcal{F}$ se le puede asociar un separoide $S = S(\mathcal{F})$. Éste captura la estructura de separación de la familia con la relación

$$A \mid B \iff \langle A\rangle \cap \langle B\rangle = \phi$$

($A \mid B$ se lee "A se separa de B").

Claramente la relación $\mid \subset \binom{2^S}{2}$, llamada *de separación*, define un separoide en los términos anteriores, a saber

$$A \dagger B \iff A \cap B = \phi \text{ y } A \not\mid B.$$

La relación de separación satisface las siguientes propiedades (una definición equivalente de separoide): si $A, B \subseteq S$ entonces

$$\begin{array}{ll} \bullet & A \mid A \implies A = \phi \\ \bullet\bullet & A \mid B \text{ y } A' \subset A \implies A' \mid B^{\cdot} \end{array}$$

Surge la siguiente pregunta: ¿cuándo se puede "realizar" un separoide con conjuntos convexos? Es decir, dado un separoide S, ¿cuándo podemos garantizar que existe una familia de conjuntos convexos $\mathcal{F}$ tal que $S \approx S(\mathcal{F})$? Aquí la respuesta es "fácil" (véase Arocha et al. 2002, Bracho & Strausz 2000 y Strausz 2003):

Teorema Básico de Representación [1.2.3]. *Todo separoide (finito) S puede ser representado por una familia de conjuntos convexos. Más aún, se puede representar con convexos compactos si y sólo si el separoide es acíclico (i.e., si $\phi \mid S$); en tal caso, la realización puede ser hecha en el espacio afín de dimensión $|S| - 1$.*

Este resultado juega un papel "polar" (cf. Björner et al. 1993 sec. 5.3) al teorema de representación topológica para *matroides orientados* de Folkman & Lawrence (1978). Los matroides orientados son separoides que cumplen además con otro par de condiciones; una abstracción de las configuraciones de puntos que usan la polaridad intrínseca del espacio euclidiano (los detalles están en el capítulo 3): si por cada punto (un vector en $\mathbb{R}^d$) tomamos su hiperplano polar, a cada configuración de puntos se le puede asociar una configuración de hiperplanos concurrentes (en el origen). Si consideramos ahora la intersección de estos últimos con la esfera unitaria, obtenemos una configuración de subesferas (de codimensión uno) que contiene toda la información combinatoria de la configuración original. Si permitimos que dichas subesferas se "enchuequen" un poco —que sigan siendo esferas, desde el punto de vista topológico— pero que conserven las dimensiones esperadas de sus intersecciones, lo que conseguimos es un matroide orientado. Más aún, el teorema de representación de Folkman & Lawrence demuestra que éstos son todos los matroides orientados. Surge la pregunta, ¿qué les pasó a los puntos polares cuando "enchuecamos" las subesferas? El teorema básico de representación responde a esta pregunta haciendo notar que los puntos "engordaron" para convertirse en conjuntos convexos. Sin embargo, la estructura combinatoria que se conserva es más general, es la de los separoides.

El teorema básico de representación nos permite introducir un invariante nuevo: la *dimensión geométrica* $\mathrm{gd}(S)$, la mínima dimensión donde se puede realizar el separoide. Es usando la noción de dimensión geométrica que los separoides encuentran su primera aplicación en la teoría de transversales geométricas (véase Arocha et al. 2002):

Teorema de Esencialidad [1.5.2]. *Sea $\mathcal{F}$ una familia de conjuntos convexos en $\mathbb{R}^{d+1}$ y sea $S = S(\mathcal{F})$ su separoide. Si $\mathrm{gd}(S) < d$ entonces $\mathcal{T}(\mathcal{F}) \hookrightarrow \mathbb{P}^d$, el espacio de hiperplanos transversales a $\mathcal{F}$, no es homotópicamente nulo.*

Para ejemplificar este resultado, considérense los 3 lados de algún triángulo en el plano y obsérvese que hay, topológicamente, tantas líneas transversales a estos 3 convexos como hay líneas por un punto. Esto es, el espacio de transversales sólo depende de la estructura combinatoria del separoide, no de su realización. En otras palabras, si cada 2 convexos comparten un punto entonces todos comparten un "punto virtual".

Surge la pregunta: si cada $d+2$ convexos admiten una d-transvesal ¿admiten todos una d-transversal virtual? El teorema de esencialidad contesta la pregunta afirmativamente para el caso en que $\mathrm{gd}(S) < d-1$ y por tanto puede ser visto como un teorema tipo Helly en el que se ha cambiado la noción de intersección por la de 0*-transversal virtual*.

Hay otros conceptos geométricos que pueden ser traducidos en términos puramente combinatorios. Se dice que un separoide S está en *posición general* si cualesquiera $\mathrm{d}(S)+1$ elementos inducen un simploide. Esto corresponde a que no se intesecten 2 a 2, que no exista una línea transversal por cada 3, que no exista un plano transversal por cada 4, etc...

Otra noción útil de índole puramente combinatorio —uno de los axiomas de matroide orientado— es la "unicidad" de las particiones de Radon. Decimos que S es un *separoide de Radon* si para cualesquiera dos particiones de Radon minimales $A \dagger B$ y $C \dagger D$,

$$A \cup B \subseteq C \cup D \implies \{A, B\} = \{C, D\}.$$

No es difícil ver que los separoides de puntos son de Radon. Tenemos además el siguiente

Teorema 1.3.4. *Sea S un separoide en posición general. Si $\mathrm{d}(S) = \mathrm{gd}(S)$ entonces S es un separoide de Radon.*

La demostración del teorema anterior, pasa por la siguiente generalización del teorema de Carathéodory (1907)

Lemma 1.3.1. *Sea $\mathcal{K} = \bigcup_{i\in I} \mathcal{K}_i \subseteq I\!R^d$ la unión de una familia de conjuntos convexos. Si $\mathbf{x} \in \langle \mathcal{K} \rangle$ entonces existe $J \subseteq I$, con $|J| \leq d+1$, tal que $\mathbf{x} \in \langle \mathcal{K}_j \rangle_{j\in J}$.*

y una aplicación inmediata a los separoides que garantiza la "buena realización" de cada partición minimal de Radon

Teorema 1.3.2. *Sea $S = \{\mathcal{K}_i\}$ la realización de un separoide. Si $A \dagger B$ es una partición de Radon minimal, entonces para cada convexo, $\mathcal{K}_a \in A$ y $\mathcal{K}_b \in B$, existe un punto $\mathbf{a}_a \in \mathcal{K}_a$ y $\mathbf{b}_b \in \mathcal{K}_b$ tal que $\langle \mathbf{a}_a \rangle \cap \langle \mathbf{b}_b \rangle \neq \phi$.*

Se puede ver entonces que la teoría de separoides permite traducir nociones geométricas a otras puramente combinatorias, y encierra así reinterpretaciones de los teoremas de Radon, Helly y Carathéodory —piedras angulares de la teoría combinatoria de los conjuntos convexos.

2. Separoides de Puntos

Uno de los problemas centrales dentro de la teoría de los matroides orientados —separoides de Radon que cumplen además el *axioma de eliminación débil*— es encontrar caracterizaciones "significativas" de los separoides de puntos. Se sabe (cf. Shor 1991) que este problema, desde el punto de vista polar de la representación topológica, es "NP-hard".

Sin embargo, desde el punto de vista geométrico intrínseco a los separoides, se pueden caracterizar aquellos separoides de puntos que están en posición general:

Teorema 2.0.1. *Un separoide en posición general es un separoide de puntos si y sólo si sus dimensiones geométrica y combinatoria coinciden.*

Como lo muestra el siguiente ejemplo, la hipótesis de la posición general no se puede quitar sin agregar algún ingrediente más. Considérese el separoide de orden tres $S = \{1, 2, 3\}$ generado por las dos particiones minimales $1 \dagger 2$ y $2 \dagger 3$. Por un lado, S se puede realizar con un segmento (representando al 2) y sus dos puntos extremos; por el otro, como $1 \mid 3$, el subseparoide $S' = \{1, 3\}$ induce un simploide de dimensión 1. Claramente S no es el simploide de dimensión 2, por tanto su dimensión geométrica y combinatoria coinciden. Sin embargo es fácil ver que S no se puede realizar con puntos —la relación $\dagger$ es transitiva en los singuletes de los separoides de puntos.

Los ejemplos pequeños de pseudolíneas sugieren la siguiente

Conjetura. *Un matroide orientado $\mathcal{M}$ es un separoide de puntos si y sólo si*

$$\mathrm{d}(\mathcal{M}) = \mathrm{gd}(\mathcal{M}).$$

Dado que no podemos caracterizar a los puntos (al menos todavía), surge la pregunta ¿cómo se ve “el espacio” de todas las configuraciones de puntos? En otras palabras, ¿podemos asociar a cada configuración un punto de algún espacio topológico? Por supuesto, estamos buscando aqui algún espacio “significativo” que nos ayude a entender la relación entre la geometría y la combinatoria de los separoides.

Una vez más, la respuesta es afirmativa. Para esto, entenderemos por una *configuración de puntos* un subconjunto finito y ordenado de $\mathbb{R}^d$, módulo la accion del grupo afín $\mathbb{A}(d)$. Es decir, dos subconjuntos $\mathcal{P} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ y $\mathcal{Q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ representan la misma configuración si y sólo si existe una transformación afín $\varphi \colon \mathbb{R}^d \to \mathbb{R}^d$ tal que $\varphi(\mathbf{p}_i) = \mathbf{q}_i$, para $i = 1, \dots, n$. Claramente, si $\mathcal{P}$ y $\mathcal{Q}$ representan la misma configuración, definen el mismo separoide (i.e., $S(\mathcal{P}) = S(\mathcal{Q})$). Por otro lado, de la definición, se antoja pensar en el espacio de todas las configuraciones de n puntos en dimensión d —que denotaremos $\mathbb{A}_n^d$— con la topología cociente de $\mathbb{R}^{n\times d}$. Tenemos el siguiente

Teorema 2.1.2. *$\mathbb{A}_n^d$ es homeomorfo a $\mathcal{G}^d(\mathbb{R}^{n-1})$, la grassmanniana de subespacios lineales de $\mathbb{R}^{n-1}$ de dimensión d.*

Este resultado, seguido del encaje de Plücker (véase e.g., Björner et al. 1993), sugiere considerar a $\mathbb{A}_n^d$ como subespacio de $\mathbb{P}^{n-2}$. Surge la pregunta ¿cómo “estratifican” los separoides a $\mathcal{G}^d(\mathbb{R}^{n-1})$? En otras palabras, ¿tiene $\mathbb{A}_n^d$ alguna descripción puramente combinatoria? Cuando la *codimensión* —la diferencia $(n-d-2)$— es pequeña, se sabe que la respuesta es afirmativa (véase más abajo) sin embargo el caso general sigue siendo un problema abierto.

Dado un separoide S se puede construir un complejo “cúbico” —un subcomplejo de algún hipercubo— cuyos vértices representan las particiones de Radon *maximales*. Como el separoide no está determinado por éstas, dicho complejo es “olvidadizo”, sin embargo, en casos importantes —matroides orientados, puntos— el complejo determina completamente al separoide. Dicho complejo será llamado *complejo de Radon* y lo definiremos más abajo. Por el momento, a manera de motivación, permítaseme mencionar que es la combinatoria del n-cubo, a traves del complejo de Radon, la que “domina” la estratificación de la grassmanniana inducida por los separoides de puntos.

La familia de subconjuntos 2^S del n-conjunto S puede ser identificada con los vértices del n-cubo $\mathcal{Q}_n$; haciendo esto, las caras del n-cubo son identificadas con los intervalos del orden parcial $\subseteq$ inducido por la contención, i.e., las caras de $\mathcal{Q}_n$ son de la forma

$$[A,B] := \{C \in 2^S : A \subseteq C \subseteq B\}.$$

Regresando al separoide S, si consideramos aquellos subconjuntos A que “no se separan de su complemento”, $A \dagger \overline{A}$, y nos fijamos en el subcomplejo de $\mathcal{Q}_n$ que inducen, el complejo resultante es lo que llamamos el *complejo de Radon* del separoide y lo denotamos por $\mathcal{R}(S)$.

Teorema 2.2.3. *Si $\mathcal{P} \subset \mathbb{R}^d$ es un separoide de puntos de orden n, entonces $\mathcal{R}(\mathcal{P})$ es, homotópicamente, una esfera de dimensión $n-d-2$. Más aún, el separoide esta en posición general si y sólo si el complejo es homeomorfo a dicha esfera.*

En particular, cuando tenemos $d+2$ puntos en dimensión d, el complejo es una 0-esfera —dos subcubos antípodas— y podemos dar la siguiente descripción combinatoria.

Teorema 2.3.1. *El espacio de $d+2$ puntos en dimensión d, módulo el grupo afín, es*

$$\mathbb{A}^d_{d+2} = (\mathcal{Q}_n \setminus \{\phi, \overline{\phi}\})^* / \{A, \overline{A}\}.$$

También podemos contar cuantos de estos son *polítopos* —cuando cada punto se separa de su complemento— para exhibir una nueva prueba de

Teorema 2.4.1 (Grümbaum 1967). *Existen exactamente $\lfloor \frac{1}{4} d^2 \rfloor$ tipos de politopos (convexos) con $d+2$ vértices en $\mathbb{R}^d$.*

El caso $d=1$ es igualmente simple. Ya el caso $n=d+3$ es suficientemente complicado; la descripcion combinatoria de la estratificación de la grassmanniana es (véase también García-Colín 2003)

Teorema 2.3.3. *Las facetas de $\mathbb{A}^d_{d+3}$ son los ciclos antipodales de $\mathcal{Q}_n \setminus \{\phi, \overline{\phi}\}$ de longitud $2n$. Dos facetas se intersectan en una colínea si y sólo si sus dos cíclos correspondientes difieren en exactamente dos puntos antípodas.*

En este caso no es tan fácil contar todos los tipos de politopos —una fórmula explícita puede llevar 6 líneas de texto para escribirse (cf. Lloyd 1970)— sin embargo el complejo de Radon muestra fácilmente que (véase Montellano-Ballesteros & Strausz 2003)

Teorema. *Existen exactamente $\nu(2n) - \lceil \frac{n}{2} \rceil$ politopos convexos con $n = d+3$ vértices en $\mathbb{R}^d$ en posición general, donde $\nu(2n)$ denota el número de collares bicoloreados antipodales de tamaño $2n$.*

Para $n > d+3$ ni el método anterior, ni ninguno que se conozca, sirve para contar tipos de politopos. La principal obstrucción es que aparecen "esferas" dentro de $\mathcal{Q}_n$ que no corresponden a ninguna configuración de puntos. La estructura que preservan estas esferas, al parecer, es la de los matroides orientados. En esta dirección tenemos el siguiente

Teorema 3.2.5. *Una gráfica $\mathcal{G}$ es la gráfica de circuitos de un matroide orientado uniforme si y sólo si es una gráfica antipodal de orden $2\binom{n}{d+2}$ y existe un encaje i-métrico $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ en el $(n-d-2)$-dual del n-cubo.*

3. Universalidad

Dados dos separoides S, T, una función $\varphi: S \to T$ será llamada un *morfismo de separoides* si, para todo $A, B \in 2^T$

$$A \mid B \implies \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

Un morfismo será llamado *homomorfismo* si cumple además que, para todo $\alpha, \beta \in 2^S$

$$\alpha \dagger \beta \text{ minimal} \implies \varphi(\alpha) \dagger \varphi(\beta) \text{ minimal}.$$

Denotaremos por $S \longrightarrow T$ la existencia de algún homomorfismo. La relación $S \leq T \iff S \longrightarrow T$ define un preorden en la classe $\mathbb{S}$ de todos los separoides. Si identificamos además a aquellos separoides tales que $T \leq S$ y $S \leq T$, conseguimos una clase parcialmente ordenada.

Si denotamos por σ^d al simploide de dimensión d, y por K_n al separoide *completo* —de tamaño máximo— de orden n, es fácil probar que

Proposición 4.0.

1. $|S| < n \implies K_n \not\longrightarrow S$
2. $K_1 \longrightarrow S \iff S \not\longrightarrow K_0$
3. $S \approx \sigma^d \iff S \not\longrightarrow K_1$
4. $S \approx \sigma^d \iff \forall T \neq K_0, S \longrightarrow T.$

Por ejemplo, el enunciado 2 dice que, *el filtro principal de $K_1 = \sigma^0$ es el complemento del ideal principal de K_0* que no es otra cosa que decir que *un separoide tiene algún elemento si y sólo si no se mapea en el vacío*. En la literatura, (σ^0, K_0) es llamado un *par dual* (cf. Nešetřil & Tardif 2000).

De hecho, el orden de los homomorfismos $(\mathbb{S}, \longrightarrow)$ es una latiz que tiene a la *suma* y al *producto* de separoides, como supremo e ínfimo, respectivamente. Dichas operaciones existen y satisfacen las propiedades universales esperadas:

$$S \longrightarrow P \times Q \iff S \longrightarrow P \text{ y } S \longrightarrow Q,$$

$$P + Q \longrightarrow S \iff P \longrightarrow S \text{ y } Q \longrightarrow S.$$

Diremos que un separoide es *conexo* si no puede ser expresado como la suma (la unión ajena) de dos separoides

$$T \longrightarrow T_0 + T_1 \implies T \longrightarrow T_0 \text{ o } T \longrightarrow T_1.$$

En estos términos, podemos expresar el teorema de Radon como

Teorema 4.2.1. *$\mathcal{P} \subset \mathbb{E}^n$ es un separoide de puntos de orden $\mathrm{d}(\mathcal{P}) + 2$ si y sólo si*

$$\mathcal{P} \not\longrightarrow K_1 \text{ y } \mathcal{P} \longrightarrow K_2 + \sigma,$$

donde σ es un simploide. Más aún, $\mathcal{P}$ está en posición general si y sólo si $\sigma = \phi$. En otras palabras, los elementos de $\mathbb{R}^d_{d+2}$ son los separoides bipartitos.

El orden de los homomorfismos es también un orden denso

Teorema 4.3.2. *Si $S < T$ entonces existe un $\mathcal{P}$ tal que $S < S + \mathcal{P} \times T < T$.*

La estructura de orden de los separoides es en cierta forma, muy parecida al orden de los homomorfismos de gráficas. Los dos son ordenes universales para la teoría de conjuntos.

Teorema 4.4.1. *Dado cualquier conjunto parcialmente ordenado $(C, \leq)$ existe un "funtor" $f: C \to \mathbb{S}$ tal que, $a, b \in C$*

$$a \leq b \iff f(a) \longrightarrow f(b).$$

Más aún, el teorema anterior —cuando se extiende la noción de homomorfismo al infinito— se puede extender a clases parcialmente ordenadas y por tanto (cf. Nešetřil & Strausz 2002)

Teorema 4.4.3. *Toda categoría puede ser representada como una subcategoría de los separoides y sus homomorfismos.*

4. Hiperseparoides

Asi como los separoides "codifican" el teorema de Radon, los *hiperseparoides* lo hacen con el teorema de Tverberg

Teorema (Tverberg 1966). *Si $\mathcal{P} \subset \mathbb{E}^d$ es suficientemente grande, a saber $|\mathcal{P}| \geq (k-1)(d+1)+1$, entonces puede ser dividido en k partes disjuntas cuyos cascos convexos se intersectan.*

Claramente, para $k = 2$ el teorema de Tverberg es el de Radon. El teorema de representación junto con el teorema de Tverberg implican inmediatamente que

Teorema 4.5.2. $|S| \geq (k-1)(\mathrm{gd}(S)+1)+1 \implies S \longrightarrow K_k.$

Esto motiva la siguiente definición: un *k-separoide* es un sitema de familias de subconjuntos $\mathcal{T} \subset \binom{2^S}{k}$ que relaciona conjuntos disjuntos —no tiene "loops"— y es un filtro en el orden parcial canónico —el heredado de $(2^S \times \ldots \times 2^S, \subseteq)$.

El teorema de representación para separoides puede ser generalizado a

Teorema 4.5.5. *Todo k-separoide acíclico S pude ser representado por una familia de cuerpos convexos $\{A_i\}_{i \in S}$ en $\mathbb{E}^{|S|-1}$ y sus particiones de Tverberg son*

$$\{A_i\}_{i=1}^k \in \mathcal{T} \iff \begin{cases} A_i \cap A_j = \phi & \text{si } i \neq j \text{, y} \\ \bigcap_{i=1}^k \langle A_i \rangle \neq \phi & . \end{cases}$$

Surgen las preguntas ¿cuándo podemos garantizar que un k-separoide es representable con puntos del euclidiano? ... ¿será cierto que todo k-separoide "uniforme", con $\mathrm{gd}(S) = \mathrm{d}(S)$, es de puntos?

Preface

This Ph.D. thesis is concerned with the application (and generalization) of the classical theorems of Helly, Radon and Carathéodory which stands as the origin of the *Combinatorial Geometry of Convex Sets*. More precisely, the combinatorial structure defined by the separations of a (finite) family of convex sets in the Euclidean d-space, will be developed. Namely, two subfamilies are said to be *separated* if there exists a hyperplane that leaves them on opposite sides of it or, equivalently, if their convex hulls do not intersect. The most basic —and trivial— properties of this relation on the subsets of the family, are the axioms of a *separoid*.

These three theorems were discovered in the first quarter of the last century and can be formulated as follows:

- **Helly's Theorem.** *Let $\mathcal{K}$ be a family of convex sets in $\mathbb{R}^d$. If every $d+1$ (and fewer) members of $\mathcal{K}$ have a common point, then there is a common point to all members.*
- **Radon's Theorem.** *Let X be a set of $d+2$ or more points in $\mathbb{R}^d$. Then X contains two disjoint subsets whose convex hulls have a common point.*
- **Carathéodory's Theorem.** *Let X be a set in $\mathbb{R}^d$ and p a point in the convex hull of X. Then there is a subset Y of X consisting of $d+1$ or fewer points such that p lies in the convex hull of Y.*

The reader will find some variations and generalizations of these results in the following pages. The aim of this thesis is to develop some branches of *Combinatorial Geometry* from a particular point of view: separoids. It is, at the same time, a survey and a basic reference to the subject.

☕ I use paragraphs as this one to easily differentiate the main text from aside information and comments. Such paragraphs contain material which, eventhough it is not essencial for the main line of research, it suplements the theory and points out some bibliographical items.

As a result of this research some papers have (and will) be published. Most of it was made together with my supervisor Javier Bracho and in collaboration with Jorge Luis Arocha and Luis Montejano. Also, there are some parts that were elaborated with Juan José Montellano-Ballesteros and Deborah Oliveros. Nevertheless, many other people have helped in developing this theory; Victor Neumann-Lara, who gave me the basis to find the Radon complex of a separoid; Francisco Larrion, who taught me everything I know about category theory and helped me generalize the Representation Theorem for **all** separoids; Eugene Schepin, who put us on the road of such a theorem for the acyclic case; Karoly Böröczky, who refined Theorem 2.0.1; Jaroslav Nešetřil, who suggested the approach of Chapter 4, and many other friends who have read and commented some or all of these results.

> Many of these results and proofs are in: Strausz 1998; Arocha, Bracho, Montejano, Oliveros & Strausz 1999; Bracho & Strausz 2001; Montellano & Strausz 2001; Strausz 2001; Nešetřil & Strausz 2002 and Strausz 2002.

I have classified all this material in four chapters: *Separoids, Configurations Oriented Matroids* and *Homomorphisms.* In the first one, you will find the basis of the theory and all the results that can be applied to the other three. The second one deals with geometric examples of separoids, in particular the point separoids are studied in detail. The third chapter is devoted to apply all previously developed material to the Theory of Oriented Matroids —the most explored area of separoids. In the fourth one, a new categorical approach is adopted and the *universality and density* of separoids homomorphisms is proved. I added an introduction to introduce the theory in the basis of examples so the reader will find there some specific pictures to think on while the theory is developed. Appendix B contains a large bibliography about the subject.

Finally, I want to thank Merari and the rest of my family, including all those who live —and lived— here in Hungary. They gave me all the emotional support I needed to end it.

Dino
Budapest 2002

Contents

In other words, it's supposed to be a panacea that satisfies everybody, at the risk of satisfying nobody.

—DONALD E. KNUTH, *The TEXbook* (1983)

0
Introduction

Separoids provide a broad setting to describe those combinatorial properties that arise from *families of convex sets* and the separations they define. Mathematical objects which appear to be totally different, such as *configurations of points, arrengments of affine subspaces, directed and undirected graphs, oriented matroids, convex polytopes* and *separation axioms of topological spaces,* find a common generalization in the language of separoids.

Separoids arise in the context of *geometric transversal theory* in an attempt to answer the question: *How does the space of hyperplanes transversal to a family of convex sets in $\mathbb{E}^d$ looks like?* As already pointed out by Goodman, Polack & Wenger the existence of a transversal hyperplane depends on the existence of a suitable oriented matroid. We found that *the space of all such hyperplanes is essential (as a subset of $\mathbb{P}^{d-1}$) if the geometric dimension of the underlying separoid is less than $d-1$.* Also in terms of the geometric dimension, those separoids that arise from a configuration of points in *general position* have been characterized: *A general position separoid is a point separoid if and only if its combinatorial dimension and its geometric dimension are equal.*

Further research lead us to an equivalent version of the Basic Sphericity Theorem (Folkman & Lawrence 1978): *The Radon complex of an oriented matroid is homologically equivalent to a sphere. Moreover, if the matroid is uniform, the complex is homeomorphic to such a sphere.* This result was the first step to reach the characterization of the *cocircuit graphs* for uniform oriented matroids.

Oriented matroids are separoids which satisfies a couple of extra properties (they will be formally defined in Chapter 3). Folkman and Lawrence introduced oriented matroids as a combinatorial description of *sphere systems.* Las Vergnas used oriented matroids to describe a purely combinatorial setting of *convexity* and, at the same time, Bland described how oriented matroids can be used to encode the basics properties of *linear programs* also in a purely combinatorial level. Independently, Dreiding, Dress, Haegi and Wirth introduced the equivalent notion of *chirotopes* to describe chirality of molecules in organic chemistry. All these approaches took place between the late 1960's and the early 1980's. Nevertheless, geometric objects that where studied much earlier turned out to be equivalent to oriented matroids. For instance, *arrangements of pseudolines* as already studied by Levi 1926, Ringel 1956 and Grümbaum 1969, are equivalent to oriented matroids in dimension 2. It is not my objective in the present to describe the theory of oriented matroids in its full extent but most of the theory developed here can (and will) be applied to it.

☕ An introduction to the subject can be found in the book of Björner, Las Vergnas, Sturmfels, White & Ziegler 1993, the survey of Bokowski 1993 and the Ph.D thesis of Richter-Gebert 1992.

Finally, separoids had been studied from a categorical point of view to prove that the *homomorphisms order is universal,* viz., *any partially ordered class can be embedded into the homomorphisms order of separoids.*

0. Some Motivating Examples

Let me introduce the theory in an informal way by giving some basic examples to have some specific pictures in mind when the theory is developed.

Consider a $(d+1) \times n$ matrix $\mathcal{M} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (\mathbb{R}^{d+1})^n$ and the n-set $X = \{1, \ldots, n\}$. Each of the n columns of $\mathcal{M}$ is interpreted as a vector in the real vector space $\mathbb{R}^{d+1}$. If these vectors span the space, the minimal linear dependences yield the *circuits of a matroid of rank* $d+1$. Such linear dependencies look like

$$\sum_{i \in X} \lambda_i \mathbf{x}_i = \mathbf{0} \quad \text{with} \quad \lambda_i \in \mathbb{R}, \text{ not all zero}$$

and the sets $C = \{i : \lambda_i \neq 0\}$ corresponding to the *minimal* ones are the circuits of the matroid. The *associated separoid* is the family of pairs $A \dagger B$ given by

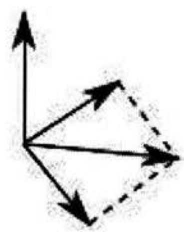
$$A = \{i : \lambda_i < 0\}, \quad B = \{i : \lambda_i > 0\},$$

for all the minimal dependencies among the $\mathbf{x}_i$.

Interesting vector configurations to be studied from this point of view are given, for example, by the vertices of polytopes and by the root systems of semisimple Lie algebras.

For a more specific example, let $\mathbf{x}_1, \ldots, \mathbf{x}_4$ be the vectors in $\mathbb{R}^3$ given by the columns of the matrix

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$



From $\mathcal{M}$ we get the separoid of rank 3 on $X = \{1, 2, 3, 4\}$, for which the linear dependence $\mathbf{x}_1 - \mathbf{x}_3 - \mathbf{x}_4 = \mathbf{0}$ translates into the circuit $1 \dagger 34$.

Every vector configuration in $\mathbb{R}^{d+1} \setminus \{\mathbf{0}\}$ corresponds to a *point configuration* in a d-dimensional affine space. For this, choose a linear form ℓ such that $\ell(\mathbf{x}_i) \neq 0$ for all i, define

$$\mathbb{A}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} : \ell(\mathbf{x}) = 1\}$$

as a model of affine space, and associate to each vector $\mathbf{x}_i$ the point $\mathbf{p}_i = \frac{1}{\ell(\mathbf{x}_i)} \mathbf{x}_i \in \mathbb{A}^d$. Here, vectors $\mathbf{x}_i$ with $\ell(\mathbf{x}_i) < 0$ determine "reoriented affine points".

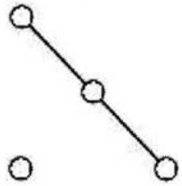
These "negative points" are somewhat annoying to have to deal with, although sometimes unavoidable. However, if the vector configuration does not contain any positive linear dependence ($\sum \lambda_i \mathbf{x}_i = \mathbf{0}$ with $\lambda_i \geq 0$), then we can choose ℓ such that $\ell(\mathbf{x}_i) > 0$ for all i, which results in an honest affine point configuration. This corresponds to the situation where the separoid is *acyclic:* it does not contain a circuit of the form $\phi \dagger B$.

Now, every affine point configuration gives rise to an acyclic separoid whose *minimal Radon partitions* (the circuits of the matroid) are given by the minimal affine dependences

$$\sum_{i \in X} \lambda_i \mathbf{p}_i = \mathbf{0} \quad \text{with} \quad \sum_{i \in X} \lambda_i = 0.$$

Starting from the vector configuration in $I\!R^3$ discussed in the example above, with $\ell(\mathbf{x}) = \sum x_i$ and dropping the first coordinate, we obtain the point configuration

$$\mathcal{P} = \begin{pmatrix} \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}.$$



Now, from $\mathcal{P}$ we obtain the separoid of *dimension* 2 on $X = \{1,2,3,4\}$, for which the affine dependence $\mathbf{p}_1 - \frac{1}{2}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_4 = \mathbf{0}$ (with $1 - \frac{1}{2} - \frac{1}{2} = 0$) translates into the minimal Radon partition $1 \dagger 34$. Observe that from the **affine** point of view, the **linear** dependence $\mathbf{p}_1 + \mathbf{p}_2 - \frac{1}{2}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_4 = \mathbf{0}$ is **not** a minimal Radon partition of the separoid.

Since the parameters λ_i of such minimal dependences are unique up to a common scalar, those equations can be rewritten as

$$-\sum_{i \in A} \lambda_i \mathbf{p}_i = \sum_{i \in B} \lambda_i \mathbf{p}_i,$$

$$-\sum_{i \in A} \lambda_i = \sum_{i \in B} \lambda_i = 1,$$

and therefore we can redefine the relation as

$$A \dagger B \iff \langle A \rangle \cap \langle B \rangle \neq \phi,$$

where $\langle . \rangle$ denotes the *convex hull* operator. In this new context, we are including **all** such partitions —not only the minimal ones.

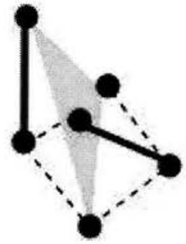
In the previous example we have two more *Radon partitions:* $12 \dagger 34$ and $1 \dagger 234$, so the separoid is

$$(\{1,2,3,4\}; 1 \dagger 34, 1 \dagger 234, 12 \dagger 34).$$

What happens if we delete the (minimal) partition $1 \dagger 34$ *but keep the other two?* It is not hard to see that the separoid is not any more the separoid of an affine configuration of points (nor a linear one). However it is the separoid of a *family of (convex) segments* in the affine 3-space: A family of four convex sets given by

$$\mathcal{F} = \{\langle \mathbf{a}_1 \mathbf{b}_1 \rangle, \langle \mathbf{a}_2 \mathbf{b}_2 \rangle, \mathbf{p}_3, \mathbf{p}_4\},$$

satisfies the desired properties if



$$\begin{array}{ll} \mathbf{a}_1 = (0,1,1) & \mathbf{b}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \\ \mathbf{a}_2 = (0,0,0) & \mathbf{b}_2 = (1,0,0) \\ \mathbf{p}_3 = (0,1,0) & \mathbf{p}_4 = (0,0,1). \end{array}$$

1. Main Concepts

From the previous examples, many concepts can be introduced. First of all, it must be clear now that not every separoid arises from a family of points, so it is natural to ask: *Which separoids arise from such a family?* We will call this kind of objects *point separoids*. A combinatorial characterization of these separoids is still unknown, but we will study them deeply on Chapter 2. An obstruction of $\mathcal{F}$ to be a point separoid is that it contains two different minimal Radon partitions on the same set. If this is not the case, we will call the separoid a *Radon separoid*, so $\mathcal{P}$ is a Radon separoid and $\mathcal{F}$ is not.

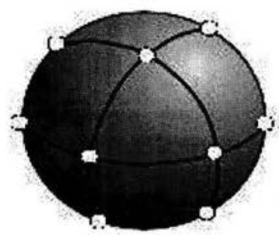
These two separoids are quite similar —all Radon partitions of $\mathcal{F}$ are contained also in $\mathcal{P}$. If that is the case, then we will say that there exists a morphism $\mathcal{F} \longrightarrow \mathcal{P}$, and call the class of all separoid morphisms, the *separoid category*.

If we restrict ourselves to point separoids over the same n-set and of the same dimension d, the natural partial order given by

$$\mathcal{P} > \mathcal{P}' \iff \mathcal{P} \longrightarrow \mathcal{P}'$$

describes a stratification of a manifold known as the *Grassmanninan*. This poset will be denoted by $\mathbb{A}_n^d$ and proved to be homeomorphic to $\mathcal{G}^d(\mathbb{R}^{n-1})$, the space of all d-subspaces of the vector space $\mathbb{R}^{n-1}$.

It will be proved that $\mathbb{A}_4^2$ is the face lattice of the hemicuboctahedron —a well known polyhedron homeomorphic to the projective plane $\mathbb{P}^2 = \mathcal{G}^2(\mathbb{R}^3)$.



The separoid $\mathcal{F}$ cannot be *realized* as a point configuration, but it was realized as a family of convex sets. It is also natural to ask: *Which separoids can be realized as a family of convex sets?* In contrast with point separoids, *separoids of convexes* can easily be characterized: They are **all** separoids. Therefore, each separoid has as an invariant the minimal dimension where it can be realized; this number will be called the *geometric dimension* of the separoid. The geometric dimension of $\mathcal{F}$ is 3 while its *combinatorial* one is 2.

This invariant will be useful to study the space of *hyperplane transversals* to a family of convex sets. Observe that there is no line transverse to the four points of $\mathcal{P}$ and that there are two planes transverse to the elements of $\mathcal{F}$, that is, the space of hyperplanes transverse to $\mathcal{P}$ is empty and that of $\mathcal{F}$ consists of two points.

Also, the geometric dimension will be used to characterize those point separoids which are in *general position* —separoids (of dimension d) whose minimal Radon partitions consist of exactly $d+2$ points. Separoid $\mathcal{F}$ is in general position and $\mathcal{P}$ is not.

An *oriented matroid* is a matroid whose (ordered) bases have received an orientation compatible with the so called *Grassmann-Plüker relations*. Although not every matroid is orientable, every point separoid is an oriented matroid. However not every oriented matroid is a point separoid, i.e., oriented matroids are more general than point separoids. On the other hand, every oriented matroid is a Radon separoid but not the other way around, so separoids are even more general than oriented matroids. They will be studied in Chapter 3.

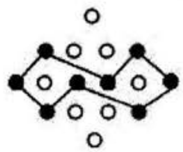
Recall the vector configuration $\mathcal{M}$ associated to $\mathcal{P}$. If the 3-sets of X are ordered lexicografically, their orientations can be extracted from the matrix $\mathcal{M}$ as the signs of the determinants of its minors, i.e., if (ijk) is a 3-subset of X we assign to it the sign given by

$$\chi(ijk) = \text{sgn}\,|\mathbf{x}_i\mathbf{x}_j\mathbf{x}_k|.$$

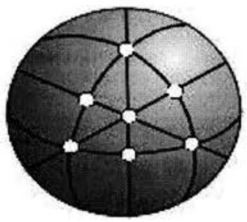
If we do this with all 3-sets $(123, 124, 134, 234)$, we obtain the list $+\ -\ 0\ +$. This list encodes the whole separoid $\mathcal{P}$ in a very compact way —it is a list of $\binom{n}{d+1}$ elements of $\{-, 0, +\}$. Unfortunately, such a compact code is not known for all kinds of separoids. It would be nice to find one!

2. One more example

As our final example, take four points in the line and consider all pairs $A \dagger B$ of disjoint subsets $A, B \subset X = \{1, 2, 3, 4\}$ such that $\langle A\rangle \cap \langle B\rangle \neq \phi$. Now draw an edge between a pair of such subsets $A \sim A'$ whenever they differ in only one element $|A \triangle A'| = 1$. The resulting graph is a cycle of length eight. It will be called the *Radon complex* of the separoid. Observe that all linear orders on four elements are in a one-to-one correspondence with this kind of cycles inside the 4-cube. We are thinking about the vertices of $\mathcal{Q}_4$ as the family of subsets 2^X of the 4-set X.



It will be proved that $\mathbb{R}^1_4$ is the projective polyhedron depicted below.



... in contrast to the matroid case oriented matroids carry information about the topology and the convexity of the underlying configurations.

—JÜRGEN RICHTER-GEBERT, *New Construction Methods for Oriented Matroids* (1992)

1
Separoids

Separoids are combinatorial objects that capture the structure arising from a family of convex sets in $\mathbb{E}^d$, where some subfamilies are naturally separated from others. Namely, two subfamilies are said to be *separated* if there exists a hyperplane that leaves them on opposite sides of it —the axioms of a separoid are simply the obvious properties of this relation.

0. Basic Notions

A *separoid* $S = (X, |)$ over the base set $X \neq \phi$ is a relation $|\subseteq 2^X \times 2^X$ on the subsets of X with the following properties: If $A, B \subset X$, then

$$\begin{array}{cc} \circ & A \mid B \implies B \mid A, \\ \circ\circ & A \mid A \implies A = \phi, \\ \circ\circ\circ & A \mid B \text{ and } A' \subset A \implies A' \mid B. \end{array}$$

So we say that a separoid is a *symmetric, quasi-antireflexive, ideal* relation on the family of subsets. The elements of $|$ are called *separations* and, when speaking of a separation $A \mid B$, it is said that "A is *separated from* B". A separoid is *acyclic* if the empty set is separated from the base one, i.e. if $\phi \mid X$. The separations with the empty set are called *trivial* separations and, in the sequel, almost all separoids are finite and acyclic. Observe that it is enough to know *maximal separations* to reconstruct the separoid —they encode the whole information.

It is easy to see that an ideal relation is quasi-antireflexive if and only if

$$\circ\circ \qquad A \mid B \implies A \cap B = \phi.$$

Now, let S and T be two separoids over the base sets X and Y respectively. A *separoid morphism* $S \longrightarrow T$ is a function $\varphi: X \to Y$ with the property that for all $A, B \subseteq Y$,

$$A \mid B \implies \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

A *separoid category* is defined with such morphisms between separoids. Two separoids are isomorphic if there exists a bijective morphism from one onto the other whose inverse function is also a morphism.

Given a subset $X' \subseteq X$ of the base set of a separoid S, the *induced separoid* is defined as the restriction of $|$ to X'. An *embedding* is an injective morphism between separoids such that it is an isomorphism between the domain and the induced separoid of the image. The *order* is the number of elements in X.

There is a notion of dimension on separoids which is easily and intrinsically determined. The d*-dimensional simploid* $\sigma = \sigma^d$ is a separoid of order $d+1$ such that every subset is separated from its complement, which by the third condition yields $A \mid B \iff A \cap B = \phi$. A simploid can be realized by the vertex set of a simplex, hence its name —Figure 1.**a**, on page 10, represents σ^2.

The *(combinatorial) dimension* of a separoid, denoted by $\mathsf{d}(S)$, is the maximum dimension of its induced simploids.

With the definition of dimension at hand, it is quite easy to translate into separoid terms the classic Radon's theorem; they capture the combinatorial essence of it (cf. Danzer, Grümbaum & Klee 1963).

0.1. Lemma (Radon). *Let $S = (X, |)$ be a d-dimensional separoid, then every subset $Y \subseteq X$ of cardinality greater than or equal to $d+2$ contains two disjoint subsets $A, B \subset Y$ such that they are not separated from each other.*

Proof. It follows immediately from the fact that Y is not a simploid. ⊛

A *Radon partition* consists of two non-separated disjoint sets, and it will be denoted by $A \dagger B$. Each part (A and B) is called a *Radon component* and the union $A \cup B$ is known as the *support* of the partition. Considering the Radon partitions of a separoid $S = (X, |)$ as a relation $\dagger \subset 2^X \times 2^X$, it has the following properties:

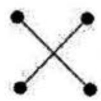
- • $A \dagger B \implies B \dagger A$,
- •• $A \dagger B \implies A \cap B = \phi$,
- ••• $A \dagger B$ and $C \subseteq X \setminus A \implies A \dagger B \cup C$.

This leads to an equivalent definition of a separoid. The separations can be reconstructed with the obvious definition; $A \mid B$ iff $A \cap B = \phi$ and there are no subsets $A' \subseteq A$ and $B' \subseteq B$ such that $A' \dagger B'$.

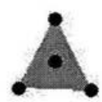
A *minimal Radon partition* is a Radon partition $A \dagger B$ for which each component is minimal under contention, i.e.,

$$A' \subset A \implies A' \mid B \qquad \text{and} \qquad B' \subset B \implies A \mid B'.$$

The set of all minimal Radon partitions of a given separoid determines it and will be denoted by MRP, so $A \dagger B \in MRP$ means that $A \dagger B$ is a minimal Radon partition.



Many authors have observed that the Radon's theorem can be settled in a more precise way (cf. Eckhoff 1993): *Let X be a set of $d+2$ points in $\mathbb{R}^d$ in general position. Then X has a unique partition in two disjoint subsets whose convex hulls have a common point. Moreover, this point is also unique.* This motivates the next definition.



A *Radon* separoid is a separoid with the property that for all $A \dagger B, C \dagger D \in MRP$ such that $A \cup B \subseteq C \cup D$ it follows that $\{A, B\} = \{C, D\}$, i.e., the elements of MRP are incomparable.

A separoid is said to be in *general position* if every subset $A \subset X$ of cardinality $d+1$ is an induced simploid.

0.2. Lemma (general position). *Let S be a d-dimensional separoid in general position. If $A \dagger B \in MRP$ is a minimal Radon partition, then the cardinality of the support $A \cup B$ is at least $d+2$.*

Proof. The cardinality of the support cannot be smaller because every subset $\sigma \subset S$ of cardinality $d+1$ or less is an induced simploid. ⊛

1. Examples

As in any interesting category, the important part of it are not the axioms themselves but the examples we think of when developing the theory. Here are some of them.

The Objects:

1. Consider a subset $X \subset \mathbb{E}^d$ of the d-dimensional Euclidean space and define the following relation

$$A \mid B \iff \langle A \rangle \cap \langle B \rangle = \phi,$$

where $\langle A \rangle$ denotes the convex hull of A. If X is finite, the pair $\mathcal{P} = (X, |)$ is an acyclic separoid and will be called a *point separoid.* In fact, the name of separoids arises as a generalization of the fact that $A \mid B$ is a non-trivial separation if and only if there exists a hyperplane strictly separating $\langle A \rangle$ from $\langle B \rangle$. Theorem 2.0.1 characterizes an important class of point separoids.

2. Consider a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ and define the separoid $S(\mathcal{F})$ as above, this is, two subsets of the family $A, B \subset \mathcal{F}$ are separated if there exists a hyperplane that leaves all members of A strictly on one side of it and those of B on the other. If $\mathcal{F}$ is finite and the elements of $\mathcal{F}$ are compact, then $S(\mathcal{F}) = (\mathcal{F}, |)$ is an acyclic separoid and will be called a *separoid of convex sets.* The Geometric Representation Theorem (Theorem 2.3) proves that every finite acyclic separoid S is isomorphic to a separoid of convex sets in $\mathbb{R}^d$, where $d = |S| - 1$. There is also a Representation Theorem (Theorem 2.2) for the non-acyclic case but with non-compact convex sets and in a huge dimension —as big as the number of separations.

3. Consider an oriented matroid $\mathcal{M} = (E, \mathcal{L})$ and identify it with the subset $\mathcal{L} \subseteq \{-, 0, +\}^E$ of its *covectors* in the usual manner. Let $\mathcal{T} = \mathcal{T}(\mathcal{L})$ be the set of *topes,* maximal covectors, and define the following relation $| \subseteq 2^E \times 2^E$ on the subsets of E: $A, B \subset E$ are separated, $A \mid B$, if and only if there exists a tope $T \in \mathcal{T}$ such that $A \subseteq T^+$, and $B \subseteq T^-$. The pair $S(\mathcal{M}) = (E, |)$ is a separoid. In Chapter 3 this example will be studied in further detail, in particular it will be shown that the oriented matroid can be reconstructed from its separoid, and hence that separoids generalize oriented matroids.

4. Edelman (1984) has defined a complex which encodes the separoid of an oriented matroid. He considers the set

$$\Gamma(\mathcal{T}) := \{X \in \{-, 0, +\}^E : X \leq T \text{ and } T \in \mathcal{T}\},$$

where $\mathcal{T}$ denotes the topes of an oriented matroid and $\leq$ denotes the *conformal relation,* i.e., $X \leq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. Clearly a signed vector $X \in \Gamma$ is in Edelman's complex if and only if $X^+ \mid X^-$. He uses the Basic Sphericity Theorem to prove that such a complex has the homotopy type of a sphere. Theorem 3.2.3 is a direct consequence of this result —it is some how the dual version of it.

5. As a special case of oriented matroids, a separoid can be defined from a digraph $D = (V, E)$. Let the set of edges be the base set and define two subsets of it $A, B \subset E$ to be separated $A \mid B$ iff for every circuit of the graph in which the arrows in one direction are contained in A, the arrows in the other direction are not contained in B. $S(D) = (E, |)$ is a separoid, and it is acyclic if and only if D is so —hence the name.

6. Consider a graph $G = (V, E)$ and define two vertices $u, v \in V$ to be a minimal Radon partition $u \dagger v$ if and only if they form an edge $uv \in E$. The pair $S(G) = (V, \dagger)$ is also a separoid. This turns out to be a functoral embedding and, since graphs endowed with homomorphisms are a *universal* category, the universality of separoids follows (Theorem 4.4.1).

7. Consider a topological space $T = (X, \tau)$ and define two subsets $A, B \subset X$ to be separated if and only if there exist disjoint neighborhoods of them, i.e. if there exist $\alpha, \beta \in \tau$ such that $A \subseteq \alpha, B \subseteq \beta$ and $\alpha \cap \beta = \phi$. This is clearly an acyclic separoid.

8. All acyclic separoids on three elements arise from one of the eight families of convex sets in Figure 1. Those labeled **a, b, e** and **h** are the point separoids of order 3; in fact, they come from the four essentially different oriented matroids with three elements. They will be denoted by σ^2, Λ_3, $K_2{+}\sigma^0$ and K_3, respectively.

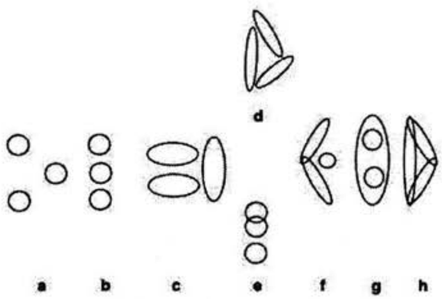


Figure 1. *The acyclic separoids of order 3*

The Morphisms:

9. Consider a family of convex sets $\mathcal{F}$, choose a point in each of its elements to construct a point separoid $\mathcal{P}$ and define the obvious bijection $\varphi: \mathcal{P} \to \mathcal{F}$. This is a morphism since every hyperplane that separates A from B, subsets of $\mathcal{F}$, also separates their respective points $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$.

10. Consider a family of convex sets $\mathcal{F}$ in $I\!R^d$ and let $\pi: I\!R^d \to I\!R^e$ be an affine projection. The obvious bijection $\hat{\pi}: \mathcal{F} \to \pi(\mathcal{F})$ is a morphism between their separoids $S(\mathcal{F}) \longrightarrow S(\pi(\mathcal{F}))$.

11. Consider the embedding $G \mapsto S(G)$ suggested by Example 6. If $\varphi: G \longrightarrow H$ is a graph homomorphism, the same map $\varphi: S(G) \to S(H)$ is a morphism of separoids (see Section 4.1).

12. Consider a family of convex sets $\mathcal{F}$ and give them a coloration $\varsigma: \mathcal{F} \to \{c_1, \ldots, c_k\}$. If we denote by $\mathcal{F}' = \{\langle \varsigma^{-1}(c_i) \rangle\}$ the convex hulls of the color classes' family, the obvious map $\mathcal{F} \longrightarrow \mathcal{F}'$ is a morphism. This is a key ingredient in our study of Tverberg's theorem (Section 4.5).

13. Consider a family of convex sets $\mathcal{F}$ and fatten them up. If $\mathcal{F}'$ denotes the new family, the obvious bijection $\mathcal{F} \longrightarrow \mathcal{F}'$ is a morphism.

14. Consider a family of convex sets $\mathcal{F} = \{K_1, \ldots, K_n\}$ with a hyperplane H transversal to all of them. If we denote by $\mathcal{F}' = \{H \cap K_1, \ldots, H \cap K_n\}$ the separoid of the intersections, then the obvious bijection $\mathcal{F}' \longrightarrow \mathcal{F}$ is also a morphism.

15. *Strong* and *weak maps* of oriented matroids are both examples of morphisms between their respective separoids.

16. In Figure 1, bijective morphisms go from left to right between every pair of separoids. Observe that there is no bijective morphism between those separoids labeled **d** and **e**.

2. The Geometric Dimension

This section introduces a basic invariant in separoid theory. It will be show that Example 2 is in fact the most general example, i.e. when thinking in separoids, we can always have in mind a family of convex sets and use all the intuition that comes from this picture without loss of generality. Let us start this section with some general facts of the separoid's category.

Given two separoids S and T over the sets X and Y respectively, their *product* $S \times T$ is defined as a separoid over the set $X \times Y$ with its two canonical projections π_X, π_Y and two subsets of it $A, B \subseteq X \times Y$ are separated iff at least one projection is, i.e.,

$$A \mid B \iff \pi_X(A) \mid \pi_X(B) \quad \text{or} \quad \pi_Y(A) \mid \pi_Y(B).$$

Clearly, this definition implies that the projections π_X, π_Y are separoid morphisms. To prove that this is the categorical product, we have to show that it satisfies the *universal property* of the product, this is

$$\begin{tikzcd} & U \arrow[dl, "\varphi"'] \arrow[d, dotted, "\xi"] \arrow[dr, "\psi"] & \\ S & S \times T \arrow[l, "\pi_S"] \arrow[r, "\pi_T"'] & T \end{tikzcd}$$

2.1. Lemma (the product). *Given two morphisms $\varphi: U \longrightarrow S$ and $\psi: U \longrightarrow T$, there exists a unique morphism $\xi: U \longrightarrow S \times T$ such that $\varphi = \pi_S\xi$ and $\psi = \pi_T\xi$.*

Proof. The category of sets (with functions) gives as unique candidate the function $\xi = (\varphi, \psi)$ so we only have to check that in fact this is a morphism of separoids. For this, let $A \mid B$ be a separation in $S \times T$ and suppose, without loss of generality, that $\pi_S(A) \mid \pi_S(B)$. Since $\varphi = \pi_S\xi$, it follows that $\varphi\xi^{-1}(A) = \pi_S(A)$ and $\varphi\xi^{-1}(B) = \pi_S(B)$. Therefore, since φ is a morphism,

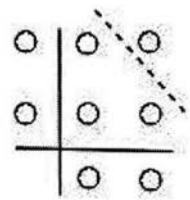
$$\varphi^{-1}\varphi\xi^{-1}(A) \mid \varphi^{-1}\varphi\xi^{-1}(B).$$

Now, since $\xi^{-1}(A) \subseteq \varphi^{-1}\varphi\xi^{-1}(A)$ and $\xi^{-1}(B) \subseteq \varphi^{-1}\varphi\xi^{-1}(B)$, we conclude that

$$\xi^{-1}(A) \mid \xi^{-1}(B),$$

and therefore ξ is a morphism. ⚽

Once the product has been defined for two separoids, the definition for a finite number of separoids $\prod_{i=1}^{m} S_i$ is obvious.



This product has a geometric counterpart. Let $\mathcal{S}$ and $\mathcal{T}$ be separoids of convex sets in $\mathbb{R}^s$ and $\mathbb{R}^t$, respectively. The *geometric product* $\mathcal{S} \otimes \mathcal{T}$ is a family of convex set in $\mathbb{R}^s \times \mathbb{R}^t$ whose elements are of the form $\mathcal{K}_s \times \mathcal{K}_t$, where $\mathcal{K}_s \in \mathcal{S}$ and $\mathcal{K}_t \in \mathcal{T}$. In general, it is not the case that the separoid of $\mathcal{S} \otimes \mathcal{T}$ is isomorphic to its combinatorial counterpart $\mathcal{S} \times \mathcal{T}$ however, in some special cases, if the convex sets are "big enough", $\mathcal{S} \otimes \mathcal{T}$ is a realization of $\mathcal{S} \times \mathcal{T}$.

2.2. Representation Theorem. *Every separoid S can be represented with a family of convex sets in $\mathbb{R}^m$, where m is the number of separations in S.*

Proof. Given a separoid S and a separation in it $A \mid B$, a *characteristic morphism* $\chi_{A|B}: S \to \mathcal{B}$ exists

$$\chi_{A|B}(x) = \begin{cases} +, & \text{if } x \in A \\ -, & \text{if } x \in B \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{B}$ denotes the separoid defined in the set $\{-, 0, +\}$ with unique separation $- \mid +$. It is not hard to prove that S can be embedded into the product of as many copies of $\mathcal{B}$ as separations S has

$$\chi: S \longrightarrow \prod_{A|B \in S} \mathcal{B}.$$

The existence of such a morphism is given by the previous lemma. In order to see that χ can be made injective, consider two different elements $x \neq y \in S$.

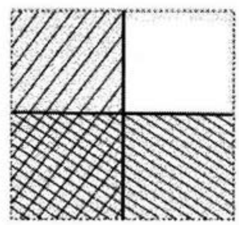
In the one hand, if there exists a separation $A \mid B$, where $x \in A$, we have then that $x \mid B$. Therefore, $\chi_{x|B}(x) \neq \chi_{x|B}(y)$ and therefore $\chi(x) \neq \chi(y)$.

On the other hand, if there are no separations with the elements x and y, both are mapped to the element $\mathbf{0} = (0, \ldots, 0) \in \Pi\mathcal{B}$. We can then take as many copies of this element $\mathbf{0}$ to map each element x with such a property. Equivalently, we can identify all of them as a single element; after the realization below, we can then consider as many copies of the ambient space —which will represents the element $\mathbf{0}$— to realize the original separoid. Observe that such a construction does not change the number of separations of the separoid.

Finally, to see that χ^{-1} is also a morphism observe that, if $A \mid B$, there is a projection $\pi: \prod \mathcal{B} \longrightarrow \mathcal{B}$ such that $\chi_{A|B} = \pi\chi$ and therefore $\chi(A) \mid \chi(B)$.

The end of the proof is to show how to realize $\prod_{i=1}^m \mathcal{B}$ as a family of convex sets; the restriction to $\chi(S)$ will then realize S.

For, in the real line, let $\mathcal{B}$ be mapped as follows:



$$\begin{array}{ccc} - & \mapsto & \mathbb{R}^-, \\ 0 & \mapsto & \mathbb{R}, \\ + & \mapsto & \mathbb{R}^+. \end{array}$$

Clearly this realizes the separoid $\mathcal{B}$. The product of m copies of it can be realized in $\mathbb{R}^m$ by the geometric product of these convex sets: since the convex sets in the geometric product are "big enough", all separations can be made with hyperplanes which are parallel to some of the linear hyperplanes spanned by some $m-1$ canonic vectors.

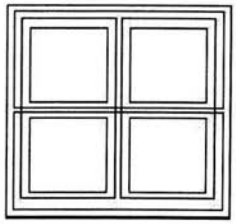


Fig 2. $\mathcal{B} \times \mathcal{B}$

It can be proved that, in the acyclic case, to compactify the realization of $\mathcal{B}^m$, it is enough to take the intervals $[-1,-\epsilon]$, $[-1,1]$ and $[\epsilon,1]$, with $0<\epsilon\leq\frac{1}{2m-1}$ (see Figure 2).

The *geometric dimension* of a separoid can now be defined as the minimum dimension of the Euclidian space where the given separoid S can be realized as a separoid of convex sets; we denote it by gd(S). There are not known algorithms to calculate this invariant and it is conjectured that it is, at least, an NP-hard problem. It is important to give better upper bounds of gd(S) than that implicitly given in the theorem; in particular, we know that, in the acyclic case, it grows at most linearly with respect to the order:

2.3. Geometric Representation Theorem. *Every acyclic separoid of order n can be represented by a family of convex polytopes in the $(n-1)$-dimensional affine space, and therefore*

$$\textit{gd}(S) \leq |S|-1.$$

Proof. Let $(S,\dagger)$ be an acyclic separoid (i.e., $A\dagger B \implies |A||B|>0$). To each element $i\in S=\{1,\ldots,n\}$ and each (minimal Radon) partition $A\dagger B\in MRP$ such that $i\in A$, we assign a point of $I\!R^n$

$$\rho^i_{A\dagger B} := \mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right], \tag{1}$$

(where $\{\mathbf{e}_j\}$ denotes the canonical basis) and realize each element $i\in S$ as the convex hull of all such points

$$i\mapsto \mathcal{K}_i := \langle \rho^i_{A\dagger B} : i\in A \text{ and } A\dagger B\in MRP\rangle.$$

Observe that these convex sets are in the $(n-1)$-dimensional affine subspace spanned by the basis, becouse (1) is, in fact, an affine combination.

To see that this family of convex polytopes realizes the separoid observe that, in the one hand, for each partition $A\dagger B$, the vertex set of the two simplices $\langle\mathbf{e}_a : a\in A\rangle$ and $\langle\mathbf{e}_b : b\in B\rangle$ "moves" —half of the way each— to realize such a partition intersecting one another precisely in their baricenter. That is, let $A\dagger B$ be fixed; in order to prove that

$$\langle\mathcal{K}_a : a\in A\rangle \cap \langle\mathcal{K}_b : b\in B\rangle \neq \phi,$$

it is enough to prove that $\langle\rho^a_{A\dagger B} : a\in A\rangle\cap\langle\rho^b_{B\dagger A} : b\in B\rangle\neq\phi$ because $\rho^a_{A\dagger B}\in\mathcal{K}_a$ and therefore $\langle\rho^a_{A\dagger B} : a\in A\rangle\subset\langle\mathcal{K}_a : a\in A\rangle$ (analogously with B).

Now, if we let $\rho : I\!R^n\to I\!R^n$ be the translation

$$\rho(\mathbf{x}) = \mathbf{x} + \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b - \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right],$$

we have that $\rho^a_{A\dagger B} = \rho(\mathbf{e}_a)$ and the baricenter of $\langle \rho^a_{A\dagger B} : a \in A\rangle$ is

$$\frac{1}{|A|}\sum_{a\in A}\rho^a_{A\dagger B} = \frac{1}{|A|}\sum_{a\in A}\rho(\mathbf{e}_a) = \rho\left(\frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right) = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

Analogously, using that $\rho^b_{B\dagger A} = -\rho(-\mathbf{e}_b)$, we have that

$$\frac{1}{|B|}\sum_{b\in B}\rho^b_{B\dagger A} = \frac{1}{2}\left[\frac{1}{|B|}\sum_{b\in B}\mathbf{e}_b + \frac{1}{|A|}\sum_{a\in A}\mathbf{e}_a\right]$$

and therefore

$$\langle \rho^a_{A\dagger B} : a \in A\rangle \cap \langle \rho^b_{B\dagger A} : b \in B\rangle \neq \phi.$$

On the other hand, given a separation $\alpha \mid \beta$, define the affine extension of the equations

$$\psi_{\alpha|\beta}(\mathbf{e}_j) = \begin{cases} -1 & j\in\alpha \\ 1 & j \in \beta \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } j = 1,\dots,n.$$

Now, it is enough to prove that for every $i \in \alpha$ (resp. β), we have that $\psi_{\alpha|\beta}(\rho^i_{A\dagger B}) < 0$ (resp. > 0) for all $A \dagger B$ such that $i \in A$. For this, observe that, if $i \in \alpha \cap A$ (and $A \dagger B$) then, since $\psi_{\alpha|\beta}(\mathbf{e}_j) = 0$ for all $j \in (A\cup B)\setminus(\alpha\cup\beta)$,

$$\psi_{\alpha|\beta}\left(\rho^i_{A\dagger B}\right) = \psi_{\alpha|\beta}\left(\mathbf{e}_i + \frac{1}{2}\left[\frac{1}{|B|}\Big(\sum_{B\cap\alpha}\mathbf{e}_b + \sum_{B\cap\beta}\mathbf{e}_b\Big) - \frac{1}{|A|}\Big(\sum_{A\cap\alpha}\mathbf{e}_a + \sum_{A\cap\beta}\mathbf{e}_a\Big)\right]\right) =$$

$$= -1 + \frac{(|B\cap\beta| - |B\cap\alpha|)}{2|B|} + \frac{(|A\cap\alpha| - |A\cap\beta|)}{2|A|} \leq -1 + \frac{1}{2} + \frac{1}{2} = 0.$$

Equality holds if and only if $B\cap\beta = B$ and $A\cap\alpha = A$ leading to a contradiction.⚽

We end this section showing how to prove that the combinatorial dimension bounds the geometric dimension.

2.4. Lemma (dimension). *For any separoid S, its combinatorial dimension is not greater than its geometric dimension, i.e., d$(S) \leq$ gd(S).*

Proof. Let S be d-dimensional with geometric dimension $g =$ gd(S), and suppose that $g < d$. Let $\mathcal{S}$ be a family of convex sets in $\mathbb{R}^g$ that realizes S. Since S is d-dimensional, it contains a d-dimensional simploid $\sigma \subseteq \mathcal{S}$ of order $d+1$. Choose a point for each convex set of σ. This set of points consists of $g+2$ or more points in $\mathbb{R}^g$ and, by the classic Radon's theorem, there exists a partition of them in two subsets whose convex hulls intersect. Therefore they are not separated. This contradicts the fact that σ was a simploid. ⚽

3. General Results

In this section we settle some general results on separoids that will be needed somewhere else. We start with a new "convex version" of the well known Carathéodory's theorem (cf. Danzer et al. 1963 and see also Eckhoff 1993):

3.1. Lemma (Carathéodory). *Let $X = \bigcup_{i\in I}\mathcal{K}_i \subseteq I\!R^d$, be the union of some convex sets $\mathcal{K}_i$. If $\mathbf{x} \in \langle X\rangle$ is a point in the convex hull of X, then there exists a subset $J \subseteq I$ with $|J| \leq d+1$ and, for every $j \in J$, a point $\mathbf{x}_j \in \mathcal{K}_j$ such that $\mathbf{x}$ is a convex combination of the points $\mathbf{x}_j$.*

Proof. By Carathéodory's theorem, we need at most $d+1$ points of X to express $\mathbf{x}$ as a convex combination of them. It is easy to see that, if two (or more) of these are on the same convex set $\mathcal{K}_j$, they can be replaced by a single point $\mathbf{x}_j \in \mathcal{K}_j$ which is a convex combination of them. Therefore we need at most one point in each convex.

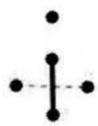
With this lemma at hand, it is easy to see how to "realize" each minimal Radon partition of a separoid.

3.2. Theorem. *Let $\mathcal{S}$ be a separoid of convex sets. Given a minimal Radon partition $A \dagger B$, there exists a point on each convex set of the support, $\mathbf{a}_i \in \mathcal{K}_i \in A$ and $\mathbf{b}_j \in \mathcal{K}_j \in B$, such that*

$$\langle \mathbf{a}_i : \mathcal{K}_i \in A\rangle \cap \langle \mathbf{b}_j : \mathcal{K}_j \in B\rangle \neq \phi.$$

Proof. If $\mathbf{x} \in \langle A\rangle \cap \langle B\rangle \neq \phi$, by Carathéodory's lemma, we need at most $d+1$ elements of A, $\mathcal{K}_i \in A$, and at most one point in each of them $\mathbf{a}_i \in \mathcal{K}_i$ to express $\mathbf{x}$ as a convex combination of them. By the minimality of the partition, it is clear that we need at least one point in each convex of A. The same argument works for B and we are done.

We will also use a "continuous version" of Radon's original proof.



3.3. Lemma (continuous Radon). *Let $\mathbf{z}_i(t) = (1-t)\mathbf{x}_i + t\mathbf{y}_i$ with $t \in [0,1]$, be $d+2$ segments in $I\!R^d$. If their respective extreme points, $\{\mathbf{x}_i\}$ and $\{\mathbf{y}_i\}$, are different point separoids in general position, there exists a $t \in (0,1)$ such that the separoid $\{\mathbf{z}_i(t)\}$ is not in general position.*

Proof. It is easy to see that, for every $t \in [0,1]$ there exists a solution for the following equations

$$\sum \lambda_i(t)\mathbf{z}_i(t) = 0, \quad \sum \lambda_i(t) = 0, \quad \sum |\lambda_i(t)| = 2,$$

and moreover the $\lambda_i(t)$ can be chosen to be continuous. Since the points $\mathbf{x}_i = \mathbf{z}_i(0)$ are in general position, such a solution for $t = 0$ is unique and every $\lambda_i(0)$ is non-zero. Such a solution leads to a unique Radon partition, the positives vs. the negatives

$$\sum_{\lambda_i(0)>0} \lambda_i(0)\mathbf{x}_i = -\sum_{\lambda_i(0)<0} \lambda_i(0)\mathbf{x}_i,$$

$$\sum_{\lambda_i(0)>0} \lambda_i(0) = - \sum_{\lambda_i(0)<0} \lambda_i(0) = 1$$

or, in separoid notation

$$\{\mathbf{x}_i : \lambda_i(0) > 0\} \dagger \{\mathbf{x}_i : \lambda_i(0) < 0\}.$$

The same argument works for $t = 1$, but by hypothesis it yields a "different" partition

$$\{\mathbf{y}_i : \lambda_i(1) > 0\} \dagger \{\mathbf{y}_i : \lambda_i(1) < 0\}.$$

Here, *different* means that there is a j such that $\lambda_j(0)$ and $\lambda_j(1)$ have different signs (while others have the same), then there exists a $t \in (0,1)$ such that $\lambda_j(t) = 0$. For that t, $\{\mathbf{z}_i(t)\}$ is not in general position. ⚽

We close this section with a beautiful theorem of separoids that will be used to characterize point separoids in general position.

3.4. Theorem. *If a separoid is in general position and its geometric dimension is equal to its dimension, then it is a Radon separoid.*

Proof. Let S be a d-dimensional separoid in general position. If its geometric dimension is equal to its dimension, it can be realized as a family $\mathcal{S}$ of convex sets in $\mathbb{R}^d$. Not to be a Radon separoid would imply that there exists a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ with two "essentially different" ways of choosing points on each convex set of it. This is, suppose that S **is not** a Radon separoid. Then there are subsets of $A, B, C, D \subseteq S$ such that $A \dagger B$, $C \dagger D \in MRP$, $A \cup B \subseteq C \cup D$ and $\{A, B\} \neq \{C, D\}$. Since S is in general position, the support $S' := A \cup B$ has at least $d + 2$ elements. Since $C \dagger D$ is minimal, applying Theorem 1 and the classic Radon's theorem, its support has at most $d + 2$ elements. Then, without loss of generality, we may suppose that $|S| = |A \cup B| = d + 2$.

Using again Theorem 1, two configuration of points can be defined, two points on each convex set, in such a way that they realize the two Radon partitions. Considering the line segment that join each couple —inside each convex set— and applying the continuous Radon lemma, we conclude that S **is not** in general position ↯ ⚽

4. The Radon Complex

We will associate to each separoid a "cubic" complex which "lives" in the boundary complex of the n-cube. We will consider the Radon components of a separoid. They can be identified with some vertices of the n-cube and the *Radon complex* of the separoid will be defined as the induced complex of these vertices.

Let $\mathcal{Q}_n$ denote the n-cube (see Figure 3). Its vertices $V(\mathcal{Q}_n)$ will be identified with the family of subsets 2^E of the n-set E. Its faces are intervals of the natural contention partial order defined in 2^E, i.e., each face is of the form

$$[A, B] := \{X \subseteq E : A \subseteq X \subseteq B\}.$$

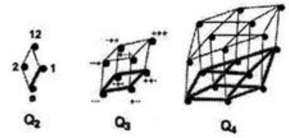


Fig 3. $\mathcal{Q}_n$, $n = 2, 3, 4$

In fact, this definition leads to an n-ball, but in the sequel the n-cube will be thought of as an $(n-1)$-sphere so the face $[\phi, E]$ is dropped out.

We call *generalized cotopes* the Radon partitions of the form $A \dagger \overline{A}$, where $\overline{A} = S \setminus A$ denotes the complement. Given a separoid $\mathcal{S}$, for each generalized cotope $A \dagger \overline{A}$, take the vertex $A \in V(\mathcal{Q}_n)$; the complex induced by all such vertices is what we call the *Radon complex* of the separoid and we denote it by $\mathcal{R} = \mathcal{R}(\mathcal{S})$. Here, by *induced* we mean that a face of $\mathcal{Q}_n$ is in the complex if and only if all of its vertices are. Some small Radon complexes are shown in Figure 4.

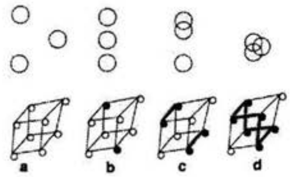


Fig 4. *The Radon complex of the four points separoids on three elements*

It follows from the definition that

4.1. Lemma (faces). *If $A \dagger B$ is a Radon partition of $\mathcal{S}$ then $[A, \overline{B}]$ is a face of $\mathcal{R}(\mathcal{S})$.*

Proof. Let $A \dagger B$ be a Radon partition of a separoid $\mathcal{S} = (X, |)$. It is clear that for all $C \subseteq \overline{B}$ we have that $(A \cup C) \dagger B$, therefore every vertex of $[A, \overline{B}]$ is a Radon partition's component of the given separoid $\mathcal{S}$. ⚽

The converse of this lemma is not true in general, this is, there exists a separoid $\mathcal{S}$ such that $[A, \overline{B}]$ is a face of $\mathcal{R}(\mathcal{S})$ and $A \dagger B$ is not a Radon partition. Therefore the generalized cotopes do not determine the separoid. In fact, there are many separoids which yield the same Radon complex (cf. Figure 1 and 4 and observe that, while there are eight acyclic separoids on 3 elements, there are only four possible Radon complex in $\mathcal{Q}_3$).

However, in some important cases the separoid can be reconstructed from its Radon complex. In particular oriented matroids, and therefore point separoids, are completely encoded by the Radon complex.

The faces of $\mathcal{R}$ with maximal dimension (that are not contained in a bigger face) are called *facets*. We say that two facets $X, Y \in \mathcal{R} \subset \mathcal{Q}_n$ are *incident* if their intersection is not empty, and *adjacent* if it is as big as possible, i.e., they are the only two covers of their intersection in the face lattice of $\mathcal{R}$. The graph with the set of facets as vertices and adjacent facets as edges is called the *circuit graph* of the separoid. Its vertices are called *circuits* and if every circuit comes from a minimal Radon partition, the separoid is said to be *full*, this is, full separoids are those for which the converse of the previous lemma holds. It will be shown that: *Oriented matroids and point separoids, are full separoids.*

We will say that a separation $A \mid B$ is *maximum* if the union of its parts $A \cup B = S$ is the separoid itself. We say that a separation $A \mid B$ *conforms* to the separation $C \mid D$ if $A \subseteq C$ and $B \subseteq D$.

4.2. Lemma (full separoids). *Let S be a separoid. If every separation conforms to a maximum separation then S is a full separoid.*

Proof. Let $[A, \overline{B}]$ be a face of $\mathcal{R}(S)$ and denote by $C = \overline{B} \setminus A$ the difference of those subsets. Clearly every vertex of such a face are of the form $A \cup C'$, for some $C' \subseteq C$. Then, since they are vertices of the complex, for all $C' \subseteq C$ we have that $A \cup C' \dagger \overline{A \cup C'}$, this is, the set $\{A \cup C' : C' \subseteq C\}$ is a subset of the components of S.

Now, in order to search for a contradiction, suppose that $A \mid B$. The hypothesis says that this separation conforms to a maximum one. Denote by C_a, respectively C_b, those elements of C which are on the same side of A, respectively B, so $C = C_a \cup C_b$. From this definition follows that $A \cup C_a \mid B \cup C_b$. But as previously settled, $A \cup C_a \dagger \overline{A \cup C_a}$ $\lightning$

It remains an open question to find necessary and sufficient conditions, in terms of Radon partitions, to characterize full separoids.

If a separoid S is in general position, all the facets of its Radon complex have the same dimension, say k, so its circuit graph is a subgraph of the *k-dual of the n-cube* denoted by $\mathcal{Q}_n^k$ and defined as follows ($k > 0$): the vertices of $\mathcal{Q}_n^k$ are the k-subcubes of $\mathcal{Q}_n$ and two of them are adjacent if their respective subcubes intersect in a $(k-1)$-subcube. From now on, we denote the faces of $\mathcal{Q}_n$ by the standard signed vectors, this is, each face $[A, B]$ is denoted by $X \in \{-, 0, +\}^E$ where

$$X_i = \begin{cases} + & \text{if } i \in A, \\ 0 & \text{if } i \in B \setminus A, \\ - & \text{otherwise.} \end{cases}$$

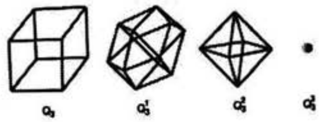


Fig 5. *The three k-duals of the 3-cube ($k = 1, 2, 3$)*

We call *antipodal automorphism* the function which sends each vector X to its opposite $-X$. Observe that if every circuit of a separoid has the same support size (say $n - k$), the 1-skeleton of the dual poset of the Radon complex of such a separoid is a subgraph of $\mathcal{Q}_n^k$ and it is closed by the antipodal automorphism.

5. Transversal Theory

In this section we will study, from the separoids point of view, the space of **all** hyperplanes transversal to a family of convex sets $\mathcal{S} = \{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$ in the Euclidian space $\mathbb{E}^{d+1}$.

Let $\mathcal{S}$ be a separoid of convex set in $\mathbb{E}^{d+1}$. An affine subspace $\mathcal{H} \hookrightarrow \mathbb{E}^{d+1}$ of dimension d is called a *hyperplane transversal* if the intersection with each convex set of $\mathcal{S}$ is non-empty. We are interested in the set of all such hyperplanes. This can be considered as a subset of the open manifold $\mathcal{G}^d := \mathcal{G}^d(\mathbb{E}^{d+1})$ which consists of all affine hyperplanes of $\mathbb{E}^{d+1}$. It is ease to see that $\mathbb{P}^d = \mathcal{G}^d(\mathbb{R}^{d+1})$, the *Grassmannian* of linear d-subspaces of $\mathbb{R}^{d+1}$, is an homotopical retraction of $\mathcal{G}^d$ —just identify each pencil of parallels with its linear representative— and therefore they are homotopically equivalent $\mathcal{G}^d \approx \mathbb{P}^d$.

Let us denote by

$$\mathcal{T}(\mathcal{S}) := \{\mathcal{H} \in \mathbb{P}^d : \exists \mathbf{v} \in \mathbb{R}^{d+1} \forall \mathcal{K} \in \mathcal{S}, \mathcal{K} \cap (\mathcal{H} + \mathbf{v}) \neq \phi\}$$

the *space of hyperplanes transversal* to $\mathcal{S}$.

5.1. Proposition. *Let $\mathcal{S}$ be a separoid of convex sets in $\mathbb{E}^{d+1}$. Then*

$$\mathcal{T}(\mathcal{S}) \neq \phi \implies \mathsf{d}(\mathcal{S}) < d + 1.$$

Proof. If $\mathsf{d}(\mathcal{S}) = d + 1$, $\mathcal{S}$ contains a simploid σ of order $d + 2$ and there is not a hyperplane transversal to it. The existence of such a hyperplane would contradict, via the Radon's theorem, that σ is a simploid. ⚽

We say that $\mathcal{T} = \mathcal{T}(\mathcal{S})$ is an *essential* subspace of $\mathcal{G}^d$ if it is not homotopically equivalent to a point inside $\mathbb{P}^d$, i.e., $\mathcal{T}$ cannot be continuously contracted to a point in its ambient space $\mathbb{P}^d$.

5.2. Theorem. *Let $\mathcal{S}$ be a separoid of convex sets in $\mathbb{E}^{d+1}$. If $\mathsf{gd}(\mathcal{S}) < d$, then $\mathcal{T} = \mathcal{T}(\mathcal{S})$ is an essential subspace of $\mathcal{G}^d$.*

Proof. Let $\mathcal{S}$ be a separoid of convex sets in $\mathbb{E}^{d+1}$, suppose that $\mathsf{gd}(\mathcal{S}) = d - k$ and let $\mathcal{F}$ be a realization of $\mathcal{S}$ in $\mathbb{R}^{d-k}$. Choose a point in each convex of $\mathcal{F}$ to construct the point separoid $\mathcal{P}$ and define the obvious morphism $\varphi : \mathcal{P} \longrightarrow \mathcal{S}$.

Now, let us denote by $\mathcal{T}^\perp := \{\mathbf{v} \in \mathbb{S}^d : \mathbf{v}^\perp \in \mathcal{T}\}$ the closed set of all unit vectors for which there exists a hyperplane transversal to $\mathcal{S}$ orthogonal to it, and let $\overline{\mathcal{T}^\perp} = \mathbb{S}^d \setminus \mathcal{T}^\perp$ be its relative complement.

For every $\mathbf{u} \in \overline{\mathcal{T}^\perp}$ we may choose continuously a hyperplane $\mathcal{H}_\mathbf{u} \in \mathcal{G}^d$ orthogonal to $\mathbf{u}$ such that $\mathcal{H}_\mathbf{u} = \mathcal{H}_{-\mathbf{u}}$ and it separates two non-empty subsets of the given separoid $\mathcal{S}$. Denote by $\mathcal{H}_\mathbf{u}^+$ the closed semispace determined by $\mathcal{H}_\mathbf{u}$ that has $\mathbf{u}$ as normal. A straight forward argument proves that the function

$$p(\mathbf{u}) := \sum_{\mathcal{K} \subset \mathcal{H}_\mathbf{u}^+} d(\mathcal{K}, \mathcal{H}_\mathbf{u})$$

is never zero and depends continuously of $\mathbf{u} \in \mathbb{S}^d$.

Let the function $f\colon \overline{\mathcal{T}^\perp} \to \mathbb{R}^{d-k}$ be defined as

$$f(\mathbf{u}) := \sum_{\mathcal{K} \subset \mathcal{H}^+_{\mathbf{u}}} \frac{d(\mathcal{K}, \mathcal{H}_{\mathbf{u}})}{p(\mathbf{u})} \varphi^{-1}(\mathcal{K}).$$

In order to find a contradiction, suppose that $\mathcal{T}$ is **not** essential in $\mathcal{G}^d$. Then, $\mathcal{T}^\perp$ is contained in a subset of $\mathbb{S}^d$ homotopic to the 0-sphere and, by the Alexander's duality, $\overline{\mathcal{T}^\perp}$ contains a $(d-1)$-pseudosphere. Therefore, due to Borsuk-Ulam's theorem, there most be a point $\mathbf{u}_0 \in \overline{\mathcal{T}^\perp}$ such that

$$f(-\mathbf{u}_0) = f(\mathbf{u}_0).$$

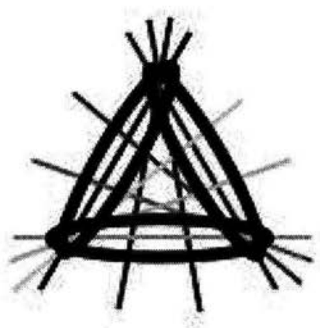
Let $A := \{\mathcal{K} \in \mathcal{S} : \mathcal{K} \subset \mathcal{H}^+_{\mathbf{u}_0}\}$ and $B := \{\mathcal{K} \in \mathcal{S} : \mathcal{K} \subset \mathcal{H}^+_{-\mathbf{u}_0}\}$ be the subsets of $\mathcal{S}$ separated by the hyperplane $\mathcal{H}_{\mathbf{u}_0}$. Since φ is a separoids morphism, then

$$\varphi^{-1}(A) \mid \varphi^{-1}(B).$$

On the other hand, observe that $f(\mathbf{u}_0)$ is a convex combination of points of $\varphi^{-1}(A)$ and also $f(-\mathbf{u}_0)$ is a convex combination of points of $\varphi^{-1}(B)$. Therefore,

$$f(\mathbf{u}_0) = f(-\mathbf{u}_0) \in \langle \varphi^{-1}(A) \rangle \cap \langle \varphi^{-1}(B) \rangle \neq \phi,$$

which is a contradiction ↯ ⚽



An ease consequence is the following **Corollary (Helly).** *Let $\mathcal{S}$ be a separoid of convex sets. If every two members of $\mathcal{S}$ have a common point, then $\mathcal{T}(\mathcal{S})$ has the same homotopy type as if there were a common point to all of them.*

Hasta que no me concedas esto con plena convicción, querido lector, no sigas leyendo.

—ALBERT EINSTEIN *Sobre la teoría especial y la teoría general de la relatividad* (1917)

2
Configurations

In this chapter we will concentrate in a very specific class of separoids. We will study those separoids which arise from an affine configuration of points. Let us start with a guide example. We will think on it all across the chapter so it is a good idea that the reader takes all the time (s)he needs to analyze it.

Consider the vertices of a regular pentagon and identify them with the numbers $\mathcal{P} = \{0, 1, 2, 3, 4\}$ in some of its two cyclic orders. Due to Radon's theorem, there should by two disjoint subsets $A, B \subset \mathcal{P}$ whose convex hulls intersect, or in separoids notation, $A \dagger B$. In fact, there are ten such pairs:

$$\begin{array}{ccccc} 02 \dagger 13 & 13 \dagger 24 & 24 \dagger 03 & 03 \dagger 14 & 14 \dagger 02 \\ 024 \dagger 13 & 013 \dagger 24 & 124 \dagger 03 & 023 \dagger 14 & 134 \dagger 02 \end{array}$$

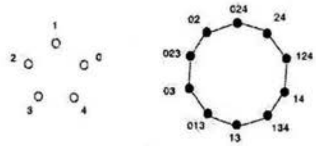


Fig 6. *The regular pentagon and its Radon complex*

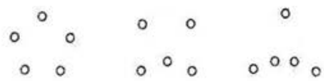


Fig 7. *Five points in the plane in general position*

Observe how $02 \dagger 13 \implies 024 \dagger 13$ and $02 \dagger 134$. More over, observe that for each minimal Radon partition $A \dagger B$ there exists a unique $d \in A \cup B$ such that $(A \setminus d) \dagger (B \setminus d \cup c)$, where $c = \overline{A \cup B}$.

If we draw an edge between a pair of Radon partitions whenever one implies the other, the resulting graph is a cycle of order 10. In fact this graph is isomorphic to the Radon complex of the separoid.

It is very easy to see that all regular pentagons in the plane are affinely equivalent. This is, given two regular pentagons, there is an affine transformation of the space that sends one onto the other —just translate the center of one into the other, scale it to reach the same size and rotate if necessary so the five points coincide— therefore we can identify them to say that they *represent the same configuration of points.* However not every two pentagons represents the same configuration —even their separoids are isomorphic— because an affine transformation of the plane is determined by the image of a triangle.

Let us give a step back —in fact two steps back— to analyze $\mathbb{A}_3^1$ in full detail. If we have three points in the line in general position, its separoid is of the form $a \dagger bc$ and —since we are dealing with affine transformations— with out loose of generality we may suppose that b and c are represented by 0 and 1, respectively. The relative position of a between b and c is parametrized with a number in the interval $(0, 1)$ so, we may think in the space of configurations of three points in the line in general position as the union of three open intervals. Now, if we miss the general position the separoid gets the form $a \dagger b$ or, equivalently, $a \dagger c$. This configuration is "rigid" —it can not be continuously transformed without changing the separoid— and there are three of them. Observe that this configurations are the limit of the previous described intervals. Therefore, the space $\mathbb{A}_3^1$ of **all** configurations (modulo affine transformations) of three points in the line is a cycle of order 3 and it is homeomorphic to $\mathbb{P}^1 \cong \mathbb{S}^1$.

The reader most be aware to distinguish between these two examples: the former cycle (of length 10) was associated to a particular configuration, the regular pentagon; the latter cycle (of length 3) was associated to the "space" of **all** configurations of 3 points in the line.

0. Uniform Point Separoids

Point separoids are those separoids which can be realized by a configuration of points in some Euclidian space. They are extremely difficult to characterize from a purely combinatorial point of view. In fact, it is known that the *stretchability problem* —a polar version in dimension 2— is NP-hard (cf. Shor 1991). However, from the geometric point of view intrinsic to separoids, we can characterize those point separoids in general position.

0.1. Theorem. *Let $\mathcal{S}$ be a separoid in general position. $\mathcal{S}$ is a point separoid if and only if its dimension and its geometric dimension are equal.*

Proof. The necessity is clear. For the sufficiency, consider $\mathcal{S}$ as a separoid of convex sets in $\mathbb{R}^d$, where $d = \mathrm{d}(\mathcal{S})$. Choose a point in each convex set, denote by $\mathcal{P}$ the point separoid that they define, and let

$$\varphi: \mathcal{P} \longrightarrow \mathcal{S}$$

be the obvious morphism (see Example 9). We will show that, in fact, this is an isomorphism of separoids.

In the one hand, by construction, we have that for every $A, B \subset \mathcal{S}$,

$$A \mid B \implies \varphi^{-1}(A) \mid \varphi^{-1}(B).$$

On the other hand, let $A \dagger B \in MRP$ be a minimal Radon partition of $\mathcal{S}$. Since $\mathcal{S}$ is a separoid in general position, the cardinality of the support is $\#(A \cup B) \geq d+2$. Then the preimage of this union consists of $d+2$ or more points in $\mathbb{R}^d$ and by the classic Radon's theorem there exists a Radon partition $D \dagger E$ of $\varphi^{-1}(A \cup B)$ in $\mathcal{P}$. Since φ is a morphism, $\varphi(D) \dagger \varphi(E)$ is a Radon partition of $A \cup B$. Finally, due to Theorem 1.3.4, $\mathcal{S}$ is a Radon separoid and, without loss of generality $\varphi(D) = A$ and $\varphi(E) = B$. Therefore $\varphi^{-1}(A) \dagger \varphi^{-1}(B)$. Since MRP generates all Radon partitions, it follows that for every $A, B \subset \mathcal{S}$,

$$A \dagger B \implies \varphi^{-1}(A) \dagger \varphi^{-1}(B).$$

Thus, φ is an isomorphism of separoids and $\mathcal{S}$ is a point separoid. ⚽

This result is sharp. The hypothesis of general position cannot be dropped without adding a new ingredient. The separoid $\mathcal{B}$ used in the proof of the Representation Theorem is a 1-dimensional separoid in general position, it can be realized in the line but it is not a point separoid. However, the small examples of non-stretchable pseudolines arrengments suggest the following (cf. Theorem 3.1.2)

0.2. Conjecture. *An oriented matroid is coordinatizable if and only if its dimension (its rank minus one) is equal to its geometric dimension.*

1. The Grassmannian

In the study of point separoids is quite difficult to avoid their algebraic properties. There are plenty of them. In particular, the *Grassman-Plüker* relations and the *Grassmann variety* it self appears naturally in this context.

We had review some examples of *configurations of points,* finite subsets of the affine space $\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \subset \mathbb{A}^d$ —we will always suppose that the points spans affinely their ambient space. Given a configuration $\mathcal{P}$, a linear function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^d$ can be defined as the linear extension of the equations

$$\varphi(\mathbf{e}_i) = \mathbf{p}_i, \quad i = 1, \ldots, n,$$

where $\{\mathbf{e}_i\}$ denotes the canonical basis of $\mathbb{R}^n$. Given two such functions φ, φ', it will be said that they represent *the same* configuration if there exists an affine transformation $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\psi\varphi(\mathbf{e}_i) = \varphi'(\mathbf{e}_i), \quad i = 1, \ldots, n,$$

i.e., two configurations are the same if one is the (ordered) image of the other by an affine transformation of the ambient space.

We will call *space of configurations,* and denote it by $\mathbb{A}^d_n$, to the set of **all** configurations with n points in dimension d (modulo $\mathbb{A}(d)$, the affine group). This set will be provided of structure and will be described in some detail.

The first example is the case $n = d + 1$ and the space of configurations consists of a single point that represents the simploid σ^d. The object starts to be more interesting when $n \geq d + 2$. It is the case where Radon's theorem applies. It guarantees that there exists a partition $A \dagger \overline{A}$ of $\mathcal{P}$ and we can consider the set of components $\mathcal{C} = \{A \subset \mathcal{P} : A \dagger \overline{A}\}$ that is a subset of the vertices of the n-cube —the family of subsets of $\mathcal{P}$— that induces a polytopal complex $\mathcal{R}(\mathcal{P}) = \mathcal{Q}_n[\mathcal{C}] = \{[A, \overline{B}] \in \mathcal{Q}_n : A \dagger B \text{ in } \mathcal{P}\}$ known as the Radon complex of $\mathcal{P}$. We will prove later in this chapter that the Radon complex of a point separoid is homotopically equivalent to the $(n - d - 2)$-sphere.

The space which consists of the k-subspheres of the n-sphere in known as the *Grassmann variety* (or, for short, *grassmannian*) and it will be carefully defined bellow but, by the moment —as a motivation— let us say that this space is homeomorphic to the space of configurations.

In the classic literature, grassmannians are defined as the set of all subspaces, with a fixed dimension, of a vector space. Here we will refer exclusively to subspaces of $\mathbb{R}^n$ and the the grassmannian of k-subspaces (k-planes) of $\mathbb{R}^n$ will be denoted by $\mathcal{G}^k = \mathcal{G}^k(\mathbb{R}^n)$.

This set can be provided with a natural topology. Every k-plane is the kernel of some linear function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ and two such functions have the same kernel if and only if there exists a linear transformation $\psi \in GL(n-k)$ that makes the following diagram commutative

$$\begin{tikzcd}
K \arrow[r, hook, "i"] & \mathbb{R}^n \arrow[d, "\varphi"'] \arrow[dr, "\varphi'"] & \\
 & \mathbb{R}^{n-k} \arrow[r, tail, "\psi"] & \mathbb{R}^{n-k}
\end{tikzcd}$$

Then, the grassmannian inherits the topology of the space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n-k})$ modulo the linear transformations of $\mathbb{R}^{n-k}$, i.e.,

$$\mathcal{G}^k \hookrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{n-k})/GL(n-k).$$

Observe that: in particular projective spaces $\mathbb{P}^{n-1}$, defined as lines —or hyperplanes— of $\mathbb{R}^n$, are a special case of grassmannians $\mathcal{G}^1 = \mathbb{P}^{n-1}$; and that, as there is a natural duality between lines and hyperplanes —orthogonal complementation— there is a duality between grassmannians $\mathcal{G}^k \cong \mathcal{G}^{n-k}$.

But we want to work with the affine group $\mathbb{A}(d)$, not the linear one, so a step further is required. The next step is to prove that the grassmannian can be embedded also in the space of linear functions modulo the affine transformations, but with a bit of difference in the dimensions,

$$\mathcal{G}^d \hookrightarrow \mathcal{L}(\mathbb{R}^{n+1}, \mathbb{R}^d)/\mathbb{A}(d).$$

This will be a consequence of Theorem 1 below. We will denote by $\Pi = \mathbf{1}^\perp$ the hyperplane of $\mathbb{R}^n$ orthogonal to the vector $\mathbf{1} = (1, 1, \ldots, 1)$ with all its coordinates equal to one, or equivalently

$$\Pi = \left\{\mathbf{x} \in \mathbb{R}^n : \sum x_i = 0\right\},$$

where x_i denotes the i-th coordinate of $\mathbf{x}$.

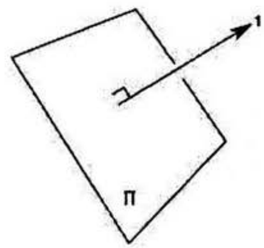


Fig 8. *The hyperplane* Π

1.1. Theorem. *Two functions $\varphi, \varphi' \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)/\mathbb{A}(d)$ represents the same configuration if and only if their respective kernels intersects the hyperplane Π in the same subspace.*

$$\begin{tikzcd} K \cap \Pi \arrow[r, hook, "i"] & \mathbb{R}^n \arrow[d, "\varphi"'] \arrow[dr, "\varphi'"] & \\ & \mathbb{A}^d \arrow[r, tail, "\psi"] & \mathbb{A}^d \end{tikzcd}$$

Proof. Let $\mathbf{p}_i = \varphi(\mathbf{e}_i)$ and $\mathbf{p}'_i = \varphi'(\mathbf{e}_i)$ be two configurations of n points in $\mathbb{A}^d$ and let K and K' denote their respective kernels. It will be proved that

$$K' \cap \Pi = K \cap \Pi$$

if and only if there exists an affine transformation $\psi: \mathbb{A}^d \to \mathbb{A}^d$ that sends one onto the other:

$$\mathbf{p}'_i = \psi \mathbf{p}_i.$$

This will induce a bijection between the space of configurations and the $(n-d-1)$-subspaces of the linear space Π of dimension $n-1$.

Necessity. It is clear that is enough to prove that $K' \cap \Pi \subset K \cap \Pi$, this is, it is enough to prove that

$$\sum \lambda_i = 0 \text{ and } \sum \lambda_i \mathbf{p}'_i = \mathbf{0} \implies \sum \lambda_i \mathbf{p}_i = \mathbf{0}.$$

In fact, if $\psi\mathbf{x} = M\mathbf{x} + \mathbf{v}$ denotes the affine isomorphism, then

$$\mathbf{0} = \sum \lambda_i \mathbf{p}_i' = \sum \lambda_i (M\mathbf{p}_i + \mathbf{v}) = M\left(\sum \lambda_i \mathbf{p}_i\right) + \left(\sum \lambda_i\right)\mathbf{v} = M\left(\sum \lambda_i \mathbf{p}_i\right).$$

Since M is invertible, then $\sum \lambda_i \mathbf{p}_i = \mathbf{0}$.

Sufficiency. In seek of simplicity, we will suppose that the first $d+1$ points of the configuration $\mathbf{p}_i$ spans affinely the space —the general case is totally analogous but working with subindexed set of indexes— therefore there exists an affine function $\psi \colon \mathbb{A}^d \to \mathbb{A}^d$ defined as the affine extension of the equations

$$\mathbf{p}_i' = \psi\mathbf{p}_i, \quad i = 1, \ldots, d+1.$$

It will be proved that the rest of the points satisfies the same equation.

For this, let $\mathbf{p}_j$ be any other point. Since the points $\{\mathbf{p}_1, \ldots, \mathbf{p}_{d+1}\}$ spans the space, there exist numbers $\{\lambda_1, \ldots, \lambda_{d+1}\}$ such that $\sum \lambda_i = 1$ and $\mathbf{p}_j = \sum \lambda_i \mathbf{p}_i$. By hypothesis we have that

$$\sum \lambda_i - 1 = 0 \quad \text{and} \quad \sum \lambda_i \mathbf{p}_i - \mathbf{p}_j = \mathbf{0} \implies \sum \lambda_i \mathbf{p}_i' - \mathbf{p}_j' = \mathbf{0},$$

and therefore

$$\mathbf{p}_j' = \sum \lambda_i \mathbf{p}_i' = \sum \lambda_i \psi \mathbf{p}_i = \psi \sum \lambda_i \mathbf{p}_i = \psi \mathbf{p}_j.$$

This concludes the proof.

With this result at hand, we obtain the desired topology for $\mathbb{A}_n^d$.

1.2. Corollary. *The points on the Grassmann variety $\mathcal{G}^d(\mathbb{R}^{n-1})$ are in one-to-one correspondence with the configurations of n vectors in $\mathbb{R}^d$ modulo the action of the affine group,*

$$\mathbb{A}_n^d \cong \mathcal{G}^d(\mathbb{R}^{n-1}).$$

Proof. The subspace Π can be identified with $\mathbb{R}^{n-1}$ and every $(n-d-1)$-subspace of it can be extended to the kernel of a linear function $\varphi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$. Theorem 1 guarantees that the correspondence of each configuration with its kernel is well defined and is a bijection of the points of $\mathcal{G}^{n-d-1}(\mathbb{R}^{n-1})$ with the points of the space of configurations. Finally, the duality of grassmannians gives us a bijection of those with $\mathcal{G}^d(\mathbb{R}^{n-1})$.

We will take now a closer and more concrete look to the Grassmann variety by describing an explicit embedding into projective space, called the *Plüker embedding* of $\mathcal{G}^d(\mathbb{R}^n)$. This dissertation will take us a little bit out of the scope of this work, but it will aloud us to ask some questions which are in this moment in research. In particular, we would like to study the *space of configurations of affine subspaces.* It is a natural generalization of the previous when we think on points as 0-dimensional affine subspaces. If we denote by ${}_\ell\mathbb{A}_n^d$ the space of **all** configurations of n subspaces, each of dimension ℓ, into the d-dimensional affine space, then the previous studied space $\mathbb{A}_n^d$ can be denoted by ${}_0\mathbb{A}_n^d$, and, as we will see, in the general case we may have an embedding of the form

$$ {}_\ell\mathbb{A}_n^d \hookrightarrow \frac{\left({}_0\mathbb{A}_{d+2}^{\ell+1}\right)^n}{\sim} \cong \frac{\left(\mathcal{G}^{\ell+1}(\mathbb{R}^{d+1})\right)^n}{\sim}, $$

where $\sim$ is denoting an equivalent relation which will be manufactured later... and when I am saying later, I mean **later** because at the moment it is not known how to build it even its existence is pretty obvious.

Let $\bigwedge_k \mathbb{R}^n$ denote the k-fold *Grassmann (or exterior) product* of the vector space $\mathbb{R}^n$. The elements of $\bigwedge_k \mathbb{R}^n$ are called *antisymmetric tensors.* We may think on it as the $\binom{n}{k}$-dimensional space (over $\mathbb{R}$) which has the canonical basis

$$ \{\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}. $$

The product of k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is given by

$$ \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \begin{vmatrix} v_{1i_1} & \cdots & v_{ki_1} \\ \vdots & & \vdots \\ v_{1i_k} & \cdots & v_{ki_k} \end{vmatrix} \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_k}. $$

and the basic property of this product is that, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent if and only if $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k \neq \mathbf{0}$. Also, if two k-subspaces $\mathcal{H}, \mathcal{H}' \hookrightarrow \mathbb{R}^n$ are equipped with basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and $\mathbf{b}'_1, \ldots, \mathbf{b}'_k$ respectively, we have that $\mathcal{H} = \mathcal{H}'$ if and only if there exists a non-zero scalar $c \in \mathbb{R}^*$ such that

$$ \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_k = c \cdot \mathbf{b}'_1 \wedge \ldots \wedge \mathbf{b}'_k. $$

Therefore we have the following embedding

$$ \mathcal{G}^k(\mathbb{R}^n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}. $$

Let us see a concrete example of how this can be used to study configurations of affine subspaces. Consider a line $\ell \hookrightarrow \mathbb{A}^3$ spanned affinely by the points $\mathbf{a}$ and $\mathbf{b}$, and think on $\mathbb{A}^3 \hookrightarrow \mathbb{R}^4$ as the hyperplane $\{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x} \cdot \mathbf{e}_4 = 1\}$ of the vectors which have the fourth coordinate equal to 1. So, we can denote by $\mathbf{a} = (a_1, a_2, a_3, 1)$ and by $\mathbf{b} = (b_1, b_2, b_3, 1)$ the affine basis of ℓ. Observe that

$$ \mathbf{a} \wedge \mathbf{b} = \begin{pmatrix} a_1b_2 - a_2b_1 \\ a_1b_3 - a_3b_1 \\ a_2b_3 - a_3b_2 \\ a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \times \mathbf{b} \\ \mathbf{a} - \mathbf{b} \end{pmatrix} \in \mathbb{R}^6. $$

Since every change of basis makes this assignments differ by a non-zero scalar, we can safely define a function $\varphi\colon \{\ell : \ell \hookrightarrow \mathbb{A}^3\} \to \mathbb{P}^5$ from the space of lines in the affine space to the projective 5-space by

$$ \varphi(\ell) = \begin{bmatrix} \mathbf{a} \times \mathbf{b} \\ \mathbf{a} - \mathbf{b} \end{bmatrix}. $$

Now, consider a family of lines $\mathcal{L} = \{\ell_1, \ldots, \ell_k\} \in {}_1\mathbb{A}^3_k$. We will say that $\mathcal{L}$ is *dependent* if the set $\{\varphi(\ell) : \ell \in \mathcal{L}\}$ is linearly dependent in $\mathbb{P}^5$, i.e. if $\varphi(\ell_1) \wedge \ldots \wedge \varphi(\ell_k) = \mathbf{0}$.

1.3. Proposition. *Three lines in $\mathbb{A}^3$ are dependent if they are concurrent and coplanar.*

Proof. Let ℓ_1, ℓ_2 and ℓ_3 be three lines in the affine space with $\mathbf{p} = \bigcap \ell_i$ a common point and let them be coplanar. Let $\mathbf{a} \in \ell_1$ and $\mathbf{b} \in \ell_2$ be points which completes a basis of their respective lines, i.e., both of them are different from $\mathbf{p}$. Since the three lines are coplanar, the line spanned by $\mathbf{a}$ and $\mathbf{b}$ intersects ℓ_3 in a point, say $\mathbf{c}$, which is different from $\mathbf{p}$ and therefore, completes a basis of it. More over, this point is an affine combination of the form

$$\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}, \text{ where } \lambda + \mu = 1.$$

From here it is easy to see that

$$\varphi(\ell_1) \wedge \varphi(\ell_2) \wedge \varphi(\ell_3) = \begin{bmatrix} \mathbf{p} \times \mathbf{a} \\ \mathbf{p} - \mathbf{a} \end{bmatrix} \wedge \begin{bmatrix} \mathbf{p} \times \mathbf{b} \\ \mathbf{p} - \mathbf{b} \end{bmatrix} \wedge \begin{bmatrix} \mathbf{p} \times \mathbf{c} \\ \mathbf{p} - \mathbf{c} \end{bmatrix} = 0.$$

Just recall that the Grassmann product is associative, antisymmetric and that it satisfies $\mathbf{x} \wedge \mathbf{x} = \mathbf{0}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$ of the space.

The necessity is also true. More over, it can be proved that four skew lines in $\mathbb{A}^3$ are dependent if and only if they form part of the ruler of a quadratic form.

We end this section with the dissertation started at the previous "coffee cup" sign above. Let ${}_\ell S^d_n$ denote the set of all families of n affine ℓ-subspaces in $\mathbb{R}^d$. As we saw, each such a subspace has associated a point in $\mathcal{G}^{\ell+1}(\mathbb{R}^{d+1})$, and each point there has associated a point configuration, i.e., an element of $\mathbb{A}^{\ell+1}_{d+2}$. Since we have n of such subspaces, we can assign n configurations of points to the family at hand. Therefore we have an embedding

$${}_\ell S^d_n \hookrightarrow \left(\mathbb{A}^{\ell+1}_{d+2}\right)^n.$$

Now, if we make a group act in $\mathbb{R}^d$, say the affine one, we naturally have an equivalent relation ${}_\ell\mathbb{A}^d_n = {}_\ell S^d_n / \mathbb{A}(d)$ and this most induce some relation on the previous product of grassmannians

$${}_\ell\mathbb{A}^d_n \hookrightarrow \left(\mathbb{A}^{\ell+1}_{d+2}\right)^n / \sim$$

...but we do not know yet how it works. It would be nice to find out!

2. The Radon Complex of a Point Separoid

In this section we will give a geometric proof of the fact: *the Radon complex of a point separoid is homotopically equivalent to a sphere.* In order to achieve this result, we will need to review some concepts of *Combinatorial Convexity*. We will be dealing with the *polytopal complexes* that bounds the n*-cube* and its dual, the n*-octahedron*. Both are the convex hull of a finite set of points in $\mathbb{E}^d$ and therefore, each has associated a finite poset, its *face lattice*. As we will see, these lattices can encode in full a point separoid but, first of all, let us prove that point separoids are full, this is

2.1. Lemma (fullness of points). *Let $\mathcal{P}$ be a point separoid. An interval $[A, B] \in \mathcal{Q}_n$ is a face of $\mathcal{R}(\mathcal{P})$, the Radon complex of the separoid, if and only if $A \dagger \overline{B}$ is a Radon partition of the separoid.*

Proof. The sufficiency is the face lemma 1.4.1. For the necessity, let $[A, B]$ be a face of $\mathcal{R}(\mathcal{P})$ and denote by $C = B \setminus A$ the difference of those subsets. Clearly every vertex of such a face are of the form $A \cup C'$, for some $C' \subseteq C$. Then, since they are vertices of the complex, for all $C' \subseteq C$ we have that $A \cup C' \dagger \overline{A \cup C'}$, this is, the set $\{A \cup C' : C' \subseteq C\}$ is a subset of the components of $\mathcal{P}$.

Now, in order to search for a contradiction, suppose that $A \mid \overline{B}$. This is equivalent to the existence of a hyperplane $\mathcal{H} \hookrightarrow \mathbb{A}^d$ that separates the convex hulls of A and $\overline{B}$. It is easy to see that, ones this hyperplane exists, it can be chosen in such a way that it does not contains any of the points. Denote by C_a, respectively C_b, those points of C which are on the same side of A, respectively $\overline{B}$, so $C = C_a \cup C_b$. From this definition follows that $\mathcal{H}$ separates $A \cup C_a$ from $\overline{B} \cup C_b$. But as previously settled, $A \cup C_a \dagger \overline{A \cup C_a}$ ↯ ⚽

Observe that an analogous argument proves that *if every separation can be extended to a maximum separation then the separoid is full* (cf. Lemma 1.4.2).

In the following, we will suppose some familiarity with concepts as *abstract* and *geometric simplicial complex*, *geometric realization* $|\mathcal{K}|$ of an *abstract complex* $\mathcal{K}$, *homeomorphism* denoted by $\cong$, *boundary* $\partial\mathcal{K}$, and *relative interior* $\mathcal{K}^\circ$. But do not worry to much with this... any time I say "polytope" you may think in the n-octahedron (also known as the n-crosspolytope), the convex hull of the canonic basis of $\mathbb{R}^n$ and its negatives; and every time I say "polytopal complex" you may think in a subset of its boundary.

☕ An introduction to this concepts, can be found in Ewald's "Combinatorial Convexity and Algebraic Geometry" (1996), Munkres' "Elements of Algebraic Topology" (1984), or to the classic Spanier's "Algebraic Topology" (1966).

Given two simplices, σ and τ, there is a ball defined by the union of all segments whose extreme points are, one in each simplex; it is called the *join* and denoted by $\sigma * \tau$. Observe that the join is again a simplex if and only if the union of both vertex sets is affinely independent.

2.2. Lemma (join). *Let $\tau_0, \tau_1 < \sigma$ be two faces of a simplex, and ℓ an affine subspace. If ℓ intersects the interior of both simplices, it intersects the interior of their join, i.e.*

$$\ell \cap \tau_i^\circ \neq \phi \implies \ell \cap (\tau_0 * \tau_1)^\circ \neq \phi.$$

Proof. Let $\mathbf{a} \in \tau_0^\circ \cap \ell$ and $\mathbf{b} \in \tau_1^\circ \cap \ell$ be points in the intersections of ℓ with the interiors of the two faces. Then, the segment $\langle \mathbf{a}, \mathbf{b} \rangle$ is contained in ℓ and its interior in $(\tau_0 * \tau_1)^\circ$. ⚽

Now, given a polytopal complex $\mathcal{K}$, its *baricentric subdivision* $\mathcal{K}'$ can easily obtained from its face lattice as follows: its vertices $(\mathcal{K}')^{(0)}$ are the elements of the poset (the faces of $\mathcal{K}$); and its k-faces $(\mathcal{K}')^{(k)}$ are the chains of length k in the poset, this is $\langle b_0, \ldots, b_k \rangle \in (\mathcal{K}')^{(k)} \iff b_0 < \cdots < b_k \in \mathcal{K}$.

☕ It is ease to see that the baricentric subdivision of dual polytopes is the same.

The realization of the baricentric subdivision is known to be homeomorphic to the space it self. More over, if the baricenter of a face is aloud to move in the interior of the face it represent, with out going out of such a interior, the space and its combinatorial properties do not change. In particular, if an affine subspace ℓ intersects the polytopal complex, the following realization of the subdivision can be defined

$$|\mathcal{K}'|^{(0)} := \left\{ \mathbf{b}_\sigma : \mathbf{b}_\sigma = \begin{cases} b(\sigma^\circ \cap \ell) & \text{if } \sigma^\circ \cap \ell \neq \phi \\ b(\sigma) & \text{otherwise} \end{cases} \right\},$$

$$\langle \mathbf{b}_{\sigma_0}, \ldots, \mathbf{b}_{\sigma_k} \rangle \in |\mathcal{K}'|^{(k)} \iff \sigma_0 < \cdots < \sigma_k \in \mathcal{K},$$

where each σ is a face of $\mathcal{K}$ and

$$b(\sigma) = \frac{1}{\#\sigma^{(0)}} \sum_{\mathbf{v} \in \sigma^{(0)}} \mathbf{v}$$

denotes the usual baricenter.

In the proof of Theorem 2.2.4 we will be dealing with two different ways to intersect an affine subspace ℓ with a polytopal complex: the *usual* one which considers **only** those faces of the complex intersected in the interior; and what we call *fat* intersection which considers **all** faces "touched" by the subspace. Both intersections are to be considered as subcomplexes of the baricentric subdivision. We will prove that the usual intersection is a retract of the fat one. For this, let us denote by $\mathcal{K}[V]$ the subcomplex of $\mathcal{K}$ *induced* by a subset $V \subseteq \mathcal{K}^{(0)}$ of its vertices. This is, the subcomplex induced by V consists of V it self, and of every face of $\mathcal{K}$ such that all its vertices are in V.

With all these at hand, denote by

$$\mathcal{K} \sqcap \ell := \mathcal{K}'[\mathbf{b}_\sigma : \sigma \cap \ell \neq \phi]$$

and by

$$\mathcal{K} \mathring{\sqcap} \ell := \mathcal{K}'[\mathbf{b}_\sigma : \sigma^\circ \cap \ell \neq \phi]$$

the fat and the usual intersections, respectively. Clearly the realization of the usual intersection is the intersection of the realization, i.e., $|\mathcal{K} \mathring{\sqcap} \ell| = |\mathcal{K}| \cap \ell$. In fact it is the first baricentric subdivision of the "geometric intersection", $\mathcal{K} \mathring{\sqcap} \ell = (\mathcal{K} \cap \ell)'$.

2.3. Theorem. *Let $\mathcal{K}$ be a polytopal complex, and ℓ an affine subspace that intersects it in the interior. Then $|\mathcal{K} \mathring{\sqcap} \ell|$ is a strong retract of $|\mathcal{K} \sqcap \ell|$.*

Proof. Let $f: \mathcal{K} \sqcap \ell \to \mathcal{K} \mathring{\sqcap} \ell$ be defined as $f(b_\sigma) = b_\tau$, where

$$\tau = \max\{\tau < \sigma : \tau^\circ \cap \ell \neq \phi\}.$$

First of all, observe that the join lemma guarantees that f is well defined and, to see that it is a simplicial map, observe that $\sigma_0 < \sigma_1$ implies

$$\{\tau < \sigma_0 : \tau^\circ \cap \ell \neq \phi\} \subset \{\tau < \sigma_1 : \tau^\circ \cap \ell \neq \phi\}$$

and therefore $f(\sigma_0) < f(\sigma_1)$.

Clearly f is the identity map on $\mathcal{K} \mathring{\sqcap} \ell$.

Now, for each $t \in [0, 1]$, let $f_t: |\mathcal{K} \sqcap \ell| \to |\mathcal{K} \mathring{\sqcap} \ell|$ be defined as follows: first, if $\mathbf{b}_\sigma \in (\mathcal{K} \sqcap \ell)^{(0)}$ is a vertex,

$$f_t(\mathbf{b}_\sigma) = (1 - t)\mathbf{b}_\sigma + t f(\mathbf{b}_\sigma).$$

Observe that, since $\mathbf{b}_\sigma$ and $f(\mathbf{b}_\sigma)$ are points of $|\sigma|$, then $f_t(\mathbf{b}_\sigma)$ is also a point of $|\sigma|$. Finally, extend linearly the function to the rest of the domain, this is, if $\mathbf{x} \in |\mathcal{K} \sqcap \ell|$ is any other point, then $\mathbf{x} \in |\langle \mathbf{b}_{\sigma_0}, \ldots, \mathbf{b}_{\sigma_k} \rangle| \subset |\sigma_k|$ and $\mathbf{x}$ can be denoted as a convex combination $\mathbf{x} = \sum \lambda_i \mathbf{b}_{\sigma_i}$. Therefore f_t can be defined by

$$f_t(\mathbf{x}) = \sum \lambda_i f_t(\mathbf{b}_{\sigma_i}).$$

So, $f_t: |\mathcal{K} \sqcap \ell| \searrow |\mathcal{K} \mathring{\sqcap} \ell|$ is a strong retract. ⚽

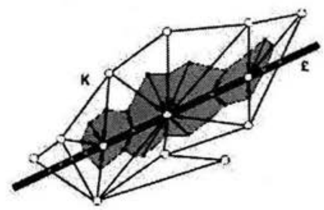


Fig 9. *The fat intersection*

We are almost ready for the main theorem of this section. The next step is to give two descriptions of the n-octahedron; a geometric one and a combinatorial one. They will be used to define a duality of it with the n-cube.

Let us provide $\mathbb{R}^n$ with an unusual metric —known as Manhattan norm— that assigns to each vector the sum of the absolute value of its coordinates $\|\mathbf{x}\| := \sum |x_i|$. In this space, the unitary sphere turns out to be the n-octahedron

$$|\mathcal{O}_n| := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\} \cong \mathbb{S}^{n-1}.$$

It is the boundary of the convex set of a finite set of points, the canonical basis and its negatives

$$|\mathcal{O}_n| = \partial\langle \pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n \rangle,$$

therefore it is a polytopal complex. More over, it is a simplicial complex.

The n-octahedron has also the following combinatorial description: each face σ can be identified with an n-vector $(z_1, \ldots, z_n) \in \{-1, 0, 1\}^n$ that indicates which vertices —which canonicals, or its negatives— are incident to it, in such a way that each face can be realized with the simplex

$$|\sigma| = \langle z_i \mathbf{e}_i : z_i \neq 0 \rangle.$$

It is well known that this complex is dual of the n-cube and therefore there is a function between their faces that realizes such a duality. We will denote it by

$$\delta: \mathcal{O}_n \to \mathcal{Q}_n,$$

$$\delta\left(\langle z_i \mathbf{e}_i : z_i \neq 0 \rangle\right) := \left[\{i : z_i = 1\}, \{i : z_i \neq -1\}\right].$$

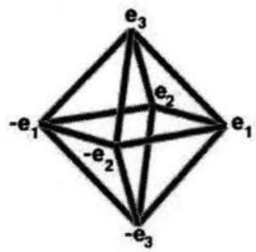


Fig 10. *The 3-octahedron*

Theorem 4 plays an important role on the next chapter. It was the basic guide of our intuition to develop the graph theoretical characterization of uniform oriented matroids in terms of their circuits. Both, the technics in the proof and the statement it self, serves as pictures to "see" points separoids —and more generally, oriented matroids. In fact, this theorem generalizes that of Radon and it was basically his technic who lead us to the proof. Although the statement was conjectured since the first days of the theory, more that a chandelier ago, we had not found a "purely combinatorial" proof of it. Always the geometry and topology had play an important role... well it was supposed to be like that, the final statement talks about homotopy of spheres so it is not rare that the topology has to play some role in the proof.

We will see also how the proof of Theorem 4 suggests a characterization of **all** point separoids. This will be used to give explicitly the stratification of $\mathbb{A}_n^d$ for $n \leq d+3$ and for $d = 1$.

2.4. Theorem. *Let $\mathcal{P}$ be a d-dimensional point separoid of order n. Then, the $(n-d-2)$-sphere is an homotopical retraction of its Radon complex $\mathcal{R} = \mathcal{R}(\mathcal{P})$,*

$$\mathcal{R} \searrow \mathbb{S}^{n-d-2}.$$

More over, if the separoid is in general position, then such homotopy is in fact an homeomorphism,

$$\mathcal{R} \cong \mathbb{S}^{n-d-2}.$$

Proof. Let $\mathcal{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_n) \in (\mathbb{A}^d)^n$ be a configuration of points, $\mathcal{S} = \mathcal{S}(\mathcal{P})$ its separoid and $\mathcal{R} = \mathcal{R}(\mathcal{S})$ its Radon complex. We will identify the configuration with the intersection of the kernel $K = \varphi^{-1}(\mathbf{0})$ of its linear function $\varphi: \mathbb{R}^n \to \mathbb{R}^d$ (where $\varphi(\mathbf{e}_i) = \mathbf{p}_i$), and the hyperplane

$$\Pi = \mathbf{1}^{\perp} = \left\{\mathbf{x} \in \mathbb{R}^n : \sum x_i = 0\right\}.$$

This $(n-d-1)$-subspace of Π will be denoted by $\ell = K \cap \Pi$. Theorem 1.1 guarantees that this assignment is well defined and, modulo affine transformations, is one-to-one.

Give to $I\!R^n$ the structure of a (Manhattan) normed space and denote by

$$|\mathcal{O}| = \left\{ \mathbf{x} \in I\!R^n : \sum |x_i| = 2 \right\}$$

the sphere of radius 2 centered at the origin (observe that here we are working with a radius 2 sphere —because a technicality that will be clear later— but the previous dissertation on the n-octahedron can be applied to it completely). Recall the definition of the fat intersection

$$\mathcal{O} \sqcap \ell = \mathcal{O}'[\sigma \in \mathcal{O} : \sigma \cap \ell \neq \phi]$$

and define the complex of its dual faces

$$\Re := \{\delta(\sigma) \in \mathcal{Q}_n : \sigma \in \mathcal{O} \text{ and } \sigma \cap \ell \neq \phi\},$$

where $\delta \colon \mathcal{O} \to \mathcal{Q}_n$ is the previously defined duality function.

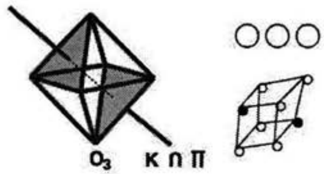


Fig 11. *Theorem 2.4*

Clearly $\Re' = \mathcal{O} \sqcap \ell$. Observe also that, since ℓ is a subspace of dimension $n-d-1$, then $\mathcal{O} \cap \ell$ is a sphere of dimension $n-d-2$. Now, due to Theorem 3, $\mathcal{O} \cap \ell$ is a strong retract of $\mathcal{O} \sqcap \ell$ and therefore $\Re$ has the homotopy type of the $(n-d-2)$-sphere

$$\Re \searrow \mathbb{S}^{n-d-2}.$$

Claim. *$\Re$ is equal to $\mathcal{R}$*

• Since point separoids are full separoids —$A \dagger B$ is a Radon partition if and only if $[A, \overline{B}]$ is a face of $\mathcal{R}$ (Lemma 1)— it is enough to prove that $[A, \overline{B}]$ is a face of $\Re$ if and only if $\langle A \rangle \cap \langle B \rangle \neq \phi$. For this, let $\sigma \in \mathcal{O}$ be a face of the n-octahedron and $(z_i) \in \{-1, 0, 1\}^n$ its corresponding signed vector. Then σ has associated the 3-partition of $\mathcal{P}$ given by $A = \{\mathbf{p}_i \in \mathcal{P} : z_i = 1\}$, $B = \{\mathbf{p}_i \in \mathcal{P} : z_i = -1\}$ and $C = \overline{A \cup B} = \{\mathbf{p}_i \in \mathcal{P} : z_i = 0\}$ and, by the definition of $\delta \colon \mathcal{O}_n \to \mathcal{Q}_n$, we have that $\delta(\sigma) = [A, A \cup C]$. Therefore, it is enough to prove that

$$\sigma \cap \ell \neq \phi \iff \langle A \rangle \cap \langle B \rangle \neq \phi.$$

For, let $\mathbf{x} \in \sigma \cap \ell$, then

$$\sum x_i \mathbf{p}_i = \mathbf{0}, \quad \sum x_i = 0 \quad \text{and} \quad \sum |x_i| = 2.$$

The first equation is due to $\mathbf{x} \in K$, the second because $\mathbf{x} \in \Pi$ (all these since $\mathbf{x} \in \ell = K \cap \Pi$) and the third one because $\mathbf{x} \in \mathcal{O}$. More over, since $\mathbf{x} \in \sigma$, we are aloud to write

$$\frac{1}{2}\mathbf{x} = \sum \lambda_i (z_i \mathbf{e}_i)$$

as a convex combination ($\sum \lambda_i = 1$ and $\lambda_i \geq 0$) of the canonic basis vectors or its corresponding negatives. Combining these ($x_i = 2 z_i \lambda_i$) we have that

$$\sum_{\mathbf{p}_i \in A} \lambda_i \mathbf{p}_i = \sum_{\mathbf{p}_i \in B} \lambda_i \mathbf{p}_i$$

and

$$\sum_{\mathbf{p}_i \in A} \lambda_i = \sum_{\mathbf{p}_i \in B} \lambda_i = 1.$$

This last happens if and only if $\langle A \rangle \cap \langle B \rangle \neq \phi$. Since all previous steps can be followed the other way around, we have concluded the proof of the claim, and therefore $\mathcal{R} \searrow \mathbb{S}^{n-d-2}$. •

For the case of general position, observe that $\mathcal{R}$ has a face $[A, B]$ of dimension greater than $n-d-2$ if and only if $\#(B \setminus A) > n-d-2$ and this is equivalent to the existence of a partition $A \dagger \overline{B}$ where $\#(A \cup \overline{B}) < d+2$. If the separoid $\mathcal{S}$ is in general position this last is impossible (Lemma 1.0.2). Since all facets have dimension $n-d-2$ we have that

$$|\mathcal{O} \sqcap \ell| = |\mathcal{O} \cap \ell|.$$

Therefore $\mathcal{R}$ is homeomorphic to the $(n-d-2)$-sphere $\mathcal{R} \cong \mathbb{S}^{n-d-2}$.

Observe that the last argument of the proof implies that *the points are in general position if and only if ℓ is "in general position" with respect to $\mathcal{O}$.* Also, if we think on full separoids as filters in the face lattice of the n-octahedron, following the same technic as above we can prove that

2.5. Corollary (point separoids). *A full separoid $\mathcal{S} \subset \mathcal{O}_n$ is a point separoid if and only if there exists a hyperplane $\mathcal{H} \hookrightarrow I\!R^n$ such that $\mathcal{S} \searrow \mathcal{H} \cap \Pi \cap \mathcal{O}_n$.*

Other application of this theorem, besides Theorem 0.1 is

2.6. Corollary (uniform sphereicity). *Let $\mathcal{S}$ be a general position separoid of n convex sets in $I\!R^d$. Then*

$$\mathsf{d}(\mathcal{S}) = d \implies \mathcal{R}(\mathcal{S}) \cong \mathbb{S}^{n-d-2}.$$

3. Concluding; the stratification of $\mathbb{A}_n^d$

We had been working in three different categories: point configurations, thought of as linear functions or matrices; separoids, thought of as the combinatorial structure they are; subcomplexes of the n-cube, defined this as the family of subsets of a given finite set.

Given a separoid $\mathcal{S}$, a "cubic" complex $\mathcal{R}(\mathcal{S}) \hookrightarrow \mathcal{Q}_n$, its Radon complex, can be defined identifying its vertices $V(\mathcal{R})$ with the components of the separoid $\mathcal{C}(\mathcal{S}) = \{A \subset \mathcal{S} : A \dagger \overline{A}\}$ and looking for the induced subcomplex of the n-cube $\mathcal{R}(\mathcal{S}) = \mathcal{Q}_n[\mathcal{C}]$. In general it is not possible to reconstruct the separoid from its Radon complex but, if the separoid is a point separoid, this complex encodes the full separoid and it is always an sphere.

On the other hand, given a configuration of points $\mathcal{P} \in \mathbb{A}_n^d$ it can be thought of as a matrix and identified with the intersection of its kernel, the hyperplane $\Pi = \mathbf{1}^\perp$ and the n-octahedron $\mathcal{O}_n$. Such an intersection defines the dual of the Radon complex of the separoid associated to the configuration. A natural question is *when does a cubic complex is the Radon complex of a point separoid?* It seem natural to ask for some sphericity but, as we will see on the next chapter, it is not enough. In general, this question is still open but in the following we will analyze the cases where $n \leq d+3$. In the way, we will find useful to study also the case $d = 1$.

We now want to bring together all these to see how the n-cube imposes a stratification to the grassmannian, via the Radon complex of point configurations.

Since the reader was supposed to work with $\mathbb{A}_4^2$, $\mathbb{A}_4^1$ and $\mathbb{A}_5^2$, and we already described $\mathbb{A}_3^1$ at the beginning of this chapter, we jump to study $\mathbb{A}_5^3$. It is an example of $(5 = 3+2)$

The case $n = d + 2$.

$d+2$ points in general position induce a unique (Radon) partition that can be interpreted as a 0-subsphere of the $(d+2)$-cube —a pair of antipode vertices— and the other way around; given a subset and its complement it is easy to construct a configuration with such subsets as its unique partition. This induce a one-to-one relation between pairs of antipode vertices of the n-cube (different from the empty set and the total one) and the d-dimensional point separoids of order $n = d+2$ in general position.

When the general position is lost, The Radon complex is "fatten" and gets faces of bigger dimension; edges, squares, cubes, etc. Each of these faces has associated an interval $[A, A \cup C]$ in the family of subsets and this represents a partition of the form $A \dagger B \cup C$, where $A \dagger B$ is the minimal Radon partition of the configuration. The dimension of the "fat" is given by the cardinal of C.

To get a picture of the fenomena, think on five points in $\mathbb{E}^3$ in general position and all of them being vertices of their convex hull. They lead a unique partition of the form $abc \dagger de$ and its Radon complex consists of two antipodal vertices. Now move continuously one of the points, say d, and push it to the closest face of the tetrahedra defined by the other four, this is, the triangle defined by abc.

While it is on the exterior of such a tetrahedra, the separoid do not change (even the configuration does).

Ones it reaches the (relative) interior of the triangle abc, the new separoid is defined by the minimal partition $abc \dagger d$ and its Radon complex consists of two antipodal edges. If you move the point in the interior of the triangle, while the configuration changes, the separoid does not change. If you move it to the other side, the interior of the tetrahedra, the separoid became in general position and the unique partition $abce \dagger d$ leads again to two antipodal vertices in its Radon complex. But, if you move, inside the triangle, to reach an edge of it, say ab, the separoid changes and defines a new minimal partition $ab \dagger d$. Its Radon complex turns now to be a pair of antipodal squares. Analogously, if this point reaches an other point, say a, its Radon complex became a pair of antipodal cubes. Observe that there is no way to put two antipodal 4-cubes inside $\mathcal{Q}_5$ with out including the empty set and inducing, by the union of the vertices, the full 5-cube.

We have the following easily generalizable description of this fenomena: with out lose of generality, we may suppose that the first $(d+1)$ of the points are the vertices of a tetrahedra (a d-simplex); the (hyper)planes defined by each three (d) of them leads a partition of the space in open sets (and points). Then the fifth $(d+2$-th) point can be localized by the position it occupies in the "polytope" this partition defines —each open set can be named with a signed vector that corresponds to the signs of the baricentric coordinates in terms of the ordered basis determined by the the first four $(d+1)$ points— and each of this regions define a Radon partition $A \dagger B \cup C$ —the positives vs. the negatives in the previous mentioned signed vector. If the fifth point lies in the plane generated by the first three (d), the baricentric coordinates of it contains a zero in the fourth term (the cardinality of C is 1); if it lies in a line, contains two zeros $(\#C = 2)$; and so on. We have then the following *types* of partitions: in general position —represented in the grassmannian by facets of dimension 3 (d)— $a \dagger bcde$ and $ab \dagger cde$; degeneracy of first grade —faces of dimension 2 $(d-1)$— $a \dagger bcd$ and $ab \dagger cd$; of second grade —edges— $a \dagger bc$; and finally when two points are equal $a \dagger b$. This is, the space of configurations $\mathbb{A}_5^3$ consists of two types of facets (5 tetrahedra and 10 prisms), two types of faces (20 triangles and 15 squares), one type of edges (30 of them) and one type of vertices (10 of them).

- The 2-skeleton of $\mathbb{A}_5^3$ contains a regular polyhedra that plays the roll of one of the platonic solids but in the projective space. If you consider only the fifteen squares of it, you get a polyhedra with schläfli symbol $\{4,6\}$ (it is made of squares, six in each vertex) and it can be embedded in $\mathbb{P}^3$ in such a way that all of its automorphisms (the group S_5) are realized with isometric projectivities. As you may already proved, $\mathbb{A}_4^2$ is the hemicuboctahedron whose symmetry group is $S_4 \ldots$

- For another presentation of the $\{4,6\}$ see Strausz 1996 and Bracho & Strausz 2001.

3.1. Theorem. *The space of configurations of n points in dimension $n-2$, modulo the action of the affine group, is*

$$\mathbb{A}_n^{n-2} = \mathcal{O}_n \cap \mathbf{1}^{\perp}/\{-\mathbf{x},\mathbf{x}\} = \left(\mathcal{Q}_n \setminus \{\phi,\overline{\phi}\}\right)^* /\{A,\overline{A}\}.$$

Proof. Since two configurations are equivalent (modulo the action of the affine group) if and only if the intersection of their kernels with $\mathcal{O}_n \cap \mathbf{1}^{\perp}$ is the same and because such intersection is a 0-dimensional sphere, the configurations are in a one-to-one relation with the pairs of antipodal points of $\mathcal{O}_n \cap \mathbf{1}^{\perp}$. The duality with the n-cube has already been defined. ⚽

We will also give a combinatorial description for $\mathbb{A}_{d+3}^{d}$ but, in the study of the case $n = d+3$, it will be important to have a good knowledge of

The case $d = 1$.

The facets of $\mathbb{A}_n^1$ are in a one-to-one correspondence with the linear orders of n elements (modulo reversing all the elements in the order), and two of them are adjacent if and only if they differ in a permutation of two consecutive members. It is ease to see that the facets are always simplices —with out loose of generality, the ending points of the configuration, say a and z, are represented by 0 and 1 so the configuration is parametrized by a sequence of numbers $0 < b < c < \cdots < x < y < 1$. On the other hand, the vertices of $\mathbb{A}_n^1$, since are rigid configurations, are pairs of accumulated points. This is, the vertices of the simplices are of the form

$$\begin{gathered} a \mid bcd\ldots yz, \\ ab \mid cd\ldots yz, \\ abc \mid d\ldots yz, \\ \ldots \\ abcd\ldots y \mid z. \end{gathered}$$

where the represented linear order is $abcd\ldots yz$. We know also that this space $\mathbb{A}_n^1 \cong \mathbb{P}^{n-1}$ most be homeomorphic to the projective space of dimension $n-1$.

Let us see this in a more concrete example, four point in the line: in the projective plane, take four points in general position —you can start, e.g., with $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ and $[1,1,1]$— which will represent the four configuration with a unique (maximal) separation of the form $a \mid bcd$. Now, draw the six lines that they define by pairs. In the intersection of these lines appear three new points. They represent those configurations with a unique separation of the form $ab \mid cd$. There are two kind of edges; six which makes adjacent two vertices of the first kind —characterized by a Radon partition of the form $bc \dagger ad$— and twelve that joins vertices of different kinds —characterized by a Radon partition of the form $b \dagger acd$. Observe that there are not edges between two vertices of the second kind. Finally, there are 12 triangular facets which consist each, of two vertices of the first kind and one of the second. All this dissertation can be resumed in the following

3.2. Theorem. *The space of configurations of n points in the line, modulo the action of the affine group, is*

$$\mathbb{A}_n^1 = \left(\mathcal{Q}_n^\star \setminus \{\phi, \overline{\phi}\}\right) / \{A, \overline{A}\},$$

where $\mathcal{Q}_n^\star$ denotes the first baricentric subdivision of the n-simplex.

Proof. The combinatorial structure of open sets defined by the $\binom{n}{n-2}$ hyperplanes spanned by n points in general position in the $(n-2)$-projective space is isomorphic to the star subdivision of the n-hemicube minus two antipodal vertices; the poset relation is given by $A < B \iff A \subset \partial B$.

The case $n = d+3$.
As a consequence of Theorem 2.4, we have that each configuration of $n = d+3$ points in general position in dimension d gives place to the embedding $C_{2n} \hookrightarrow \mathcal{Q}_n$ of a cycle into the cube, its Radon complex. The fullness of point separoids (Lemma 2.1) implies that two such separoids are equal if and only if their Radon complexes are. Now, by Theorem 3.2.3, each antipodal cycle $C_{2n} \hookrightarrow \mathcal{Q}_n$ can be associated to an oriented matroid of codimension 1, and therefore a point separoid. Finally, observe that the condition of acyclicity is equivalent to say that the Radon complex of the separoid does not contains the empty set (neither the total). We have then that

3.3. Theorem. *The facets of the space of configurations of n points in dimension $n-3$, modulo the action of the affine group, are*

$$\left(\mathbb{A}_n^{n-3}\right)^{(2n-6)} = \left\{C_{2n} \hookrightarrow \mathcal{Q}_n \setminus \{\phi, \overline{\phi}\} : A \in C_{2n}^{(0)} \implies \overline{A} \in C_{2n}^{(0)}\right\},$$

and two facets represented by the cycles C and C' are adjacent if and only if

$$|V(C) \,\Delta\, V(C')| = 2.$$

Analyzing in detail $\mathbb{A}_{d+3}^d$ for the cases $d = 0, 1, 2, 3$, in particular we can prove that

- In $\mathcal{Q}_3 \setminus \{\phi, \overline{\phi}\}$ there is only one cycle of order 6.
- In $\mathcal{Q}_4 \setminus \{\phi, \overline{\phi}\}$ there is, essentially, only one kind of antipodal cycles of order 8, and there are as many as linear orders with four elements.
- In $\mathcal{Q}_5 \setminus \{\phi, \overline{\phi}\}$ there are three kind of antipodal cycles of order 10.

They correspond to the configurations depicted in Figure 7.

This leads to complete the combinatorial description of this space of configurations. But one question remains open: *how can we reconstruct the configuration from its Radon complex?*

For this special case, we have a construction that we know it functions in all small cases ($n = 3, 4, 5, 6$) but we do not have yet a prove for the general case... however we are convinced that this will be the case.

To describe the construction, let us go back to the first example of this chapter: the regular pentagon. Observe that, given the cycle $C_{10} \hookrightarrow \mathcal{Q}_5$ and a vertex $A \in C_{10}^{(0)}$ of it, each path from A to its antipode defines a linear order of the base set $X = \{0, 1, 2, 3, 4\}$. In the example, if we start with the set $A = 02$, one of the two paths to its antipode is $02, 024, 24, 124, 14, 134$. This path defines naturally the lineal order 40123, the elements that change in each step of it. If we forget for a moment some element, say 4, the remaining information is: a minimal Radon partition $02 \dagger 13$ and a linear order 0123.

Recall the triangulation $\mathbb{A}_4^1$. In this, there are four special vertices (those that represents the configurations of the form $a \mid bcd$) and each facet represents a linear order. If you add a fifth point in the baricenter of the region that represents the order 0123 and we apply a projective transformation such that the four special vertices realizes the partition $02 \dagger 13$ in the affine plane $\Pi + \mathbf{e}_i$, parallel to Π thru the canonic basis, we obtain the desired configuration.

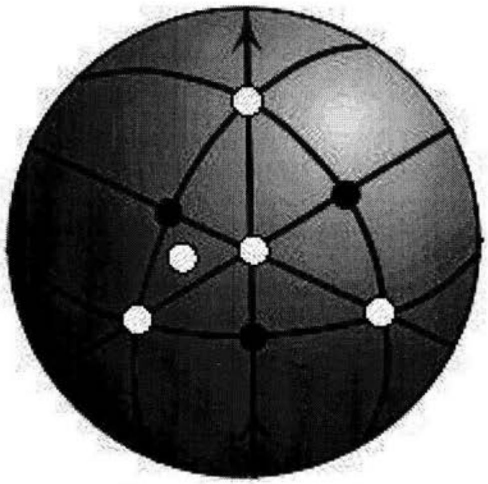


Figure 12. *The realization of the pentagon*

4. One last application

We close this chapter with an application of the Radon complex of point separoids. We say that a separoid $\mathcal{S}$ is *polytopal* if every member of it is separated from its complement $x \in \mathcal{S} \implies x \mid \overline{x}$. If a point separoid is polytopal, all its points are vertices of its convex hull, and therefore it is a polytope —hence the name.

One of the main problems in *Combinatorial Convexity* is to classify convex polytopes and there had been developed lots of tools to reach this. We will see how the Radon complex of polytopal point separoids help us to count the different *types* of polytopes in the ease case $n = d + 2$.

The following result is usually settled using *Gale diagrams* which I will avoid, but it can be found, e.g., in the second chapter's last section of Ewald 1996.

4.1. Theorem (Grümbaum 67). *There are precisely $\left\lfloor \frac{1}{4} d^2 \right\rfloor$ combinatorial types of d-polytopes with $d + 2$ vertices.*

Proof. Let $\mathcal{P} \in \mathbb{P}^d_{d+2}$ be a polytopal point separoid. By Theorem 2.4, its Radon complex $\mathcal{R} = \mathcal{R}(\mathcal{P})$ is homotopically equivalent to the 0-sphere and therefore is the union of two intervals $[A, B]$ and $[\overline{A}, \overline{B}]$. Since it is polytopal, neither of these intervals contains a singleton (neither a subset of cardinality $d + 1$). With this extra condition we have that

$$\mathcal{R} \hookrightarrow \mathcal{Q}_{d+2} \setminus \left\{ \phi, \binom{\mathcal{P}}{1}, \binom{\mathcal{P}}{d+1}, \mathcal{P} \right\},$$

where $\mathcal{P} = \overline{\phi}$ is identified with the base set and $\binom{\mathcal{P}}{k} = \{A \subset \mathcal{P} : |A| = k\}$ denotes the family of k-subsets of it. If $[A, B] \cong \mathcal{Q}_0$, there are $\left\lceil \frac{d-1}{2} \right\rceil$ essentially different ways to embed $\mathcal{R}$; if $[A, B] \cong \mathcal{Q}_1$, there are $\left\lceil \frac{d-2}{2} \right\rceil$; if $[A, B] \cong \mathcal{Q}_2$, there are $\left\lceil \frac{d-3}{2} \right\rceil$; $\ldots$; if $[A, B] \cong \mathcal{Q}_{d-1}$, there is one ($= \left\lceil \frac{1}{2} \right\rceil$) way to embed $\mathcal{R}$. Therefore we have the sum

$$\left\lfloor \frac{d^2}{4} \right\rfloor = \left\lceil \frac{d-1}{2} \right\rceil + \left\lceil \frac{d-2}{2} \right\rceil + \cdots + \left\lceil \frac{1}{2} \right\rceil,$$

and we are done.

If you are not too ambitious, it can be a pleasure to realize that you have rediscovered something previously known, because at least you know that you were on the right track.

—I.M. GEL'FAND (¿1980's?)

3

Oriented Matroids

Oriented matroid theory was introduced in the 1960's when J. Folkman and J. Lawrence proved that every oriented matroid can be thought of as a family of oriented pseudospheres. In particular, they proved that the natural partial order associated to an oriented matroid is the face lattice of a sphere —this last result is known as the Basic Sphericity Theorem. This result has many applications and we will use, as our intuition source, a slightly different version of it: *The Radon complex of an oriented matroid is a sphere.* It will be shown that this is a direct consequence of Edelman's theorem (1984) and Alexander's duality.

One of the main bricks of the theory is the classic Radon's theorem (1921). Oriented matroids encode minimal Radon partitions in terms of *circuits*, a set of signed vectors $\mathcal{C} \subset \{-,0,+\}^E$ with some properties; in particular, they define a separoid which encodes all the information. As we saw, the family of all signed vectors has associated a natural poset which turns out to be the face lattice of the n-cube. The Radon complex of an oriented matroid is a "cubic" complex (an ideal in the face lattice of the n-cube) whose vertices are identified with those subsets non-separated of its complement. Since oriented matroids leads to *full* separoids, this complex captures the structure of the oriented matroid. In Theorem 2.2 we will prove that the dual poset of such a complex, can be characterized via a natural combinatorial metric associated to the 1-skeleton of it.

We follow here ideas explored by K. Fukuda and K. Handa (1993) but in the more general context of *separoids*: Symmetric ideals (or filters) defined by an antichain in the face lattice of the n-cube (or the n-crosspolytope). Fukuda & Handa characterized every *tope graph* $\mathcal{T} = \mathcal{T}(\mathcal{M})$ of an oriented matroid of rank 3 —dimension 2— showing that they are those antipodal planar graphs which can be embedded in the n-cube preserving their graph distance. The planarity of $\mathcal{T}$ induces a dual graph $\mathcal{G} = \mathcal{T}^*$ which can be proved to be the *cocircuit graph* of the oriented matroid. No characterization is known of the cocircuit (or tope) graph in the general case, but Theorem 2.6 gives necessary and sufficient conditions for *uniform* oriented matroids with arbitrary rank. We basically settle that *a graph $\mathcal{G}$ is the cocircuit graph of a uniform oriented matroid of order n and rank r if and only if it is of order $2\binom{n}{r+1}$, it is antipodal and it can be embedded "metrically" in the $(n-r-1)$-dual of the n-cube.*

In the way to reach this, we will apply all the theory developed in the previous chapters. In particular we settle that *oriented matroids can be represented by families of convex sets* and characterize those uniform matroids which can be realized as point separoids.

0. The Cryptomorphism

In this chapter oriented matroids, and separoids, will be handled as families of signed vectors. Thus some notation and definitions have to be introduced.

Let E be any set with n elements and denote by $\mathcal{O}_E = \{-,0,+\}^E$ the set of (signed) vectors with n entries in $\{-,0,+\}$. Given a signed vector $X = (X_e)_{e\in E}$, the set $X^\pm := \{e \in E : X_e \neq 0\}$ is called the *support* of X. The *zero set* of X is the complement of its support, $X^0 := E \setminus X^\pm = \{e \in E : X_e = 0\}$. Its *positive* and *negative* sets are $X^+ := \{e \in E : X_e = +\}$ and $X^- := \{e \in E : X_e = -\}$, respectively. The *opposite* $-X$ is defined by $(-X)_e = -(X_e)$.

In the family of signed vectors $\mathcal{O}_E$ a partial order can be defined as

$$X \leq Y \iff X^+ \subseteq Y^+ \text{ and } X^- \subseteq Y^-.$$

If $X \leq Y$, it will be said that X *conforms to* Y.

☕ This poset is the face lattice of the n-crosspolytope $\mathcal{O}_n$ and dual of the n-cube $\mathcal{Q}_n$ —hence the notation.

With all this at hand, a separoid $S = (E, |)$ can be encoded with signed vectors as follows: $\mathcal{S} \subseteq \mathcal{O}_E$ is a separoid if

$$\begin{array}{lcr} (S1) & X \in \mathcal{S} \implies -X \in \mathcal{S}, & \text{(symmetry)} \\ (S3) & X \in \mathcal{S} \text{ and } X' \leq X \implies X' \in \mathcal{S}. & \text{(it is an ideal)} \end{array}$$

The separations can be reconstructed with the obvious definition:

$$X \in \mathcal{S} \iff X^+ \mid X^-.$$

Recall that it suffices to know maximal separations to reconstruct the whole separoid —they encode the whole information of it.

To define separoid morphisms in this context, the set $\mathcal{O}_E$ can be interpreted as the family of functions of the form $\alpha: E \to \{-,0,+\}$ where, given one such a function $\alpha = X \in \mathcal{O}_E$, its applications are denoted by $\alpha(e) = X_e$. Also, if F is any other m-set —together with its family $\mathcal{O}_F$— and $\varphi: E \to F$ is any function, the *cofunction* $\varphi^*: \mathcal{O}_F \to \mathcal{O}_E$ can be defined in the usual way: if $\beta \in \mathcal{O}_F$ then $\varphi^*\beta \in \mathcal{O}_E$ is defined as

$$(\varphi^*\beta)(e) = \beta(\varphi(e)).$$

Now, given two separoids $\mathcal{S} \subseteq \mathcal{O}_E$ and $\mathcal{F} \subseteq \mathcal{O}_F$, a *separoid morphism*, denoted by $\mathcal{S} \longrightarrow \mathcal{F}$, is a function $\varphi: E \to F$ such that

$$\beta \in \mathcal{F} \implies \varphi^*\beta \in \mathcal{S}.$$

Analogously to the former definition of a separoid, the Radon partitions of a separoid $S = (E, \dagger)$ can be encoded with signed vectors: $\mathcal{S} \subseteq \mathcal{O}_E$ are the Radon partitions of a separoid if

$$\begin{array}{lcr} (R1) & X \in \mathcal{S} \implies -X \in \mathcal{S}, & \text{(symmetry)} \\ (R3) & X \in \mathcal{S} \text{ and } X \leq X' \implies X' \in \mathcal{S}. & \text{(it is a filter)} \end{array}$$

Once again, recall that the minimal Radon partitions encode the whole information of the separoid.

An *oriented matroid* $\mathcal{M} = (E, \mathcal{C})$ of order $n = |E|$ is a set of signed vectors, $\mathcal{C} \subseteq \mathcal{O}_E$, with the following properties:

$$\begin{array}{ll}
(C1) & \mathbf{0} \notin \mathcal{C} \\
(C2) & X \in \mathcal{C} \implies -X \in \mathcal{C} \\
(C3) & X, Y \in \mathcal{C} \text{ and } X^{\pm} \subseteq Y^{\pm} \implies X = \pm Y \\
(C4) & X, Y \in \mathcal{C} \text{ and } X_e = -Y_e \neq 0 \implies \text{ there exists } Z \in \mathcal{C} \\
& \text{such that } Z^+ \subseteq X^+ \cup Y^+, Z^- \subseteq X^- \cup Y^- \text{ and } Z_e = 0
\end{array}$$

The elements of $\mathcal{C}$ are known as the *circuits* of the matroid.

Given an oriented matroid $\mathcal{M}$ the set of its circuits $\mathcal{C}$ can be identified, in a one to one fashion, with the MRP set of a separoid on the same base set E. We have the following obvious cryptomorphism.

0.1. Theorem. *The minimal Radon partitions MRP of a separoid $\mathcal{S}$ are the circuits of an oriented matroid if and only if*

$$\begin{array}{ll}
(M1) & \phi \dagger \phi \notin MRP, \\
(M3) & \mathcal{S} \text{ is a Radon separoid,} \\
(M4) & A \dagger B, A' \dagger B' \in MRP \text{ and } x \in A \cap B' \implies \\
& \exists A'' \dagger B'' \in MRP : A'' \subseteq A' \cup A \setminus x \text{ and } B'' \subseteq B \cup B' \setminus x.
\end{array}$$

Given the circuits of an oriented matroid, its *vectors,* $\mathcal{V} = \mathcal{V}(\mathcal{M})$, can be reconstructed by an operation known as *composition,* defined as

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{otherwise,} \end{cases}$$

via the following: $\mathcal{V} \supseteq \mathcal{C}$ is the minimal superset of $\mathcal{C}$ closed by composition, i.e., $X, Y \in \mathcal{V} \implies X \circ Y \in \mathcal{V}$. Observe that $\mathcal{V} \subset \mathcal{S}$, i.e., since vectors close circuits by composition and separoids by conformal relation, in general there are more Radon partitions in the separoid than vectors in the oriented matroid. Therefore, generalized cotopes (maximal Radon partitions) effectively generalize *cotopes,* maximal vectors.

Recall the example of Section 0.1; the configuration $\mathcal{P}$ contains only two circuits $1 \dagger 34 = (+, 0, -, -)$ and its opposite. From the oriented matroid point of view these two circuits are all the vectors, but for the separoid there are four more Radon partitions $1 \dagger 234, 12 \dagger 34$ and its opposites.

The *topes* $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{O}_E$ of the oriented matroid are the maximal separations and its *covectors* $\mathcal{L}(\mathcal{M}) \subseteq \mathcal{O}_E$ are those separations which composed with topes give topes, i.e.,

$$X \in \mathcal{L} \iff \forall T \in \mathcal{T} : X \circ T \in \mathcal{T}.$$

Observe once again that not every separation is a covector, this is, there are more separations than covectors in an oriented matroid.

It can be proved that the covectors of an oriented matroid $\mathcal{M}$ are the vectors of another one known as the *dual* oriented matroid $\mathcal{M}^*$. More over, $(\mathcal{M}^*)^* = \mathcal{M}$.

There are different axiomatizations of oriented matroids; in terms of topes, vectors, covectors, circuits and cocircuits. They are rigorously treated in Chapter 3 of Björner et al. (1993).

1. Representations of oriented matroids

"Oriented matroids can be thought of as a combinatorial abstraction of point configurations over the reals" –so reads the opening remark of Björner et al.'s basic reference book. However, they are more general than that, and one of the basic problems in the area is to give meaningful characterizations of those oriented matroids that do arise from point configurations, they are called *linear* or *realizable*. They can also be thought of, by polarity, as a combinatorial abstraction of configurations of oriented hyperplanes, or of oriented (codimension 1) subsphere arrangements on a sphere. From this point of view, it is remarkable that all oriented matroids can be realized if the spheres are let to "wiggle" a bit, that is, if they are not asked to be geometrically flat but only that they keep the topological behavior of spheres, they are then called "pseudospheres arrangements".

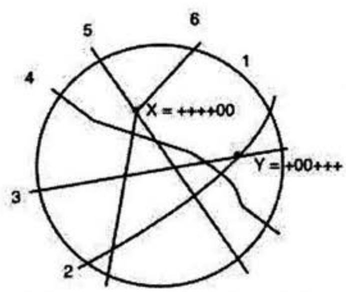


Fig 13. *An oriented matroid.*

Thinking again in terms of points, there should be an analogue of the extra freedom that comes from "wiggling" hyperplanes...

The Representation Theorems (1.2.2 y 1.2.3), besides Theorem 0.1, implies that

1.1. Theorem. *Every oriented matroid of order n can be represented with a family of convex sets in some Euclidian space. More over, if the separoid is acyclic, such a representation can be done in the $(n-1)$-dimensional affine space.*

...this result plays the dual role of the Topological Representation Theorem due to Folkman & Lawrence. This is, when hyperplanes "wiggle" to became pseudohyperplanes, their dual points "fatten" to became convex bodies. But then the natural combinatorial abstraction becomes more general —becomes a separoid.

An oriented matroid is said to be *uniform* if its separoid is in general position. Its *rank* is the dimension of the separoid plus one. Theorem 2.0.1 implies that

1.2. Theorem. *A uniform oriented matroid is linear if and only if its geometric dimension equals its rank minus one.*

Now, since Edelman's complex (cf. Example 4) is a sphere and it is the complement of the Radon complex, due to Alexander's duality we have that

1.3. Theorem. *The Radon complex of an oriented matroid is homotopically equivalent to a sphere.*

This result was our intuition guide to the following section, but first we have to introduce some definitions.

2. The circuit graph of an oriented matroid

Given two signed vectors X and Y, the *separator* of X and Y, is the set

$$S(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}.$$

Two signed vectors X, Y with the same support size ($|X^{\pm}| = |Y^{\pm}| < n$) will be said to be *adjacent* if there exist $i, j \in E$ such that $X_k = Y_k$ for all $k \notin \{i, j\}$, $X_i = 0 \neq Y_i$ and $Y_j = 0 \neq X_j$.

This notion of adjacency defines a graph G_n with vertex set the family of all signed vectors. It leads naturally to the definition of *moving a zero* from one place to another (non-zero place) which is a step of a walk in the graph. Therefore the distance in G_n from one vector to other is the minimum number of moves of zeros needed to reach the destination vector. This motivates the following definition: the *traversen* of two signed vectors X, Y is

$$T(X,Y) = \{e \in E : X_e = 0 \neq Y_e \text{ or } Y_e = 0 \neq X_e\}.$$

2.1. Remark. *X and Y are adjacent in G_n if and only if*

$$S(X,Y) = \phi \quad \textit{and} \quad |T(X,Y)| = 2.$$

This notion will be interpreted in three different settings: as adjacency on the circuit graph of an oriented matroid; as adjacency in the k-dual graph of the n-cube; and as adjacency of k-subcubes of the n-cube.

Every oriented matroid $\mathcal{M} = (E, \mathcal{C})$ has associated a graph $\mathcal{G} = \mathcal{G}(\mathcal{M})$ whose vertices are the circuits of the matroid and two of them $X, Y \in \mathcal{C}$ are adjacent if $X \circ Y = Y \circ X$ and for every $Z \leq X \circ Y$, it follows that $Z \in \{X, Y\}$. This graph is what we call the *circuit graph* of the oriented matroid —in the literature this graph is studied via the dual oriented matroid so it is better known as *cocircuit graph*.

Previous attempts to understand the cocircuit graph of an oriented matroid can be found in: Fukuda & Handa 1993; Babson, Finschi & Fukuda 1999; and Finschi & Fukuda 2000.

It is well known that the cocircuit graph of an oriented matroid is the 1-skeleton of the cell decomposition induced by the pseudospheres that realize the oriented matroid via the Topological Representation Theorem therefore, if the matroid is uniform, two (co)circuits X, Y are adjacent if and only if

$$|S(X,Y)| = 0 \quad \text{and} \quad |T(X,Y)| = 2.$$

From this follows

2.2. Theorem. *Let $\mathcal{M}$ be an oriented matroid, $\mathcal{S}$ its associated separoids and $\mathcal{R}$ its Radon complex. If $\mathcal{M}$ is uniform then $\mathcal{G}(\mathcal{M})$ is the 1-skeleton of $\mathcal{R}^*$, the dual of its Radon complex.*

This is the first step to reach the characterization of the cocircuit graphs of oriented matroids. We will analyze different ways of walking with this notion of adjacency, and develop a series of metrical restrictions on such paths. Recall the definition of $\mathcal{Q}_n^k$, the k-dual of the n-cube (Section 1.4).

2.3. Lemma (metric). *The graph distance in $\mathcal{Q}_n^k$ ($k > 0$) is, for $X \neq Y$*

$$d_{\mathcal{Q}_n^k}(X,Y) = \begin{cases} |S(X,Y)| + 1 & \textit{if } X^{\pm} = Y^{\pm}, \\ |S(X,Y)| + \frac{1}{2}|T(X,Y)| & \textit{otherwise.} \end{cases}$$

Proof. Let $X, Y \in V(\mathcal{Q}_n^k)$. First of all, we exhibit a XY-path with the desired length —this will show that the distance in $\mathcal{Q}_n^k$ is at most that of the statement. There are four cases:

Case 1 ($S(X,Y) = \phi$ and $T(X,Y) = \phi$). This condition is equivalent to $X = Y$.

Case 2 ($S(X,Y) = \phi$ and $T(X,Y) \neq \phi$). Let $T_0(X,Y) = \{i \in E : X_i = 0 \neq Y_i\}$ and analogously $T_0(Y,X) = \{i \in E : Y_i = 0 \neq X_i\}$.

Clearly $T(X,Y) = T_0(X,Y) \cup T_0(Y,X)$ and, since X and Y have the same support size, $|T_0(X,Y)| = |T_0(Y,X)|$. Let us give an arbitrary (but fixed) linear order in both previously defined sets: $T_0(X,Y) = (\tau_1, \ldots, \tau_{|T_0(X,Y)|})$ and $T_0(Y,X) = (\pi_1, \ldots, \pi_{|T_0(Y,X)|})$. Now, let $\{Z^1, Z^2, \ldots, Z^{\frac{1}{2}|T(X,Y)|}\}$ be defined as follows:

$$(Z^m)_i = \begin{cases} Y_i & \text{if } i \in \{\tau_1, \ldots, \tau_m, \pi_1, \ldots, \pi_m\}, \\ X_i & \text{otherwise.} \end{cases}$$

Observe that

$$S(X, Z^1) = S(Z^1, Z^2) = \cdots = S(Z^{\frac{1}{2}|T(X,Y)|-1}, Y) = \phi,$$

$$|T(X, Z^1)| = |T(Z^1, Z^2)| = \cdots = |T(Z^{\frac{1}{2}|T(X,Y)|-1}, Y)| = 2,$$

and $Z^{\frac{1}{2}|T(X,Y)|} = Y$. Therefore, by the remark, $(X, Z^1, Z^2, \ldots, Z^{\frac{1}{2}|T(X,Y)|} = Y)$ is a XY-path and its length is $\frac{1}{2}|T(X,Y)|$.

Case 3 ($S(X,Y) \neq \phi$ and $T(X,Y) \neq \phi$). Let us give an arbitrary (but fixed) linear order in the separator: $S(X,Y) = (\sigma_1, \dots, \sigma_{|S(X,Y)|})$, and let

$$\{Z^1, Z^2, \dots, Z^{|S(X,Y)|}\}$$

be defined as follows:

$$(Z^m)_i = \begin{cases} Y_i & \text{if } i \in \{\tau_1, \sigma_1, \dots, \sigma_{m-1}\}, \\ 0 & \text{if } i = \sigma_m, \\ X_i & \text{otherwise.} \end{cases}$$

Observe that,

$$S(X, Z^1) = S(Z^1, Z^2) = \cdots = S(Z^{|S(X,Y)|-1}, Z^{|S(X,Y)|}) = \phi,$$

$$|T(X, Z^1)| = |T(Z^1, Z^2)| = \cdots = |T(Z^{|S(X,Y)|-1}, Z^{|S(X,Y)|})| = 2.$$

Moreover, $S(Z^{|S(X,Y)|}, Y) = \phi$ and

$$|T(Z^{|S(X,Y)|}, Y)| = |T(X,Y) \setminus \{\tau_1\} \cup \{\sigma_{S(X,Y)}\}| = |T(X,Y)|.$$

Now, construct a $Z^{|S(X,Y)|}Y$-path as in the previous case. This completes the XY-path of the desired length.

Case 4 ($S(X,Y) \neq \phi$ and $T(X,Y) = \phi$). Let $i_0 \in X^0 = Y^0$ be arbitrary (but fixed) and let Z^1 defined as follows

$$(Z^1)_i = \begin{cases} 0 & \text{if } i = \sigma_1, \\ + & \text{if } i = i_0, \\ X_i & \text{otherwise.} \end{cases}$$

Observe that $S(Z^1, Y) = S(X,Y) \setminus \{\sigma_1\}$ and $T(Z^1, Y) = \{\sigma_1, i_0\}$, therefore the previous cases applies.

To end the proof, we have to show that the distance in $\mathcal{Q}_n^k$ is at least that of the statement. We do it by induction.

Let $d\colon V(\mathcal{Q}_n^k) \times V(\mathcal{Q}_n^k) \to \mathbb{N}$ be the following function

$$d(X,Y) = \begin{cases} |S(X,Y)| + 1 & \text{if } X^\pm = Y^\pm, \\ |S(X,Y)| + \frac{1}{2}|T(X,Y)| & \text{otherwise.} \end{cases}$$

By the remark it follows that $d(X,Y) = 1$ if and only if $d_{\mathcal{Q}_n^k}(X,Y) = 1$. Let suppose that for every X, Y and for every $m < m_0$, we have that $d(X,Y) = m$ if and only if $d_{\mathcal{Q}_n^k}(X,Y) = m$. Let $(X, Z^1, \dots, Z^{m_0} = Y)$ be a geodesic XY-path (of minimum length). We want to prove that $d(X,Y) \leq m_0$ so, suppose that $d(X,Y) > m_0$.

Since the path is geodesic, it follows that

$$d_{\mathcal{Q}_n^k}(X,Y) = d_{\mathcal{Q}_n^k}(X, Z^1) + d_{\mathcal{Q}_n^k}(Z^1, Y)$$

which by hypothesis implies that $m_0 = 1 + d(Z^1, Y)$ and so $d(X,Y) > 1 + d(Z^1, Y)$. If we denote as

$$\delta_{XY} = \begin{cases} 1 & \text{if } X^\pm = Y^\pm, \\ 0 & \text{otherwise,} \end{cases}$$

we can write $d(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)| + \delta_{XY}$ including in one equation both cases of its definition. Recall that $X^{\pm} = Y^{\pm}$ if and only if $T(X,Y) = \phi$.

With this notation at hand we have that

$$|S(X,Y)| + \frac{1}{2}|T(X,Y)| + \delta_{XY} > 1 + |S(Z^1,Y)| + \frac{1}{2}|T(Z^1,Y)| + \delta_{Z^1Y}.$$

Since X is adjacent to Z^1, there exist $i,j \in E$ such that for all $\ell \notin \{i,j\}$ we have that $X_\ell = (Z^1)_\ell$, $X_i = 0 \neq (Z^1)_i$ and $X_j \neq 0 = (Z^1)_j$. Then $S(X,Y)$ and $S(Z^1,Y)$, and respectively $T(X,Y)$ and $T(Z^1,Y)$, differs only in the ith and jth coordinates. This motivates the following notation: Given $F \subseteq E$, let $S_F(X,Y) = F \cap S(X,Y)$ and $T_F(X,Y) = F \cap T(X,Y)$. Therefore we have that

$$|S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| + \delta_{XY} > 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)| + \delta_{Z^1Y}.$$

We consider two cases:

Case 1 ($T_{ij}(X,Y) = \phi$). Since $X_i = 0 \neq (Z^1)_i$ and $X_j \neq 0 = (Z^1)_j$, then $Y_i = 0$ and $Y_j \neq 0$ therefore $\{i,j\} \subset T_{ij}(Z^1,Y)$ and $i \notin S_{ij}(X,Y)$. But

$$2 \geq |S_{ij}(X,Y)|+\frac{1}{2}|T_{ij}(X,Y)|+\delta_{XY} > 1+|S_{ij}(Z^1,Y)|+\frac{1}{2}|T_{ij}(Z^1,Y)|+\delta_{Z^1Y} \geq 2$$

an obvious contradiction $\lightning$

Case 2 ($T_{ij}(X,Y) \neq \phi$). Clearly, in this case, $\delta_{XY} = 0$. Then

$$|S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| > 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)|.$$

Since $X_i = 0$ then $i \notin S_{ij}(X,Y)$, and then $j \in S_{ij}(X,Y)$. Therefore $j \in T_{ij}(Z^1,Y)$ which implies that

$$1 + \frac{1}{2} \geq |S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| > 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)| \geq 1 + \frac{1}{2}$$

a new contradiction $\lightning$ This concludes the proof. ⚽

By a graph *embedding* $G \hookrightarrow H$ is meant an injective function $i\colon V(G) \to V(H)$ of its vertices that sends edges to edges. Moreover, in such a case, we will identify the vertices of the domain with those of its image. In fact we will refer to the vertices of the domain with the name of their respective image. In particular, if a graph is embedded in $\mathcal{Q}_n^k$, the vertices of the graph will be denoted by those signed vectors of their images. As usual, an embedding is said to be *isometric* if the graph distance of the domain is preserved by its image.

2.4. Lemma (weak elimination). *Let $G \hookrightarrow \mathcal{Q}_n^k$ be an isometric embedding such that $X^{\pm} = Y^{\pm}$ if and only if $X = \pm Y$. Given $X, Y \in V(G)$ two non-antipodal vertices ($X \neq \pm Y$) and an element in its separator $e \in S(X,Y)$, there exists a vertex $Z \in V(G)$ such that $e \in Z^0$, $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$.*

Proof. Since changing the separator $S(X,Y)$ in a XY-path from one sign to the other requires to move a sign to zero and, after that, to the other sign, then for every element in the separator there exists a vertex Z in the path with a zero in that position. It remains to prove that this vertex works.

Let $(X, Z^1, Z^2, \ldots Z^m = Y)$ be a geodesic path in G. It follows that

$$d_{\mathcal{Q}_n^k}(X,Y) = 1 + d_{\mathcal{Q}_n^k}(Z^1,Y)$$

which by Lemma 3 implies that

$$|S(X,Y)| + \frac{1}{2}|T(X,Y)| + \delta_{XY} = 1 + |S(Z^1,Y)| + \frac{1}{2}|T(Z^1,Y)| + \delta_{Z^1Y}.$$

By an analogous argument to that in proof of Lemma 3, it is easy to see that

$$|S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| + \delta_{XY} = 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)| + \delta_{Z^1Y},$$

where $X_i = 0 \neq (Z^1)_i$ and $X_j \neq 0 = (Z^1)_j$. Since $X \neq \pm Y$ then $\delta_{XY} = 0$ and, because $X_i = 0$ we have that $i \notin S_{ij}(X,Y)$ and therefore

$$1 + \frac{1}{2} \geq |S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| = 1 + |S_{ij}(Z^1,Y)| + \frac{1}{2}|T_{ij}(Z^1,Y)| + \delta_{Z^1Y}$$

which implies that $|S_{ij}(Z^1,Y)| = \delta_{Z^1Y} = 0$.

In particular this implies that two antipodal vectors belongs to a geodesic path if and only if they are the extreme points of it.

Observe the following contradiction; if $(Z^1)_i \neq Y_i$ then $Y_i = 0 = X_i, i \in T_{ij}(Z^1,Y)$ and so

$$1 \geq |S_{ij}(X,Y)| + \frac{1}{2}|T_{ij}(X,Y)| = 1 + \frac{1}{2}|T_{ij}(Z^1,Y)| \geq 1 + \frac{1}{2} \lightning$$

Therefore $(Z^1)_i = Y_i$.

Since for every $\ell \notin \{i,j\}, X_\ell = (Z^1)_\ell$ and $(Z^1)_j = 0$, we have that

$$(Z^1)^+ \subseteq X^+ \cup Y^+ \quad \text{and} \quad (Z^1)^- \subseteq X^- \cup Y^-.$$

Finally, since $Z^1 \neq -Y$ then the previous argument works all over along the path, i.e.

$$(Z^{m+1})^+ \subseteq (Z^m)^+ \cup Y^+ \subseteq X^+ \cup Y^+$$

and

$$(Z^{m+1})^- \subseteq (Z^m)^- \cup Y^- \subseteq X^- \cup Y^-,$$

therefore we have that every Z in the path has the desired property. ⚽

2.5. Theorem 17. *Let $\mathcal{G}$ be a graph. If there exists an antipodal embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ and*

$$X \neq \pm Y \in V(\mathcal{G}) \implies d_{\mathcal{G}}(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)|, \qquad (*)$$

then $\mathcal{G}$ is the circuit graph of an oriented matroid.

Proof. First of all observe that the metric Lemma 3 implies that the extra condition $(*)$ is true if and only if the embedding is isometric and there are not two non-antipodal vertices with the same support.

Let $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ be an antipodal embedding with the property $(*)$. We want to construct an oriented matroid such that its circuit graph is $\mathcal{G}$. Let $\mathcal{S}$ be the following separoid over the base set $E = \{1, \ldots, n\}$: for every $X \in V(\mathcal{G})$, which corresponds to the face $[A, B] < \mathcal{Q}_n$, define a minimal Radon partition as $A \dagger \overline{B}$. We have to prove that the set of all such partitions are the circuits of a uniform oriented matroid. The axiom (C1) is trivial. Since $\mathcal{G}$ is closed by the antipodal automorphism, the relation is symmetric and axiom (C2) follows. (C3) is equivalent to say that $\mathcal{S}$ is a Radon separoid, a direct consequence of $(*)$. In order to prove (C4) we apply the weak elimination Lemma 4 to see that in a geodesic path from a vertex X to a vertex $Y \neq \pm X$ there should exists another vertex Z with the desired property. ⚽

However the condition of isometry is too strong to be necessary as Figure 13 shows. In it, the vertices X and Y are non-antipodal (in fact we are depicting only the projective half of the oriented matroid) and $|S(X,Y)| + \frac{1}{2}|T(X,Y)| = 2$ but $d_{\mathcal{G}}(X,Y) = 3$.

So, let us introduce the weaker (and therefore more general) concept of metric embedding; an embedding $G \hookrightarrow H$ is said to be *metric* if for every pair of vertices $X, Y \in V(G)$ there exists an XY-path P such that

$$Z \in V(P) \implies d_H(Z,Y) \leq d_H(X,Y).$$

Such a path P will be called a *metric path*. Observe that, if an embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ **is not** metric then for every XY-path P there exists a vertex $Z \in V(P)$ such that

$$|S(Z,Y)| > |S(X,Y)| \quad \text{or} \quad |T(Z,Y)| > |T(X,Y)|.$$

The existence of a metric embedding is necessary.

2.6. Theorem. *Let $\mathcal{G}$ be the circuit graph of an uniform oriented matroid then the natural embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ is a metric embedding.*

Proof. We analyze two cases: first, suppose that $X, Y \in V(\mathcal{G})$ have empty separator and they do not have common zeros, i.e., $S(X,Y) = \phi = X^0 \cap Y^0$. Then their composition $\tau = X \circ Y \in \mathcal{T}$ is a tope of the oriented matroid.

In the topological representation this tope is the ball that results of intersecting a number of closed semispaces —hence the name— and we may suppose, with a little abuse of notation, that

$$\tau = \bigcap H_i^+ = + + \cdots +,$$

where $H_i^+ = \{V \in \mathcal{V}(\mathcal{M}) : V_i \in \{0, +\}\}$. Since the boundary of such a ball $\partial\tau$ is connected and it contains both X and Y, there exists a geodesic XY-path

$P \subset \partial\tau$ in it. As we walk into this path from X to Y, neither the separator nor the traversen increase. This is, for every $Z \in V(P)$ we have that

$$S(Z,Y) = \phi \quad \text{and} \quad |T(Z,Y)| \leq |T(X,Y)| = 2k.$$

Therefore P is a metric path. More over, it is an i-metric path (see the definition below).

In the case where the separator or the set of common zeros of X and Y are non-empty, we use an inductive argument in the matroid

$$\mathcal{M}' = \mathcal{M} \setminus S(X,Y)/(X^0 \cap Y^0)$$

to find there a metric path P'. It is easy to see that if $P \subset \mathcal{G}$ is the subdivision of P' that comes from "putting back the separator and the common zeros" of X and Y then P is a metric path. ⚽

However, this metric condition is to weak to be sufficient. In order to prove the sufficiency we should be able to prove a generalization of the weak elimination Lemma 4 but this kind of embedding allow us to construct metric paths of the form

$$\begin{array}{rccl} X = & 0 & + & xxxxx \\ Z = & - & 0 & xxxxx \\ \vdots & & & \\ Y = & + & - & yyyyy \end{array}.$$

In such a path, when giving the step from X to Z, neither the separator nor the traversen increase but we are "walking with the wrong direction". This is, $Z^- \not\subset X^- \cup Y^-$.

Therefore, in order to find a necessary and sufficient condition, we have to strength ones more our concept of metric. We say that an embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ is *i-metric* if for every pair of vertices $X, Y \in V(\mathcal{G})$ there exists an XY-path $P = (X, Z^1, \ldots, Z^m = Y)$ in $\mathcal{G}$ in which every step takes "the right direction". This is, for every pair of adjacent vertices, if $(Z^\ell)_i = (Z^{\ell+1})_j = 0 \neq (Z^\ell)_j$ —if we are moving a zero from i to j— then $(Z^{\ell+1})_i \in \{X_i, Y_i\}$ and therefore, for every $Z \in P$

$$S(Z,Y) \subseteq S(X,Y) \quad \text{and} \quad |T(Z,Y)| \leq |T(X,Y)|.$$

Such a path is called an *i-metric path*. This concept allow us to generalize Lemma 4.

2.7. Lemma (i-metric paths). *Let $\mathcal{G} \hookrightarrow \mathcal{Q}_n^k$ be an embedding. If $P \subset \mathcal{G}$ is an i-metric path from X to Y then for every $e \in S(X,Y)$ there exists $Z \in V(P)$ such that $e \in Z^0$, $Z^- \subseteq X^- \cup Y^-$ and $Z^+ \subseteq X^+ \cup Y^+$.*

Proof. Since changing an element of the separator the separator $S(X,Y)$, while walking in an XY-path, from one sign to the other requires to move the sign to zero and, after that, to the other sign, then for every element in the separator there exists a vertex Z in the path with a zero in that position. Such a vertex satisfies the extra sign conditions because we are in an i-metric path. ✪

2.8. Theorem. *A graph $\mathcal{G}$ is the circuit graph of a d-dimensional uniform oriented matroid of order $n > d+2$ if and only if it is of order $2\binom{n}{d+2}$ and there exist an antipodal embedding $\mathcal{G} \hookrightarrow \mathcal{Q}_n^{n-d-2}$ with the following properties: if $X \neq \pm Y \in \mathcal{G}$,*

$(G1)$ $d_{\mathcal{Q}_n^{n-d-2}}(X,Y) = |S(X,Y)| + \frac{1}{2}|T(X,Y)|$, *(uniqueness)*
$(G2)$ *There exists an i-metric XY-path.* *(weak elimination)*

Proof. The necessity is proved by an analogous argument to that in the proof of Theorem 6. Just observe that the path $P \subset \partial\tau$ is also an i-metric path. The sufficiency is analogous to that of proof of Theorem 5 but using the previous lemma in the i-metric path instead of weak elimination lemma in a geodesic one. Finally, the condition of uniformity is equivalent to the condition on the order. ✪

This theorem leads to a new axiomatization of uniform oriented matroids. On the other hand, the hypothesis of uniformity cannot be dropped without a new ingredient because the circuit graph of a non-uniform oriented matroid may not be embedable in $\mathcal{Q}_n^k$. We believe that there should be a notion of distance in the first baricentric subdivision of the n-cube that leads to a similar theorem but for the general (non-uniform) case.

Finally, putting together Theorem 1.3 and Theorem 8, we have the following

2.9. Corollary. *The Radon complex of a uniform oriented matroid of order n and dimension d, is homeomorphic to the $(n-d-2)$-sphere.*

Jon Folkman began working on oriented matroids by 1967, in an attempt to prove the lower bound conjecture for polytopes by generalizing it. Tragically, he died before publishing his theory. His notes resided with Victor Klee and Ray Fulkerson. Later, when Klee discovered that his doctoral student, Jim Lawrence, was already thinking along similar lines, and had made substantial progress, Klee gave him the notes. Lawrence completed the theory in his doctoral thesis (1975), and later published the results in a joint paper with Folkman (Folkman & Lawrence 1978).

—ANDERS BJÖRNER, MICHEL LAS VERGNAS, BERND STURMFELS, NEIL WHITE & GÜNTER M. ZIEGLER *Oriented Matroids* (1993)

4
Homomorphisms

Given a category, if two objects are identified $S \sim T$ when there exist morphisms

$$S \underset{\psi}{\overset{\varphi}{\rightleftarrows}} T$$

a partially ordered class is obtained when we define $S \leq T \iff S \longrightarrow T$. Its elements are called *color classes*. The category is called *dense* if for every $S < T$ there exists a P such that $S < P < T$. We are going to introduce a dense category on the class of separoids.

Since the constant function is a separoid morphism, the category of morphisms collapses into a single color class. In the light of this, we introduce a kind of morphisms that, in the case of injective functions —the most used until now— coincide with the original concept, but it is a bit more restrictive; we call them *homomorphisms* because they resemble homomorphisms of relational systems. We prove that the homomorphism category of separoids is *universal*, i.e., any partially ordered class —hence the existence of morphisms in any category— can be represented by the existence of separoids homomorphisms.

The reader is encouraged to take a look to Hell & Nešetřil 1990, Nešetřil 2000 and Nešetřil & Tardif 2000 to read more about this "structural" approach to the study of some categories.

0. Basic notions

Let us start with a review of some basic definitions. We do this in order to generalize some concepts to infinite separoids —this is necessary to the universality theorem.

A *separoid* is a relation $\dagger \subseteq 2^S \times 2^S$ defined on the family of subsets of a set S with three simple properties: for every $A, B \subseteq S$

$$\begin{array}{lc} \circ & A \dagger B \implies B \dagger A \\ \circ\circ & A \dagger B \implies A \cap B = \phi \\ \circ\circ\circ & A \dagger B \text{ and } C \subseteq S \setminus A \implies A \dagger B \cup C \end{array}$$

The separoid is identified with the set S. An element $A \dagger B$ is called a *Radon partition* and the union of its parts $A \cup B$ is called the *support* of the partition. The *order* of the separoid is the cardinal $|S|$ and the *size* is half of the Radon partitions $\frac{1}{2}|\dagger|$. The separoid is *acyclic* if $A \dagger B \implies |A||B| > 0$. A *separation* $A \mid B$ is a pair of disjoin sets that are not a Radon partition.

It is very easy to see that a separoid S of order $n \in I\!N$ can be defined as an antipodal filter (cf. Chapter 3)

$$S \subseteq \mathcal{O}_n = (\{-, 0, +\}^n, \prec)$$

in the face lattice of the n-crosspolytope (or by duality, as an antipodal ideal of the n-cube $S \subseteq \mathcal{Q}_n$). Observe that it is enough to know *minimal* Radon partitions to reconstruct all Radon partitions, therefore we can concentrate on the study of them. In particular, when defining an operation, it is enough to define some (minimal Radon) partitions and close the separoid to became a filter. To emphasize this, with a little abuse of notation, we will denote $A \dagger B \in \mathcal{S}$ to mean that "$A \dagger B$ is a minimal Radon partition of the separoid S." In other words, $\mathcal{S}$ will denote a set of generators of the antipodal filter $(S, \prec)$.

Given two finite separoids S and T, a separoid *homomorphism* $\varphi: S \longrightarrow T$ is a function that sends minimal Radon partitions into minimal Radon partitions, i.e., for every $A, B \subseteq S$

$$A \dagger B \in \mathcal{S} \implies \varphi(A) \dagger \varphi(B) \in \mathcal{T}.$$

Clearly these functions defines a category on the class of finite separoids. In fact it is a concrete category. This is also a subcategory of separoids with *morphisms* in the sense of Chapter 1, this is, the preimage of separations are separations: for every $C, D \subseteq T$

$$C \mid D \implies \varphi^{-1}(C) \mid \varphi^{-1}(D).$$

In order to generalize homomorphisms of separoids to infinite sets, we have to give a meaning to minimal Radon partitions. However, in contrast with the finite case, there exist non-trivial infinite separoids with out any minimal Radon partition. To see this, consider the following separoid: $\dagger \subset 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, where $1 \dagger A \iff 1 \notin A$ and $|\mathbb{N} \setminus A| \in \mathbb{N}$, i.e., the singleton of 1 forms a Radon partition with every set A which does not contains it and is the complement of a finite set. Clearly this defines a separoid but, in this separoid there is not such a thing as minimal Radon partitions and therefore the previous definition does not make sense in this context. This motivates the following definition. In it, we think on separoids $\dagger \subset \mathcal{O}^S$ as subsets of the *generalized crosspolytope* $\mathcal{O}^S = \{f: S \to \mathcal{O}\}$ (where $\mathcal{O} = \{-, 0, +\}$) ordered naturally by $f \preceq g \iff f^{-1}(-) \subseteq g^{-1}(-)$ and $f^{-1}(+) \subseteq g^{-1}(+)$, with the obvious properties (cf. Section 3.0)

$$\begin{array}{ll} \circ & f \in \dagger \implies -f \in \dagger \\ \circ\circ\circ & f \in \dagger \text{ and } f \prec g \implies g \in \dagger \end{array}$$

(we denote $A \dagger B \iff \exists f \in \dagger : f^{-1}(-) = A$ and $f^{-1}(+) = B$).

Given two separoids $\dagger \subset \mathcal{O}^S$ and $\ddagger \subset \mathcal{O}^T$, a function $\varphi: S \longrightarrow T$ will be called an *homomorphism* if the following two conditions holds:

- $f \in \mathcal{O}^T \setminus \ddagger \implies \varphi^*(f) \in \mathcal{O}^S \setminus \dagger$,
- $f, g \in \ddagger, f \prec g$ and $\varphi^*(g) \in \dagger \implies \exists h \in \dagger : h \prec \varphi^*(g)$,

where $\varphi^*: \mathcal{O}^T \to \mathcal{O}^S$ denotes the usual cofunction $\varphi^*(g) = g \circ \varphi$. Informally, this can be read as follows: φ is an homomorphism if it is a morphism and, the preimage of non-minimal Radon partitions are not minimal. Observe that, in the previous definition, it may be that $h \neq \varphi^*(f) \notin \dagger$.

Two separoids are *isomorphic* $S \approx T$ if there is a bijective homomorphism between them whose inverse function is also a homomorphism. If $S \subseteq T$ is a subset of a separoid $\dagger \subseteq 2^T \times 2^T$, the *induced* separoid $T[S]$ is the restriction

$\dagger \subseteq 2^S \times 2^S$ and an *embedding* $S \hookrightarrow T$ is an injective homomorphism that is an isomorphism between the domain and the induced separoid of its image. Observe that these notions do not change if we replace morphisms by homomorphisms.

Finite separoids have an intrinsic notion of dimension which is easy to determine.

The *d-dimensional simploid* is the separoid of order $d+1$ and size 0 and it will be denoted by σ^d. The *dimension* of a separoid S is the maximum dimension of its induced simploids

$$\mathrm{d}(S) = \max_{\sigma^d \hookrightarrow S} d.$$

It is said that S is a *general position* separoid if every subset with $\mathrm{d}(S)+1$ elements induces a simploid. S is called *complete* if for every $i, j \in S$ follows that $i \dagger j$. The complete separoid of order n is denoted by K_n. We will adopt the conventions $\sigma^{-1} = K_0 = \phi$ and $\sigma^0 = K_1 = \{\bullet\}$.

From now on, will denote by $S \longrightarrow T$ the fact that there exists an homomorphism, and by $S \not\longrightarrow T$ the other case. Also, as mention in the first paragraph, we write

$$S \sim T \iff S \rightleftarrows T.$$

This last defines an equivalence relation and, in its *color classes*, a partially ordered class called the *homomorphisms order*:

$$S \leq T \iff S \longrightarrow T.$$

It is easy to see now that the homomorphisms order, do not collapses. Indeed we have the following ease-to-check facts (Proposition 2 is an example of a *duality pair*. It will play the main role in Section 4 where we prove that, indeed, it is the only duality pair in the homomorphisms order).

0.1. Proposition. $|S| < n \implies K_n \not\longrightarrow S$

0.2. Proposition. $K_1 \longrightarrow S \iff S \not\longrightarrow K_0$

0.3. Proposition. $S \approx \sigma^d \iff S \longrightarrow K_1$

0.4. Proposition. $S \approx \sigma^d \iff \forall T \neq K_0,\ S \longrightarrow T$

Proposition 1 can be read: there are no homomorphisms in every direction, i.e., the homomorphisms order do not collapses; Proposition 2 says that, in the homomorphisms order, the *principal filter* generated by K_1 is equal to the complement of the *principal ideal* generated by K_0; Proposition 3 settles that the color class of the singleton is constituted by all simploids; and Proposition 4 settles that K_1 is the only *cover* of the *bottom element* K_0. All of them implies, in one way or the other, that the homomorphisms order is not trivial.

1. The homomorphisms lattice

The homomorphisms order is in fact a lattice. The category of separoids homomorphisms has products $\times$ and sums $+$ and they play the role of the meet (infimum) and the joint (supremum), respectively.

$$\begin{tikzcd}
& T+P & \\
T \arrow[ur] & & P \arrow[ul] \\
& T\times P \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

They satisfy the categoric properties of products and coproducts:

- $S \longrightarrow P\times T \iff S \longrightarrow P$ and $S \longrightarrow T$,
- $P+T \longrightarrow S \iff P \longrightarrow S$ and $T \longrightarrow S$,

and, in the finite case, they have the following internal definitions.

Given two separoids P and T, their *product* is a separoid defined in the cartesian product $P\times T$, with projections π and τ respectively, such that for every $A,B \subseteq P\times T$

$$A \dagger B \in \mathcal{P}\times\mathcal{T} \iff \pi(A)\dagger\pi(B)\in\mathcal{P} \text{ and } \tau(A)\dagger\tau(B)\in\mathcal{T}.$$

Given two separoids P and T, their *sum* is a separoid defined in the disjoin union $P\cup T$ such that for every $A,B\subseteq P\cup T$

$$A\dagger B\in\mathcal{P}+\mathcal{T} \iff A\cap P\dagger B\cap P\in\mathcal{P} \text{ xor } A\cap T\dagger B\cap T\in\mathcal{T}.$$

There is also a notion of *exponentiation* but it deserves a more detailed analysis. For this, let us introduce the notion of a *pseudoseparoid*; a relation $\not\dagger \subseteq 2^S\times 2^S$ which satisfies the first and the third conditions of a separoid. That is, we do not ask for the related subsets $A \not\dagger B$ to be disjoint. As an example, consider the relation of being "non-separated":

$$A\not\dagger B \iff A\dagger B \text{ or } A\cap B\neq\phi.$$

The pairs of related subsets with non-empty intersection will be called *loops*. So, a separoid is a pseudoseparoid with out loops.

Now, consider the following construction.

Given two separoids S and T, their *power* (or exponentation) is a pseudoseparoid defined in the family of functions $S^T=\{f\colon T\rightarrow S\}$ such that for every $F,G\subseteq S^T$

$$F\not\dagger G\in\mathcal{S}^{\mathcal{T}} \iff \forall A\dagger B\in\mathcal{T},\ F(A)\dagger G(B)\in\mathcal{S},$$

where, $F(A)=\{f(a); f\in F \text{ and } a\in A\}$ and analogously with $G(B)$.

It is not hard to see that, the power S^T of two separoids is a separoid only if $T \not\rightarrow S$. That is, if the power does not contain any loop then no function $f: T \to S$ is an homomorphism.

The power satisfies the categoric property of exponentation when $T \not\rightarrow S$

- $T \times P \longrightarrow S \iff P \longrightarrow S^T.$

The following results were first isolated in the context of relational systems in Nešetřil & Tardif 2000. Here we will generalize them to any category. These results will appear in Nešetřil & Strausz 2002.

We will denote by $S << T$ the fact that $S < T$ and there is no P such that $S < P < T$, i.e.,

$$S << T \iff S \leq P \leq T \text{ implies } S \sim P \text{ or } P \sim T.$$

The pair (S, T) is called a *gap*. So, a dense order is an order with out gaps.

Also we denote $S \rightarrow = \not\rightarrow T$ if for all P we have that (cf. Proposition 0.2)

$$S \longrightarrow P \iff P \not\rightarrow T.$$

The pair (S, T) is called a *duality pair*. That is, (S, T) is a duality pair if, in the homomorphism order, the filter generated by S is equal to the complement of the ideal generated by T.

We say that the separoid T is *connected* if it cannot be expressed as the sum of other two separoids, i.e.,

$$T \longrightarrow T_0 + T_1 \implies T \longrightarrow T_0 \text{ or } T \longrightarrow T_1.$$

1.1. Lemma (Duality pairs). *$T \rightarrow = \not\rightarrow S$ implies that*

- *T is connected, and*
- *$T \times S << T$.*

Proof. If T is not connected then $T \approx T_0 + T_1$ and $T \not\rightarrow T_i$. Therefore $T_i \longrightarrow S$ and then $T \longrightarrow S$ which is a contradiction. Now, suppose that $T \times S \longrightarrow P \longrightarrow T$. If $T \not\rightarrow P$ then $P \longrightarrow S$ and $P \longrightarrow T \times S$. Therefore $P \sim T$ or $P \sim T \times S$ which concludes the proof.

1.2. Theorem (Characterization of gaps). *If there is a gap $P << Q$, with Q not-connected, there exists another gap $S << T$ where T is connected. Furthermore, $Q \sim T + P$ and $S \approx T \times P$.*

$$\begin{tikzcd}
& Q \sim T + P & \\
T \arrow[ur] & & P \arrow[ul] \\
& S \approx T \times P \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. First, let $Q = T_1 + \cdots + T_k$, where each T_i is connected. Clearly $P \longrightarrow P + T_i \longrightarrow Q$ and then $P \sim P + T_i$ or $P + T_i \sim Q$. Since $Q \not\longrightarrow P$, there exists a $T = T_i$ such that $T \not\longrightarrow P$ and therefore $P + T \not\longrightarrow P$ and $P + T \sim Q$. Finally, let R be such that $P \times T \longrightarrow R \longrightarrow T$. Since $P \longrightarrow P + R \longrightarrow T$, then $P \sim P + R$ or $P + R \sim T$. Therefore, if $T \not\longrightarrow R$, then $R \longrightarrow P$ and $R \longrightarrow T \times P$ which concludes the proof. ⚽

2. A comment on Radon's theorem

If we restrict more our homomorphisms to consider only those φ which do not allow any Radon partition (not only the minimal ones) to collapse, i.e.,

$$A \dagger B \implies \varphi(A) \dagger \varphi(B) \implies \varphi(A) \cap \varphi(B) = \phi,$$

we can characterize Radon's theorem in the following

2.1. Theorem. *$P \subset \mathbb{E}^n$ is a point separoid of order $|P| = \mathrm{d}(P) + 2$ if and only if*

$$P \not\longrightarrow K_1 \quad \textit{and} \quad P \longrightarrow K_2 + \sigma,$$

where σ is a simploid. Furthermore, $\sigma = \phi$ if and only if P is in general position.

Proof. A separoid S is a point separoid of order $\mathrm{d}(S) + 2$ if and only if it is determined by a unique minimal Radon partition $A \dagger B$ (cf. Theorem 2.2.4). Let $C = S \setminus (A \cup B)$ be the complement of the support and give it an arbitrary (but fixed) linear order $C = (c_0, \ldots, c_d)$. Now, let $K_2 = \{a, b\}$, where $a \dagger b$, and $\sigma^d = \{c'_0, \ldots, c'_d\}$. Clearly the function $\varphi \colon S \to K_2 + \sigma^d$, where

$$\varphi(s) = \begin{cases} a & \text{if } s \in A, \\ b & \text{if } s \in B, \\ c'_i & \text{if } s = c_i, \end{cases}$$

is a *strong* homomorphism of separoids. More over, if this is the case, S is in general position if and only if $A \cup B = S$. ⚽

However, in this subcategory there is not any more a meaningful notion of product which made out of the projections, strong homomorphisms. To see this, consider the separoids $P_3 = \{0, 1, 2\}$ where $0 \dagger 12$, and $K_2 = \{a, b\}$ where $a \dagger b$. Let us denote by $P_3 \times K_2 = \{0a, 0b, 1a, 1b, 2a, 2b\}$ the elements of the product and by π and κ the two projections. If $A \dagger B$ implies that $\pi(A) \dagger \pi(B)$ and $\kappa(A) \dagger \kappa(B)$ then the natural candidates to A and B are $A = \{0a\}$, $B = \{1b, 2b\}$. This would imply that $A \dagger B \cup \{0b\}$ but

$$\pi(0a) \cap \pi(0b, 1b, 2b) = \{0\} \cap \{0, 1, 2\} = \{0\} \neq \phi,$$

therefore $P_3 \times K_2 \approx \sigma^5 \sim K_1$. So, every pair of separoids meets on the singleton.

3. On density

The following result is due to Welzl 1982 and Perles & Nešetřil 1990. See also Nešetřil 2001.

3.1. Theorem (Density of graphs). *The class of all color classes of (undirected) graphs is dense, with the unique exception of the pairs* (K_0, K_1) *and* (K_1, K_2).

And the idea of the proof is explained; Let G_1, G_2 graphs such that $G_1 \longrightarrow G_2$ but $G_2 \not\longrightarrow G_1$. For every graph H we have that $G_1 \longrightarrow G_1 + (H \times G_2) \longrightarrow G_2$. If H has odd-girth and chromatic numbers big enough, the opposite arrows does not exist and we are done. Such a graph H exists due to a theorem of Erdös (1959). ⚽

The following settles that the trivial gap $K_0 << K_1$ is the only gap on finite separoids.

3.2. Theorem. *Let S and T be finite separoids. If $S < T \not\to K_1$ and T is connected, then there exists a separoid P such that $S < S + (P \times T) < T$.*

Proof. Clearly, since $S \longrightarrow T$, for every P we have that $S \longrightarrow S+(P\times T) \longrightarrow T$. So, we want a separoids P for which the opposite arrows do not exist. In this case a separoid P can explicitly be constructed. Let n and n' denote the orders of S and T respectively. Let P be the separoid of order $|P| = 2n'n^{n'}$ and Radon partitions as follows: for every $A, B \subseteq P$

$$A \dagger B \iff |A| \geq n' \leq |B| \quad \text{and} \quad A \cap B = \phi.$$

Observe that $\mathrm{d}(P) = 2(n'-1)$ and P is in general position.

Since T is connected and $T \not\longrightarrow S$, every homomorphism $T \longrightarrow S+(P\times T)$ most be an homomorphism $T \longrightarrow P\times T$ which, followed by the projection, would lead an homomorphism $\varphi: T \longrightarrow P$. Since $|T| = n'$ and the supports in P have at least $2n'$ elements, then $P[\varphi(T)] \approx \sigma^d$ (for some $d < n'$) which contradicts the fact that $T \not\to K_1$. Therefore, such an homomorphism φ does not exists.

Now, every homomorphism $S+(P\times T) \longrightarrow S$ restricts to an homomorphism $\varphi: P \times T \longrightarrow S$. For every $p \in P$ there is a function $\varphi_p: T \to S$ defined as $\varphi_p(t) = \varphi(p,t)$ (such functions does not have to be homomorphisms). Since there are at most $|S^T| = n^{n'}$ different functions, there exists a subset $P' \subseteq P$ of order $|P'| = 2n'$ such that for every $p, p' \in P'$ we have that $\varphi_p = \varphi_{p'}$. Let $A, B \in \binom{P'}{n'}$ such that $A \cup B = P'$ and then $A \dagger B \in \mathcal{P}$.

Since $T \not\longrightarrow S$ there there exists a Radon partition $\alpha \dagger \beta \in \mathcal{T}$ such that

$$\varphi_{p'}(\alpha) \mid \varphi_{p'}(\beta) \qquad (\text{or} \ \ \varphi_{p'}(\alpha) \cap \varphi_{p'}(\beta) \neq \phi).$$

But $\varphi_{p'}(\alpha) = \varphi(p' \times \alpha) = \varphi(A \times \alpha)$ and $\varphi_{p'}(\beta) = \varphi(B \times \beta)$, therefore we have also that

$$\varphi_{p'}(\alpha) \dagger \varphi_{p'}(\beta),$$

an obvious contradiction. Hence the homomorphism φ does not exists and we are done. ⚽

3.3. Corollary. *The class of all color classes of separoids is dense, with the unique exception of the pair* (K_0, K_1).

Proof. On the one hand, due to the duality pairs Lemma 1.1,

$$T \to = \not\to S \implies T \times S << T,$$

and therefore, by Proposition 0.2, we have that $K_0 << K_1$.

On the other hand, due to the characterization of gaps Theorem 1.2, there is a gap only if there is a **connected** gap. Therefore, since there are no other connected gap (Theorem 2), there are no other gap at all and we are done. ⚽

4. On universality

The functor $\Psi\colon \mathbf{GRA} \hookrightarrow \mathbf{SEP}$ that maps each (simple) graph $G = (V, E)$ to a separoid $S = (V, \dagger)$, where $i \dagger j \in \mathcal{S} \iff ij \in E$, is an order embedding. A straight forward argument shows that

$$G \longrightarrow H \iff \Psi(G) \longrightarrow \Psi(H).$$

Since **GRA** is a *set-universal partial order* it follows that

4.1. Theorem. *The homomorphisms order* **SEP** *is a set-universal partial order. Explicitly: For any partially ordered* **set** X *there exists an injective mapping* $\iota\colon X \hookrightarrow \mathbf{SEP}$ *such that, for all* $x, y \in X$

$$x \leq y \iff \iota(x) \leq \iota(y).$$

In this direction, we can formulate the following: *is any partially ordered* **class X** *representable by* **SEP***?* The analogous question for graphs cannot be formalized in set theory, i.e., the principle

$\mathcal{P}(\mathbf{X})$: **X** cannot be represented by **GRA**

is an axiom independent from ZFC. However, for separoids the history is different.

4.2. Theorem. *The homomorphisms order of hypergraphs can be embedded into that of separoids. Explicitly: there exists an injective functor* $\Phi\colon \mathbf{HG} \to \mathbf{SEP}$ *which maps each (simple) hypergraph* H *to a separoid* $\Phi(H)$ *and*

$$H \longrightarrow G \iff \Phi(H) \longrightarrow \Phi(G).$$

Proof. Let $\Phi\colon \mathbf{HG} \hookrightarrow \mathbf{SEP}$ be the function which assigns to each (simple) hypergraph (without isolated pints) $H = (V, E)$ the separoid $S = (V \cup E, \dagger)$, whose minimal Radon partitions are $U \dagger e \in \mathcal{S} \iff U = e \in E$. A straight forward argument shows that this function is injective. More over, if $\varphi\colon V \to V'$ is an homomorphism of hypergraphs (the image of edges are edges) that sends the hypergraph $H = (V, E)$ to the hypergraph $G = (V', E')$, it defines a function in the edges (denoted again by $\varphi\colon E \to E'$) and therefore a function in their union

$$\varphi\colon V \cup E \to V' \cup E'.$$

To see that this function is a separoid homomorphism $\Phi(H) \longrightarrow \Phi(G)$, observe that each minimal Radon partition $U \dagger e$ is mapped to the minimal Radon partition $\varphi(U) \dagger \varphi(e)$.

To the turn, let $\varphi: V \cup E \to V' \cup E'$ be a separoid homomorphism $\Phi(H) \longrightarrow \Phi(G)$. First observe that $\varphi(V) \subseteq V'$; for, let $v \in V$ a vertex and $v \in U = e \in E$ an edge that contains it. Since $U \dagger e$ then $\varphi(U) \dagger \varphi(e)$ and therefore $\varphi(v) \in \varphi(U) \subseteq V'$. That is, φ restricts in a function from V to V'. Now, observe that such a restriction is an homomorphism; for, let $U = e \in E$ be an edge then $U \dagger e$ and therefore $\varphi(U) \dagger \varphi(e)$. This implies that $\varphi(e) \in E'$ and therefore φ defines an homomorphism of hypergraps and we had proved that, as desired,

$$H \longrightarrow G \iff \Phi(H) \longrightarrow \Phi(G).$$

Since **HG** is a *class-universal partial order* it follows that

4.3. Corollary. *The homomorphisms order* **SEP** *is a class-universal partial order. Explicitly: For any partially ordered* **class X** *there exists an injective mapping* $\iota: \mathbf{X} \hookrightarrow \mathbf{SEP}$ *such that, for all* $x, y \in \mathbf{X}$

$$x \leq y \iff \iota(x) \leq \iota(y).$$

5. Hyperseparoids

In the remainder of this chapter the focus is put in a famous generalization of Radon's theorem:

5.1. Theorem (Tverberg 1966). *Let* $P \subset \mathbb{E}^d$ *be a set of* $(k-1)(d+1)+1$ *points. Then* P *can be divided into* k *pairwise disjoint sets* $P = P_1 \cup \cdots \cup P_k$ *whose convex hulls have a common point:*

$$\bigcap \langle P_i \rangle \neq \phi.$$

The partition $P = P_1 \cup \cdots \cup P_k$ will be called a *Tverberg partition*.

In Eckhoff's 1993 (sec. 9.3) it can be found more about Tverberg's theorem and its relatives. To the references there, I should add those of Bárány & Onn 1997, Matoušek 1999, Kalai 2000 and Sarkaria 2000.

Clearly, Tverberg's theorem reduces to Radon's when $k = 2$, and for $k = 1$ it is trivial. However, even for $k = 3$, it is a hard —and deep— result. The simplest proof known to me is based in a variant of Sarkaria's (1992) argument and uses the colorful version of Charathéodory's theorem due to Bárány (1982). It seems that, contrasting Radon's theorem which only depends on the affine structure of $\mathbb{R}^d$, Tverberg's theorem is deeply tied to the metric (and topological) properties of the Euclidian d-space.

A simple consequence of Tverberg's theorem is the following

5.2. Corollary. *If S is a separoid of order $(k-1)(\mathrm{gd}(S)+1)+1$, then there exists a morphism $\varsigma\colon S \longrightarrow K_k$ such that, for each minimal Radon partition $i \dagger j$ in K_k, follows that $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$.*

Proof. Let us denote by $K_k = \{1, \dots, k\}$ the elements of the complete separoid of order k and let S be a separoid of $(k-1)(d+1)+1$ convex sets in $\mathbb{E}^d$, where $d = \mathrm{gd}(S)$. For any choice $\varphi\colon P \longrightarrow S$, due to Theorem 1, there exists a partition $P = P_1 \cup \cdots \cup P_k$ such that $\bigcap \langle P_i \rangle \neq \phi$. Clearly the function $\varsigma\colon S \to K_k$ defined as $\varsigma(s) = i \iff \varphi^{-1}(s) \in P_i$ has the desired property. ⚽

Observe that this result is far from imply Theorem 1 (cf. the two realizations of K_3 given in Figure 1 and Figure 2). A naïve first look may suggest that it is weaker to ask for the existence of a k-partition whose convex hulls are isomorphic to K_k than to ask for such a partition whose convex hulls have a common point —think on the vertices of a regular hexagon and perturb them a bit— and in this direction we may be tempted to reduce Tverberg's number, say to $(k-1)(d+1)$. However it is ease to see that the six points in the plane given by the vertices of a regular pentagon and it baricenter, cannot be partitioned in three sets such that the convex hulls of the parts are isomorphic to K_3.

Another direction may be to try to prove (or disprove) the following

5.3. Conjecture. *If S is a separoid of order $(k-1)(\mathrm{d}(S)+1)+1$, then there exists a morphism $\varsigma\colon S \longrightarrow K_k$ such that, for each minimal Radon partition $i \dagger j$ in K_k, follows that $\varsigma^{-1}(i) \dagger \varsigma^{-1}(j)$.*

The rest of this section is a first attempt to understand the combinatorial structure of "Tverberg's partitions". For this, let me first give some "esoteric" names to all acyclic separoids of order 3 (modulo isomorphism) and show their bijective morphisms —the names are intended to remind us their "shape"— (cf. Figure 1):

$$\begin{tikzcd}
& & & \nabla \arrow[dr, tail, two heads] & & & \\
\sigma^2 \arrow[r, tail, two heads] & \Lambda_3 \arrow[r, tail, two heads] & \Pi \arrow[ur, tail, two heads] \arrow[dr, tail, two heads] & & A \arrow[r, tail, two heads] & H \arrow[r, tail, two heads] & K_3 \\
& & & \boxed{K_2+\sigma^0} \arrow[ur, tail, two heads] & & &
\end{tikzcd}$$

Diagram 1. *The acyclic separoids of order 3 and their epimorphisms.*

Observe that only σ^2, Λ_3, $K_2+\sigma^0$ and K_3 are point separoids.

Now, consider a separoid $(S, \dagger)$ of convex sets in $\mathbb{E}^d$. If we give a 3-coloration of its elements $\varsigma\colon S \twoheadrightarrow \{0,1,2\}$ and consider the convex hulls of each coloration class, then we are constructing a morphism onto one of these eight separoids of order 3. These morphisms satisfies the extra property that the preimage of minimal Radon partitions are Radon partitions. Such morphisms will be called *cromomorphisms*.

Let see how this works for the point separoids of order 4 and dimension 2. There are four of them. It is easy to see that we have the following combinations (where the number on each dashed arrow counts the number of different cromomorphisms $|\varsigma\colon S \longrightarrow T|$):

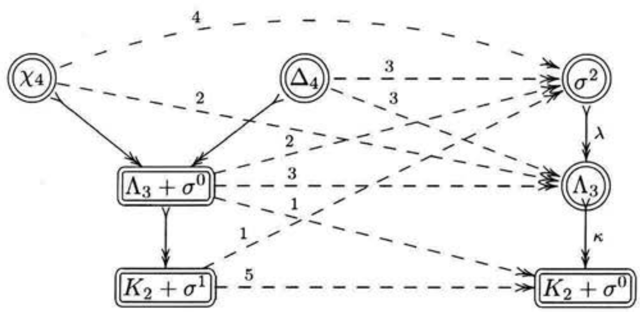


Diagram 2. *The 3-cromomorphisms of 4 point in the plane.*

Here, χ_4 and Δ_4 denote the separoids of order four with unique Radon partitions of the form $12 \dagger 34$ and $1 \dagger 234$, respectively.

Observe that these cromomorphisms does not commute with the epimorphisms λ and κ.

This example suggested the following Tverberg-type theorem for transversals. It basically says that, for a point separoid of order $d+2$, there is always a cromomorphism onto the simploid σ^2 and, there is a cromomorphism onto Λ_3 or onto $K_2 + \sigma^0$.

5.4. Theorem. *Let $d > 1$. If S is the separoid of $d+2$ points in $\mathbb{E}^d$, then*

$$|\varsigma\colon S \longrightarrow \sigma^2| \left(|\varsigma\colon S \longrightarrow \Lambda_3| + |\varsigma\colon S \longrightarrow K_2 + \sigma^0| \right) > 0.$$

Proof. Given $d+2$ points $X \subset \mathbb{E}^d$, due to Radon's theorem, its separoid $S = (X, \dagger)$ is determined by a unique minimal partition $A \dagger B$. To construct a cromomorphism onto σ^2, take an element in each part $a \in A$, $b \in B$ and give any separation of the complement $\alpha \mid \beta$. It is easy to see that the function

$$\varsigma(x) = \begin{cases} 0 & x \in \{a, b\}, \\ 1 & x \in \alpha, \\ 2 & x \in \beta, \end{cases}$$

has the desired properties. Therefore, the first factor is non-zero.

If there is some element in the complement of $A \cup B$ (i.e., the separoid is not in general position), say $C = X \setminus (A \cup B)$, then the function (cf. Theorem 10)

$$\varsigma(x) = \begin{cases} 0 & x \in A, \\ 1 & x \in B, \\ 2 & x \in C, \end{cases}$$

is clearly a cromomorphism onto $K_2 + \sigma^0$ and the second factor is non-zero. If not, A or B has more than one element, say A. Let $A_0 \cup A_1$ be a partition of A. It is easy to see that the function

$$\varsigma(x) = \begin{cases} 0 & x \in A_0, \\ 1 & x \in A_1, \\ 2 & x \in B, \end{cases}$$

is a cromomorphism onto Λ_3 and therefore the second factor is non-zero and we are done. ⚽

Observe how the fact that the second factor is never zero implies, in the case $k = 3$ and $\ell = 1$, Stangeland's (1978) generalization of Tverberg's theorem:

5.5. Corollary. *Let $X \subset \mathbb{E}^d$ be a set of $(k - \ell - 1)(d - \ell + 1) + \ell + 1$ points. If $k = 3$ and $\ell = 1$, there exists a 3-partition of the set $X = X_0 \cup X_1 \cup X_2$ and a line L such that*

$$\langle X_i \rangle \cap L \neq \phi, \qquad \text{for } i = 1, 2, 3.$$

Proof. Any realization of Λ_3 or $K_2 + \sigma^0$ have a line transversal. ⚽

It seems that, while the existence of a Tverberg partition depends on the realization, the existence of a cromomorphism onto K_k do not (see Figure 14).

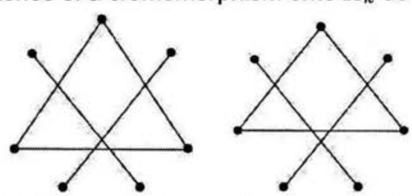


Figure 14. *Two configurations of seven points in the plane.*

These observations motivates the following definition

A *k-separoid* is a relational system $\dagger \subseteq 2^S \times \cdots \times 2^S$ (k times) defined on a family of subsets with the following properties, for $A_i \subseteq S$, $i = 1, \ldots, k$

∘ $A_1 \dagger \cdots \dagger A_k \implies A_{\pi(1)} \dagger \cdots \dagger A_{\pi(k)}$

∘∘ $A_1 \dagger \cdots \dagger A_k \implies A_i \cap A_j = \phi, \quad 1 \leq i < j \leq k$

∘∘∘ $A_1 \dagger \cdots \dagger A_k$ and $B \subseteq S \setminus \bigcup A_i \implies A_1 \dagger \cdots \dagger A_k \cup B$

where π is any permutation of the indices. The elements of such a relational system will be called *Tverberg partitions*. Clearly separoids are 2-separoids. As before, we identify the k-separoid with the given set S. We say that the separoid is *acyclic* if $A_1 \dagger \cdots \dagger A_k \implies \prod |A_i| > 0$.

The following discussion can be made in a more general context —for all $k \in I\!N$— but, in order to keep things simple, we will restrict to the case $k = 3$.

Given three pairwise disjoint subsets of a 3-separoid which are not a Tverberg partition, we say that they are a *3-separation* and denote it by $\alpha \mid \beta \mid \gamma$.

5.6. Theorem. *Every acyclic 3-separoid of order n can be represented with a family of convex polytopes and their Tverberg partitions in the $(n-1)$-dimensional affine space.*

Proof. Let S be a 3-separoid. For each Tverberg partitions $A \dagger B \dagger C$ and each element $i \in A$, we assign a point of $I\!R^n$

$$\rho^i_{A\dagger B\dagger C} = \mathbf{e}_i + \frac{1}{3}\left[\frac{1}{|A|}\sum \mathbf{e}_a + \frac{1}{|B|}\sum \mathbf{e}_b + \frac{1}{|C|}\sum \mathbf{e}_c\right] - \frac{1}{|A|}\sum \mathbf{e}_a,$$

and realize each element $i \in S$ as the convex hull of all such points

$$i \mapsto \langle \rho^i_{A\dagger B\dagger C} : i \in A \text{ and } A \dagger B \dagger C\rangle.$$

These convex polytopes "live" in the $(n-1)$-dimensional affine subspace of $I\!R^n$ spanned by the basis.

The construction is made to guarantee that the Tverberg partitions are preserved, i.e., for each partition $A \dagger B \dagger C$ the vertices of the simplices $\langle \mathbf{e}_a : a \in A\rangle$, $\langle \mathbf{e}_b : b \in B\rangle$ and $\langle \mathbf{e}_c : c \in C\rangle$ moves to realize such a partition intersecting precisely in their baricenter, therefore

$$\langle \rho^a\rangle \cap \langle \rho^b\rangle \cap \langle \rho^c\rangle \neq \phi.$$

On the other hand, to prove that also the 3-separations $\alpha \mid \beta \mid \gamma$ are preserved, we use the following well-known fact: compact convex sets $\mathcal{K}_1, \ldots, \mathcal{K}_n$ in $I\!R^d$ have no point in common if and only if there are open semispaces $\ell^1_+, \ldots, \ell^n_+$ such that $\mathcal{K}_i \subset \ell^i_+$ for every i and $\bigcap \ell^i_+ = \phi$. The case $n = 2$ is the basic separation theorem and the general case follows by induction.

Define the affine extension $\psi = \psi_{\alpha|\beta|\gamma}\colon I\!R^n \to I\!R^2$ of the following equations, for $j = 1, \ldots, n$,

$$\psi(\mathbf{e}_j) = \begin{cases} \mathbf{u} & \text{if } j \in \alpha, \\ \mathbf{v} & \text{if } j \in \beta, \\ \mathbf{w} & \text{if } j \in \gamma, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \frac{1}{2}\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}.$$

It follows from the definition, and with a little abuse of the notation, that

$$\psi(\rho^i) = \psi(\mathbf{e}_i) \;-\frac{2}{3|A|}\begin{pmatrix}|A\cap\alpha|\\|A\cap\beta|\\|A\cap\gamma|\end{pmatrix}\cdot\begin{pmatrix}\mathbf{u}\\\mathbf{v}\\\mathbf{w}\end{pmatrix} + \frac{1}{3|B|}\begin{pmatrix}|B\cap\alpha|\\|B\cap\beta|\\|B\cap\gamma|\end{pmatrix}\cdot\begin{pmatrix}\mathbf{u}\\\mathbf{v}\\\mathbf{w}\end{pmatrix} + \frac{1}{3|C|}\begin{pmatrix}|C\cap\alpha|\\|C\cap\beta|\\|C\cap\gamma|\end{pmatrix}\cdot\begin{pmatrix}\mathbf{u}\\\mathbf{v}\\\mathbf{w}\end{pmatrix}.$$

Let us denote by $\psi_\alpha = \psi(\rho^i)$ when $i \in \alpha$ and analogously with β and γ.

If we have that $\psi_\alpha\cdot\mathbf{u} > 0$ and $\psi_\beta\cdot\mathbf{v} > 0$ and $\psi_\gamma\cdot\mathbf{w} > 0$, we are done (the semispaces $\psi^{-1}(\mathbf{u}_+^\perp)$, $\psi^{-1}(\mathbf{v}_+^\perp)$ and $\psi^{-1}(\mathbf{w}_+^\perp)$ will do). So let us suppose, with out loose of generality, that $\psi_\alpha\cdot\mathbf{u} = 0$. Since

$$\psi_\alpha\cdot\mathbf{u} = 1 - \frac{2|A\cap\alpha| - (|A\cap\beta| + |A\cap\gamma|)}{3|A|} + \frac{|B\cap\alpha| - \frac{1}{2}(|B\cap\beta| + |B\cap\gamma|)}{3|B|} + \frac{|C\cap\alpha| - \frac{1}{2}(|C\cap\beta| + |C\cap\gamma|)}{3|C|} \geq 0,$$

we have that $\psi_\alpha\cdot\mathbf{u} = 0$ if and only if $A\subseteq\alpha$ and $B\subseteq\beta\cup\gamma$ and $C\subseteq\beta\cup\gamma$. In such a case, we have also that

$$\psi_\beta\cdot\mathbf{v} = 1 + \frac{1}{3} + \frac{1}{3}\left[\left(\frac{|B\cap\beta|}{|B|} + \frac{|C\cap\beta|}{|C|}\right) - \frac{1}{2}\left(\frac{|B\cap\gamma|}{|B|} + \frac{|C\cap\gamma|}{|C|}\right)\right] \geq 1$$

and, analogously, $\psi_\gamma\cdot\mathbf{w}\geq 1$. Then we can pick any small number $0<\epsilon<1$, define the semispaces

$$\ell_+^\alpha = \{\mathbf{x}\in I\!R^2 : \mathbf{x}\cdot\mathbf{u} > -\epsilon\},$$
$$\ell_+^\beta = \{\mathbf{x}\in I\!R^2 : \mathbf{x}\cdot\mathbf{v} > 1-\epsilon\},$$
$$\ell_+^\gamma = \{\mathbf{x}\in I\!R^2 : \mathbf{x}\cdot\mathbf{w} > 1-\epsilon\},$$

and their preimage $\psi^{-1}(\ell_+^\alpha)$, $\psi^{-1}(\ell_+^\beta)$ and $\psi^{-1}(\ell_+^\gamma)$ will do the work, concluding the proof. ⚽

Every 3-separoid has associated a 2-separoid in a natural way: each Tverberg partition $A\dagger B\dagger C$, implies the Radon partitions $A\dagger B$, $A\dagger C$ and $B\dagger C$. This separoid is already realized with the construction of Theorem 6. However, we miss some structure; e.g., consider the point separoid of five points in the line in general position, and give the points the linear order $(1,2,3,4,5)$. This configuration has two Tverberg partitions: $14\dagger25\dagger3$ and $15\dagger24\dagger3$. If we apply the previous construction, in the final family of convex sets we will miss some Radon partitions, for example $13 \dagger 2$. To correct this 'anomaly', we can go one step further in our generalization of separoids with the following natural definition.

A *hyperseparoid* is a collection of families of subsets $\mathcal{T} \subseteq 2^{2^S}$ with the following three properties: for all $A_i \subseteq S$, $i = 1, \ldots, k$

$$\circ \quad \{A_1, \ldots, A_k\} \in \mathcal{T} \implies A_i \cap A_j = \phi$$
$$\circ\circ \quad \{A_1, \ldots, A_k\} \in \mathcal{T} \implies \{A_1, \ldots, A_{k-1}\} \in \mathcal{T}$$
$$\circ\circ\circ \quad \{A_1, \ldots, A_k\} \in \mathcal{T} \text{ and } B \subseteq S \setminus \bigcup A_i \implies \{A_1, \ldots, A_k \cup B\} \in \mathcal{T}$$

The elements of $\mathcal{T}$ are the Tverberg partitions. The hyperseparoid is *acyclic* if $\{\phi\} \notin \mathcal{T}$. From the second and third axioms follows that it is enough to know the *principal* partitions; those partitions $\{A_1, \ldots, A_k\}$ where k is maximal and each A_i is minimal. The *morphisms* and *homomorphisms* can be defined analogously as before.

Clearly, we can combine the Geometric Representation Theorem and Theorem 6 (in its general version —for $k \geq 3$) to conclude that

5.7. Corollary. *Every acyclic hyperseparoid can be represented by a family of convex polytopes, and its Tverberg partitions, in some affine space.*

6. Remarks and open problems

Hyperseparoids seems to be "the right concept" to study Tverberg's Theorem from a purely combinatorial point of view, but this will have to be done some where else... Here I will formulate some questions which may guide such a further development.

Let us start with the most challenge (and may be difficult) one. In the spirit of Theorems 2.0.1 and 4.2.1,

Problem 1. *Find necessary and sufficient conditions for a hyperseproid to be a point separoid.*

In the light of Shor's theorem (1991), it may be that problem **1** remains NP-hard, however it may have a simple solution as the following argument suggest. Consider a realization of a full Radon hyperseparoid S with convex sets as "thin" as possible; if each convex set is a point, we are done. If there exist a convex set $\mathcal{K} \in S$ with dimension greater that 0, it will contain at least one segment $\langle \mathbf{a}, \mathbf{b} \rangle \subseteq \mathcal{K}$. The extreme points of such a segment, have to be participating in two different principal partitions, say $\mathbf{a} \dagger A_1 \dagger \cdots \dagger A_k$ and $\mathbf{b} \dagger B_1 \dagger \cdots \dagger B_k$, which are "far" each from the other... they are "separated". So it may be sufficient to ask for a condition of the form *if* $\mathbf{a} \dagger A_1 \dagger \cdots \dagger A_k$ *and* $\mathbf{b} \dagger B_1 \dagger \cdots \dagger B_k$ *are principal then* $A_i \dagger B_j \setminus A_i$ *or* $B_j \dagger A_i \setminus B_j$, in order to guarantee that S is a point separoid (see Figure 15).

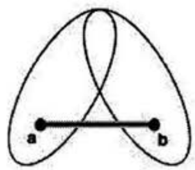


Figure 15. *A "minimal" segment whose extreme points are "separated".*

The next problem has to do with an invariant which may be called *Tverberg dimension*. Given a hyperseparoid S, define $\mathrm{d}_k(S)$ as the minimum natural number d such that every subset $X \subseteq S$ of cardinality $(k-1)(d+1)+1$ contains a k-partition $A_1 \dagger \cdots \dagger A_k$. Clearly, $\mathrm{d}(S) = \mathrm{d}_2(S)$ and $\mathrm{d}_k(S) \leq \mathrm{gd}(S) \leq |S|-1$, but no more can be said, at least in principle (see Figure 16).

Problem 2. *Find necessary and sufficient conditions to guarantee that*

$$\mathrm{d}(S) = \mathrm{d}_2(S) \leq \mathrm{d}_3(S) \leq \ldots \leq \mathrm{gd}(S) \leq |S|-1.$$

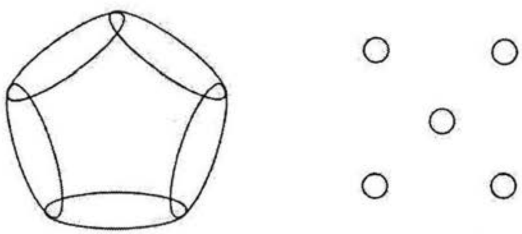


Figure 16. *Two separoids with different values of* $\mathrm{d}_2(S)$ *and* $\mathrm{d}_3(S)$.

Finally, let me present a problem whose character may look more technical. For each separoid S, define the infinite vector $\Upsilon(S) \in I\!N^{\aleph}$ whose coordinates are indexed by finite separoids (modulo isomorphism) and each of these,

$$\Upsilon(S)_T = |\varsigma: S \longrightarrow T|,$$

counts the number of homomorphisms (cromomorphisms, strong morphisms).

> This definition has to be contrasted with that of Lovász 1971 where he proved that, with the arrows in the opposite direction, such a vector characterizes each object of a relational system. See also Nešetřil 1999.

Problem 3. *Is it true that* $S \approx T$ *(or* $S \sim T$*) if and only if* $\Upsilon(S) = \Upsilon(T)$*?*

If we restrict to finite families of separoids, the answer may be negative as the following (and last) diagram shows. In it, χ_5 and Δ_5 denotes the general position point separoids with Radon partitions $12 \dagger 345$ and $1 \dagger 2345$, respectively.

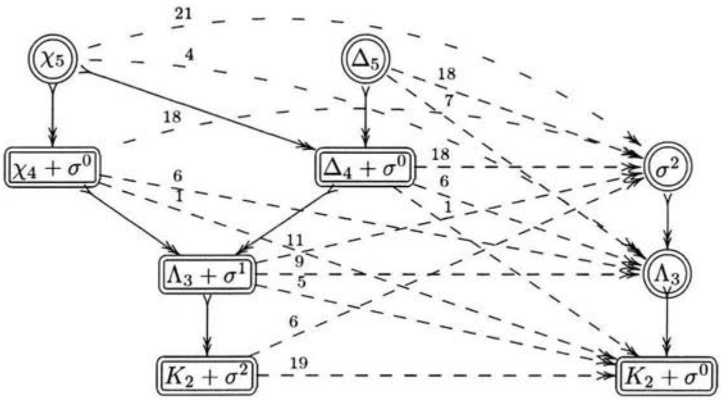


Diagram 3. *The 3-cromomorphisms of 5 point in the space.*

A mathematical statement is just a story you tell about some devices. Some of those stories are clever, some are stupid; some of those stories are true, some other are false. Doing mathematics is telling clever stories which are true.

—FRANCIS BORCEUX *Handbook of Categorical Algebra* (1994)

B
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What we have to learn to do we learn by doing.
—ARISTOTLE, *Ethica Nicomachea II* (c. 325 B.C.)